# RESIDUALLY FAITHFUL MODULES AND THE COHEN-MACAULAY TYPE OF IDEALIZATIONS

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Abstract. The Cohen-Macaulay type of idealizations of maximal Cohen-Macaulay modules over Cohen-Macaulay local rings is explored. There are two extremal cases, one of which is closely related to the theory of Ulrich modules [2, 9, 10, 14], and the other one is closely related to the theory of residually faithful modules and the theory of closed ideals [3].

## 1. Introduction

The purpose of this paper is to explore the behavior of the Cohen-Macaulay type of idealizations of maximal Cohen-Macaulay modules over Cohen-Macaulay local rings, mainly in connection with their residual faithfulness.

Let $R$ be a commutative ring and $M$ an $R$-module. We set $A = R \oplus M$ as an additive group and define the multiplication in $A$ by

$$(a, x){\cdot}(b, y) = (ab, ay + bx)$$

for $(a, x), (b, y) \in A$. Then, $A$ forms a commutative ring, which we denote by $A = R \ltimes M$ and call the idealization of $M$ over $R$ (or, the trivial extension of $R$ by $M$). Notice that $R \ltimes M$ is a Noetherian ring if and only if so is the ring $R$ and the $R$-module $M$ is finitely generated. If $R$ is a local ring with maximal ideal $\mathfrak{m}$, then so is the idealization $A = R \ltimes M$, and the maximal ideal $\mathfrak{n}$ of $A$ is given by $\mathfrak{n} = \mathfrak{m} \times M$.

The notion of the idealization was introduced in the book [20] of Nagata, and we now have diverse applications in several directions (see, e.g., [1, 8, 13]). Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d$. We set

$$\mathrm{r}(R) = \ell_R\left(\mathrm{Ext}_R^d(R/\mathfrak{m}, R)\right)$$

and call it the Cohen-Macaulay type of $R$ (here $\ell_R(*)$ denotes the length). Then, as is well-known, $R$ is a Gorenstein ring if and only if $\mathrm{r}(R) = 1$, so that the invariant $\mathrm{r}(R)$ measures how different the ring $R$ is from being a Gorenstein ring. In the current paper,

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we are interested in the Cohen-Macaulay type $\mathrm{r}(R \ltimes M)$ of $R \ltimes M$, for a maximal Cohen-Macaulay (MCM for short) $R$-module $M$, that is a finitely generated $R$-module $M$ with $\operatorname{depth}_R M = \dim R$. In the researches of this direction, one of the most striking results is, of course, the characterization of canonical modules obtained by I. Reiten [21]. She showed that $R \ltimes M$ is a Gorenstein ring if and only if $R$ is a Cohen-Macaulay local ring and $M$ is the canonical module of $R$, assuming $(R, \mathfrak{m})$ is a Noetherian local ring and $M$ is a non-zero finitely generated $R$-module. Motivated by this result, our study aims at explicit formulae of the Cohen-Macaulay type $\mathrm{r}(R \ltimes M)$ of idealizations for diverse MCM $R$-modules $M$.

Let us state some of our main results, explaining how this paper is organized. Throughout, let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring, and $M$ a MCM $R$-module. Then, we have in general

$$\mathrm{r}_R(M) \le \mathrm{r}(R \ltimes M) \le \mathrm{r}(R) + \mathrm{r}_R(M)$$

(here $\mathrm{r}_R(M) = \ell_R\left(\operatorname{Ext}_R^d(R/\mathfrak{m}, M)\right)$ denotes the Cohen-Macaulay type of $M$), which we shall confirm in Section 2 (Theorem 2.2). As is shown in Example 2.3 and Proposition 2.4, the difference $\mathrm{r}(R \ltimes M) - \mathrm{r}_R(M)$ can be arbitrary among the interval $[0, \mathrm{r}(R)]$. We explore two extremal cases; one is the case of $\mathrm{r}(R \ltimes M) = \mathrm{r}_R(M)$, and the other one is the case of $\mathrm{r}(R \ltimes M) = \mathrm{r}(R) + \mathrm{r}_R(M)$.

The former case is exactly the case where $M$ is a residually faithful $R$-module and closely related to the preceding research [3]. To explain the relationship more precisely, for $R$-modules $M$ and $N$, let

$$t = t_N^M : \operatorname{Hom}_R(M, N) \otimes_R M \to N$$

denote the $R$-linear map defined by $t(f \otimes x) = f(x)$ for all $f \in \operatorname{Hom}_R(M, N)$ and $x \in M$. With this notation, we have the following, which we will prove in Section 3. Here, $\mu_R(*)$ denotes the number of elements in a minimal system of generators.

**Theorem 1.1.** *Let $M$ be a MCM $R$-module and suppose that $R$ possesses the canonical module $\mathrm{K}_R$. Then*

$$\mathrm{r}(R \ltimes M) = \mathrm{r}_R(M) + \mu_R(\operatorname{Coker} t_{\mathrm{K}_R}^M).$$

As a consequence, we get the following, where the equivalence between Conditions (2) and (3) is due to [3, Proposition 5.2]. Remember that a MCM $R$-module $M$ is said to be *residually faithful*, if $M/\mathfrak{q}M$ is a faithful $R/\mathfrak{q}$-module for some (eventually, for every) parameter ideal $\mathfrak{q}$ of $R$ (cf. [3, Definition 5.1]).

**Corollary 1.2** (cf. [3, Proposition 5.2])**.** *Let $M$ be a MCM $R$-module and suppose that $R$ possesses the canonical module $\mathrm{K}_R$. Then the following conditions are equivalent.*

(1) $\mathrm{r}(R \ltimes M) = \mathrm{r}_R(M)$.

(2) *The homomorphism $t_{\mathrm{K}_R}^M : \operatorname{Hom}_R(M, \mathrm{K}_R) \otimes_R M \to \mathrm{K}_R$ is surjective.*

(3) *$M$ is a residually faithful $R$-module.*

In Section 3, we will also show the following, where $\Omega\mathrm{CM}(R)$ denotes the class of the (not necessarily minimal) first syzygy modules of MCM $R$-modules.

**Theorem 1.3.** *Let $M \in \Omega\mathrm{CM}(R)$. Then*

$$\mathrm{r}(R \ltimes M) = \begin{cases} \mathrm{r}_R(M) & \text{if } R \text{ is a direct summand of } M, \\ \mathrm{r}(R) + \mathrm{r}_R(M) & \text{otherwise.} \end{cases}$$

In Section 4, we are concentrated in the latter case where $\mathrm{r}(R \ltimes M) = \mathrm{r}(R) + \mathrm{r}_R(M)$, which is closely related to the theory of Ulrich modules ([2, 9, 10, 14]). In fact, the equality $\mathrm{r}(R \ltimes M) = \mathrm{r}(R) + \mathrm{r}_R(M)$ is equivalent to saying that $(\mathfrak{q} :_R \mathfrak{m})M = \mathfrak{q}M$ for some (and hence every) parameter ideal $\mathfrak{q}$ of $R$, so that all the Ulrich modules and all the syzygy modules $\Omega_R^i(R/\mathfrak{m})$ $(i \geq d)$ satisfy the above equality $\mathrm{r}(R \ltimes M) = \mathrm{r}(R) + \mathrm{r}_R(M)$ (Theorems 4.1, 4.3), provided $R$ is not a regular local ring (here $\Omega_R^i(R/\mathfrak{m})$ is considered in a minimal free resolution of $R/\mathfrak{m}$).

In Section 5, we give the bound of $\sup \mathrm{r}(R \ltimes M)$, where $M$ runs through certain MCM $R$-modules. In particular, when $d = 1$, we get the following (Corollary 5.2).

**Theorem 1.4.** *Suppose that $(R, \mathfrak{m})$ is a Cohen-Macaulay local ring of dimension one and multiplicity $e$. Let $\mathcal{F}$ be the set of $\mathfrak{m}$-primary ideals of $R$. Then*

$$\sup_{I \in \mathcal{F}} \mathrm{r}(R \ltimes I) = \begin{cases} 1 & \text{if } R \text{ is a DVR}, \\ \mathrm{r}(R) + e & \text{otherwise.} \end{cases}$$

In Section 6, we focus our attention on the case where $\dim R = 1$. The main objectives are the trace ideals and closed ideals. The notion of closed ideals was introduced by [3], where one finds a beautiful theory of closed ideals. As for the theory of trace ideals, we refer to [6, 18] for the recent progress. In Section 6, we compute the Cohen-Macaulay type $\mathrm{r}(R \ltimes I)$ for fractional trace or closed ideals $I$ over a one-dimensional Cohen-Macaulay local ring $R$, in terms of the numbers of generators of $I$ together with the Cohen-Macaulay type $\mathrm{r}_R(I)$ of $I$ as an $R$-module.

In what follows, unless otherwise specified, $(R, \mathfrak{m})$ denotes a Cohen-Macaulay local ring with $d = \dim R \geq 0$. When $R$ possesses the canonical module $\mathrm{K}_R$, for each $R$-module $M$ we denote $\mathrm{Hom}_R(M, \mathrm{K}_R)$ by $M^\vee$. Let $\mathrm{Q}(R)$ be the total ring of fractions of $R$. For $R$-submodules $X$ and $Y$ of $\mathrm{Q}(R)$, let

$$X : Y = \{a \in \mathrm{Q}(R) \mid aY \subseteq X\}.$$

If we consider ideals $I, J$ of $R$, we set $I :_R J = \{a \in R \mid aJ \subseteq I\}$; hence

$$I :_R J = (I : J) \cap R.$$

For each finitely generated $R$-module $M$, let $\mu_R(M)$ (resp. $\ell_R(M)$) denote the number of elements in a minimal system of generators (resp. the length) of $M$. For an $\mathfrak{m}$-primary ideal $\mathfrak{a}$ of $R$, we denote by

$$\mathrm{e}^0_{\mathfrak{a}}(M) = \lim_{n\to\infty} d!{\cdot}\frac{\ell_R(M/\mathfrak{a}^n M)}{n^d}$$

the multiplicity of $M$ with respect to $\mathfrak{a}$.

## 2. The Cohen-Macaulay type of general idealizations

In this section, we estimate the Cohen-Macaulay type of idealizations for general maximal Cohen-Macaulay modules over Cohen-Macaulay local rings. We begin with the following observation, which is the starting point of this research.

**Proposition 2.1.** *Let $(R, \mathfrak{m})$ be a (not necessarily Noetherian) local ring and let $M$ be an $R$-module. We set $A = R \ltimes M$ and denote by $\mathfrak{n} = \mathfrak{m} \times M$ the maximal ideal of $A$. Then*

$$(0) :_A \mathfrak{n} = ([(0) :_R \mathfrak{m}] \cap \mathrm{Ann}_R M) \times [(0) :_M \mathfrak{m}]\,.$$

*Therefore, when $R$ is an Artinian local ring, $(0) :_A \mathfrak{n} = (0) \times [(0) :_M \mathfrak{m}]$ if and only if $\mathrm{Ann}_R M = (0)$.*

*Proof.* Let $(a, x) \in A$. Then $(a, x){\cdot}(b, y) = 0$ for all $(b, y) \in \mathfrak{n} = \mathfrak{m} \times M$ if and only if $ab = 0, ay = 0$, and $bx = 0$ for all $b \in \mathfrak{m}, y \in M$. Hence, the first equality follows. Suppose that $R$ is an Artinian local ring. Then, since $I = \mathrm{Ann}_R M$ is an ideal of $R$, $I \neq (0)$ if and only if $[(0) :_R \mathfrak{m}] \cap I \neq (0)$, whence the second assertion follows. ☐

We now assume, throughout this section, that $(R, \mathfrak{m})$ is a Cohen-Macaulay local ring with $d = \dim R \geq 0$. We say that a finitely generated $R$-module $M$ is a *maximal Cohen-Macaulay* (MCM for short) $R$-module, if $\mathrm{depth}_R M = d$.

**Theorem 2.2.** *Let $M$ be a MCM $R$-module and $A = R \ltimes M$. Then*

$$\mathrm{r}_R(M) \leq \mathrm{r}(A) \leq \mathrm{r}(R) + \mathrm{r}_R(M).$$

*Let $\mathfrak{q}$ be a parameter ideal of $R$ and set $\overline{R} = R/\mathfrak{q}$, $\overline{M} = M/\mathfrak{q}M$. We then have the following.*

1. *$\mathrm{r}(A) = \mathrm{r}_R(M)$ if and only if $\overline{M}$ is a faithful $\overline{R}$-module.*
2. *$\mathrm{r}(A) = \mathrm{r}(R) + \mathrm{r}_R(M)$ if and only if $(\mathfrak{q} :_R \mathfrak{m})M = \mathfrak{q}M$.*

*Proof.* We set $\overline{A} = A/\mathfrak{q}A$. Therefore, $\overline{A} = \overline{R} \ltimes \overline{M}$. Since $A$ is a Cohen-Macaulay local ring and $\mathfrak{q}A$ is a parameter ideal of $A$, we have $\mathrm{r}(A) = \mathrm{r}(\overline{A})$, and by Proposition 2.1 it follows that

$$\begin{aligned}
\mathrm{r}(A) = \ell_{\overline{A}}((0) :_{\overline{A}} \mathfrak{n}) &= \ell_{\overline{A}}\big(([(0) :_{\overline{R}} \mathfrak{m}] \cap \mathrm{Ann}_{\overline{R}}\overline{M}\big) \times [(0) :_{\overline{M}} \mathfrak{m}])\\
&= \ell_{\overline{R}}([(0) :_{\overline{R}} \mathfrak{m}] \cap \mathrm{Ann}_{\overline{R}}\overline{M}) + \ell_{\overline{R}}((0) :_{\overline{M}} \mathfrak{m})\\
&= \ell_{\overline{R}}([(0) :_{\overline{R}} \mathfrak{m}] \cap \mathrm{Ann}_{\overline{R}}\overline{M}) + \mathrm{r}_R(M)\\
&\leqslant \ell_{\overline{R}}((0) :_{\overline{R}} \mathfrak{m}) + \mathrm{r}_R(M)\\
&= \mathrm{r}(R) + \mathrm{r}_R(M).
\end{aligned}$$

Hence, $\mathrm{r}_R(M) \le \mathrm{r}(A) \le \mathrm{r}(R) + \mathrm{r}_R(M)$, so that by Proposition 2.1, $\mathrm{r}(A) = \mathrm{r}_R(M)$ if and only if $\overline{M}$ is a faithful $\overline{R}$-module. We have $\mathrm{r}(A) = \mathrm{r}(R) + \mathrm{r}_R(M)$ if and only if $(0) :_{\overline{R}} \overline{\mathfrak{m}} \subseteq \mathrm{Ann}_{\overline{R}}\overline{M}$, and the latter condition is equivalent to saying that $\mathfrak{q} :_R \mathfrak{m} \subseteq \mathfrak{q}M :_R M$, that is $(\mathfrak{q} :_R \mathfrak{m})M = \mathfrak{q}M$. □

The following shows the difference $\mathrm{r}(A) - \mathrm{r}_R(M)$ in Theorem 2.2 can be arbitrary among the interval $[0, \mathrm{r}(R)]$. Notice that $\mathrm{r}(R \ltimes R) = \mathrm{r}(R)$.

**Example 2.3.** Let $\ell \ge 2$ be an integer and $S = k[[X_1, X_2, \dots, X_\ell]]$ the formal power series ring over a field $k$. Let $\mathfrak{a} = \mathbb{I}_2(\mathbb{M})$ denote the ideal of $S$ generated by the maximal minors of the matrix $\mathbb{M} = \begin{pmatrix} X_1 & X_2 & \dots & X_{\ell-1} & X_\ell \\ X_2 & X_3 & \dots & X_\ell & X_1^q \end{pmatrix}$ with $q \ge 2$. We set $R = S/\mathfrak{a}$. Then $R$ is a Cohen-Macaulay local ring of dimension one. For each integer $2 \le p \le \ell$, we consider the ideal $I_p = (x_1) + (x_p, x_{p+1}, \dots, x_\ell)$ of $R$, where $x_i$ denotes the image of $X_i$ in $R$. Then $\mathrm{r}(R \ltimes I_p) = (\ell - p + 1) + \mathrm{r}_R(I_p)$, and

$$\mathrm{r}_R(I_p) = \begin{cases} \ell & \text{if } p = 2 \\ \ell - 1 & \text{if } p \ge 3 \end{cases}$$

for each $2 \le p \le \ell$.

*Proof.* Let $\mathfrak{m}$ denote the maximal ideal of $R$. We set $I = I_p$ and $x = x_1$. It is direct to check that $I^2 = xI$, where we use the fact that $q \ge 2$. In particular, $\mathfrak{m}^2 = x\mathfrak{m}$. We consider the exact sequence

$$(E) \quad 0 \to R/I \xrightarrow{\iota} I/xI \to I/(x) \to 0,$$

where $\iota(1) = x \bmod xI$, and get $\mathrm{Ann}_R I/xI = I$, since $I^2 = xI$. Therefore, $\mathrm{Ann}_{R/(x)} I/xI = I/(x)$. Because $I/(x) \subseteq \mathfrak{m}/(x) = (0) :_{R/(x)} \mathfrak{m}$, we get

$$\ell_R([(0) :_{R/(x)} \mathfrak{m}] \cap \mathrm{Ann}_{R/(x)} I/xI) = \ell_R(I/(x)) = \ell - p + 1,$$

whence

$$\mathrm{r}(R \ltimes I) = (\ell - p + 1) + \mathrm{r}_R(I)$$

by Theorem 2.2. Because $(x_2, x_3, \dots, x_{p-1}) \cdot (x_p, x_{p+1}, \dots, x_\ell) \subseteq xI$, the above sequence $(E)$ remains exact on the socles, so that

$$\mathrm{r}_R(I) = \mathrm{r}(R/I) + \mathrm{r}_R(I/(x)).$$

Therefore, $\mathrm{r}_R(I) = \ell$ if $p = 2$, and $\mathrm{r}_R(I) = (p-2) + (\ell - p + 1) = \ell - 1$ if $p \ge 3$. □

Assume that $R$ is not a regular local ring and let $0 \le n \le \mathrm{r}(R)$ be an integer. Then, we suspect if there exists a MCM $R$-module $M$ such that $\mathrm{r}(R \ltimes M) = n + \mathrm{r}_R(M)$. When $R$ is the semigroup ring of a numerical semigroup, we however have an affirmative answer.

**Proposition 2.4.** *Let $a_1, a_2, \ldots, a_\ell$ be positive integers such that* $\mathrm{GCD}(a_1, a_2, \cdots, a_\ell) = 1$. *Let $H = \langle a_1, a_2, \ldots, a_\ell \rangle$ be the numerical semigroup generated by $\{a_i\}_{1 \le i \le \ell}$. Let $k[[t]]$ denote the formal power series ring over a field $k$ and consider, inside of $k[[t]]$, the semigroup ring*

$$R = k[[t^{a_1}, t^{a_2}, \ldots, t^{a_\ell}]]$$

*of $H$ over $k$. We set $e = \min\{a_i \mid 1 \le i \le \ell\}$ and assume that $e > 1$, that is $R$ is not a* DVR. *Let $r = \mathrm{r}(R)$. Then, for each integer $0 \le n \le r$, $R$ contains a non-zero ideal $I$ such that $\mathrm{r}(R \ltimes I) = n + \mathrm{r}_R(I)$.*

*Proof.* Let $\mathfrak{m}$ be the maximal ideal of $R$ and set $B = \mathfrak{m} : \mathfrak{m}$. Then $B = R : \mathfrak{m}$ since $R$ is not a DVR, and

$$(t^e) :_R \mathfrak{m} = (t^e) : \mathfrak{m} = t^e(R : \mathfrak{m}) = t^e B.$$

We denote by $\mathrm{PF}(H) = \{\alpha_1 < \alpha_2 < \cdots < \alpha_r\}$ the pseudo-Frobenius numbers of $H$. Hence, $B = R + \sum_{1 \le i \le r} Rt^{\alpha_i}$, so that $(t^e) :_R \mathfrak{m} = (t^e) + (t^{\alpha_i + e} \mid 1 \le i \le r)$. Let $1 \le p \le r$ be an integer and set $I = (t^e) + (t^{\alpha_j + e} \mid p \le j \le r) \subseteq (t^e) :_R \mathfrak{m}$. Let $\alpha_0 = 0$. We then have the following.

**Claim 1.** *Let $0 \le i \le r$ and $p \le j \le r$ be integers. Then $t^{\alpha_i + e} t^{\alpha_j + e} \in t^e I$. Consequently, $I^2 = t^e I$.*

*Proof.* Assume that $t^{\alpha_i + e} t^{\alpha_j + e} \notin t^e I$. Then $t^{\alpha_i + \alpha_j + e} \notin I$. On the other hand, since $t^{\alpha_i} t^{\alpha_j} \in B = \mathfrak{m} : \mathfrak{m}$, we get $\alpha_i + \alpha_j = \alpha_k + h$ for some $0 \le k \le r$ and $h \in H$. If $h > 0$, then $\alpha_i + \alpha_j \in H$, so that $t^{\alpha_i + \alpha_j + e} \in I$, which is impossible. Therefore, $h = 0$, and $\alpha_k - \alpha_j = \alpha_i \ge 0$, so that $k \ge j \ge p$. Hence, $t^{\alpha_i + \alpha_j + e} = t^{\alpha_k + e} \in I$. This is a contradiction. $\square$

We now consider the exact sequence $0 \to R/I \to I/t^e I \to I/(t^e) \to 0$, and get that $\mathrm{Ann}_R\, I/t^e I = I$. Hence

$$\mathrm{Ann}_{R/(t^e)}\, I/t^e I = I/(t^e) \subseteq (0) :_{R/(t^e)} \mathfrak{m}.$$

Therefore, $\mathrm{r}(R \ltimes I) = \ell_R(I/(t^e)) + \mathrm{r}_R(I) = n + \mathrm{r}_R(I)$, where $n = r - p + 1$. For $n = 0$, just take $I = R$. $\square$

**Remark 2.5.** With the same notation as in the proof of Proposition 2.4, let $\mathrm{K}_R$ denote the canonical module of $R$ and consider the ideal $I = (t^e) + (t^{\alpha_j + e} \mid p \le j \le r)$. Then, because $I^2 = t^e I$ and $\mathfrak{m} I = \mathfrak{m} t^e$, by [8, Proposition 6.1] $R \ltimes I^\vee$ is an almost Gorenstein local ring, where $I^\vee = \mathrm{Hom}_R(I, \mathrm{K}_R)$. Since $\mathrm{Ann}_R\, I^\vee / t^e I^\vee = \mathrm{Ann}_R\, I/t^e I$, we get

$$\mathrm{r}(R \ltimes I^\vee) = (r - p + 1) + \mathrm{r}_R(I^\vee) = (r - p + 1) + \mu_R(I),$$

so that $\mathrm{r}(R \ltimes I^\vee) = 2r - 2p + 3$.

**Corollary 2.6.** *With the same notation as in Proposition* 2.4*, assume that* $a_1 < a_2 < \cdots < a_\ell$*, and that* $H$ *is minimally generated by* $\ell$ *elements with* $\ell = a_1 \geq 2$*, that is* $R$ *has maximal embedding dimension* $\ell \geq 2$*. Let* $2 \leq p \leq \ell$ *be an integer and set* $I_p = (t^{a_1}) + (t^{a_p}, t^{a_{p+1}}, \ldots, t^{a_\ell})$*. Then* $\mathrm{r}(R \ltimes I_p) = (\ell - p + 1) + \mathrm{r}_R(I_p)$*, and*

$$\mathrm{r}_R(I_p) = \begin{cases} \ell & \text{if } p = 2 \\ \ell - 1 & \text{if } p \geq 3 \end{cases}$$

*for each* $2 \leq p \leq \ell$*.*

*Proof.* Let $e = a_1$ and $r = \mathrm{r}(R)$. Hence $\mathrm{r}(R) = e-1$. Let $1 \leq i, j \leq \ell$ be integers. Then $i = j$ if $a_i \equiv a_j \bmod e$, because $H$ is minimally generated by $\{a_i\}_{1 \leq i \leq \ell}$. Therefore, $\mathrm{PF}(H) = \{a_2 - e < a_3 - e < \cdots < a_e - e\}$, so that $\mathrm{r}(R \ltimes I_p) = (e-p+1) + \mathrm{r}_R(I_p)$ by Proposition 2.4. To get $\mathrm{r}_R(I_p)$, by the proof of Example 2.3 it suffices to show that $\mathfrak{m} \cdot (t^{a_p}, t^{a_{p+1}}, \ldots, t^{a_\ell}) \subseteq t^{a_1} I$, which follows from Claim 1 in the proof of Proposition 2.4. $\square$

In the following two sections, Sections 3 and 4, we explore the extremal cases where $\mathrm{r}(R \ltimes M) = \mathrm{r}_R(M)$ and $\mathrm{r}(R \ltimes M) = \mathrm{r}(R) +_\mathrm{R}$ (M), respectively.

## 3. Residually faithful modules and the case where $\mathrm{r}(R \ltimes M) = \mathrm{r}_R(M)$

Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with $d = \dim R \geq 0$. In this section, we consider the case of Theorem 2.2 (1), that is $\mathrm{r}(R \ltimes M) = \mathrm{r}_R(M)$. Let us begin with the following.

**Definition 3.1.** Let $M$ be a MCM $R$-module. We say that $M$ is *residually faithful*, if $M/\mathfrak{q}M$ is a faithful $R/\mathfrak{q}$-module for some parameter ideal $\mathfrak{q}$ of $R$.

With this definition, Theorem 2.2 (1) assures the following.

**Proposition 3.2.** *Let* $M$ *be a MCM* $R$*-module. Then the following conditions are equivalent.*

(1) $\mathrm{r}(R \ltimes M) = \mathrm{r}_R(M)$.
(2) $M$ *is a residually faithful* $R$*-module.*
(3) $M/\mathfrak{q}M$ *is a faithful* $R/\mathfrak{q}$*-module for every parameter ideal* $\mathfrak{q}$ *of* $R$*.*

For $R$-modules $M$ and $N$, let

$$t = t_N^M : \mathrm{Hom}_R(M, N) \otimes_R M \to N$$

denote the $R$-linear map defined by $t(f \otimes m) = f(m)$ for all $f \in \mathrm{Hom}_R(M, N)$ and $m \in M$. With this notation, we have the following.

**Theorem 3.3.** *Let* $M$ *be a MCM* $R$*-module and suppose that* $R$ *possesses the canonical module* $\mathrm{K}_R$*. Let* $C = \mathrm{Coker}\ t_{\mathrm{K}_R}^M$*. Then*

$$\mathrm{r}(R \ltimes M) = \mathrm{r}_R(M) + \mu_R(C).$$

*Proof.* We set $K = \mathrm{K}_R$ and $A = R \ltimes M$. Let us make the $R$-module $M^\vee \times K$ into an $A$-module on which the $A$-action is defined by

$$(a, m) \circ (f, x) = (af, f(m) + ax)$$

for each $(a, m) \in A$ and $(f, x) \in M^\vee \times K$. Then $M^\vee \times K \cong \mathrm{Hom}_R(A, K)$ as an $A$-module. Therefore, $\mathrm{K}_A = M^\vee \times K$, the canonical module of $A$ ([5, Section 6, Augmented rings] or [7, Section 2]). Let $\mathfrak{n} = \mathfrak{m} \times M$ denote the maximal ideal of $A$ and $L = \mathrm{Im}\, t^M_{\mathrm{K}_R}$. Then, since $\mathfrak{n}(M^\vee \times \mathrm{K}_R) = \mathfrak{m}M^\vee \times (L + \mathfrak{m}\mathrm{K}_R)$, we get

$$\begin{aligned}
\mathrm{r}(A) &= \mu_A(\mathrm{K}_A) \\
&= \ell_A([M^\vee \times K]/[\mathfrak{m}M^\vee \times (L + \mathfrak{m}K)] \\
&= \ell_R([M^\vee \oplus K]/[\mathfrak{m}M^\vee \oplus (L + \mathfrak{m}K)] \\
&= \ell_R(M^\vee/\mathfrak{m}M^\vee) + \ell_R(K/(L + \mathfrak{m}K)) \\
&= \mu_R(M^\vee) + \mu_R(C) \\
&= \mathrm{r}_R(M) + \mu_R(C).
\end{aligned}$$

□

Theorem 3.3 covers [3, Proposition 5.2]. In fact, we have the following, where the equivalence of Conditions (1) and (3) follows from Proposition 3.2, and the equivalence of Conditions (1) and (2) follows from Theorem 3.3.

**Corollary 3.4** (cf. [3, Proposition 5.2])**.** *Let $M$ be a MCM $R$-module and suppose that $R$ possesses the canonical module $\mathrm{K}_R$. Then the following conditions are equivalent.*

(1) $\mathrm{r}(R \ltimes M) = \mathrm{r}_R(M)$.
(2) *The homomorphism $t^M_{\mathrm{K}_R} : \mathrm{Hom}_R(M, \mathrm{K}_R) \otimes_R M \to \mathrm{K}_R$ is surjective.*
(3) *$M$ is a residually faithful $R$-module.*

We note one example of residually faithful modules $M$ such that $M \ncong R, \mathrm{K}_R$.

**Example 3.5** ([12, Example 7.3])**.** Let $k[[t]]$ be the formal power series ring over a field $k$ and consider $R = k[[t^9, t^{10}, t^{11}, t^{12}, t^{15}]]$ in $k[[t]]$. Then $\mathrm{K}_R = R + Rt + Rt^3 + Rt^4$ and $\mu_R(\mathrm{K}_R) = 4$. Let $I = R + Rt$. Then the homomorphism $t^I_{\mathrm{K}_R} : \mathrm{Hom}_R(I, \mathrm{K}_R) \otimes_R I \to \mathrm{K}_R$ is an isomorphism of $R$-modules, so that $I$ is a residually faithful $R$-module, but $I \ncong R, \mathrm{K}_R$, since $\mu_R(I) = 2$.

Here we notice that Corollary 3.4 recovers the theorem of Reiten [21] on Gorenstein modules. In fact, with the same notation as in Corollary 3.4, suppose that $R \ltimes M$ is a Gorenstein ring and let $\mathfrak{q}$ be a parameter ideal of $R$. Then, since $\mathrm{r}(R \ltimes M) = 1$, Corollary 3.4 implies that $\overline{M} = M/\mathfrak{q}M$ is a faithful module over the Artinian local ring $\overline{R} = R/\mathfrak{q}$ with $\mathrm{r}_{\overline{R}}(\overline{M}) = 1$. Therefore, $\overline{M}$ is the injective envelope $\mathrm{E}_{\overline{R}}(R/\mathfrak{m})$ of the residue class field $R/\mathfrak{m}$ of $\overline{R}$, so that $M \cong \mathrm{K}_R$ is the canonical module (that is a Gorenstein module of rank one) of $R$.

Residually faithful modules enjoy good properties. Let us summarize some of them.

**Proposition 3.6.** *Let $M$ be a MCM $R$-module. Then the following assertions hold true.*

(1) *Let $a \in \mathfrak{m}$ be a non-zerodivisor of $R$. Then $M$ is a residually faithful $R$-module if and only if so is the $R/(a)$-module $M/aM$.*
(2) *Let $(S, \mathfrak{n})$ be a Cohen-Macaulay local ring and let $\varphi : R \to S$ denote a flat local homomorphism of local rings. Then $M$ is a residually faithful $R$-module if and only if so is the $S$-module $S \otimes_R M$. Therefore, $M$ is a residually faithful $R$-module if and only if so is the $\widehat{R}$-module $\widehat{M}$, where $\widehat{*}$ denotes the $\mathfrak{m}$-adic completion.*
(3) *Suppose that $M$ is a residually faithful $R$-module. Then $M$ is a faithful $R$-module and $M_\mathfrak{p}$ is a residually faithful $R_\mathfrak{p}$-module for every $\mathfrak{p} \in \operatorname{Spec} R$.*

*Proof.* (1) This directly follows from Proposition 3.2.

(2) We set $n = \dim S/\mathfrak{m}S$ and $L = S \otimes_R M$. Firstly, suppose that $n = 0$. Let $\mathfrak{q}$ be a parameter ideal of $R$ and set $\mathfrak{a} = \mathrm{Ann}_R M/\mathfrak{q}M$. Then $\mathfrak{a}S = \mathrm{Ann}_S(L/\mathfrak{q}L)$. If $\mathfrak{a} = \mathfrak{q}$, then $\mathfrak{q}S = \mathrm{Ann}_S L/\mathfrak{q}L$, so that $L$ is a residually faithful $S$-module, since $\mathfrak{q}S$ is a parameter ideal of $S$. Conversely, suppose that $L$ is a residually faithful $S$-module. We then have $\mathfrak{a}S = \mathfrak{q}S$ by Proposition 3.2, so that $\mathfrak{a} = \mathfrak{q}$, and $M$ is a residually faithful $R$-module.

We now assume that $n > 0$ and that Assertion (2) holds true for $n-1$. Let $g \in \mathfrak{n}$ and suppose that $g$ is $S/\mathfrak{m}S$-regular. Then $g$ is $S$-regular and the composite homomorphism

$$R \to S \to S/gS$$

remains flat and local, so that $M$ is a residually faithful $R$-module if and only if so is the $S/gS$-module $L/gL$. Since $\dim S/(gS + \mathfrak{m}S) = n-1$, the latter condition is, by Assertion (1), equivalent to saying that $L$ is a residually faithful $S$-module.

(3) Let $a_1, a_2, \ldots, a_d$ be a system of parameters of $R$. We then have by Proposition 3.2

$$\mathrm{Ann}_R M \subseteq \mathrm{Ann}_R M/(a_1^n, a_2^n, \ldots, a_d^n)M = (a_1^n, a_2^n, \ldots, a_d^n)$$

for all $n > 0$. Therefore, $M$ is a faithful $R$-module. Let $\mathfrak{p} \in \operatorname{Spec} R$ and choose $P \in \operatorname{Min}_{\widehat{R}} \widehat{R}/\mathfrak{p}\widehat{R}$. Then, $\mathfrak{p} = P \cap R$, and we get a flat local homomorphism $R_\mathfrak{p} \to \widehat{R}_P$ of local rings such that $\dim \widehat{R}_P/\mathfrak{p}\widehat{R}_P = 0$. Therefore, to see that $M_\mathfrak{p}$ is a residually faithful $R_\mathfrak{p}$-module, by Assertion (1) it suffices to show that $\widehat{M}_P$ is a residually faithful $\widehat{R}_P$-module. Consequently, because $\widehat{M}$ is a residually faithful $\widehat{R}$-module by Assertion (1), passing to the $\mathfrak{m}$-adic completion $\widehat{R}$ of $R$, without loss of generality we may assume that $R$ possesses the canonical module $\mathrm{K}_R$. Then, the current assertion readily follows from Corollary 3.4, because

$$\mathrm{K}_{R_\mathfrak{p}} = (\mathrm{K}_R)_\mathfrak{p} = \left(\operatorname{Im} t^M_{\mathrm{K}_R}\right)_\mathfrak{p} = \operatorname{Im} t^{M_\mathfrak{p}}_{\mathrm{K}_{R_\mathfrak{p}}}.$$

□

By Proposition 3.6, we have the following.

**Corollary 3.7.** *Let $M$ be a MCM $R$-module. If* $\mathrm{r}(R \ltimes M) = \mathrm{r}_R(M)$*, then* $\mathrm{r}(R_\mathfrak{p} \ltimes M_\mathfrak{p}) = \mathrm{r}_{R_\mathfrak{p}}(M_\mathfrak{p})$ *for every* $\mathfrak{p} \in \operatorname{Spec} R$.

**Corollary 3.8.** *Let $M$ be a MCM $R$-module, and suppose that $R$ possesses the canonical module* $\mathrm{K}_R$*. If $M$ is a residually faithful $R$-module, then so is* $M^\vee$.

*Proof.* We may assume that $d > 0$ and that our assertion holds true for $d-1$. Let $a \in \mathfrak{m}$ be a non-zerodivisor of $R$ and let $\overline{*}$ denote the reduction mod $(a)$. We then have $\overline{M^\vee} \cong \mathrm{Hom}_{\overline{R}}(\overline{M}, \overline{\mathrm{K}_R}) = \overline{M}^\vee$, where we identify $\overline{\mathrm{K}_R} = \mathrm{K}_{\overline{R}}$. Because by Proposition 3.6 (3), $\overline{M}$ is a residually faithful $\overline{R}$-module, by the hypothesis of induction we have $\overline{M}^\vee = \mathrm{Hom}_{\overline{R}}(\overline{M}, \mathrm{K}_{\overline{R}})$ is a residually faithful $\overline{R}$-module, whence Proposition 3.6 (1) shows that $M^\vee$ is a residually faithful $R$-module. $\square$

Suppose that $R$ possesses the canonical module $\mathrm{K}_R$. Then, certain residually faithful $R$-modules $M$ satisfy the condition $\mathrm{Hom}_R(M, \mathrm{K}_R) \otimes_R M \cong \mathrm{K}_R$, as we show in the following. Recall that a finitely generated $R$-module $C$ is called *semidualizing*, if the natural homomorphism $R \to \mathrm{Hom}_R(C, C)$ is an isomorphism and $\mathrm{Ext}^i_R(C, C) = (0)$ for all $i > 0$. Hence, the canonical module is semidualizing, and all the semidualizing $R$-modules satisfy the hypothesis in Theorem 3.9, because semidualizing modules are Cohen-Macaulay.

**Theorem 3.9.** *Suppose that $R$ possesses the canonical module* $\mathrm{K}_R$ *and let $M$ be a MCM $R$-module. If* $R \cong \mathrm{Hom}_R(M, M)$ *and* $\mathrm{Ext}^i_R(M, M) = (0)$ *for all* $1 \le i \le d$*, then the homomorphism*

$$M^\vee \otimes_R M \xrightarrow{t} \mathrm{K}_R$$

*is an isomorphism of $R$-modules, where* $t = t^M_{\mathrm{K}_R}$.

*Proof.* Notice that $M$ is a residually faithful $R$-module. In fact, the assertion is clear, if $d = 0$. Suppose that $d > 0$ and let $f \in \mathfrak{m}$ be a non-zerodivisor of $R$. We set $\overline{R} = R/(f)$ and denote $\overline{*} = \overline{R} \otimes_R *$. Then, since $f$ is regular also for $M$, we have $\mathrm{Ext}^i_R(M, \overline{M}) = \mathrm{Ext}^i_{\overline{R}}(\overline{M}, \overline{M})$ for all $i \in \mathbb{Z}$, and it is standard to show that $\overline{R} \cong \mathrm{Hom}_{\overline{R}}(\overline{M}, \overline{M})$ and that $\mathrm{Ext}^i_{\overline{R}}(\overline{M}, \overline{M}) = (0)$ for all $1 \le i \le d-1$. Therefore, by induction on $d$, we may assume that $\overline{M}$ is a residually faithful $\overline{R}$-module, whence Proposition 3.6 (1) implies that so is the $R$-module $M$.

We now consider the exact sequence

$$(E) \quad 0 \to X \to M^\vee \otimes_R M \xrightarrow{t} \mathrm{K}_R \to 0$$

of $R$-modules, where $t = t^M_{\mathrm{K}_R}$. If $d = 0$, then because

$$\mathrm{Hom}_R(M^\vee \otimes_R M, \mathrm{K}_R) = \mathrm{Hom}_R(M, M^{\vee\vee}) = \mathrm{Hom}_R(M, M),$$

taking the $\mathrm{K}_R$-dual of $(E)$, we get the exact sequence

$$0 \to R \to \mathrm{Hom}_R(M, M) \to X^\vee \to 0.$$

Hence $X^\vee = (0)$ because $R \cong \mathrm{Hom}_R(M, M)$, so that $M^\vee \otimes_R M \xrightarrow{t} \mathrm{K}_R$ is an isomorphism. Suppose that $d > 0$ and let $f \in \mathfrak{m}$ be $R$-regular. We denote $\overline{*} = R/(f) \otimes_R *$. Then since $f$ is $\mathrm{K}_R$-regular, we get from Exact sequence $(E)$

$$(\overline{E}) \quad 0 \to \overline{X} \to \overline{M^\vee \otimes_R M} \xrightarrow{\overline{t}} \overline{\mathrm{K}_R} \to 0.$$

Because $\overline{\mathrm{K}_R} = \mathrm{K}_{\overline{R}}$, $\overline{M^\vee \otimes_R M} = \overline{M}^\vee \otimes_{\overline{R}} \overline{M}$, and $\overline{t} = t_{\overline{M}}^{\mathrm{K}_{\overline{R}}}$, by induction on $d$ we see in the above exact sequence $(\overline{E})$ that $\overline{X} = (0)$, whence $X = (0)$ by Nakayama's lemma. Therefore, $M^\vee \otimes_R M \xrightarrow{t} \mathrm{K}_R$ is an isomorphism. $\square$

Therefore, we have the following, which guarantees that the converse of Theorem 3.9 also holds true, if $R_\mathfrak{p}$ is a Gorenstein ring for every $\mathfrak{p} \in \mathrm{Spec}\, R \setminus \{\mathfrak{m}\}$. See [11, Proposition 2.4] for details.

**Corollary 3.10** ([11, Proposition 2.2])**.** *With the same hypothesis of Theorem* 3.9*, one has* $\mathrm{r}(R) = \mathrm{r}_R(M){\cdot}\mu_R(M)$*. Consequently, the following assertions hold true.*

(1) *If* $\mathrm{r}(R)$ *is a prime number, then* $M \cong R$ *or* $M \cong \mathrm{K}_R$*.*
(2) *If* $R$ *is a Gorenstein ring, then* $M \cong R$*.*

Let us note the following.

**Proposition 3.11.** *Suppose that* $R$ *is an integral domain, possessing the canonical module* $\mathrm{K}_R$*. Let* $M$ *be a MCM* $R$*-module and assume that* $\mathrm{r}(R \ltimes M) = 2$*. If* $\mathrm{Ext}_R^i(M, M) = (0)$ *for all* $1 \le i \le d$*, then*

$$M \cong \mathrm{K}_R^{\oplus 2} \quad or \quad M^\vee \otimes_R M \cong \mathrm{K}_R.$$

*Therefore, if* $\mathrm{r}(R)$ *is a prime number and* $M$ *is indecomposable, then* $\mathrm{r}(R) = 2$ *and* $M \cong R$*.*

*Proof.* Let $C = \mathrm{Coker}\, t_{\mathrm{K}_R}^M$. Then, $\mathrm{r}_R(M) = \mu_R(C) = 1$, or $\mathrm{r}_R(M) = 2$ and $C = (0)$, since $\mathrm{r}(R \ltimes M) = \mathrm{r}_R(M) + \mu_R(C)$ by Theorem 3.3. If $\mathrm{r}_R(M) = 1$, then $M^\vee \cong R$, since the cyclic module $M^\vee$ is of dimension $d$ and $R$ is an integral domain. Therefore, $M \cong \mathrm{K}_R$, so that $\mathrm{r}(R \ltimes M) = 1$, which is impossible. Hence, $\mathrm{r}_R(M) = 2$, and $M$ is, by Proposition 3.2, a residually faithful $R$-module. Let us take a presentation

$$0 \to X \to R^{\oplus 2} \to M^\vee \to 0$$

of $M^\vee$. If $X = (0)$, then $M \cong \mathrm{K}_R^{\oplus 2}$. Suppose that $X \ne (0)$. Then, $X$ is a MCM $R$-module, and taking the $\mathrm{K}_R$-dual of the presentation, we get the exact sequence

$$0 \to M \to \mathrm{K}_R^{\oplus 2} \to X^\vee \to 0.$$

Let $F = \mathrm{Q}(R)$. Then $F \otimes_R X^\vee \ne (0)$, since $X^\vee$ is a MCM $R$-module. Consequently, $F \otimes_R M \cong F$, that is $\mathrm{rank}_R M = 1$, because $F \otimes_R \mathrm{K}_R \cong F$. Hence, in the canonical exact sequence

$$(E) \quad 0 \to L \to M^\vee \otimes_R M \xrightarrow{t} \mathrm{K}_R \to 0,$$

$F \otimes_R L = (0)$, because $\mathrm{rank}_R M = 1$. Consequently, because the $R$-module $L$ is torsion, taking the $\mathrm{K}_R$-dual of the sequence $(E)$ we get the isomorphism

$$R = \mathrm{K}_R^\vee \to [M^\vee \otimes M]^\vee = \mathrm{Hom}_R(M, M).$$

Thus, $M^\vee \otimes_R M \cong \mathrm{K}_R$ by Theorem 3.9.

If $M$ is indecomposable and $\mathrm{r}(R)$ is a prime number, we then have $M \cong R$ or $M \cong \mathrm{K}_R$, while $\mathrm{r}(R \ltimes M) = 2$, so that $M \cong R$ and $\mathrm{r}(R) = 2$. □

The following result is essentially due to [24, Lemma 3.1] (see also [16, Proof of Lemma 2.2]). We include a brief proof for the sake of completeness.

**Lemma 3.12.** *Let $M$ be a MCM $R$-module and assume that there is an embedding*

$$(E) \quad 0 \to M \to F \to N \to 0$$

*of $M$ into a finitely generated free $R$-module $F$ such that $N$ is a MCM $R$-module. Then the following conditions are equivalent.*

(1) *$M$ is a residually faithful $R$-module.*

(2) *$M \not\subseteq \mathfrak{m}F$.*

(3) *$R$ is a direct summand of $M$.*

*Proof.* (3) ⇒ (1) and (2) ⇒ (3) These are clear.

(1) ⇒ (2) Let $\mathfrak{q}$ be a parameter ideal of $R$. Then, since $N$ is a MCM $R$-module, Embedding (E) gives rise to the exact sequence

$$0 \to M/\mathfrak{q}M \to F/\mathfrak{q}F \to N/\mathfrak{q}N \to 0.$$

Notice that $\mathrm{Ann}_{R/\mathfrak{q}}\mathfrak{m}{\cdot}(F/\mathfrak{q}F) \neq (0)$ because $\dim R/\mathfrak{q} = 0$, and we have $M/\mathfrak{q}M \not\subseteq \mathfrak{m}{\cdot}(F/\mathfrak{q}F)$. Thus $M \not\subseteq \mathfrak{m}F$. □

Let $\Omega\mathrm{CM}(R)$ denote the class of MCM $R$-modules $M$ such that there is an embedding $0 \to M \to F \to N \to 0$ of $M$ into a finitely generated free $R$-module with $N$ a MCM $R$-module. With this notation, we have the following.

**Theorem 3.13.** *Let $M \in \Omega\mathrm{CM}(R)$. Then*

$$\mathrm{r}(R \ltimes M) = \begin{cases} \mathrm{r}_R(M) & \textit{if } R \textit{ is a direct summand of } M, \\ \mathrm{r}(R) + \mathrm{r}_R(M) & \textit{otherwise.} \end{cases}$$

*Proof.* We may assume that $R$ is not a direct summand of $M$. Let us choose an embedding

$$0 \to M \to F \to N \to 0$$

of $M$ into a finitely generated free $R$-module $F$ such that $N$ is a MCM $R$-module. Let $\mathfrak{q}$ be a parameter ideal of $R$ and set $I = \mathfrak{q} :_R \mathfrak{m}$. Then, since $M \subseteq \mathfrak{m}F$ by Lemma 3.12, we have from the exact sequence

$$0 \to M/\mathfrak{q}M \to F/\mathfrak{q}F \to N/\mathfrak{q}N \to 0$$

that $I{\cdot}(M/\mathfrak{q}M) \subseteq (I\mathfrak{m}){\cdot}(F/\mathfrak{q}F) = (0)$. Therefore, $IM \subseteq \mathfrak{q}M$, so that $\mathrm{r}(R \ltimes M) = \mathrm{r}(R) + \mathrm{r}_R(M)$ by Theorem 2.2 (2). $\square$

If $R$ is a Gorenstein ring, every MCM $R$-module $M$ belongs to $\Omega\mathrm{CM}(R)$, so that Theorem 3.13 yields the following.

**Corollary 3.14.** *Let $R$ be a Gorenstein ring and $M$ a MCM $R$-module. Then the following conditions are equivalent.*
(1) $\mathrm{r}(R \ltimes M) = \mathrm{r}_R(M)$.
(2) *$R$ is a direct summand of $M$.*

## 4. Ulrich modules and the case where $\mathrm{r}(R \ltimes M) = \mathrm{r}(R) + \mathrm{r}_R(M)$

Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 0$. In this section, we study the other extremal case of Theorem 2.2 (2), that is $\mathrm{r}(R \ltimes M) = \mathrm{r}(R) + \mathrm{r}_R(M)$. We already have a partial answer by Theorem 3.13, and the following also shows that over a non-regular Cohen-Macaulay local ring $(R, \mathfrak{m}, k)$, there are plenty of MCM $R$-modules $M$ such that $\mathrm{r}(R \ltimes M) = \mathrm{r}(R) + \mathrm{r}_R(M)$.

Let $\Omega_R^i(k)$ denote, for each $i \geq 0$, the $i$-th syzygy module of the simple $R$-module $k = R/\mathfrak{m}$ in its minimal free resolution. Notice that, thanks to Theorem 3.13, the crucial case in Theorem 4.1 is actually the case where $i = d$.

**Theorem 4.1.** *Suppose that $R$ is not a regular local ring. Then $(\mathfrak{q} :_R \mathfrak{m}){\cdot}\Omega_R^i(k) = \mathfrak{q}{\cdot}\Omega_R^i(k)$ for every $i \geq d$ and for every parameter ideal $\mathfrak{q}$ of $R$. Therefore*

$$\mathrm{r}(R \ltimes \Omega_R^i(k)) = \mathrm{r}(R) + \mathrm{r}_R(\Omega_R^i(k))$$

*for all $i \geq d$.*

*Proof.* We may assume that $d > 0$ and that the assertion holds true for $d-1$. Choose $a \in \mathfrak{m} \setminus \mathfrak{m}^2$ so that $a$ is a non-zerodivisor of $R$. We set $\overline{R} = R/(a)$ and $\overline{\mathfrak{m}} = \mathfrak{m}/(a)$. We then have, for each $i > 0$, the isomorphism

$$\Omega_R^i(k)/a{\cdot}\Omega_R^i(k) \cong \Omega_{\overline{R}}^{i-1}(k) \oplus \Omega_{\overline{R}}^i(k).$$

We now choose elements $a_2, a_3, \ldots, a_d$ of $\mathfrak{m}$ so that $\mathfrak{q}_0 = (a, a_2, a_3, \ldots, a_d)$ is a parameter ideal of $R$ and set $\overline{\mathfrak{q}_0} = \mathfrak{q}_0/(a)$. Then, by the hypothesis of induction, we have

$$(\overline{\mathfrak{q}_0} :_{\overline{R}} \overline{\mathfrak{m}}){\cdot}\Omega_{\overline{R}}^i(k) = \overline{\mathfrak{q}_0}{\cdot}\Omega_{\overline{R}}^i(k)$$

for all $i \geq d-1$, so that

$$(\overline{\mathfrak{q}_0} :_{\overline{R}} \overline{\mathfrak{m}}){\cdot} \left[\Omega_R^i(k)/a{\cdot}\Omega_R^i(k)\right] = \overline{\mathfrak{q}_0}{\cdot} \left[\Omega_R^i(k)/a{\cdot}\Omega_R^i(k)\right]$$

for all $i \geq d$. Hence, because $\overline{\mathfrak{q}_0} :_{\overline{R}} \overline{\mathfrak{m}} = (\mathfrak{q}_0 :_R \mathfrak{m})/(a)$,

$$(\mathfrak{q}_0 :_R \mathfrak{m}){\cdot}\Omega_R^i(k) = \mathfrak{q}_0{\cdot}\Omega_R^i(k)$$

for all $i \geq d$. Therefore, by Theorem 2.2 (2), $(\mathfrak{q} :_R \mathfrak{m}){\cdot}\Omega_R^i(k) = \mathfrak{q}{\cdot}\Omega_R^i(k)$ for every parameter ideal $\mathfrak{q}$ of $R$, because $\Omega_R^i(k)$ is a MCM $R$-module. $\square$

Let us pose one question.

**Question 4.2.** Suppose that $R$ is not a regular local ring. Does the equality

$$(\mathfrak{q} :_R \mathfrak{m}){\cdot}\Omega_R^i(k) = \mathfrak{q}{\cdot}\Omega_R^i(k)$$

hold true for every $i \geq 0$ and for every parameter ideal $\mathfrak{q}$ of $R$? As is shown in Theorem 4.1, this is the case, if $i \geq d = \dim R$. Hence, the answer is affirmative, if $d = 2$ ([4]).

Let $M$ be a MCM $R$-module. Then we say that $M$ is an *Ulrich $R$-module with respect to* $\mathfrak{m}$, if $\mu_R(M) = \mathrm{e}_{\mathfrak{m}}^0(M)$ (see [2], where the different terminology MGMCM (maximally generated MCM module) is used). Ulrich modules play an important role in the representation theory of local and graded algebras. See [9, 10] for a generalization of Ulrich modules, which later we shall be back to. Here, let us note that a MCM $R$-module $M$ is an Ulrich $R$-module with respect to $\mathfrak{m}$ if and only if $\mathfrak{m}M = \mathfrak{q}M$ for some (hence, every) minimal reduction $\mathfrak{q}$ of $\mathfrak{m}$, provided the residue class field $R/\mathfrak{m}$ of $R$ is infinite (see, e.g., [13, Proposition 2.2]). We refer to [17, Theorem A] for the ample existence of Ulrich modules with respect to $\mathfrak{m}$ over certain two-dimensional normal local rings $(R, \mathfrak{m})$.

**Theorem 4.3.** *Suppose that $R$ is not a regular local ring and let $M$ be a MCM $R$-module. We set $A = R \ltimes M$. If $M$ is an Ulrich $R$-module with respect to $\mathfrak{m}$, then $\mathrm{r}_R(M) = \mu_R(M)$ and $\mathrm{r}(A) = \mathrm{r}(R) + \mathrm{r}_R(M)$, so that $(\mathfrak{q} :_R \mathfrak{m})M = \mathfrak{q}M$ for every parameter ideal $\mathfrak{q}$ of $R$. When $R$ has maximal embedding dimension in the sense of* [22]*, the converse is also true.*

*Proof.* Enlarging the residue class field of $R$ if necessary, we may assume that $R/\mathfrak{m}$ is infinite. Let us choose elements $f_1, f_2, \ldots, f_d$ of $\mathfrak{m}$ so that $\mathfrak{q} = (f_1, f_2, \ldots, f_d)$ is a reduction of $\mathfrak{m}$. Then, $\mathfrak{q}$ is a parameter ideal of $R$, and $\mathfrak{m}M = \mathfrak{q}M$, since $M$ is an Ulrich $R$-module with respect to $\mathfrak{m}$ ([13, Proposition 2.2]). We then have $\mathrm{r}_R(M) = \mu_R(M)$, and $\mathfrak{q} :_R \mathfrak{m} \subseteq \mathfrak{m}$, because $R$ is not a regular local ring. Hence, $(\mathfrak{q} :_R \mathfrak{m})M = \mathfrak{q}M$, because

$$\mathfrak{q}M \subseteq (\mathfrak{q} :_R \mathfrak{m})M \subseteq \mathfrak{m}M = \mathfrak{q}M.$$

Thus, $\mathrm{r}(A) = \mathrm{r}(R) + \mathrm{r}_R(M)$ by Theorem 2.2.

Assume that $R$ has maximal embedding dimension and we will show that the converse also holds true. We have $\mathfrak{m}^2 = \mathfrak{q}\mathfrak{m}$ for some parameter ideal $\mathfrak{q}$ of $R$, so that $\mathfrak{m} = \mathfrak{q} :_R \mathfrak{m}$, because $R$ is not a regular local ring. If $\mathrm{r}(A) = \mathrm{r}(R) + \mathrm{r}_R(M)$, we then have

$$\mathfrak{m}M = (\mathfrak{q} :_R \mathfrak{m})M = \mathfrak{q}M$$

by Theorem 2.2 (2), whence $M$ is an Ulrich $R$-module with respect to $\mathfrak{m}$. $\square$

**Remark 4.4.** Unless $R$ has maximal embedding dimension, the second assertion in Theorem 4.3 is not necessarily true. For example, let $(R, \mathfrak{m})$ be a one-dimensional Gorenstein local ring. Assume that $R$ is not a DVR. Then $\mathrm{r}(R \ltimes \mathfrak{m}) = 3 = \mathrm{r}(R) + \mathrm{r}_R(\mathfrak{m})$ (see Proposition 6.7 and Corollary 6.8 below), while $\mathfrak{m}$ is an Ulrich $R$-module with respect to $\mathfrak{m}$ itself if and only if $\mathfrak{m}^2 = a\mathfrak{m}$ for some $a \in \mathfrak{m}$. The last condition is equivalent to saying that $\mathrm{e}(R) = 2$.

We note one more example, for which the both cases $\mathrm{r}(R \ltimes M) = \mathrm{r}(R) + \mathrm{r}_R(M)$ and $\mathrm{r}(R \ltimes M) = \mathrm{r}_R(M)$ are possible, choosing different MCM modules $M$.

**Example 4.5.** Let $R = k[[X, Y, Z]]/(Z^2 - XY)$, where $k[[X, Y, Z]]$ denotes the formal power series ring over a field $k$. Then, the indecomposable MCM $R$-modules are $\mathfrak{p} = (x, z)$ and $R$, up-to isomorphisms (here, by $x, y, z$ we denote the images of $X, Y, Z$ in $R$, respectively). Since $\mathfrak{p}$ is an Ulrich $R$-module with respect to $\mathfrak{m}$, by Theorem 4.3 we have $\mathrm{r}(R \ltimes \mathfrak{p}) = 1 + \mathrm{r}_R(\mathfrak{p}) = 3$. Let $M$ be an arbitrary MCM $R$-module. Then, $M \cong \mathfrak{p}^{\oplus \ell} \oplus R^{\oplus n}$ for some integers $\ell, n \geq 0$, and $M/\mathfrak{q}M$ is a faithful $R/\mathfrak{q}$-module for the parameter ideal $\mathfrak{q} = (x, y)$ if and only if $n > 0$. Therefore, $\mathrm{r}(R \ltimes M) = \mathrm{r}_R(M) = 2\ell + n$ if $n > 0$, while $\mathrm{r}(R \ltimes M) = 1 + \mathrm{r}_R(M) = 1 + 2\ell$ if $n = 0$ (see Theorem 2.2).

The generalized notion of Ulrich ideals and modules was introduced by [9]. We briefly review the definition. Let $I$ be an $\mathfrak{m}$-primary ideal of $R$ and $M$ a MCM $R$-module. Suppose that $I$ contains a parameter ideal $\mathfrak{q}$ as a reduction. We say that $M$ is an *Ulrich $R$-module* with respect to $I$, if $\mathrm{e}_I^0(M) = \ell_R(M/IM)$ and $M/IM$ is a free $R/I$-module. Notice that the first condition is equivalent to saying that $IM = \mathfrak{q}M$ and that the second condition is automatically satisfied, when $I = \mathfrak{m}$. We say that $I$ is an *Ulrich ideal* of $R$, if $I \supsetneq \mathfrak{q}$, $I^2 = \mathfrak{q}I$, and $I/I^2$ is a free $R/I$-module. Notice that when $\dim R = 1$, every Ulrich ideal of $R$ is an Ulrich $R$-module with respect itself. Ulrich modules and ideals are closely explored by [6, 9, 10, 14], and it is known that they enjoy very specific properties. For instance, the syzygy modules $\Omega_R^i(R/I)$ $(i \geq d)$ for an Ulrich ideal $I$ are Ulrich $R$-modules with respect to $I$.

**Theorem 4.6.** *Let $I$ be an Ulrich ideal of $R$ and $M$ an Ulrich $R$-module with respect to $I$. We set $\ell = \mu_R(M)$ and $m = \mu_R(I)$. Then*

$$\mathrm{r}(R \ltimes M) = \mathrm{r}(R) + \mathrm{r}_R(M) = \mathrm{r}(R/I){\cdot}(\ell + m - d).$$

*Proof.* Let $\mathfrak{q}$ be a parameter ideal of $R$ such that $I^2 = \mathfrak{q}I$. Then $IM = \mathfrak{q}M$ because $\mathrm{e}_I^0(M) = \ell_R(M/IM)$, while $M/IM \cong (R/I)^{\oplus \ell}$ as an $R/I$-module. Therefore, since $\mathrm{Ann}_{R/\mathfrak{q}} M/\mathfrak{q}M = I/\mathfrak{q}$ and $I/\mathfrak{q} \cong (R/I)^{\oplus(m-d)}$ as an $R/I$-module ([9, Lemma 2.3]), we have by Proposition 2.1

$$\mathrm{r}(R \ltimes M) = \mathrm{r}_R(I/\mathfrak{q}) + \ell{\cdot}\mathrm{r}(R/I) = \mathrm{r}(R/I){\cdot}(m-d) + \ell{\cdot}\mathrm{r}(R/I) = \mathrm{r}(R) + \mathrm{r}_R(M),$$

where the last equality follows from the fact that $\mathrm{r}(R) = (m-d){\cdot}\mathrm{r}(R/I)$ (see [14, Theorem 2.5]). $\square$

**Corollary 4.7.** *Suppose that $d = 1$ and let $I$ be an Ulrich ideal of $R$ with $m = \mu_R(I)$. Then* $\mathrm{r}(R \ltimes I) = (2m-1){\cdot}\mathrm{r}(R/I)$.

We note a few examples.

**Example 4.8.** Let $k[[t]]$ be the formal power series ring over a field $k$.

(1) Let $R = k[[t^3, t^7]]$. Then $\mathcal{X}_R = \{(t^6 - at^7, t^{10}) \mid 0 \neq a \in k\}$ is exactly the set of Ulrich ideals of $R$. For all $I \in \mathcal{X}_R$, $R/I$ is a Gorenstein ring, so that $\mathrm{r}(R \ltimes I) = 3$ by Proposition 4.7.

(2) Let $R = k[[t^6, t^{13}, t^{28}]]$. Then the following families consist of Ulrich ideals of $R$ ([6, Example 5.7 (3)]):
   (i) $\{(t^6 + at^{13}) + \mathfrak{c} \mid a \in k\}$,
   (ii) $\{(t^{12} + at^{13} + bt^{19}) + \mathfrak{c} \mid a, b \in k\}$, and
   (iii) $\{(t^{18} + at^{25}) + \mathfrak{c} \mid a \in k\}$,
   
   where $\mathfrak{c} = (t^{24}, t^{26}, t^{28})$. We have $\mu_R(I) = 3$ and $R/I$ is a Gorenstein ring for all ideals $I$ in these families, whence $\mathrm{r}(R \ltimes I) = 5$.

Suppose that $\dim R = 1$. If $R$ possesses maximal embedding dimension $v$ but not a DVR, then for every Ulrich ideal $I$ of $R$, $R/I$ is a Gorenstein ring, and $I$ is minimally generated by $v$ elements ([6, Corollary 3.2]). Therefore, by Corollary 4.7, we get the following.

**Corollary 4.9.** *Suppose that* $\dim R = 1$ *and that $R$ is not a* DVR. *If $R$ has maximal embedding dimension $v$, then* $\mathrm{r}(R \ltimes I) = 2v - 1$ *for every Ulrich ideal $I$ of $R$.*

## 5. Bounding the supremum $\sup \mathrm{r}(R \ltimes M)$

Let $r > 0$ be an integer and set

$$\mathcal{F}_r(R) = \{M \mid M \text{ is an } R\text{-submodule of } R^{\oplus r} \text{ and a maximal Cohen-Macaulay } R\text{-module}\}.$$

We are now interested in the supremum $\sup_{M \in \mathcal{F}_r(R)} \mathrm{r}(R \ltimes M)$ and get the following.

**Theorem 5.1.** *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of multiplicity $e$ and let $M \in \mathcal{F}_r(R)$. Then $\mathrm{r}(R \ltimes M) \le \mathrm{r}(R) + re$. When $\mathfrak{m}$ contains a parameter ideal $\mathfrak{q}$ of $R$ as a reduction and $R$ is not a regular local ring, the equality holds if and only if $M$ is an Ulrich $R$-module with respect to $\mathfrak{m}$, possessing rank $r$.*

*Proof.* Enlarging the residue class filed $R/\mathfrak{m}$ of $R$ if necessary, without loss of generality we may assume that $\mathfrak{m}$ contains a parameter ideal $\mathfrak{q}$ of $R$ as a reduction. We then have

$$re \ge \mathrm{e}^0_{\mathfrak{q}}(M) = \ell_R(M/\mathfrak{q}M) \ge \ell_R((0) :_{M/\mathfrak{q}M} \mathfrak{m}) = \mathrm{r}_R(M).$$

Hence

$$\mathrm{r}(R \ltimes M) \le \mathrm{r}(R) + \mathrm{r}_R(M) \le \mathrm{r}(R) + re.$$

Consequently, if $\mathrm{r}(R \ltimes M) = \mathrm{r}(R) + re$, then $re = \mathrm{r}_R(M)$, that is $re = \mathrm{e}^0_{\mathfrak{q}}(M)$ and $\ell_R(M/\mathfrak{q}M) = \ell_R((0) :_{M/\mathfrak{q}M} \mathfrak{m})$, which is equivalent to saying that $\dim_R R^{\oplus r}/M < d$ and $\mathfrak{m}M = \mathfrak{q}M$, that is $M$ has rank $r$ and an Ulrich $R$-module with respect to $\mathfrak{m}$. Therefore, when $R$ is not a regular local ring, $\mathrm{r}(R \ltimes M) = \mathrm{r}(R) + \mathrm{r}_R(M)$ if and only if $M$ is an Ulrich $R$-module with rank $r$ (see Theorem 4.3). ☐

**Corollary 5.2.** *Suppose that $(R, \mathfrak{m})$ is a Cohen-Macaulay local ring of dimension one and multiplicity $e$. Let $\mathcal{F}$ be the set of $\mathfrak{m}$-primary ideals of $R$. Then*

$$\sup_{I \in \mathcal{F}} \mathrm{r}(R \ltimes I) = \begin{cases} 1 & \textit{if } R \textit{ is a } \mathrm{DVR}, \\ \mathrm{r}(R) + e & \textit{otherwise.} \end{cases}$$

*Proof.* We have only to show the existence of an $\mathfrak{m}$-primary ideal $I$ such that $I$ is an Ulrich $R$-module with respect to $\mathfrak{m}$ and $\mu_R(I) = e$. This is known by [2, Lemma (2.1)]. For the sake of completeness, we note a different proof. Let

$$A = \bigcup_{n>0} (\mathfrak{m}^n : \mathfrak{m}^n)$$

in $\mathrm{Q}(R)$. Then $A$ is a birational finite extension of $R$ (see [19]). Since $A \cong I$ for some $\mathfrak{m}$-primary ideal $I$ of $R$, it suffices to show that $A$ is an Ulrich $R$-module with respect to $\mathfrak{m}$ and $\mu_R(A) = e$. To do this, enlarging the residue class field $R/\mathfrak{m}$ of $R$ if necessary, we may assume that $\mathfrak{m}$ contains an element $a$ such that $Q = (a)$ is a reduction of $\mathfrak{m}$. Then $\mathfrak{m}A = aA$ because $A = R[\frac{\mathfrak{m}}{a}]$ ([19]), whence $A$ is an Ulrich $R$-module with respect to $\mathfrak{m}$. We have

$$\mu_R(A) = \ell_R(A/aA) = \mathrm{e}^0_Q(A) = \mathrm{e}^0_Q(R) = e$$

as wanted. ☐

## 6. The case where $d = 1$

In this section, we focus our attention on the one-dimensional case. Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension one, admitting a fractional canonical ideal $K$. Hence, $K$ is an $R$-submodule of $\overline{R}$ such that $K \cong \mathrm{K}_R$ as an $R$-module and $R \subseteq K \subseteq \overline{R}$, where $\overline{R}$ denotes the integral closure of $R$ in the total ring $\mathrm{Q}(R)$ of fractions of $R$. The hypothesis about the existence of fractional canonical ideals $K$ is equivalent to saying that $R$ contains an $\mathfrak{m}$-primary ideal $I$ such that $I \cong \mathrm{K}_R$ as an $R$-module and such that $I$ possesses a reduction $Q = (a)$ generated by a single element $a$ of $R$ ([8, Corollary 2.8]). The latter condition is satisfied, once $\mathrm{Q}(\widehat{R})$ is a Gorenstein ring and the field $R/\mathfrak{m}$ is infinite. We have $\mathrm{r}_R(M) = \mu_R(\mathrm{Hom}_R(M, K))$ for every MCM $R$-module $M$ ([15, Satz 6.10]). See [8, 15] for more details.

First of all, let us begin with the following review of a result of Brennan and Vasconcelos [3]. We include a brief proof.

**Proposition 6.1** ([3, Propositions 2.1, 5.2]). *Let $I$ be a fractional ideal of $R$ and set $I_1 = K : I$. Then the following conditions are equivalent.*

(1) $I : I = R$.
(2) $I_1{\cdot}I = K$.
(3) $J{\cdot}I = K$ *for some fractional ideal* $J$ *of* $R$.
(4) $I/fI$ *is a faithful* $R/fR$*-module for every parameter* $f$ *of* $R$.
(5) $I/fI$ *is a faithful* $R/fR$*-module for some parameter* $f$ *of* $R$.

*Proof.* (1) $\Leftrightarrow$ (2) This follows from the facts that $K : I_1 I = (K : I_1) : I = I : I$, and that $K : K = R$. See [15, Definition 2.4] and [15, Bemerkung 2.5 a)], respectively.

(3) $\Rightarrow$ (2) Since $JI = K$, we have $J \subseteq I_1 = K : I$, so that $K = JI \subseteq I_1 I \subseteq K$, whence $I_1 I = K$.

(2) $\Rightarrow$ (3) This is clear.

Since $I_1 \cong \mathrm{Hom}_R(I, K)$, the assertion that $I_1 I = K$ is equivalent to saying that the homomorphism $t_K^I : \mathrm{Hom}_R(I, K) \otimes_R I \to K$ is surjective. Therefore, the equivalence between Assertions (1), (4), (5) are special cases of Corollary 3.4 (see [3, Proposition 5.2] also). $\square$

We say that a fractional ideal $I$ of $R$ is *closed*, if it satisfies the conditions stated in Proposition 6.1. Thanks to Proposition 6.1 (3), we readily get the following.

**Corollary 6.2** ([3, Corollary 3.2]). *If $R$ is a Gorenstein ring, then every closed ideal of $R$ is principal.*

Assertion (2) of the following also follows from Corollary 3.14. Let us note a direct proof.

**Theorem 6.3.** *Suppose that $R$ is a Gorenstein ring and let $I$ be an $\mathfrak{m}$-primary ideal of $R$. Then the following assertions hold true.*

(1) $\mathrm{r}(R/I) \le \mathrm{r}_R(I) \le 1 + \mathrm{r}(R/I)$,
(2) $\mathrm{r}(R \ltimes I) = 1 + \mathrm{r}_R(I)$, *if* $\mu_R(I) > 1$.

*Proof.* Take the $R$-dual of the canonical exact sequence

$$0 \to I \to R \to R/I \to 0$$

of $R$-modules and we get the exact sequence

$$0 \to R \to \mathrm{Hom}_R(I, R) \to \mathrm{Ext}_R^1(R/I, R) \to 0.$$

Hence, $\mathrm{r}(R/I) \le \mathrm{r}_R(I) \le 1 + \mathrm{r}(R/I)$, because

$$\mathrm{r}_R(I) = \mu_R(\mathrm{Hom}_R(I, R)) \quad \text{and} \quad \mathrm{r}(R/I) = \mu_R(\mathrm{Ext}_R^1(R/I, R))$$

([15, Satz 6.10]). To see the second assertion, suppose that $\mu_R(I) > 1$. Let $\mathfrak{q} = (a)$ be a parameter ideal of $R$ and set $J = \mathfrak{q} :_R \mathfrak{m}$. Let us write $J = (a, b)$. We then have $J = \mathfrak{q} : \mathfrak{m}$, and $\mathfrak{m}J = \mathfrak{m}\mathfrak{q}$ by [4], because $R$ is not a DVR. On the other hand, by Corollary 6.2 we have $R \subsetneq I : I$, since $R$ is a Gorenstein ring and $I$ is not principal. Consequently

$$R \subseteq R : \mathfrak{m} \subseteq I : I,$$

since $\ell_R([R : \mathfrak{m}]/R) = 1$. Therefore, $\frac{b}{a} \in I : I$, because

$$R : \mathfrak{m} = \frac{1}{a}{\cdot}[\mathfrak{q} : \mathfrak{m}] = \frac{1}{a}{\cdot}(a,b) = R + R\frac{b}{a}.$$

Thus $bI \subseteq aI$, which shows $(\mathfrak{q} :_R \mathfrak{m})I = (a,b)I \subseteq \mathfrak{q}I$, so that

$$\mathrm{r}(R \ltimes I) = \mathrm{r}(R) + \mathrm{r}_R(I) = 1 + \mathrm{r}_R(I)$$

by Theorem 2.2 (2). □

**Remark 6.4.** In Theorem 6.3 (1), the equality $\mathrm{r}_R(I) = 1 + \mathrm{r}(R/I)$ does not necessarily hold true. For instance, consider the ideal $I = (t^8, t^9)$ in the Gorenstein local ring $R = k[[t^4, t^5, t^6]]$. Then $\mathrm{r}(R/I) = 2$. Because $t^{-4} \in R : I$, we have $1 \in \mathfrak{m}{\cdot}[R : I]$, which shows, identifying $R : I = \mathrm{Hom}_R(I, R)$ in the proof of Assertion (2) of Theorem 6.3, that $\mu_R(\mathrm{Hom}_R(I, R)) = \mu_R(\mathrm{Ext}^1_R(R/I, R))$. Hence $\mathrm{r}_R(I) = \mathrm{r}(R/I) = 2$, while $\mathrm{r}(R \ltimes I) = 3$ by Theorem 6.3 (2).

We however have $\mathrm{r}_R(I) = 1 + \mathrm{r}(R/I)$ for trace ideals $I$, as we show in the following. Let $I$ be an ideal of $R$. Then $I$ is said to be a *trace ideal* of $R$, if

$$I = \mathrm{Im}\left(\mathrm{Hom}_R(M, R) \otimes_R M \overset{t_R^M}{\to} R\right)$$

for some $R$-module $M$. When $I$ contains a non-zerodivisor of $R$, $I$ is a trace ideal of $R$ if and only if $R : I = I : I$ (see [18, Lemma 2.3]). Therefore, $\mathfrak{m}$-primary trace ideals are not principal.

**Proposition 6.5.** *Suppose that $R$ is a Gorenstein ring. Let $I$ be an $\mathfrak{m}$-primary trace ideal of $R$. Then $\mathrm{r}_R(I) = 1 + \mathrm{r}(R/I)$ and $\mathrm{r}(R \ltimes I) = 2 + \mathrm{r}(R/I)$.*

*Proof.* We have $1 \notin \mathfrak{m}{\cdot}[R : I]$, since $R : I = I : I \subseteq \overline{R}$. Therefore, thanks to the proof of Assertion (2) in Theorem 6.3, $\mathrm{r}_R(I) = 1 + \mathrm{r}(R/I)$, so that $\mathrm{r}(R \ltimes I) = 2 + \mathrm{r}(R/I)$ by Theorem 6.3 (2). □

**Example 6.6** ([6, Example 3.12])**.** Let $R = k[[t^4, t^5, t^6]]$. Then $R$ is a Gorenstein ring and

$$R,\ (t^8, t^9, t^{10}, t^{11}),\ (t^6, t^8, t^9),\ (t^5, t^6, t^8),\ (t^4, t^5, t^6),\ \left\{I_a = (t^4 - at^5, t^6)\right\}_{a \in k}$$

are all the non-zero trace ideals of $R$. We have $I_a = I_b$, only if $a = b$.

**Proposition 6.7.** *Suppose that $R$ is a not a* DVR. *Then $\mathfrak{m}$ is a trace ideal of $R$ with $\mathrm{r}_R(\mathfrak{m}) = \mathrm{r}(R) + 1$ and $\mathrm{r}(R \ltimes \mathfrak{m}) = 2{\cdot}\mathrm{r}(R) + 1$.*

*Proof.* We have $\mathfrak{m} : \mathfrak{m} = R : \mathfrak{m}$, because $R$ is not a DVR, whence $\mathfrak{m}$ is a trace ideal of $R$. We take the $K$-dual of the sequence $0 \to \mathfrak{m} \to R \to R/\mathfrak{m} \to 0$ and consider the resulting exact sequence

$$0 \to K \to K : \mathfrak{m} \to \mathrm{Ext}_R^1(R/\mathfrak{m}, K) \to 0.$$

Then, since $\mathrm{Ext}_R^1(R/\mathfrak{m}, K) \cong R/\mathfrak{m}$, we get

$$\mathrm{r}_R(\mathfrak{m}) = \mu_R(K : \mathfrak{m}) \le \mu_R(K) + 1 = \mathrm{r}(R) + 1.$$

We actually have the equality in the estimation

$$\mu_R(K : \mathfrak{m}) \le \mu_R(K) + 1.$$

To see this, it is enough to show that $\mathfrak{m}(K : \mathfrak{m}) = \mathfrak{m}K$. We have

$$K : \mathfrak{m}(K : \mathfrak{m}) = [K : (K : \mathfrak{m})] : \mathfrak{m} = \mathfrak{m} : \mathfrak{m}$$

and

$$K : \mathfrak{m}K = (K : K) : \mathfrak{m} = R : \mathfrak{m}.$$

Therefore, since $\mathfrak{m} : \mathfrak{m} = R : \mathfrak{m}$, we get $K : \mathfrak{m}(K : \mathfrak{m}) = K : \mathfrak{m}K$, so that $\mathfrak{m}(K : \mathfrak{m}) = \mathfrak{m}K$. Hence $\mathrm{r}_R(\mathfrak{m}) = \mu_R(K : \mathfrak{m}) = \mu_R(K) + 1 = \mathrm{r}(R) + 1$ as wanted. We have $\mathrm{r}(R \ltimes \mathfrak{m}) = \mathrm{r}(R) + \mathrm{r}_R(\mathfrak{m})$ by Theorem 2.2 (2), because $(\mathfrak{q} :_R \mathfrak{m}){\cdot}\mathfrak{m} = \mathfrak{q}{\cdot}\mathfrak{m}$ for every parameter ideal $\mathfrak{q}$ of $R$ ([4]; see Theorem 4.1 also), whence the second assertion follows. ☐

**Corollary 6.8.** *Let $R$ be a Gorenstein ring which is not a* DVR. *Then $R \ltimes \mathfrak{m}$ is an almost Gorenstein ring in the sense of* [8], *possessing* $\mathrm{r}(R \ltimes \mathfrak{m}) = 3$.

*Proof.* See [8, Theorem 6.5] for the assertion that $R \ltimes \mathfrak{m}$ is an almost Gorenstein ring. ☐

Let us give one more result on closed ideals.

**Proposition 6.9.** *Let $I \subsetneq R$ be a closed ideal of $R$ and set $I_1 = K : I$. Then* $\mathrm{r}(R/I) = \mu_R(I_1) = \mathrm{r}_R(I)$.

*Proof.* We consider the exact sequence $0 \to K \to I_1 \to \mathrm{Ext}_R^1(R/I, K) \to 0$. It suffices to show $K \subseteq \mathfrak{m}I_1$. We have $K : \mathfrak{m}I_1 = (K : I_1) : \mathfrak{m}$, while $(K : I_1) : \mathfrak{m} = I : \mathfrak{m} \subseteq I : I = R = K : K$. Hence $\mathfrak{m}I_1 \supseteq K$ and the assertion follows. ☐

Combining Corollary 3.4, Proposition 6.1, and Proposition 6.9, we have the following, which is the goal of this paper.

**Corollary 6.10.** *Let $I$ be a fractional ideal of $R$. Then the following conditions are equivalent.*

(1) $\mathrm{r}(R \ltimes I) = \mathrm{r}_R(I)$.
(2) *$I$ is a closed ideal of $R$.*

*When this is the case,* $\mathrm{r}(R \ltimes I) = \mathrm{r}(R/I)$, *if $I \subsetneq R$.*

We close this paper with the following example.

**Example 6.11.** Let $k$ be a field. Let $R = k[[t^3, t^4, t^5]]$ and set $I = (t^3, t^4)$. Then $I \cong \mathrm{K}_R$, and $I$ is a closed ideal of $R$ with $\mathrm{r}(R) = 2$ and $\mathrm{r}(R \ltimes I) = \mathrm{r}_R(I) = 1$. We have $\mathrm{r}(R \ltimes J) = 1 + \mathrm{r}_R(J) = 3$ for $J = (t^3, t^5)$. The maximal ideal $\mathfrak{m}$ of $R$ is an Ulrich $R$-module, and $\mathrm{r}(R \ltimes \mathfrak{m}) = 2 + \mathrm{r}_R(\mathfrak{m}) = 5$ by Theorem 4.3, since $\mathrm{r}_R(\mathfrak{m}) = \mathrm{r}(R) + 1 = 3$ by Proposition 6.7. See Corollary 2.6 for more details.

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