Periodic solution of stochastic process in the distributional sense

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Abstract

In this paper, we aim to study a stochastic process from a macro point of view, and thus periodic solution of a stochastic process in distributional sense is introduced. We first give the definition and then establish the existence of periodic solution on bounded domain. Lastly, for the case that probability density function exists, we obtain the existence periodic solutions of the probability density function corresponding to the stochastic process by using the technique of deterministic partial differential equations.

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1 Introduction

Some properties of a stochastic process are worth being studied, such as the long time behavior, periodicity, ergodicity and so on. There are many classical theories, see the books [3, 4, 5]. In this paper, we will give a new viewpoint about the periodicity of stochastic processes. We consider the stochastic process from another fact–a macro point of view. We do not consider the motion of a single particle, while we are concerned with the motion the entire system. It is well-known that the probability density function (PDF) of a stochastic process can describe the entire distribution of a system. Hence, in this paper, we consider some property of entire system–time-periodicity of PDF. The density of a stochastic process is called as Fokker-Planker equation or Kolmogorov equation, which has been studied in [6].

Now, we consider the multidimensional Fokker-Planck equation for the following SDE

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad X_0 = x \in \mathbb{R}^d,$$
(1.1)

where b is an d-dimensional vector function, σ is an $d \times m$ matrix function and B_t is an mdimensional Brownian motion, see Page 99 in [8]. Let the probability density function of (1.1) be p(t, x) if it exists, then we deduce that p(t, x) satisfies

$$\partial_t p(t,x) = \frac{1}{2} div(div(\sigma\sigma^T p)) - div(b(t,x)p)$$
(1.2)

with initial data

$$p(0,x) = p_0(x), \quad x \in \mathbb{R}^d.$$
 (1.3)

In this paper we mainly consider the property of p(t, x). There is a fact that for most of stochastic process, the PDF may not exist. Therefore, in this paper, we first give the notion of periodic solution in distributional sense for discrete time and continuous time stochastic process and then consider a special case, in which the PDF exists.

Let us recall some development of periodic solutions. Periodic solutions have been a central concept in the theory of the deterministic dynamical system starting from Poincaré's work [23]. For a random periodic dynamic system, to study the pathwise random periodic solutions is of great importance. Zhao-Zheng [30] started to study the problem and gave a definition of pathwise random periodic solutions for C^1 -cocycles. Recently, Feng et al. did some beautiful work about the periodic solutions, see [10, 11, 12]. Noting that the definition of periodic solution in [10, 11, 12, 30] is different from here, we consider the time-periodicity of entire system in distributional sense. During we prepare our paper, we find the paper of Chen et al. [7], where the existence of periodic solutions of Fokker-Planck equations is considered. They obtained the desired results by discussing the existence of periodic solutions in distributional sense for some stochastic differential equations (SDEs). More precisely, they used the properties of solutions of SDEs to study the properties of solutions of Fokker-Planck equation. They obtained the time-periodicity of PDF in the whole space. We will give another proof in viewpoint of PDEs. Moreover, the definition of periodic solution to discrete time and discrete state stochastic process will be given. The topic of periodic solutions to stochastic process in distributional sense on bounded domain is also considered in this paper. About the almost periodic solution, see [21, 27].

The rest of this paper is arranged as follows. In Sections 2, we present some known results on PDEs' theory. In Sections 3, we first give some definitions of periodic solutions to stochastic process in distributional sense, then establish the existence of periodic solution on bounded domain by using the method of [7]. For the case that the PDF exists, we obtain the existence of periodic solution of Fokker-Planck equations on bounded domain and in the whole space by using the method of deterministic partial differential equations in Section 4.

2 Some known results

In this section, we recall some known results about existence of PDF of the diffusion Itô process and the existence of periodic solution of parabolic equations.

Consider a Markov process in \mathbb{R}^d with transition probabilities P(s, x, t, B) (*B* is a Borel set in \mathbb{R}^d) is called a *diffusion process* or a *diffusion* if there is a mapping $b : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}^d$, called the *drift coefficient*, and a mapping $(x, t) \mapsto A(x, t)$ with values in the space of symmetric operator on \mathbb{R}^d , called the *diffusion coefficient* or *diffusion matrix*, such that

(i) for all $\varepsilon > 0$, $t \ge 0$ and $x \in \mathbb{R}^d$ we have

$$\lim_{h \to 0} h^{-1} P(t, x, t+h, V(x, \varepsilon)) = 0,$$

(ii) for some $\varepsilon > 0$ and all $t \ge 0, x \in \mathbb{R}^d$ we have

$$\lim_{h \to 0} h^{-1} \int_{U(x,\varepsilon)} (y-x) P(t, x, t+h, dy) = b(x, t),$$

(iii) for some $\varepsilon > 0$ and all $t \ge 0, x, z \in \mathbb{R}^d$ we have

$$\lim_{h \to 0} h^{-1} \int_{U(x,\varepsilon)} \langle y - x, z \rangle P(t, x, t+h, dy) = 2 \langle A(x, t)z, z \rangle$$

where $U(x,\varepsilon) = \{y : |x-y| < \varepsilon\}$ and $V(x,\varepsilon) = \{y : |x-y| > \varepsilon\}$. If A and b do not depend on t, then the diffusion is homogeneous. Bogachev et al. [6] obtained the following proposition.

Proposition 2.1 Suppose that relations (i)-(iii) hold locally uniformly in x and the functions $a^{ij}, b^i \ (A = (a^{ij}), b = (b_j))$ are locally bounded. Then the transition probabilities satisfy the parabolic Fokker-Planck-Kolmogorov equation

$$\partial_t \mu = \partial_{x_i} \partial_{x_j} (a^{ij} \mu) - \partial_{x_i} (b^i \mu)$$

in the sense of generalized functions. If ν is a finite Borel measure on \mathbb{R}^d and

$$\mu_t(dx) = \int_{\mathbb{R}^d} P(0, y, t, dx) \nu(dy),$$

then the measure $\mu = \mu_t(dx)dt$ gives a solution to the Cauchy problem with the initial condition $\nu|_{t=0} = \nu$.

The above proposition is concerned with the case of measure-valued solution about the Fokker-Planck equation. The next result shows that there exists PDF for a stochastic process under some assumptions. Let $D_T = \Omega \times (0,T)$, where $D \subset \mathbb{R}^d$ is an open set and T > 0 is a fixed number. Bogechev et al. [6] obtained the following result.

Proposition 2.2 [6, Theorem 6.3.1] Let μ be a locally finite Borel measure on D_T such that $a^{ij} \in L^1_{loc}(D_T, \mu)$ and

$$\int_{D_T} \left[\partial_t \phi + a^{ij} \partial_{x_i} \partial_{x_j} \phi \right] d\mu \le C(\sup_{D_T} |\phi| + \sup_{D_T} |\nabla_x \phi|)$$

for all nonnegative $\phi \in C_0^{\infty}(\Omega_T)$. Then the following assertions are true.

(i) If $\mu \ge 0$, then $(detA)^{1/(d+1)}\mu = \rho dx dt$, where $\rho \in L_{loc}^{(d+1)'}(D_T)$.

(ii) If, on every compact set in D_T , the mapping A is uniformly bounded, uniformly nondegenerate, and Hölder continuous in x uniformly with respect to t, then $\mu = \rho dx dt$, where $\rho \in L^r_{loc}(D_T)$ for every $r \in [1, (d+2)')$.

The above Proposition is the existence of probability density function in the whole space. Now, we consider the bounded domain. As stated in [8], in the simulations, we have to take x in a large but bounded domain $D \subset \mathbb{R}^d$ and we could impose absorbing boundary condition on ∂D , i.e., as long as a "particle" or a solution path reaches the boundary, it is removed from the system. The above assumptions implies the following system

$$\begin{cases} \partial_t p(t,x) = A^* p(t,x), \quad t > 0, \ x \in D, \\ p|_{\partial D} = 0, \\ p(0,x) = p_0(x), \quad x \in D, \end{cases}$$
(2.1)

where

$$A^*p = -\sum_i \frac{\partial}{\partial x_i} (b_i p) + \frac{1}{2} \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} ((\sigma \sigma^T)_{ij} p).$$

Due to the absorbing boundary condition, the particle will not come back when it reach the boundary. Thus under the absorbing boundary, it is impossible to get the existence of periodic solution to (2.1). Therefore, we must consider another case: the reflecting boundary condition [14, Section 5.1.1]. The Fokker-Planck equation of (1.1) can be written as

$$\partial_t p(t,x) = \nabla \cdot \left[\frac{1}{2} (\nabla \cdot (\sigma \sigma^T p)) - b p(t,x) \right].$$
(2.2)

The reflecting boundary condition means particles or solution paths can not leave a bounded domain D, and hence there is zero net flow of p crossing the boundary ∂D . Thus we impose the following reflecting boundary condition

$$\frac{1}{2}(\nabla \cdot (\sigma \sigma^T p)) - bp(t, x) = 0 \quad \text{on} \quad \partial D.$$
(2.3)

Integrating (2.2) over D and using the boundary condition (2.3) together with the divergence theorem, we have conservation of probability

$$\frac{\partial}{\partial t} \int_D p(t, x) = 0$$

that is to say,

$$\int_D p(t,x) = \int_D p_0(x) = 1.$$

In this case, it is possible to obtain the existence of periodic solution to (2.2)-(2.3) with initial data. In order to obtain the desired results, we recall some results about the periodic parabolic equations, see [15].

Now, we consider the periodic-parabolic eigenvalue problem

$$\begin{cases} \partial_t u + \mathcal{A}(t)u = \mu u & \text{in } D \times \mathbb{R}, \\ \mathcal{B}u = 0 & \text{on } \partial D \times \mathbb{R}, \\ u & T - \text{periodic in } t, \end{cases}$$
(2.4)

where $\mathcal{A}(t)$ is a uniformly elliptic differential operator of second order depending T-periodically on t, i.e.,

$$\mathcal{A}(t)u = \mathcal{A}(t, x, D)u = -\sum_{j,k=1}^{d} a_{jk}(t, x) \frac{\partial^2}{\partial x_j \partial x_k} u + \sum_{j=1}^{d} a_j(t, x) \frac{\partial}{\partial x_j} u + a_0(t, x) u$$

and

$$\mathcal{B}u = \begin{cases} u & Dirichlet \ b.c.,\\ \frac{\partial u}{\partial \nu} + b_0(x)u & Neumann \ or \ regular \ oblique \ derivative \ b.c..\end{cases}$$

We say $\mu \in \mathcal{C}$ (\mathcal{C} denotes complex value) is an eigenvalue if there is a nontrivial solution u (eigenfunction) of (2.4). We search in particular for an eigenvalue $\mu \in \mathbb{R}$ having a positive eigenfunction ("principal eigenvalue" μ).

In order to establish the existence of solutions of (2.4), we consider the inhomogeneous linear evolution equation

$$\begin{cases} \dot{u}(t) + \mathcal{A}(t)u = f(t), & 0 < t < T, \\ u(0) = u_0, & u_0 \in X, \end{cases}$$
(2.5)

where $f \in C^{\theta}([0,T],X)$, $0 < \theta \leq 1$ and X is a Banach space. Assume the closed linear operator \mathcal{A} in X satisfies

(i) $dom(\mathcal{A}) := dom(\mathcal{A}(t))$ is dense in X and independent of t,

(ii) $\{\lambda \in \mathcal{C} : Re\lambda \leq 0\} \subset \rho(\mathcal{A}(t)), \forall t \in [0,T], (\rho(\mathcal{A}(t)) \text{ denotes the resolvent set of operator } \mathcal{A}(t)),$

(iii) $\|(\mathcal{A}(t) - \lambda)^{-1}\| \leq \frac{c}{1+|\lambda|}, \, \forall \lambda \in \mathcal{C}, \, Re\lambda \leq 0, \, \forall t \in [0, T].$

Set $\mathcal{A} := \mathcal{A}(0)$ and take the fractional power spaces X_{α} with respect to \mathcal{A} . Assume further (iv) $\mathcal{A}(\cdot) : [0,T] \to \mathcal{L}(X_1,X)$ is Hölder continuous.

It follows from the results of Sobolevskii [25] that there exists a unique solution u of (2.5) with

$$u \in C([0,T],X) \cap C^1((0,T],X) \text{ if } u_0 \in X.$$

Moreover, there exists the evolution operator

$$U(t,s) \in \mathcal{L}(X)$$

such that the solution of (2.5) can be represented in the following form

$$u(t) = U(t,0)u_0 + \int_0^t U(t,s)f(s)ds, \quad 0 \le t \le T.$$
(2.6)

The function U is strongly continuous on the set $\triangle := \{(t,s) \in [0,T] \times [0,T] : 0 \le s \le t \le T\}$, i.e. $U(\cdot)u_0 \in C(\triangle, X)$ for each $u_0 \in X$, and satisfies

$$U(t,t) = I, \quad U(s,t)U(t,\tau) = U(s,\tau), \quad 0 \le \tau \le t \le s \le T.$$

Set K := U(T, 0). The Krein-Rutman theorem implies that r := spr(K) > 0, where spr(K) is the principal eigenvalue of K.

Let

$$\mathcal{L} := \partial_t + \mathcal{A}(t),$$

and set

$$L := the operator in \mathbb{F}_1 introduced by \mathcal{L}, \mathcal{B}$$

and the T - periodicity, with domain domL = \mathbb{F}_1

where

$$\mathbb{F}_1 := \{ w \in C^{2+\theta, 1+\frac{\theta}{2}}(\bar{D} \times \mathbb{R}) : \mathcal{B}w = 0 \text{ on } \partial D \times \mathbb{R}, w \ T - periodic \ in \ t \}.$$

Assume the following conditions hold:

(A) $\mathcal{A}(t)$ is uniformly elliptic for each $t \in \mathbb{R}$ and *T*-periodic in *t*, of given period T > 0. More precisely, assume the coefficient functions $a_{jk} = a_{kj}, a_j, a_0$ belong to the space

$$\mathbb{F} := \{ w \in C^{\theta, \frac{\theta}{2}}(\bar{D} \times \mathbb{R}) : w \ T - periodic \ in \ t \}.$$

We keep $\mathcal{B} = \mathcal{B}(x, D)$ independent of $t \in [0, T]$, such that the operator $\mathcal{A}(t)$, the realization of $(\mathcal{A}(t), \mathcal{B})$ in $L^p(D)$ (N has domain independent of t. We assume that

$$a_0(t,x) \ge 1, \quad \forall (t,x) \in [0,T] \times \overline{D}$$

Then $\{\mathcal{A}(t): 0 \leq t \leq T\}$ satisfies the hypotheses (i)-(iv). Thus, by the results of Sobolevskii [25], we get the existence of evolution operator U(t,s) for $0 \leq s \leq t \leq T$.

Now, we give the relation between the solutions of (2.4) and (2.5) with f = 0.

Proposition 2.3 Assume (A) holds. Then we have :

r := spr(K) is principal eigenvalue of K, with principal eigenfunction $u_0 \gg 0 \Leftrightarrow \mu = -\frac{1}{T} \log r$ is an eigenvalue of L with positive eigenfunction $u := u(t) = e^{\mu t} U(t, 0) u_0$.

We have the following proposition about the positivity of μ .

Proposition 2.4 Assume (A) holds. Assume further that the zero-order term of $\mathcal{A}(t)$ satisfies $a_0 \geq 0$ on $\overline{D} \times \mathbb{R}$, and that

$$a_0 \not\equiv 0 \quad on \ \bar{D} \times \mathbb{R} \quad if \ \mathcal{B} = \frac{\partial}{\partial \nu}$$

Then 0 < r < 1.

3 Definitions of periodic solutions in distributional sense

In this section, we give some definitions of periodic solutions in distributional sense, including discrete time and discrete state stochastic process (also called stochastic sequence) and continuous time and continuous state stochastic process.

We start to consider the discrete time and discrete state stochastic process. Suppose a stochastic sequence $\{X_n\}_{n\geq 1}$ defined on a complete probability space has a one-step transition probability matrix P. Following the Chapman-Kolmogorov equation, we have the N-th step transition probability matrix $P^{(N)}$ satisfying

$$P^{(N)} = P \cdot P^{(N-1)} = \dots = P^N$$

Now, we suppose each particle has m state in a particle system and the particle system has an initial distribution $(x_1^0, x_2^0, \dots, x_m^0)^T$. Consider the distribution of the system after being transferred N-step (denoted by $(x_1^N, x_2^N, \dots, x_m^N)^T$), we have

$$(x_1^N, x_2^N, \cdots, x_m^N)^T = P^{(N)}(x_1^0, x_2^0, \cdots, x_m^0)^T = P^N(x_1^0, x_2^0, \cdots, x_m^0)^T.$$

Therefore, if the following holds

$$P^{(N)}(x_1^0, x_2^0, \cdots, x_m^0)^T = (x_1^0, x_2^0, \cdots, x_m^0)^T,$$

then the particle system turn back to the initial distribution. We give the first definition of periodic solution in distributional sense.

Definition 3.1 (discrete time and discrete state stochastic process) Suppose a particle system has one-step transition probability matrix P and contains m states with the initial distribution $(x_1^0, x_2^0, \dots, x_m^0)^T$. If there exists a positive constant $N \in \mathbb{N}$ such that

$$P^{(N)}(x_1^0, x_2^0, \cdots, x_m^0)^T = (x_1^0, x_2^0, \cdots, x_m^0)^T,$$
(3.1)

then the particle system is called N-periodic system in distributional sense.

One can give some examples to satisfy (3.1). For example, suppose a particle system has five states and the initial distribution is $(\frac{1}{10}, \frac{1}{20}, \frac{7}{2}, \frac{2}{5}, \frac{1}{20})^T$. Assume that the one-step transition probability matrix is

$$P = \begin{pmatrix} a_1 & b_1 & 0 & 0 & 0\\ a_2 & b_2 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$P^{2}\left(\frac{1}{10}, \frac{1}{10}, \frac{7}{20}, \frac{2}{5}, \frac{1}{20}\right)^{T} = \left(\frac{1}{10}, \frac{1}{10}, \frac{7}{20}, \frac{2}{5}, \frac{1}{20}\right)^{T}$$

On the other hand, it is easy to see that if

$$P^{(N)} = I_m, (3.2)$$

where I_m denotes $m \times m$ identity matrix (which is called Idempotent matrix in algebra), then the equality (3.1) holds. A stochastic process is called *strong N-periodic system in distributional sense* if (3.2) holds. We remark that the number N in (3.2) is definitely equal to least common multiple of the periodicity of every particle.

For continuous time and continuous state stochastic process, we borrow the idea of [6, 7]. A stochastic process is called *T*-periodic system in distributional sense if $\mu(t+T, x) = \mu(t, x)$ for all $t \ge 0$ and $x \in \mathbb{R}^d$, where μ is defined as in Proposition 2.1.

Before we close this section, we establish the existence of periodic solution in distributional sense on bounded domain. We generalize the result of [7] to the bounded domain. We remark that the boundary of the bounded domain should be reflective. If the boundary is absorbing, then we cannot get the limit in the following sense

$$\mu_n(f) = \int_D f d\mu_n \to \int_D f d\mu = \mu(f), \quad as \quad n \to \infty,$$

where μ_n and μ are probability measure of some stochastic process on the bounded domain $D \subset \mathbb{R}^d$. The probability measure considered here keeps entirety, i.e., $\mu_n(\bar{D}) = 1$ and the limit probability measure $\mu_0(\bar{D}) = 1$. The results obtained here coincide with those in next section.

Let D be a convex domain in \mathbb{R}^d and (Ω, \mathcal{F}, P) be a complete probability space with an increasing family $\{\mathcal{F}_t\}_{t\geq 0}$ of sub- σ -fields of \mathcal{F} . Suppose an \mathcal{F}_t -adapt r-dimensional Brownian motion $B(t) = (B^1(t), \dots, B^r(t))$ with B(0) = 0 is given. Let $\sigma(t, x)$ and b(t, x) be $\mathbb{R}^d \otimes \mathbb{R}^d$ -valued and \mathbb{R}^d -valued functions, both being defined on $\mathbb{R}_+ \times \overline{D}$, respectively. Consider the stochastic differential equation with reflection

$$dX_t = b(t, X)dt + \sigma(t, X)dB + d\Phi, \quad X(0) = x,$$
(3.3)

where $x \in \overline{D}$ and $\{\Phi(t)\}$ is an associated process of $\{X(t)\}$. In [20], the authors gave the relationship between Φ and X, i.e.,

$$\Phi_t = \int_0^t \nu(X_s) d|\Phi|_s, \quad |\Phi|_t = \int_0^t \mathbf{1}_{\{X_s \in \partial D\}} d|\Phi|_s,$$

where ν is the unit outward normal to ∂D at x, and k_t stands for the total variation of k on [0, t]. In order to make the meaning of Φ_t clearly, we introduce the following spaces of functions, see [26, Page 164] for more details.

 $C(\mathbb{R}_+, \mathbb{R}^d)$ (resp. $C(\mathbb{R}_+, \overline{D})$) = the space of \mathbb{R}^d -valued (resp. \overline{D} -valued) continuous functions on \mathbb{R}_+ .

 $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$ (resp. $\mathbb{D}(\mathbb{R}_+, \overline{D})$) = the space of \mathbb{R}^d -valued (resp. \overline{D} -valued) right continuous functions on \mathbb{R}_+ with left limits.

On $C(\mathbb{R}_+, \mathbb{R}^d)$ and $C(\mathbb{R}_+, \overline{D})$ we consider the compact uniform topology. Given a function ξ in $\mathbb{D}(\mathbb{R}_+, \overline{D})$, a function Φ is said to be associated with ξ if the following three conditions are satisfied.

(i) Φ is a function in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$ with bounded variation and $\Phi(0) = 0$.

- (ii) The set $\{t \in \mathbb{R}_+ : \xi(t) \in D\}$ has $d|\Phi|$ -measure zero.
- (iii) For any $\eta \in C(\mathbb{R}_+, \overline{D}), (\eta(t) \xi(t), \Phi(dt)) \ge 0.$

Using the above properties, Tanaka proved that the following Lemma.

Proposition 3.1 [26, Lemma 2.2] Let $w, \tilde{w} \in \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$ with $w(0), \tilde{w}(0) \in \overline{D}$, and $\xi, \tilde{\xi}$ be any solutions of

$$\xi = w + \Phi, \qquad \tilde{\xi} = \tilde{w} + \tilde{\Phi},$$

respectively. Then we have

$$\begin{aligned} |\xi(t) - \tilde{\xi}(t)|^2 &\leq |w(t) - \tilde{w}(t)|^2 \\ &+ 2\int_0^t (w(t) - \tilde{w}(t) - w(s) + \tilde{w}(s), \Phi(ds) - \tilde{\Phi}(ds)). \end{aligned}$$

Tanaka [26] obtained the following result.

Proposition 3.2 [26, Theorem 4.1] If there exists a constant K > 0 such that

$$\begin{aligned} \|\sigma(t,x) - \sigma(t,y)\| &\leq K|x-y|, \quad \|b(t,x) - b(t,y)\| \leq K|x-y|, \\ \|\sigma(t,x)\| &\leq K(1+|x|^2)^{1/2}, \quad \|b(t,x)\| \leq K(1+|x|^2)^{1/2}, \end{aligned}$$

then there exists a (pathwise) unique \mathcal{F}_t -adapted solution (3.3) for any $x \in \overline{D}$.

Later, Lions-Sznitman [20] generalized the results of [26]. Now, we follow the idea of [7] to prove the existence of periodic solution in distributional sense on bounded domain. Note that the bounded domain with reflection boundary is similar to the whole space, the proof is similar to that of [7]. We only write out the difference. Due to that nothing is lost in the bounded domain, so the probability measure on \overline{D} will be always 1. Using this fact, we can obtain a similar theorem on bounded domain to [24, Theorem Page 9]. And thus Lemmas 2.3 and 2.4 in [7] hold for the bounded domain.

Let $\mathcal{P}(D)$ be the set of Borel probability measures on D. We denote the law of X on D by $\mu : \mathbb{R} \to \mathcal{P}(\overline{D})$. Assume there exists a stochastic process L such that the solution Y(t) on \mathbb{R}_+ of (3.3) satisfying

$$|Y(nT)| \le L, \quad \mathbb{E}|L|^2 < \infty, \quad n = 1, 2, \cdots, .$$
 (3.4)

We borrow symbols from the [7]. $P \circ [Y(t)]^{-1}$ denotes the distribution of Y(t). Similar to Section 2 of [7], we define the d_{BL} which means the distance of bounded and Lipschitz function.

$$\|h\|_{\infty} = \sup_{\bar{D}} |h(x)|, \quad \|h\|_{L} = \sup_{x,y\in\bar{D},x\neq y} \{\frac{|h(x) - h(y)|}{|x - y|}\}, \\\|h\|_{BL} = \max\{\|h\|_{\infty}, \, \|h\|_{L}\}, \quad d_{BL}(\mu,\nu) = \sup_{\|h\|_{BL}\leq 1} \left|\int hd(\mu - \nu)\right|$$

for all $\mu, \nu \in \mathcal{P}(\bar{D})$ and all Lipschitz continuous real-valued functions h on \bar{D} . It is easy to check that $(d_{BL}, \mathcal{P}(\bar{D}))$ is a complete metric space, see [9, Page 390] for details. The main result is the following theorem.

Theorem 3.1 Let b and σ be continuous functions which are T-periodic in the time variable and satisfy the assumptions of Proposition 3.2. If (3.4) holds and

$$\lim_{k \to \infty} \frac{1}{n_k + 1} \sum_{m=0}^{n_k} d_{BL} (P \circ [Y((m+1)T)\mathbf{1}_{A_m}]^{-1}, P \circ [Y(mT)\mathbf{1}_{A_m}]^{-1}) = 0,$$
(3.5)

where Y(t) is a solution of (3.3), A_m is defined as in (3.7) and $\{n_k\}$ is a sequence of integers tending to $+\infty$ and d_{BL} is a metric, then there exists an L^2 -bounded T-periodic solution in distribution sense of (3.3).

$$X_k(0,\omega) = Y(\chi_k(\omega)), \quad X_k(t,\omega) = Y(t + \chi_k(\omega), \omega),$$

where $\omega \in \Omega$, χ_k is a random variable independent of B_t and $Y(0,\omega)$ such that $P(\chi_k = nT) = \frac{1}{k+1}$, $n = 0, 1, \dots, k$. Due to the functions b and σ are T-periodic in time variable and the fact that Φ_t just depends on X_t , X_k is still a solution of

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)d\tilde{B}_t + d\Phi_t, \quad \tilde{B}_t = B(t + nT) - B(nT),$$

where \tilde{B}_t has the same distribution with B_t . Similar to [7], using the fact that χ_k is independent of \tilde{B}_t , we have

$$P(X_k(t) \in A_0) = \frac{1}{k+1} \sum_{n=0}^k P(Y(t+nT) \in A_0),$$
(3.6)

where $A_0 \subset \overline{D}$ is a Borel set. It follows from (3.4), (3.6) and Chebyshev's inequality that

$$P(|X_k(0,\omega)| > R) \le \frac{1}{k+1} \sum_{n=0}^k \frac{\mathbb{E}|Y(nT)|^2}{R^2} \le \frac{C}{R^2} \to 0 \text{ as } R \to \infty$$

Applying Skorokhod' Lemma ([7, Lemmas 2.3 and 2.4]), we have that in some probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ there exists a sequence $\tilde{X}_k(0, \tilde{\omega})$ $(k = 0, 1, \cdots)$ with the same distribution as $X_k(0, \omega)$ such that some subsequence $\{\tilde{X}_{n_k}(0, \tilde{\omega})\}_{k=0,1,\cdots}$ converges in probability to $\tilde{X}(0, \tilde{\omega})$. Also, we can construct random variables $X_k(\omega)$ and $X(\omega)$ on the space (Ω, \mathcal{F}, P) , whose joint distribution is the same as the joint distribution of $\tilde{X}_k(\tilde{\omega})$ and $\tilde{X}(\tilde{\omega})$. Notice that $\tilde{X}_{n_k}(0, \tilde{\omega})$ has the same distribution as $X_{n_k}(0, \omega)$, and thus we have $\mathbb{E}|\tilde{X}_{n_k}(0, \tilde{\omega})|^2 = \mathbb{E}|X_{n_k}(0, \omega)|^2$, $|\tilde{X}_{n_k}(0, \tilde{\omega})| \leq L$ and $\mathbb{E}|L|^2 < \infty$. The Vitali's theorem implies that $\mathbb{E}|\tilde{X}_{n_k}(0, \tilde{\omega}) - \tilde{X}(0, \tilde{\omega})|^2 \to 0$ as $k \to \infty$.

Let $X_{n_k}(t)$ be the solution of

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)d\tilde{B}_t + d\Phi_t$$

with initial data $\tilde{X}_{n_k}(0,\tilde{\omega}) = X_{n_k}(\tilde{\omega})$ on the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$. Note that

$$\begin{split} \tilde{X}_{n_k}(t) - \tilde{X}(t) &= \tilde{X}_{n_k}(\tilde{\omega}) - \tilde{X}(\tilde{\omega}) + \int_0^t (b(s, \tilde{X}_{n_k}(t)) - b(s, \tilde{X}(s))) ds \\ &+ \int_0^t (\sigma(s, \tilde{X}_{n_k}(t)) - \sigma(s, \tilde{X}(s))) d\tilde{B}_s + \Phi_t - \tilde{\Phi}_t, \end{split}$$

we have

$$\begin{aligned} |\tilde{X}_{n_k}(t) - \tilde{X}(t)|^2 &\leq 3|\tilde{X}_{n_k}(\tilde{\omega}) - \tilde{X}(\tilde{\omega})|^2 + 3|\int_0^t (b(s, \tilde{X}_{n_k}(t)) - b(s, \tilde{X}(s)))ds|^2 \\ &+ 3|\int_0^t (\sigma(s, \tilde{X}_{n_k}(t)) - \sigma(s, \tilde{X}(s)))d\tilde{B}_s + \Phi_t - \tilde{\Phi}_t|^2. \end{aligned}$$

Let

$$\begin{aligned} \xi(t) &= w(t) + \Phi_t, \quad \tilde{\xi}(t) = \tilde{w}(t) + \tilde{\Phi}_t, \\ w(t) &= \int_0^t \sigma(s, \tilde{X}_{n_k}(t)) d\tilde{B}_s, \quad \tilde{w}(t) = \int_0^t \sigma(s, \tilde{X}(t)) d\tilde{B}_s \end{aligned}$$

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Then applying Proposition 3.1, using Itô isometry, and noting that w(t) is a martingale with respect to $\tilde{\mathcal{F}}$, we have (also see the proof of [26, Theorem 4.1], and here we use the reason why "the reminder" disappear in (4.4) on Page 175)

$$\begin{split} \mathbb{E}|\tilde{X}_{n_{k}}(t) - \tilde{X}(t)|^{2} &\leq 3\mathbb{E}|\tilde{X}_{n_{k}}(\tilde{\omega}) - \tilde{X}(\tilde{\omega})|^{2} + 3t\mathbb{E}\int_{0}^{t} \|b(s,\tilde{X}_{n_{k}}(s)) - b(s,\tilde{X}(s))\|^{2} ds \\ &\quad + 3\mathbb{E}\Big|\int_{0}^{t} (\sigma(s,\tilde{X}_{n_{k}}(t)) - \sigma(s,\tilde{X}(s))) d\tilde{B}_{s}\Big|^{2} \\ &\quad + 6\mathbb{E}\int_{0}^{t} (w(t) - \tilde{w}(t) - w(s) + \tilde{w}(s), \Phi(ds) - \tilde{\Phi}(ds)) \\ &\leq 3\mathbb{E}|\tilde{X}_{n_{k}}(\tilde{\omega}) - \tilde{X}(\tilde{\omega})|^{2} + 3tK^{2}\mathbb{E}\int_{0}^{t} |\tilde{X}_{n_{k}}(s)) - \tilde{X}(s)|^{2} ds \\ &\quad + 6\mathbb{E}\int_{0}^{t} |\sigma(s,\tilde{X}_{n_{k}}(t)) - \sigma(s,\tilde{X}(s))|^{2} ds \\ &\leq 3\mathbb{E}|\tilde{X}_{n_{k}}(\tilde{\omega}) - \tilde{X}(\tilde{\omega})|^{2} + 3K^{2}(t+2)\mathbb{E}\int_{0}^{t} |\tilde{X}_{n_{k}}(s)) - \tilde{X}(s)|^{2} ds, \end{split}$$

where we used the fact that (independent increment of Brownian motion)

$$\mathbb{E}\int_0^t (w(t) - \tilde{w}(t) - w(s) + \tilde{w}(s), \Phi(ds) - \tilde{\Phi}(ds)) = 0.$$

By Gronwall's inequality, we have

$$\mathbb{E}|\tilde{X}_{n_k}(t) - \tilde{X}(t)|^2 \le 3\mathbb{E}|\tilde{X}_{n_k}(\tilde{\omega}) - \tilde{X}(\tilde{\omega})|^2 e^{3(t+2)K^2} \to 0 \text{ as } k \to \infty.$$

Since the uniqueness of weak solutions implies the uniqueness of laws, we have

$$P \circ [X_{n_k}(t)]^{-1} = P \circ [\tilde{X}_{n_k}(t)]^{-1} \to P \circ [\tilde{X}(t)]^{-1}$$

unformly on [0,T]. Moreover, we can replace $\tilde{X}(0,\tilde{\omega})$ on $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{P})$ by $X(0,\tilde{\omega})$ on (Ω,\mathcal{F},P) with the same law. Then the solution X(t) admits the same distribution of $\tilde{X}(t)$ by weak uniqueness of the equation (3.3). It suffices to prove that

$$P \circ [X(T)]^{-1} = P \circ [X(0)]^{-1}.$$

Denote

$$A_m = \{ \omega \in \Omega : \quad \chi_{n_k}(\omega) = mT \}, \quad m = 0, 1, \cdots, n_k,$$
(3.7)

then we have

$$P(A_m) = \frac{1}{n_k + 1}, \quad m = 0, 1, \cdots, n_k.$$

By using the above equality, we get

$$\int_{\Omega} \phi(X_{n_k}(T))dP = \frac{1}{n_k + 1} \sum_{m=0}^{n_k} \int_{A_m} \phi(Y((m+1)T))dP.$$
(3.8)

Set

$$1_{A_m}(\omega) = \begin{cases} 1, & \omega \in A_m, \\ 0, & \omega \notin A_m. \end{cases}$$

It follows from (3.5), (3.6) and (3.8) that (see the proof of [7] in Page 292 for details)

$$\begin{aligned} &d_{BL}(P \circ [X(T)]^{-1}, P \circ [X(0)]^{-1}) \\ &= \lim_{k \to \infty} d_{BL}(P \circ [X_{n_k}(T)]^{-1}, P \circ [X_{n_k}(0)]^{-1}) \quad \text{(by the definition)} \\ &= \lim_{k \to \infty} \sup_{\|\phi\|_{BL} \le 1} \left| \int_D \phi dP \circ [X_{n_k}(T)]^{-1} - \int_D \phi dP \circ [X_{n_k}(0)]^{-1} \right| \\ &= \lim_{k \to \infty} \sup_{\|\phi\|_{BL} \le 1} \left| \int_D \phi (X_{n_k}(T)) dP - \int_\Omega \phi (X_{n_k}(0)) dP \right| \\ &= \lim_{k \to \infty} \sup_{\|\phi\|_{BL} \le 1} \left| \frac{1}{n_k + 1} \sum_{m=0}^{n_k} \int_{A_m} \phi (Y((m+1)T)) dP - \frac{1}{n_k + 1} \sum_{m=0}^{n_k} \int_{A_m} \phi (Y(mT)) dP \right| \\ &= \lim_{k \to \infty} \sup_{\|\phi\|_{BL} \le 1} \left| \frac{1}{n_k + 1} \sum_{m=0}^{n_k} \int_{A_m} [\phi(Y((m+1)T)) - \phi(Y(mT))] dP \right| \\ &\leq \lim_{k \to \infty} \sup_{\|\phi\|_{BL} \le 1} \frac{1}{n_k + 1} \sum_{m=0}^{n_k} \left| \int_{A_m} [\phi(Y((m+1)T)) - \phi(Y(mT))] dP \right| \\ &\leq \lim_{k \to \infty} \frac{1}{n_k + 1} \sum_{m=0}^{n_k} d_{BL} (P \circ [Y((m+1)T)1_{A_m}]^{-1}, P \circ [Y(mT)1_{A_m}]^{-1}) \\ &= 0, \end{aligned}$$

that is to say, X(T) has the same distribution as X(0).

Define the function $z: \mathbb{R}_+ \to \bar{D}$ by

$$z(t) = Y(t - n_t T),$$

where $n_t = \max\{n \in \mathbb{N} | nT \le t\}$. Then z(t) is a *T*-periodic solution to (3.3). The proof is complete. \Box

Remark 3.1 Comparing with the assumptions in Theorem 3.1 with [7, Theorem 1.2], one can find there is a little difference from [7, Theorem 1.2]. The reason is that in (3.9), the second last equality we use an equality, which different from [7], where they used the following inequality

$$\lim_{k \to \infty} \sup_{\|\phi\|_{BL} \le 1} \frac{1}{n_k + 1} \sum_{m=0}^{n_k} \left| \int_{A_m} [\phi(Y((m+1)T)) - \phi(Y(mT))] dP \right|$$

$$\leq \lim_{k \to \infty} \sup_{\|\phi\|_{BL} \le 1} \frac{1}{n_k + 1} \sum_{m=0}^{n_k} d_{BL} (P \circ [Y((m+1)T)]^{-1}, P \circ [Y(mT)]^{-1}).$$

Noting that

$$d_{BL}(P \circ [Y((m+1)T)1_{A_m}]^{-1}, P \circ [Y(mT)1_{A_m}]^{-1}) \le d_{BL}(P \circ [Y((m+1)T)]^{-1}, P \circ [Y(mT)]^{-1}),$$

implies that the condition (3.5) is weaker than [7, (5)].

On the other hand, if $\sigma = 0$, the condition (3.5) becomes

$$\lim_{k \to \infty} \sup_{\|\phi\|_{BL} \le 1} \frac{1}{n_k + 1} \sum_{m=0}^{n_k} 1_{A_m} |Y((n_k + 1)T) - Y(0)| = 0.$$

which is an extension assumption to Halanay [2].

Similarly, if we define

$$P(A_m) = p_m,$$

then the condition (3.5) becomes

$$\lim_{k \to \infty} \sum_{m=0}^{n_k} p_m d_{BL} (P \circ [Y((m+1)T)1_{A_m}]^{-1}, P \circ [Y(mT)1_{A_m}]^{-1}) = 0.$$

For different choice of p_m , we can get the different periodic solution, and thus there are infinity periodic solutions in distributional sense.

One can also obtain the existence of periodic solution to (4.11) similar to [7, Theorem 1.1]. Base on the Theorem 3.1, one can establish the uniqueness of periodic solution under similar assumptions to [7, Theorem 1.4] and give the Lyapunov functional to verify the existence of periodic solution to (3.3), see [7, Theorem1.3]. Due to similarity, we omit the details to readers.

4 Existence of periodic solutions on bounded domain and in the whole space

In this section, we obtain some properties of PDF by considering the existence of periodic solutions to the Fokker-Planck equations. In order to establish the desired results, we divide this section into two parts.

4.1 Bounded domain with Dirichlet boundary condition

The reason why we first consider the Dirichlet boundary condition problem is the assumptions on $(\mathcal{A}, \mathcal{B})$. More precisely, in Section 2, we make the assumption that $\mathcal{B} = \mathcal{B}(x, D)$ is independent of $t \in [0, T]$. In this subsection, we assume that a stochastic process X_t satisfies

$$\begin{cases} dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \\ X_0 = x, \end{cases}$$
(4.1)

where b is an d-dimensional vector function, σ is an $d \times m$ matrix function and B_t is an m-dimensional Brownian motion. Our aim is to study the properties of PDF by considering the existence of periodic solution to the corresponding Fokker-Planck equation. Throughout this subsection, we assume the particle (in the system) will die if it touch the boundary. That is to say, the PDF of this system satisfies the following evolution equation

$$\begin{cases} \partial_t p(t,x) = \mathcal{A}^* p(t,x), & \text{ in } D \times (0,T], \\ p = 0, & \text{ on } \partial D \times (0,T], \\ p(0,x) = p_0(x), & \text{ in } D, \end{cases}$$

$$(4.2)$$

where

$$\mathcal{A}^* p = -\sum_i \frac{\partial}{\partial x_i} (b_i p) + \frac{1}{2} \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} ((\sigma \sigma^T)_{ij} p).$$

In order to get the properties of p in (4.2), we need consider the following auxiliary equation

$$\begin{cases} \partial_t u(t,x) = \mathcal{A}^* u(t,x) + \mu u, & \text{ in } D \times (0,T], \\ u = 0, & \text{ on } \partial D \times (0,T], \\ u(0,x) = u(T,x), & \text{ in } D. \end{cases}$$

$$(4.3)$$

In order to get the existence of solution of (4.3), we first consider the following initial boundary problem

$$\begin{cases} \partial_t u(t,x) = \mathcal{A}^* u(t,x) + \mu u(t,x), & in \ D \times (0,T], \\ u = 0, & on \ \partial D \times (0,T], \\ u(0,x) = u_0(x), & in \ D, \end{cases}$$
(4.4)

where $u_0(x)$ is a fixed function and will be given later. It is easy to see that there exists an evolution operator U(t, s) such that the solution of (4.4) can be represented in the following form (see section 13 in [15])

$$u(t) = U(t,0)u_0 + \mu \int_0^t U(t,s)u(s)ds.$$

If we set $v(t, x) = e^{-\mu t}u(t, x)$, then we have

$$v(t,x) = U(t,0)v_0 = U(t,0)u_0$$

that is to say,

$$u(t,x) = e^{\mu t} U(t,0) u_0.$$

It is easy to check that v(t, x) is a solution of (4.2) with $p_0 = u_0$.

We assume that

(C2) The operator \mathcal{A}^* is a uniform elliptic operator, $(\sigma\sigma^T)_{ij}(t,x) \in C^{2+\theta,1+\frac{\theta}{2}}(\bar{D}\times\mathbb{R}), b_i(t,x) \in C^{1+\theta,\frac{1+\theta}{2}}(\bar{D}\times\mathbb{R})$, and both $(\sigma\sigma^T)_{ij}(t,x)$ and $b_i(t,x)$ are T-periodic in t.

Set K = U(T, 0). Then the Krein-Rutman theorem [15, Theorem 7.2] implies that r := spr(K) > 0. Proposition 2.3 yields that $\mu > 0$ if r < 1.

Lemma 4.1 Assume that the condition (C2) holds. Assume further that

$$a_0(t,x) := -\frac{1}{2} \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} (\sigma \sigma^T)_{ij} + divb(t,x) \ge 0.$$

Then 0 < r < 1, where r := spr(K).

Proof. For completely, we give the outline of the proof. To show that r < 1, let $u_0 \in W_0^{2,p}(D)$, $u_0 \gg 0$ be a principal eigenfunction of K, i.e., $Ku_0 = ru_0$. Then $u := U(\cdot, 0)u_0$ solves

$$\begin{cases} \partial_t u(t,x) - \mathcal{A}^* u(t,x) = 0, & \text{ in } D \times (0,T], \\ u = 0, & \text{ on } \partial D \times (0,T], \\ u(0,x) = u_0(x), & \text{ in } D. \end{cases}$$

If $a_0 \ge 0$ in $\overline{D} \times [0,T]$ and $u_0 > 0$ in $W_0^{2,p}(D)$, the Propositions 13.1 and 13.3 and Remark 13.2 in [15] that $U(t,\tau)u_0 \gg 0$ in $W_0^{2,p}(D)$ for $\tau < t \le T$. Now let $v := ||u_0||_{C(\overline{D})} - u$, then v satisfies

$$\begin{cases} \partial_t v(t,x) - \mathcal{A}^* v(t,x) = a_0 \|u_0\|_{C(\bar{D})} \ge 0, & \text{ in } D \times (0,T], \\ v \ge 0, & \text{ on } \partial D \times (0,T], \\ v(0,x) \ge 0, & \text{ in } D, \end{cases}$$
(4.5)

and hence $v \gg 0$ in $W_0^{2,p}(D)$ for each $0 < t \le T$ by the Propositions 13.1 and 13.3 and Remark 13.2 in [15]. In particular,

$$r \|u_0\|_{C(\bar{D})} = \|Ku_0\|_{C(\bar{D})} = \|u(T)\|_{C(\bar{D})} < \|u_0\|_{C(\bar{D})},$$

which implies that r < 1. \Box

Theorem 4.1 Under the assumptions of Lemma 4.1, there exists a unique solution to the equation (4.3). Hence the solution of (4.2) satisfies exponential decay for any fixed point in D as time goes to infinity under the special initial data. That is to say, the solution p(t, x) has the following property:

$$p(nT, x) = e^{-\mu nT} p_0(x), \quad x \in D, \ n \in Z,$$

where $\mu > 0$ is given as in (4.3) and $p_0(x)$ satisfies $Kp_0 = rp_0$ with K = U(T, 0).

Proof. It follows from Lemma 4.1 that the principal eigenvalue of K satisfies 0 < r < 1. Proposition 2.3 implies that

$$\mu = -\frac{1}{T}\log r$$

is an eigenvalue of (4.3) with the positive eigenfunction

$$u(t) = e^{\mu t} U(t,0) u_0.$$

Take u_0 be the principal eigenfunction of K (take $p_0(x)$ be the principal eigenfunction of corresponding K in equation (4.3)). And thus we have

$$u(T) = e^{\mu T} U(T, 0) u_0 = \frac{1}{r} K u_0 = u(0).$$

That is to say, u(t) is the solution of (4.3). The uniqueness of principal eigenvalue implies the uniqueness of solution of (4.3). By *T*-periodicity of $\mathcal{A}(t)$, we have $U(t,\tau) = U(t + nT, \tau + nT)$, $n \in \mathbb{Z}$. Noting that the solution of (4.2) can be written as

$$p(t,x) = e^{-\mu t} u(t,x),$$

we have

$$p(T,x) = e^{-\mu T} u(T,x) = e^{-\mu T} u(0,x) = e^{-\mu T} p_0(x).$$

By using the properties

$$U(t,t)=I, \quad U(s,t)U(t,\tau)=U(s,\tau), \quad 0\leq \tau\leq t\leq s\leq T,$$

we have

$$p(nT,x) = U(nT,0)p_0(x) = U(nT,(n-1)T)\cdots U(T,0)p_0(x) = e^{-\mu nT}p_0(x),$$

where we used the fact that $U(T,0)p_0(x) = e^{-\mu T}p_0(x)$. \Box

Remark 4.1 In the proof of Lemma 4.1, we know that the initial data u_0 (or P_0) is special function, that is, u_0 satisfies $Ku_0 = ru_0$. Now, we give an example to show that this is possible. Consider the problem

$$\begin{cases} \partial_t u(t,x) - \Delta u(t,x) = 0, & \text{ in } D \times (0,T], \\ u = 0, & \text{ on } \partial D \times (0,T], \\ u(0,x) = u_0(x), & \text{ in } D. \end{cases}$$

Assume that u_0 satisfies (r > 0)

$$-\Delta u_0 = r u_0, \quad in \ D, \qquad u|_{\partial D} = 0,$$

and by using the fact

$$u(t,x) = e^{t\Delta_D} u_0$$

where Δ_D denotes the Laplace operator with Dirichlet boundary, then we get (by Taylor expansion)

$$u(t,x) = \sum_{i=0}^{\infty} \frac{(t\Delta_D)^i}{i!} u_0 = \sum_{i=0}^{\infty} \frac{(-tr)^i}{i!} u_0 = e^{-tr} u_0.$$

And thus the solution of the following equation

$$\left\{ \begin{array}{ll} \partial_t v(t,x) - \Delta v(t,x) = \mu v(t,x), & \mbox{ in } D \times (0,T], \\ v = 0, & \mbox{ on } \partial D \times (0,T], \\ v(0,x) = u_0(x), & \mbox{ in } D. \end{array} \right.$$

can be written as

$$v(t,x) = e^{\mu t}u(t,x) = e^{t(\mu-r)}u_0$$

If we want to get $v(T, x) = v_0(x)$, we will take $\mu = r$. Because there is no concrete value for T, we obtain that the solution u satisfies $u(t, x) = e^{-\mu t}u_0$ for all t > 0 and $x \in D$.

It follows from Theorem 4.1 that it is difficult to obtain the existence of periodic solution to linear parabolic equation. Now we turn to the nonlinear case. In 1998 Pardoux and Zhang proved in [22] a probabilistic formula for the viscosity solution of a system of semilinear PDEs with Neumann boundary condition

$$\begin{cases} \partial_t u + \mathcal{A}(u) + f(t, x, u)(t, x)) = 0, & \text{ in } D \times (0, T], \\ \frac{\partial u}{\partial \nu} + g(t, x, u) = 0, & \text{ on } \partial D \times (0, T], \\ u(T, x) = h(x), & \text{ in } D, \end{cases}$$

where D is an open connected bounded subset of \mathbb{R}^d .

In order to get the existence of periodic solution for Dirichlet problem, we need to consider the nonlinear parabolic equation

$$\begin{cases} \partial_t u(t,x) = \mathcal{A}^* u(t,x) + f(t,x,u), & in \ D \times (0,T], \\ u = 0, & on \ \partial D \times (0,T], \\ u(0,x) = u_0(x), & in \ D. \end{cases}$$
(4.6)

The assumptions on f will be given later.

We first recall some results. In [15], Hess considered the following periodic initial boundary problem

$$\begin{cases} \partial_t u(t,x) + \mathcal{A}(t)u(t,x) = f(x,t,u), & \text{ in } D \times (0,T], \\ \mathscr{B}u := \partial_\nu u + bu = 0, & \text{ on } \partial D \times (0,T], \\ u(0,x) = u(T,x), & \text{ in } D, \end{cases}$$

$$(4.7)$$

where they assumed the function b does not depend on t, and

$$\mathcal{A}(t)u = -\sum_{i,j} a_{ij}(t,x) \frac{\partial^2}{\partial x_j \partial x_k} u + \sum_i a_i(t,x) \frac{\partial}{\partial x_i} u + a_0(t,x)u.$$

They used the upper and lower solution method to prove the existence of periodic solution of (4.7). We first recall the definition of upper (lower) solution.

Definition 4.1 Let $u \in C^{1,0}([0,T] \times \overline{D}) \cap C^{2,1}([0,T) \times D)$. Such a function u is referred to as an upper (lower) solution if

$$\left\{ \begin{array}{ll} \partial_t u(t,x) + \mathcal{A}(t) u(t,x) \geq (\leq) f(x,t,u), & \mbox{ in } D \times (0,T], \\ \mathcal{B} u \geq (\leq) 0, & \mbox{ on } \partial D \times (0,T], \\ u(0,x) \geq (\leq) u(T,x), & \mbox{ in } D, \end{array} \right.$$

Let $\bar{u} \geq \underline{u}$ be the upper and lower solution of (4.7), respectively. We define $\sigma = \min_{[0,T] \times \bar{D}} \underline{u}$ and $\omega = \max_{[0,T] \times \bar{D}} \bar{u}$. Set

$$W^2_{p,\mathscr{B}}(D) = \{ u \in W^2_p(D) : [\partial_{\nu}u + b(x)u = 0 \}, \quad p > d.$$

Proposition 4.1 Suppose $\bar{u} \geq \underline{u}$ are the upper and lower solutions of (4.7), respectively. Let $f(\cdot, \cdot, u) \in C^{\alpha/2, 1+\alpha}([0, T] \times \bar{D})$ be uniformly with respect to $u \in [\sigma, \omega]$ and f(x, 0, 0) = 0 on ∂D . Fixed $p > \max\{d, 1 + \frac{d}{2}\}$. If there exists at least one $u_0 \in W^2_{p,\mathscr{B}}$ satisfying $\underline{u}(0, x) \leq u_0(x) \leq \bar{u}(0, x)$, then the problem (4.7) has at least one solution $u \in C^{1+\alpha/2, 2+\alpha}([0, T] \times \bar{D})$ and satisfies

$$\underline{u} \le u \le \bar{u} \quad on \quad [0,T] \times D.$$

The proof of Proposition 4.1 is standard and we omit it here. Now, by using the Proposition 4.1, we only need find a pair of upper and lower solution to the problem (4.7). In order to do that, we need consider the periodic parabolic eigenvalue problem

$$\begin{cases} \partial_t u(t,x) + \mathcal{A}(t)u(t,x) = \lambda u, & \text{in } D \times (0,T], \\ \mathscr{B}u = 0, & \text{on } \partial D \times (0,T], \\ u(0,x) = u(T,x), & \text{in } D. \end{cases}$$
(4.8)

It is easy to check that if $b \ge 0$ on $\partial D \times (0, T]$, then the maximum principle holds for the problem $(\mathcal{A}, \mathscr{B})$. We will need the following Lemmas.

Lemma 4.2 [15, Proposition 14.4] The principal eigenvalue λ_1 of (4.8) exists uniquely. Furthermore, if $a_0 \geq 0$, and $a_0 \not\equiv 0$ when $\mathscr{B} = \partial_{\nu}$, then $\lambda_1 > 0$. In case $a_0 = 0$ and $\mathscr{B} = \partial_{\nu}$, we have $\lambda_1 = 0$.

We want to know how the principle eigenvalue λ_1 depends on the zero-order term a_0 . Because in Fokker-Planck equations, the role of drift term is reflected in the zero-order term a_0 . In order to do that, we need the following lemma.

Lemma 4.3 [15, Proposition 16.6] For the inhomogeneous problem

$$\partial_t u(t,x) + \mathcal{A}(t)u(t,x) - \lambda u = h, \quad h \ge \neq 0, \tag{4.9}$$

where $h \in \mathbb{F}$ (see Section 2 for the definition of \mathbb{F}). Let λ_1 be the principal eigenvalue λ_1 of (4.8). Then we have

- (i) If $\lambda < \lambda_1$, then the problem (4.9) has a unique solution u and u > 0 in \mathbb{F}_1 ;
- (ii) If $\lambda \geq \lambda_1$, then the problem (4.9) has no positive solution, and no solution at all if $\lambda = \lambda_1$.

By using the above Lemma 4.3, it is easy to prove the following Lemma.

Lemma 4.4 Let $\lambda_1 = \lambda_1(a_0)$ be the principal eigenvalue of (4.8). Then $\lambda(a_0)$ is strictly increasing in a_0 .

Proof. suppose $a_0^1(t,x) \leq a_0^2(t,x)$ and $a_0^1(t,x) \neq a_0^2(t,x)$. Suppose ϕ_i is the corresponding positive eigenfunction to $\lambda_1(a_0^i)$ and $\lambda_1(a_0^i) > 0$, i = 1, 2. Denote $\mathcal{A}_0 = \mathcal{A} - a_0$. We aim to prove that $\lambda_1(a_0^1) < \lambda_1(a_0^2)$. On the contrary, we suppose $\lambda_1(a_0^1) \geq \lambda_1(a_0^2)$, then we have

$$\begin{aligned} \partial_t(\phi_1 - \phi_2) + \mathcal{A}_0(t)(\phi_1 - \phi_2) + a_0^2(\phi_1 - \phi_2) \\ &= \partial_t \phi_1 + \mathcal{A}_0(t)\phi_1 + a_0^2\phi_1 - \lambda_1(a_0^2)\phi_2 \\ &\geq \partial_t \phi_1 + \mathcal{A}_0(t)\phi_1 + a_0^1\phi_1 - \lambda_1(a_0^2)\phi_2 \\ &= \lambda_1(a_0^1)\phi_1 - \lambda_1(a_0^2)\phi_2 \\ &\geq \lambda_1(a_0^1)(\phi_1 - \phi_2), \end{aligned}$$

that is to say,

$$\partial_t(\phi_1 - \phi_2) + \mathcal{A}_0(t)(\phi_1 - \phi_2) + a_0^2(\phi_1 - \phi_2) - \lambda_1(a_0^1)(\phi_1 - \phi_2) =: h \ge \neq 0.$$

By using comparison principle, we deduce that $\phi_1 - \phi_2 > 0$ if $\lambda_1(a_0^1) > \lambda_1(a_0^2)$. We obtain a contradiction with (ii) of Lemma 4.3. \Box

Now, we use the above discussion to solve the problem (4.6).

Theorem 4.2 Suppose there exist two positive constants c_0 and M_0 satisfying

$$a_0(t,x) \ge c_0, \quad f(t,x,0) > 0, \quad f(t,x,\xi) \le 0 \text{ for } \xi \ge M_0, \quad f \in \mathbb{F}.$$

Then the problem (4.6) admits a unique solution.

Proof. The existence of periodic solution is obtained by using Proposition 4.1. We only need find a pair of upper-lower solution of (4.6). Actually, from the assumptions on a_0 and f, we see that $\bar{u} = M \ge M_0$ is an upper solution. Let ϕ be a positive eigenfunction corresponding to $\lambda_1(a_0)$, i.e., ϕ is the solution of (4.8) with $\lambda = \lambda_1(a_0)$. Take $\varepsilon > 0$ and set $\underline{u} = \varepsilon \phi$, then \underline{u} is s lower solution of (4.6). Moreover, \underline{u} and \bar{u} are the ordered upper and lower solutions of (4.6) if we choose $\varepsilon \ll 1$ and $M \gg 1$. According to Proposition 4.1, the problem (4.6) has at least one solution u satisfying $\varepsilon \phi \le u \le M$. The proof of uniqueness follows from the comparison principle. \Box

Remark 4.2 Following Lemma 4.2, we can see that if $a_0 = 0$, then the problem (4.8) with Neumann boundary admits the principle eigenvalue $\lambda_1 = 0$ and eigenfunction $\phi = 1$.

A typical example in 4.2 is $f(t,x) = f_1(t,x) - uf_2(t,x)$ with $f_i(t,x) \in (N_1, N_2)$ for all $(t,x) \in [0,T] \times \overline{D}$ and $N_1 > 0$.

The assumptions on f can be given weaker, but it is not our aim. See [28] for $f = u(h_1(t, x) - h_2(t, x)u)$ and h_i , i = 1, 2, are some functions.

4.2 Bounded domain with reflecting boundary condition

It is easy to see that there is no periodic solution to equation (4.2). Now, we consider another case. We assume that a stochastic process X_t satisfies

$$\begin{cases} dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \\ X_0 = x, \end{cases}$$
(4.10)

with the reflecting boundary condition [14, Section 5.1.1], where b is an d-dimensional vector function, σ is an $d \times m$ matrix function and B_t is an m-dimensional Brownian motion. Our aim is to study the existence of periodic solution to the corresponding Fokker-Planck equation, i.e., the existence of the following equations

$$\begin{cases} \partial_t p(t,x) = \mathcal{A}^* p(t,x), & \text{in } D \times (0,T], \\ (b \cdot \nu) p - p_{\partial D} \cdot \nu = 0, & \text{on } \partial D \times (0,T], \\ p(0,x) = p(T,x), & \text{in } D, \end{cases}$$
(4.11)

where

$$\mathcal{A}^* p = -\sum_i \frac{\partial}{\partial x_i} (b_i p) + \frac{1}{2} \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} ((\sigma \sigma^T)_{ij} p),$$

$$p_{\partial D} = (\sum_i \frac{\partial}{\partial x_i} ((\sigma \sigma^T)_{i1} p), \cdots, \sum_i \frac{\partial}{\partial x_i} ((\sigma \sigma^T)_{id} p)).$$

It follows the results of [8] that for the existence of probability density function, the necessary condition is that operator \mathcal{A}^* is a uniform elliptic operator. Throughout this section we assume that the operator \mathcal{A}^* is a uniform elliptic operator.

We first consider a special case. It is noted that most of work on the periodic parabolic problem the authors assumed the boundary function b does not depend on the time t. Because under this assumption, one can apply the standard theory of evolution equation of "parabolic type", see [1, 28]. So we assume

$$(\sigma\sigma^T)_{ij}(t,x) = \alpha(t), \quad 0 < \alpha_0 \le \alpha(t) \le \alpha_1, \quad b(t,x) = \alpha(t) \times b_0(x).$$

Then the problem (4.11) becomes

$$\begin{cases} \partial_t p(t,x) = \mathcal{A}^* p(t,x), & \text{in } D \times (0,T], \\ \partial_\nu p - b_0 p = 0, & \text{on } \partial D \times (0,T], \\ p(0,x) = p(T,x), & \text{in } D. \end{cases}$$

$$(4.12)$$

We can use the similar method to deal with the problem (4.12).

It is well known that the upper-lower method is not suitable to the linear parabolic equation. The reason is that if we find an upper solution ϕ for a linear parabolic equation, then $\lambda \phi$ will be an upper solution for any $\lambda > 0$. Hence we can not obtain the existence of non-negative solution for this linear parabolic equation. And in this subsection, we only consider the one dimensional case because it can be calculated clearly. We want to obtain the existence of periodic solution of (4.11). For simplicity, we denote $a(t, x) = (\sigma \sigma^T)(t, x)$ and D = (0, 1). Due to the operator \mathcal{A}^* is a uniform elliptic operator, we have a(t, x) > 0 for $(t, x) \in [0, T] \times [0, 1]$. The one dimensional problem will be written as

$$\begin{cases} \partial_t p(t,x) - (a(t,x)p(t,x))_{xx} + (b(t,x)p(t,x))_x = 0, & in \ D \times (0,T], \\ (a(t,x)p(t,x))_x - bp = 0, & on \ \partial D \times (0,T], \\ p(0,x) = p(T,x), & in \ D. \end{cases}$$
(4.13)

We first assume that

$$(a(t,x)p(t,x))_x - bp = 0, \text{ in } [0,T] \times [0,1],$$

Then we get

$$p(t,x) = \exp\left(\int \frac{b - a_x}{a} dx\right),\tag{4.14}$$

which implies that $p_t = 0$, i.e.,

$$\int \frac{a(b_t - a_{xt}) - a_t(b - a_x)}{a^2} dx = 0.$$
(4.15)

That is to say, the stochastic process has stationary probability measure. Summing the above discussion, we have

Theorem 4.3 Suppose (4.15) hold. Then problem (4.13) admits a solution p satisfying (4.14).

For $d \ge 2$, we can not calculate it clearly. But we guess there exists a positive periodic solution to problem (4.11). Indeed, it follows from (4.11) that

$$\int_D p_0(x)dx = \int_D p(t,x)dx, \quad \forall t > 0.$$
(4.16)

The existence of periodic solution to (4.11) is equivalent to getting p(0, x) = p(T, x) point by point for $x \in D$ from (4.16).

In 2000, Lieberman [17, 18, 19] did a series of work about the periodic solution of parabolic equation on bounded domain. Especially in [19], Lieberman obtained the existence of periodic of the following parabolic equation

$$\begin{cases} u_t - divA(t, x, u, \nabla u) + B(t, x, u, \nabla u) = 0, & in (0, T) \times \partial D, \\ A(t, x, u, \nabla u) \cdot \nu + \psi(t, x, u) = 0, & on (0, T) \times \partial D, \\ u(0, x) = u(T, x), & in D. \end{cases}$$

If b, σ satisfy the conditions of [19, Lemma 2.1], then the (4.11) will admit a periodic solution p.

4.3 Whole space

In this subsection, we consider the existence of periodic solutions of Fokker-Planck equations in the whole space. For a stochastic process X_t satisfies equation (4.1), the corresponding Fokker-Planck equation is the following form

$$\partial_t p = -\sum_i \frac{\partial}{\partial x_i} (b_i p) + \frac{1}{2} \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} ((\sigma \sigma^T)_{ij} p).$$
(4.17)

Furthermore, if the probability density p(t,x) satisfies p(t+T,x) = p(t,x), $\forall (t,x)$, then p(t,x) is call a *T*-periodic solution of (4.17).

By using the method of [13], we will obtain the existence of periodic solution of (4.17). In [13], the author considered the following periodicity problem

$$\begin{cases} u_t - \Delta u = f(t, x, u, u_x), & t > 0, \quad x \in \mathbb{R}^d, \\ u(t, x) = u(t + T, x), & t \ge 0, \quad x \in \mathbb{R}^d, \end{cases}$$

$$(4.18)$$

where $d \ge 2$, $f \in C(\mathbb{R}, \mathbb{R}^d, \mathbb{R}, \mathbb{R}^d)$, $u_x = (u_{x_1}, u_{x_2}, \cdots, u_{x_d})$, f is *T*-periodic function with respect to the time variable t, the period T > 0 is arbitrary chosen and fixed. They got the following result.

Proposition 4.2 Let $d \ge 2$, $n \in \mathbb{N}$ be fixed, T > 0 be fixed, $f \in C(\mathbb{R}, \mathbb{R}^d, \mathbb{R}, \mathbb{R}^d)$. f is T-periodic with respect to the time variable t. Also let $0 \le c_i, l_i, m_i, p_i, q_i, l_i < \infty$, $i = 1, 2, \dots, n$, be fixed constants, $0 \le k_i, n_i < \infty$, $i = 1, 2, \dots, d$, be fixed constants, $b_i(t) \in C(\mathbb{R}_+)$, $g_i(x) \in C(\mathbb{R}^d)$, $\sup_{\mathbb{R}_+} |b_i(t)| < \infty$, $\sup_{\mathbb{R}^d} |g_i(x)| < \infty$, $i = 1, 2, \dots, n$,

$$|f(t, x, u, u_x)| \le \sum_{i=1}^n \left(c_i |b_i(t)|^{p_i} + l_i |u|^{q_i} + m_i |g_i(x)|^{l_i} \right) + \sum_{i=1}^d k_i |u_{x_i}|^{n_i}$$

for every $(t, x, u, u_x) \in (\mathbb{R}, \mathbb{R}^d, \mathbb{R}, \mathbb{R}^d)$. Then the problem (4.18) has a solution $u \in \mathcal{C}^1(\mathbb{R}_+, \mathcal{C}^2(\mathbb{R}^d))$ $(\mathcal{C}^1(\mathbb{R}_+, \mathcal{C}^2(\mathbb{R}^d))$ will be defined later).

Comparing the problem (4.17) and (4.18), we see that the problem (4.17) is a linear problem and problem (4.18) will contain problem (4.17) if $(\sigma\sigma^T)_{ij} = constant$. Firstly, it is remarked that when d = 1, the results in subsection 3.2 also holds for problem (4.17). That is to say, theorem 4.3 holds for (4.17). In order to get the existence of solutions to (4.17) for $d \ge 2$, we suppose that a(t)is continuous positive *T*-periodic function, which is defined on the whole real axis \mathbb{R} . We denote $[a] = \frac{1}{T} \int_0^T a(t) dt$. For $D \subset \mathbb{R}^d$, set

$$\mathcal{C}^{1}([0,T],\mathcal{C}^{2}(D)) = \{u(t,x) : continuously - differentiable in t \in \mathbb{R}_{+}, \\ twice continuously - differentiable in x \in D, \\ u(t+T,x) = u(t,x) \text{ for } t \ge 0 \text{ and } x \in D\}.$$

Fix $0 < Q < \infty$, $0 < \varepsilon < 1$ and denote

$$F = \max\left\{\sup_{t\geq 0, x\in\mathbb{R}^{d}} divb(t, x), \sup_{t\geq 0, x\in\mathbb{R}^{d}} \sum_{i=1}^{d} |b_{i}(t, x)|, \sup_{t\geq 0, x\in\mathbb{R}^{d}} \sum_{i,j=1}^{d} |(\sigma\sigma^{T})_{ij}(t, x)|\right\},\$$

$$G = \max\left\{\sup_{0\leq t,s\leq T} \frac{e^{-[a]T}}{1-e^{-[a]T}} e^{\int_{t}^{t+s} a(r)dr}, \sup_{0\leq t\leq T} a(t), \sup_{0\leq t\leq T} e^{\int_{0}^{t} a(r)dr}\right\},\$$

$$S = (2d+1)(F^{2}+Q^{2}).$$

If we assume $\sup_{t,x} |b_i(t,x)| < \infty$, $\sup_{t,x} |(\sigma\sigma^T)_{ij}(t,x)| < \infty$, $i, j = 1, 2, \dots, d$, we have that $0 \le F < \infty$. Note that a(t) is a continuous positive *T*-periodic function, which is defined on the whole real axis \mathbb{R} , we conclude that $0 \le G < \infty$. From here and $0 < Q < \infty$ we get $0 \le S < \infty$.

The main result of this subsection is the following theorem.

Theorem 4.4 Assume that the SDEs (4.1) admits a probability density function p(t, x) satisfying (4.17). Assume further that $F < \infty$ and b_i , σ are *T*-periodic functions, then there exists a periodic solution $p(t, x) \in C^1(\mathbb{R}_+, C^2(\mathbb{R}^d))$ satisfying (4.17) with p(t + T, x) = p(t, x) for $t \ge 0$ and $x \in \mathbb{R}^d$. We choose the constants A_i , $i = 1, 2, \cdots, d$, so that $A_i > 0$ satisfying

$$\begin{split} &(A_{1}\cdots A_{d})^{2}Q + GT \Big[G(A_{1}\cdots A_{d})^{2}Q + \sum_{i=1}^{d} (A_{1}\cdots A_{i-1}A_{i+1}\cdots A_{d})^{2}SQ \\ &+ \sum_{i,j=1, i\neq j}^{d} A_{i}A_{j}(A_{1}\cdots A_{i-1}A_{i+1}\cdots A_{j-1}A_{j+1}\cdots A_{d})^{2}SQ \\ &+ \sum_{i=1}^{d} A_{i}(A_{1}\cdots A_{i-1}A_{i+1}\cdots A_{d})^{2}SQ \Big] \leq (1-\varepsilon)Q, \\ &(A_{1}\cdots A_{d})^{2}Q + (G^{3}T+1) \Big[G(A_{1}\cdots A_{d})^{2}Q + \sum_{i=1}^{d} (A_{1}\cdots A_{i-1}A_{i+1}\cdots A_{d})^{2}SQ \\ &+ \sum_{i,j=1, i\neq j}^{d} A_{i}A_{j}(A_{1}\cdots A_{i-1}A_{i+1}\cdots A_{j-1}A_{j+1}\cdots A_{d})^{2}SQ \\ &+ \sum_{i=1}^{d} A_{i}(A_{1}\cdots A_{i-1}A_{i+1}\cdots A_{d})^{2}SQ \Big] \leq (1-\varepsilon)Q, \\ &(A_{1}\cdots A_{k-1}A_{k+1}\cdots A_{d})^{2}A_{k}Q + (G^{3}T+1) \Big[GT(A_{1}\cdots A_{k-1}A_{k+1}\cdots A_{d})^{2}A_{k}Q \\ &+ \sum_{i=1, i\neq k}^{d} (A_{1}\cdots A_{i-1}A_{i+1}\cdots A_{k-1}A_{k+1}\cdots A_{d})^{2}A_{k}SQ + (A_{1}\cdots A_{i-1}A_{i+1}\cdots A_{d})^{2}SQ \\ &+ \sum_{i,j=1, i\neq j\neq k}^{d} A_{i}A_{j}A_{k}(A_{1}\cdots A_{i-1}A_{i+1}\cdots A_{k-1}A_{k+1}\cdots A_{d})^{2}SQ \\ &+ 2\sum_{j,k=1, k\neq j}^{d} A_{k}A_{j}(A_{1}\cdots A_{i-1}A_{j+1}\cdots A_{k-1}A_{k+1}\cdots A_{d})^{2}SQ \\ &+ \sum_{i=1, i\neq k}^{d} A_{i}A_{k}(A_{1}\cdots A_{i-1}A_{i+1}\cdots A_{k-1}A_{k+1}\cdots A_{d})^{2}SQ \\ &+ \sum_{i=1, i\neq k}^{d} A_{i}A_{k}(A_{1}\cdots A_{i-1}A_{i+1}\cdots A_{k-1}A_{k+1}\cdots A_{d})^{2}SQ \\ &+ (A_{1}\cdots A_{i-1}A_{i+1}\cdots A_{d})^{2}SQ \Big] \leq (1-\varepsilon)Q \quad \forall j = 1, 2, \cdots, d, \\ &A_{0} = A_{n+1} = 1. \end{split}$$

Such choice is suitable if $0 < A_i < 1, i = 1, 2, \dots, d$, is small enough. We set $A = (A_1, A_2, \dots, A_d)$ and

$$B_1 = \{ x \in \mathbb{R}^d : 0 \le x_i \le A_i, i = 1, 2, \cdots, d \}.$$

We first prove that the periodicity problem

$$\begin{cases} \partial_t p = -\sum_i \frac{\partial}{\partial x_i} (b_i p) + \frac{1}{2} \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} ((\sigma \sigma^T)_{ij} p), & t > 0, \quad x \in B_1, \\ p(t, x) = p(t + T, x), & t \ge 0, \quad x \in B_1. \end{cases}$$
(4.20)

We will use the fixed point arguments to prove the existence of solution to (4.20). Following the idea of [13], we give the following lemma.

Lemma 4.5 If $p \in C^1([0,T], C^2(B_1))$ satisfies the equation

$$\int_{x}^{A} \int_{y}^{A} p(t,z) dz dy - \frac{e^{-[a]T}}{1 - e^{-[a]T}} \int_{0}^{T} e^{\int_{t}^{t+s} a(r) dr} \left(a(t+s) \int_{x}^{A} \int_{y}^{A} p(t+s,z) dz dy + \sum_{i=1}^{d} \int_{\bar{x}_{i}}^{A} \int_{\bar{y}_{i}}^{A} (\sigma \sigma^{T})_{ii} (t+s,\hat{z}_{i}) p(t+s,\hat{z}_{i}) d\hat{z}_{i} d\hat{y}_{i} + \sum_{i,j=1, i\neq j}^{d} \int_{x}^{A} \int_{\bar{y}_{i}}^{A} (\sigma \sigma^{T})_{ij} (t+s,\hat{z}_{ij}) p(t+s,\hat{z}_{ij}) d\hat{z}_{ij} dy + \sum_{i=1}^{d} \int_{x}^{A} \int_{\bar{y}_{i}}^{A} [b_{i}(t+s,\check{z}_{i}) p(t+s,\check{z}_{i})] d\check{z}_{i} dy \Big) ds = 0,$$
(4.21)

then p(t, x) is a solution to the problem (4.20). We use the following symbols

$$\begin{split} &\int_{x}^{A} = \int_{x_{1}}^{A_{1}} \int_{x_{2}}^{A_{2}} \cdots \int_{x_{d}}^{A_{d}}, \quad \int_{\bar{x}_{i}}^{A} = \int_{x_{1}}^{A_{1}} \cdots \int_{x_{i-1}}^{A_{i-1}} \int_{x_{i+1}}^{A_{i+1}} \cdots \int_{x_{d}}^{A_{d}}, \\ &\int_{\bar{y}_{ij}}^{A} = \int_{y_{1}}^{A_{1}} \cdots \int_{y_{i-1}}^{A_{i-1}} \int_{y_{i+1}}^{A_{i+1}} \cdots \int_{y_{d}}^{A_{j-1}} \int_{y_{j+1}}^{A_{j+1}} \cdots \int_{x_{d}}^{A_{d}}, \\ &\int_{\bar{y}_{i}}^{A} = \int_{y_{1}}^{A_{1}} \cdots \int_{y_{i-1}}^{A_{i-1}} \int_{y_{i+1}}^{A_{i+1}} \cdots \int_{y_{d}}^{A_{d}}, \quad \check{z}_{i} = (z_{1}, \cdots, z_{i-1}, y_{i}, z_{i+1}, z_{d}), \\ &\hat{w}_{i} = (w_{1}, \cdots, w_{i-1}, x_{i}, w_{i+1}, \cdots, w_{d}), \quad d\hat{w}_{i} = dw_{1} \cdots dw_{i-1} dw_{i+1} \cdots dw_{n}, \quad w = y \text{ or } z, \\ &\hat{z}_{ij} = (z_{1}, \cdots, z_{i-1}, y_{i}, z_{i+1}, \cdots, z_{j-1}, y_{j}, z_{j+1}, z_{d}). \end{split}$$

Proof. Differentiating the (4.21) twice in x_1 , twice in $x_2, \dots,$ twice in x_d and using the periodicity of a, p, b and σ , we can obtain the desired result. See [13, Lemma 2.1] for more details.

Lemma 4.5 implies that the existence of solution to (4.20) is equivalent to the existence of fixed point of L_1 , i.e., $L_1(p) = p$, where

$$\begin{split} L_{1}(p) &= p(t,x) + \int_{x}^{A} \int_{y}^{A} p(t,z) dz dy - \frac{e^{-[a]T}}{1 - e^{-[a]T}} \int_{0}^{T} e^{\int_{t}^{t+s} a(r) dr} \left(a(t+s) \int_{x}^{A} \int_{y}^{A} \int_{y}^{A} \int_{y}^{A} \int_{y}^{A} \int_{y}^{A} \int_{y_{i}}^{A} (\sigma \sigma^{T})_{ii} (t+s,\hat{z}_{i}) p(t+s,\hat{z}_{i}) d\hat{z}_{i} d\hat{y}_{i} \\ &+ \sum_{i,j=1, i \neq j}^{d} \int_{x}^{A} \int_{\bar{y}_{i}}^{A} (\sigma \sigma^{T})_{ij} (t+s,\hat{z}_{ij}) p(t+s,\hat{z}_{ij}) d\hat{z}_{ij} dy \\ &+ \sum_{i=1}^{d} \int_{x}^{A} \int_{\tilde{y}_{i}}^{A} [b_{i} (t+s,\check{z}_{i}) p(t+s,\check{z}_{i})] d\check{z}_{i} dy \Big) ds. \end{split}$$

In order to get the fixed point of L_1 , we define

$$D_{1} = \{ u \in \mathcal{C}^{1}([0,T], \mathcal{C}^{2}(B_{1})) : |u| \leq Q, |u_{t}| \leq Q, |u_{x_{i}}| \leq Q, i = 1, 2, \cdots, d \}, \\ \tilde{D}_{1} = \{ u \in \mathcal{C}^{1}([0,T], \mathcal{C}^{2}(B_{1})) : |u| \leq (1+\varepsilon)Q, |u_{t}| \leq (1+\varepsilon)Q, \\ |u_{x_{i}}| \leq (1+\varepsilon)Q, i = 1, 2, \cdots, d \}.$$

In the set D_1 and \tilde{D}_1 , we define a norm as follows:

$$||u|| = \max\left\{\max_{t \in [0,T], x \in B_1} |u|, \max_{t \in [0,T], x \in B_1} |u_t|, \max_{t \in [0,T], x \in B_1} |u_{x_i}|, i = 1, 2, \cdots, d\right\}.$$

Then D_1 , \tilde{D}_1 and $\mathcal{C}^1([0,T], \mathcal{C}^2(B_1))$ are completely normed spaces with respect to this norm, see Appendix of [13]. We rewrite the operator L_1 in the following form

$$L_1(p) = M_1(p) + N_1(p),$$

where

$$\begin{split} M_{1}(p) &= (1+\varepsilon)p, \\ N_{1}(p) &= -\varepsilon p + \int_{x}^{A} \int_{y}^{A} p(t,z) dz dy - \frac{e^{-[a]T}}{1-e^{-[a]T}} \int_{0}^{T} e^{\int_{t}^{t+s} a(r) dr} \left(a(t+s) \int_{x}^{A} \int_{y}^{A} \int_{y}^{A} dt dy \right) \\ &= p(t+s,z) dz dy + \sum_{i=1}^{d} \int_{\bar{x}_{i}}^{A} \int_{\bar{y}_{i}}^{A} (\sigma\sigma^{T})_{ii} (t+s,\hat{z}_{i}) p(t+s,\hat{z}_{i}) d\hat{z}_{i} d\hat{y}_{i} \\ &+ \sum_{i,j=1, i\neq j}^{d} \int_{x}^{A} \int_{\bar{y}_{i}}^{A} (\sigma\sigma^{T})_{ij} (t+s,\hat{z}_{ij}) p(t+s,\hat{z}_{ij}) d\hat{z}_{ij} dy \\ &+ \sum_{i=1}^{d} \int_{x}^{A} \int_{\bar{y}_{i}}^{A} [b_{i} (t+s,\check{z}_{i}) p(t+s,\check{z}_{i})] d\check{z}_{i} dy \Big) ds. \end{split}$$

To obtain the operator L_1 has a fixed point in the space $\mathcal{C}^1([0,T], \mathcal{C}^2(B_1))$ we need the following lemma.

Lemma 4.6 [29, Corollary 2.4, p.3231] Let X be a nonempty closed convex subset of a Banach space Y. Suppose that T and S map X into Y such that

(i) S is continuous, S(X) resides in a compact subset of Y; (ii) $T: X \to Y$ is expansive and onto. Then there exists a point $x^* \in X$ with $Sx^* + Tx^* = x^*$.

We recall the definition of expansive operator.

Definition 4.2 [29] Let (X, d) be a metric space and M be a subset of X. The mapping $T: M \to X$ is said to be expansive, if there exists a constant h > 1 such that

$$d(Tx, Ty) \ge hd(x, y), \quad \forall x, y \in M.$$

It is easy to check the following lemma, see [13, lemma 2.3] for the details.

Lemma 4.7 The operator $M_1: D_1 \to \tilde{D}_1$ is an expansive operator and onto.

Next we prove the operator N_1 satisfies the (i) of Lemma 4.6.

Lemma 4.8 The operator $N_1: D_1 \to D_1$ is continuous and D_1 is a compact set in D_1 .

Proof. We first prove N_1 maps D_1 to D_1 . For any $p \in D_1$, by using (4.19), we have

for every $t \in [0,T]$ and every $x \in B_1$.

For $(N_1(p))_t$, we get

$$\begin{split} (N_{1}(p))_{t} &= -\varepsilon p_{t} + \int_{x}^{A} \int_{y}^{A} p_{t}(t,z) dz dy + \frac{e^{-[a]T}}{1 - e^{-[a]T}} a(t) e^{-\int_{0}^{t} a(r) dr} \int_{t}^{t+T} e^{\int_{0}^{t_{1}} a(r) dr} \\ &\times \left[a(t_{1}) \int_{x}^{A} \int_{y}^{A} p(t_{1},z) dz dy + \sum_{i=1}^{d} \int_{\bar{x}_{i}}^{A} \int_{\bar{y}_{i}}^{A} (\sigma\sigma^{T})_{ii}(t_{1},\hat{z}_{i}) p(t_{1},\hat{z}_{i}) d\hat{z}_{i} d\hat{y}_{i} \right. \\ &+ \sum_{i,j=1,i\neq j}^{d} \int_{x}^{A} \int_{\bar{y}_{i}j}^{A} (\sigma\sigma^{T})_{ij}(t_{1},\hat{z}_{ij}) p(t_{1},\hat{z}_{ij}) d\hat{z}_{ij} dy \\ &+ \sum_{i=1}^{d} \int_{x}^{A} \int_{y}^{A} \left[b_{i}(t_{1},\check{z}_{i}) p(t_{1},\check{z}_{i}) \right] d\check{z}_{i} dy \right] dt_{1} \\ &+ \left[a(t) \int_{x}^{A} \int_{y}^{A} p(t,z) dz dy + \sum_{i=1}^{d} \int_{\bar{x}_{i}}^{A} \int_{\bar{y}_{i}}^{A} (\sigma\sigma^{T})_{ii}(t,\hat{z}_{i}) p(t,\hat{z}_{i}) d\hat{z}_{i} d\hat{y}_{i} \right. \\ &+ \sum_{i,j=1,i\neq j}^{d} \int_{x}^{A} \int_{\bar{y}_{ij}}^{A} (\sigma\sigma^{T})_{ij}(t,\hat{z}_{ij}) p(t,\hat{z}_{ij}) d\hat{z}_{ij} dy \\ &+ \sum_{i=1}^{d} \int_{x}^{A} \int_{\bar{y}_{i}}^{A} \left[b_{i}(t,\check{z}_{i}) p(t,\check{z}_{ij}) d\check{z}_{i} dy \right] \end{split}$$

which implies that (using (4.19) again)

$$\begin{split} |(N_1(p))_t| &\leq \varepsilon Q + (A_1 \cdots A_d)^2 Q + (G^3 T + 1) \Big[G(A_1 \cdots A_d)^2 Q \\ &+ \sum_{i=1}^d (A_1 \cdots A_{i-1} A_{i+1} \cdots A_d)^2 S Q \\ &+ \sum_{i,j=1, i \neq j}^d A_i A_j (A_1 \cdots A_{i-1} A_{i+1} \cdots A_{j-1} A_{j+1} \cdots A_d)^2 S Q \\ &+ \sum_{i=1}^d A_i (A_1 \cdots A_{i-1} A_{i+1} \cdots A_d)^2 S Q \Big] \\ &\leq \varepsilon Q + (1 - \varepsilon) Q, \quad \forall t \in [0, T], \ x \in \mathbb{R}^d. \end{split}$$

Let $k \in \{1, 2, \cdots, d\}$ be arbitrary chosen and fixed. Then we have

$$\begin{split} (N_{1}(p))_{x_{k}} &= -\varepsilon p_{x_{k}} - \int_{\bar{x}_{k}}^{A} \int_{\hat{y}_{k}}^{A} p_{t}(t,z) dz d\hat{y}_{k} \\ &- \frac{e^{-[a]T}}{1 - e^{-[a]T}} \int_{0}^{T} e^{\int_{t}^{t+s} a(r)dr} \Big(-a(t+s) \int_{\bar{x}_{k}}^{A} \int_{\hat{y}_{k}}^{A} p(t+s,z) dz d\hat{y}_{k} \\ &+ \sum_{i=1, i \neq k}^{d} \int_{(\bar{x}_{i})_{k}}^{A} \int_{(\bar{y}_{i})_{k}}^{A} (\sigma\sigma^{T})_{ii}(t+s,\hat{z}_{i}) p(t+s,\hat{z}_{i}) d\hat{z}_{i} d\widehat{(\bar{y}_{i})_{k}} \\ &+ \int_{\bar{x}_{k}}^{A} \int_{\bar{y}_{k}}^{A} (\sigma\sigma^{T})_{kk}(t+s,\hat{z}_{k}) p(t+s,\hat{z}_{k}) d\hat{z}_{k} d\hat{y}_{k} \\ &+ \sum_{i,j=1, i \neq j \neq k}^{d} \int_{\bar{x}_{k}}^{A} \int_{(\bar{y}_{i})_{k}}^{A} (\sigma\sigma^{T})_{ij}(t+s,\hat{z}_{ij}) p(t+s,\hat{z}_{ij}) d\hat{z}_{ij} d\hat{y}_{k} \\ &+ \sum_{i,j=1, i \neq j, j=k}^{d} \int_{\bar{x}_{k}}^{A} \int_{\bar{y}_{i}}^{A} (\sigma\sigma^{T})_{ik}(t+s,\hat{z}_{ik}) p(t+s,\hat{z}_{ik}) d\hat{z}_{kj} d\hat{y}_{k} \\ &+ \sum_{i,j=1, i \neq j, i=k}^{d} \int_{\bar{x}_{k}}^{A} \int_{\bar{y}_{i}}^{A} (\sigma\sigma^{T})_{kj}(t+s,\hat{z}_{ik}) p(t+s,\hat{z}_{kj}) d\hat{z}_{kj} d\hat{y}_{k} \\ &+ \sum_{i,j=1, i \neq j, i=k}^{d} \int_{\bar{x}_{k}}^{A} \int_{\bar{y}_{k}}^{A} (\sigma\sigma^{T})_{kj}(t+s,\hat{z}_{kj}) p(t+s,\hat{z}_{kj}) d\hat{z}_{kj} d\hat{y}_{k} \\ &+ \sum_{i,j=1, i \neq j, i=k}^{d} \int_{\bar{x}_{k}}^{A} \int_{\bar{y}_{k}}^{A} (\sigma\sigma^{T})_{kj}(t+s,\hat{z}_{i}) p(t+s,\hat{z}_{kj}) d\hat{z}_{kj} d\hat{y}_{k} \\ &+ \sum_{i,j=1, i \neq j, i=k}^{d} \int_{\bar{x}_{k}}^{A} \int_{\bar{y}_{k}}^{A} (\sigma\sigma^{T})_{kj}(t+s,\hat{z}_{k}) p(t+s,\hat{z}_{k}) d\hat{z}_{kj} d\hat{y}_{k} \\ &+ \sum_{i,j=1, i \neq j, i=k}^{d} \int_{\bar{x}_{k}}^{A} \int_{\bar{y}_{k}}^{A} (\sigma\sigma^{T})_{kj}(t+s,\hat{z}_{k}) p(t+s,\hat{z}_{k}) d\hat{z}_{k} d\hat{y}_{k} \\ &+ \sum_{i,j=1, i \neq j, i=k}^{d} \int_{\bar{x}_{k}}^{A} \int_{\bar{y}_{k}}^{A} (\sigma\sigma^{T})_{kj}(t+s,\hat{z}_{k}) p(t+s,\hat{z}_{k}) d\hat{z}_{k} d\hat{y}_{k} \\ &+ \sum_{i=1, i \neq j, i=k}^{d} \int_{\bar{x}_{k}}^{A} \int_{\bar{y}_{k}}^{A} (\sigma\sigma^{T})_{kj}(t+s,\hat{z}_{k}) p(t+s,\hat{z}_{k}) d\hat{z}_{k} d\hat{y}_{k} \\ &+ \sum_{i=1, i \neq j, i=k}^{d} \int_{\bar{x}_{k}}^{A} \int_{\bar{y}_{k}}^{A} (\sigma\sigma^{T})_{kj}(t+s,\hat{z}_{k}) p(t+s,\hat{z}_{k}) p(t+s,\hat{z}_{k}) d\hat{z}_{k} d\hat{y}_{k} \\ &+ \sum_{i=1, i \neq j, i=k}^{d} \int_{\bar{y}_{k}}^{A} \int_{\bar{y}_{k}}^{A} (\sigma\sigma^{T})_{kj}(t+s,\hat{z}_{k}) p(t+s,\hat{z}_{k}) p(t+s,\hat{z}_{k}) p(t+s,\hat{z}_{k}) p(t+s,\hat{z}_{k}) p(t+s,\hat{z}_{k}) p(t+s,\hat{z}_{k}) p(t+s,\hat{z}_{k}) p(t+s,\hat{z}_{k}) p(t+s,\hat{z}_{k}) p(t+s,\hat{z})$$

Using (4.19), we obtain

$$\begin{split} |(N_{1}(p))_{x_{k}}| &\leq \varepsilon Q + (A_{1} \cdots A_{k-1}A_{k+1} \cdots A_{d})^{2}A_{k}Q \\ &+ (G^{3}T+1) \Big[GT(A_{1} \cdots A_{k-1}A_{k+1} \cdots A_{d})^{2}A_{k}Q \\ &+ \sum_{i=1, i \neq k}^{d} (A_{1} \cdots A_{i-1}A_{i+1} \cdots A_{k-1}A_{k+1} \cdots A_{d})^{2}A_{k}SQ \\ &+ (A_{1} \cdots A_{i-1}A_{i+1} \cdots A_{d})^{2}SQ \\ &+ \sum_{i, j=1, i \neq j \neq k}^{d} A_{i}A_{j}A_{k}(A_{1} \cdots A_{i-1}A_{i+1} \cdots A_{j-1}A_{j+1} \cdots A_{k-1}A_{k+1} \cdots A_{d})^{2}SQ \\ &+ 2\sum_{j, k=1, k \neq j}^{d} A_{k}A_{j}(A_{1} \cdots A_{j-1}A_{j+1} \cdots A_{k-1}A_{k+1} \cdots A_{d})^{2}SQ \\ &+ \sum_{i=1, i \neq k}^{d} A_{i}A_{k}(A_{1} \cdots A_{i-1}A_{i+1} \cdots A_{k-1}A_{k+1} \cdots A_{d})^{2}SQ \\ &+ (A_{1} \cdots A_{i-1}A_{i+1} \cdots A_{d})^{2}SQ \Big] \\ &\leq \varepsilon Q + (1-\varepsilon)Q, \ \forall t \in [0,T], \ x \in \mathbb{R}^{d}. \end{split}$$

Consequently, $N_1: D_1 \to D_1$. It follows from the above estimates that if $p_n \to p$ in sense of the topology of the set D_1 we have $N_1(p_n) \to N_1(p)$ in sense of the topology of the set D_1 . Therefore the operator $N_1: D_1 \to D_1$ is a continuous operator. It follows from the definitions of D_1 and \tilde{D}_1 that D_1 is a compact set in the space \tilde{D}_1 . \Box

Proof of Theorem 4.4 Combining the Lemmas 4.7 and 4.8, and using Lemma 4.6, we deduce that the operator L_1 has a fixed point $p^1 \in D_1$. Hence p^1 is a solution to (4.20) in D_1 . In order to get the global existence, we need to define the set

$$B_2 = \{ x \in \mathbb{R}^d : A_i \le x_i \le 2A_i, i = 1, 2, \cdots, d \}.$$

We consider the problem (4.20) in B_2 . In order to do that, we consider the operator

$$\begin{split} L_{2}(p) &= p(t,x) + \int_{x}^{A} \int_{y}^{A} p(t,z) dz dy - \frac{e^{-[a]T}}{1 - e^{-[a]T}} \int_{0}^{T} e^{\int_{t}^{t+s} a(r) dr} \left(a(t+s) \int_{x}^{A} \int_{y}^{A} \int_{y}^{A} p(t,z) dz dy + \sum_{i=1}^{d} \int_{\bar{x}_{i}}^{A} \int_{\bar{y}_{i}}^{A} (\sigma \sigma^{T})_{ii} (t+s,\hat{z}_{i}) \left[p(t+s,\hat{z}_{i}) - p^{1}(t+s,z_{1},\cdots,z_{i-1},A_{i},z_{i+1},\cdots,z_{d}) + (A_{i}-x_{i}) p_{x_{i}}^{1}(t+s,z_{1},\cdots,z_{i-1},A_{i},z_{i+1},\cdots,z_{d}) \right] d\hat{z}_{i} d\hat{y}_{i} \\ &+ \sum_{i,j=1,i\neq j}^{d} \int_{x}^{A} \int_{\bar{y}_{ij}}^{A} (\sigma \sigma^{T})_{ij} (t+s,\hat{z}_{ij}) p(t+s,\hat{z}_{ij}) d\hat{z}_{ij} dy \\ &+ \sum_{i=1}^{d} \int_{x}^{A} \int_{\bar{y}_{i}}^{A} \left[b_{i} (t+s,\check{z}_{i}) p(t+s,\check{z}_{i}) \right] d\check{z}_{i} dy \Big] ds \end{split}$$

under the sets

$$D_2 = \{ u \in \mathcal{C}^1([0,T], \mathcal{C}^2(B_2)) : |u| \le Q, |u_t| \le Q, |u_{x_i}| \le Q, \ i = 1, 2, \cdots, d \}, \\ \tilde{D}_2 = \{ u \in \mathcal{C}^1([0,T], \mathcal{C}^2(B_2)) : |u| \le (1+\varepsilon)Q, |u_t| \le (1+\varepsilon)Q, \\ |u_{x_i}| \le (1+\varepsilon)Q, \ i = 1, 2, \cdots, d \}.$$

In the set D_2 and \tilde{D}_2 , we define a norm as follows:

$$||u|| = \max\left\{\max_{t \in [0,T], x \in B_2} |u|, \max_{t \in [0,T], x \in B_2} |u_t|, \max_{t \in [0,T], x \in B_2} |u_{x_i}|, i = 1, 2, \cdots, d\right\}.$$

Similar to the operator L_1 , we define

$$L_2(p) = M_2(p) + N_2(p),$$

where

$$\begin{split} M_{2}(p) &= (1+\varepsilon)p, \\ N_{2}(p) &= -\varepsilon p + \int_{x}^{A} \int_{y}^{A} p(t,z) dz dy - \frac{e^{-[a]T}}{1 - e^{-[a]T}} \int_{0}^{T} e^{\int_{t}^{t+s} a(r) dr} \\ & \left(a(t+s) \int_{x}^{A} \int_{y}^{A} p(t+s,z) dz dy + \sum_{i=1}^{d} \int_{\bar{x}_{i}}^{A} \int_{\bar{y}_{i}}^{A} (\sigma\sigma^{T})_{ii} (t+s,\hat{z}_{i}) \left[p(t+s,\hat{z}_{i}) \right. \\ & \left. -p^{1}(t+s,z_{1},\cdots,z_{i-1},A_{i},z_{i+1},\cdots,z_{d}) \right. \\ & \left. + (A_{i}-x_{i})p_{x_{i}}^{1} (t+s,z_{1},\cdots,z_{i-1},A_{i},z_{i+1},\cdots,z_{d}) \right. \\ & \left. + \sum_{i,j=1,i\neq j}^{d} \int_{x}^{A} \int_{\bar{y}_{i}}^{A} (\sigma\sigma^{T})_{ij} (t+s,\hat{z}_{ij})p(t+s,\hat{z}_{ij}) d\hat{z}_{ij} dy \right. \\ & \left. + \sum_{i=1}^{d} \int_{x}^{A} \int_{\tilde{y}_{i}}^{A} \left[b_{i}(t+s,\check{z}_{i})p(t+s,\check{z}_{i}) \right] d\check{z}_{i} dy \right] ds. \end{split}$$

Similar to the case of L_1 , we obtain the operator L_2 has a fixed point $p^2(t, x)$ in the set D_2 , which is a solution to (4.20) in D_2 . Following Lemma 4.5, we have $p^2(t, x)$ satisfies

$$\int_{x}^{A} \int_{y}^{A} p^{2}(t,z) dz dy - \frac{e^{-[a]T}}{1 - e^{-[a]T}} \int_{0}^{T} e^{\int_{t}^{t+s} a(r) dr} \\
\left(a(t+s) \int_{x}^{A} \int_{y}^{A} p^{2}(t+s,z) dz dy + \sum_{i=1}^{d} \int_{\bar{x}_{i}}^{A} \int_{\bar{y}_{i}}^{A} (\sigma\sigma^{T})_{ii}(t+s,\hat{z}_{i}) \left[p^{2}(t+s,\hat{z}_{i}) - p^{1}(t+s,z_{1},\cdots,z_{i-1},A_{i},z_{i+1},\cdots,z_{d}) + (A_{i}-x_{i})p_{x_{i}}^{1}(t+s,z_{1},\cdots,z_{i-1},A_{i},z_{i+1},\cdots,z_{d}) \right] d\hat{z}_{i} d\hat{y}_{i} \\
+ \sum_{i,j=1,i\neq j}^{d} \int_{x}^{A} \int_{\bar{y}_{i}j}^{A} (\sigma\sigma^{T})_{ij}(t+s,\hat{z}_{ij})p^{2}(t+s,\hat{z}_{ij}) d\hat{z}_{ij} dy \\
+ \sum_{i=1}^{d} \int_{x}^{A} \int_{\bar{y}_{i}}^{A} \left[b_{i}(t+s,\check{z}_{i})p^{2}(t+s,\check{z}_{i})\right] d\check{z}_{i} dy \Big] ds = 0.$$
(4.22)

When $x_1 = A_1$, the above equality deduces that

$$\frac{e^{-[a]T}}{1-e^{-[a]T}} \int_0^T e^{\int_t^{t+s} a(r)dr} \Big(\int_{\bar{x}_1}^A \int_{\bar{y}_1}^A (\sigma\sigma^T)_{11}(t+s,A_1,z_2,\cdots,z_d) \\ [p^2(t+s,A_1,z_2,\cdots,z_d) - p^1(t+s,A_1,z_2,\cdots,z_d)] d\hat{z}_1 d\hat{y}_1 = 0.$$

Differentiating the above equality with respect to t, and using the periodicity of σ, p^1 and p^2 and $(\sigma\sigma^T)_{ij} > 0$, one can obtain

$$p^{1}(t, A_{1}, x_{2}, \cdots, x_{d}) = p^{2}(t, A_{1}, x_{2}, \cdots, x_{d}).$$

Differentiating (4.22) with respect to x_1 , after which we put $x_1 = A_1$, we get

$$\frac{e^{-[a]T}}{1-e^{-[a]T}} \int_0^T e^{\int_t^{t+s} a(r)dr} \Big(\int_{\bar{x}_1}^A \int_{\bar{y}_1}^A (\sigma\sigma^T)_{11}(t+s,A_1,z_2,\cdots,z_d) \\ [p_{x_1}^2(t+s,A_1,z_2,\cdots,z_d) - p_{x_1}^1(t+s,A_1,z_2,\cdots,z_d)] d\hat{z}_1 d\hat{y}_1 = 0.$$

Similarly, we get

$$p_{x_1}^1(t, A_1, x_2, \cdots, x_d) = p_{x_1}^2(t, A_1, x_2, \cdots, x_d).$$

As in above discussion, we can deduce that

$$p^{1}(t, x_{1}, A_{2}, \cdots, x_{d}) = p^{2}(t, x_{1}, A_{2}, \cdots, x_{d}),$$

$$p^{1}_{x_{2}}(t, x_{1}, A_{2}, \cdots, x_{d}) = p^{2}_{x_{2}}(t, x_{1}, A_{2}, \cdots, x_{d}),$$

$$\vdots$$

$$p^{1}(t, x_{1}, x_{2}, \cdots, A_{d}) = p^{2}(t, x_{1}, x_{2}, \cdots, A_{d}),$$

$$p^{1}_{x_{d}}(t, x_{1}, x_{2}, \cdots, A_{d}) = p^{2}_{x_{d}}(t, x_{1}, x_{2}, \cdots, A_{d}).$$

The function

$$p(t,x) = \begin{cases} p^1(t,x), & t \ge 0, \ x \in B_1\\ p^2(t,x), & t \ge 0, \ x \in B_2 \end{cases}$$

is a solution to the following problem

$$\begin{cases} \partial_t p = -\sum_i \frac{\partial}{\partial x_i} (b_i p) + \frac{1}{2} \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} ((\sigma \sigma^T)_{ij} p), & t > 0, \quad x \in B_1 \cup B_2, \\ p(t, x) = p(t + T, x), & t \ge 0, \quad x \in B_1 \cup B_2. \end{cases}$$

Repeating the above steps, using partitioning of \mathbb{R}^d into cubes, we obtain a periodic solution to (4.17). The proof is complete. \Box

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