

Several Metric Properties of Level Curves

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ABSTRACT. This article establishes several remarkably simple identities relating certain metric invariants of level curves of real and complex functions. In particular, we relate lengths of level curves to their curvature and to the gradient field of the function. Some geometric and analytic applications of the results are shown.

1 Introduction

This article examines certain metric invariants of level curves of a real two-variable function and of a complex analytic function. In particular, we relate lengths of level curves to their curvature and to the gradient field.

We review some terminologies before stating the main result. A *Morse function* on an open subset of \mathbb{R}^n is a real-valued C^2 function whose critical points are nondegenerate (i.e., with nonsingular Hessian). Near a critical point, a Morse function behaves as a nondegenerate quadratic form (i.e., its second-degree Taylor approximation) and thus has no other critical points. A *regular value* of a Morse function is a number which is not the image of a critical point¹, whereas a *critical value* is the image of a critical point.

Let f be a Morse function on an open connected subset of \mathbb{R}^2 . Suppose that a and b are values of f such that $f^{-1}([a, b])$ is compact. For $t \in [a, b]$, let $L(t)$ be the length of the level- t curve $f^{-1}(t)$; $L(t)$ is well-defined even if t is a critical value. At each *regular point* (i.e., noncritical point) on $f^{-1}(t)$, install a unit tangent \mathbf{T} and a unit normal \mathbf{N} by letting $\mathbf{N} = -\nabla f / |\nabla f|$ and requiring that the frame (\mathbf{T}, \mathbf{N}) be positively-oriented. Then, the *signed curvature* κ of $f^{-1}(t)$ at all regular points is defined by the equation $d\mathbf{T}/ds = \kappa\mathbf{N}$, where s is arc length (with its positive direction induced by \mathbf{T}). Thereby, κ is defined at all regular points of $f^{-1}([a, b])$.

With the notations introduced, we now state our main result.

Main Result: *If a and b are values of f such that $f^{-1}([a, b])$ is compact, then, for $t \in [a, b]$,*

$$\frac{d}{dt} \iint_{f^{-1}([a, t])} |\nabla f| dA = L(t) = L(a) + \iint_{f^{-1}([a, t])} \kappa dA.$$

By $\iint_{f^{-1}([a, t])} \kappa dA$, we mean either the (proper) Riemann integral of κ when $f^{-1}([a, t])$ is free of critical points, or the improper Riemann integral of κ otherwise; the latter case will be addressed in detail.

These two identities are established in §2 and their counterparts for complex analytic functions are given in §3. A number of applications are shown in §4, including in §4.1 instances of curve evolution and a characterization of circles and in §4.2 parallel results for level surfaces. Two technical details are relegated to §5.

¹As a seemingly peculiar (but useful for certain differential-topological purposes) consequence of the definition, any number *not* in the range of the function *is* a regular value. However, we will only be concerned with regular values that are attained by the function.

2 Level Curves of Real Functions

2.1 Preliminaries

Let Ω be an open connected subset of \mathbb{R}^2 and f be a Morse function (defined in §1) on Ω . The notions of *regular value*, *critical value*, and *regular point* of f have been introduced in §1.

We shall mean, by a *regular* (resp. *critical*) *level* of f , a nonempty level set $f^{-1}(t)$ with t a regular (resp. critical) value. Near a regular point P , the set $f^{-1}(f(P))$ is locally a C^2 curve (by the implicit function theorem). Therefore, each component of a regular level is globally a C^2 curve, and a compact regular level is a disjoint union of simple closed curves.

Let $[a, b]$ be an interval of regular values attained by f such that $f^{-1}([a, b])$ is compact. As basic facts in differential topology, $f^{-1}(t)$ is diffeomorphic to $f^{-1}(a)$ for $t \in [a, b]$ and $f^{-1}([a, b])$ is diffeomorphic to $f^{-1}(a) \times [a, b]$. The idea is to start a flow originating from $f^{-1}(a)$ following the gradient field and to control the flow's speed so that points on $f^{-1}(a)$ all “drift” to points of the same f -value at the same time; details are given in §2.2.

Suppose that $f^{-1}(t)$ is compact and consider its length $L(t)$. We explain that $L(t)$ is well-defined even if t is a critical value. Near a critical point $P \in f^{-1}(t)$, a local C^2 coordinate system $(u(x, y), v(x, y))$ exists such that $u(P) = 0 = v(P)$ and $f(x, y) = t + q(u, v)$ where q is a nondegenerate quadratic form; see [14, pp. 54-58]. A neighborhood of P on $f^{-1}(t)$ is thus diffeomorphic to a neighborhood of the origin on $q^{-1}(0)$, which is sufficiently well-behaved to admit length. If $f^{-1}([a, b])$ is compact, then L is continuous on $[a, b]$; its continuity at a regular value t_0 is due to the diffeomorphism between $f^{-1}([t_0 - \epsilon, t_0 + \epsilon])$ and $f^{-1}(t_0) \times [t_0 - \epsilon, t_0 + \epsilon]$, whereas its continuity at a critical value is due to the good behavior of $q^{-1}(\delta)$ as δ varies over $(-\epsilon, \epsilon)$.

Recall from §1 our choice of \mathbf{N} and \mathbf{T} at regular points of $f^{-1}(t)$; i.e., $\mathbf{N} := -\nabla f / |\nabla f|$ and (\mathbf{T}, \mathbf{N}) is a positively-oriented frame. At regular points on $f^{-1}(t)$, the *signed* curvature κ and the frame (\mathbf{T}, \mathbf{N}) are (by definition) related by

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N} \quad \text{and} \quad \frac{d\mathbf{N}}{ds} = -\kappa\mathbf{T},$$

where s is arc length (with its positive direction induced by \mathbf{T}). For a simple closed component C of a regular level,

$$\int_C \kappa ds = \pm 2\pi$$

where the plus sign is in force iff \mathbf{T} induces the positive orientation² on C ; see [7, pp. 36-37].

Henceforth, we will regard κ as a function on the set of regular points in Ω , i.e., $\kappa(P)$ is the signed curvature at P of the curve $f^{-1}(f(P))$.

It is a fact (shown in [5, p. 125]) that

$$\kappa = \frac{f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2}{|\nabla f|^3}; \tag{1}$$

²The positive orientation on a simple closed curve C is the one following which the traversal of C gives a positive winding number around any interior point in the Jordan domain bounded by C ; see [7, pp. 392-396]. If C is convex, the positive orientation on C is counterclockwise.

it is a matter of computation to verify that

$$\operatorname{div} \left(\frac{\nabla f}{|\nabla f|} \right) = \kappa. \quad (2)$$

As κ becomes unbounded near a critical point P , its improper Riemann integral $\iint_D \kappa dA$ over a small disc D around P nonetheless converges. We relegate this matter to §5, in which we prove the integrability of κ under a hypothesis that generalizes the Morse condition; before then, we take this fact for granted.

2.2 Length in Relation to Curvature and Gradient

For a number of results, we assume the following hypothesis.

Hypothesis (†) f is a C^2 Morse function on an open connected subset Ω of \mathbb{R}^2 ; a and b are values attained by f such that $f^{-1}([a, b])$ is compact. (As Ω is connected, each $t \in [a, b]$ is also a value attained by f .)

We begin with a key lemma, upon which many of our results rest.

Lemma 1 *Assume Hypothesis (†). If a function g is continuous on the set of regular points in $f^{-1}([a, b])$ and is Riemann-integrable on $f^{-1}([a, b])$, then*

$$\iint_{f^{-1}([a, b])} g dA = \int_a^b \left(\int_{f^{-1}(t)} \frac{g}{|\nabla f|} ds \right) dt,$$

where the line integral $\int_{f^{-1}(t)} (g/|\nabla f|) ds$ is only defined for t a regular value.³

When $[a, b]$ is free of critical values, this formula is (somewhat informally) shown in [5, p. 298-300]. We shall sketch a proof based on a differential-topological construction, which will be of use in a later argument.

Proof. There are two cases, according as whether $[a, b]$ contains a critical value. *Case 1.* Suppose that $[a, b]$ contains no critical value. For each $p \in f^{-1}(a)$, let $t \mapsto H(p, t)$ be the flow (i.e. the integral curve) for the field $\nabla f/|\nabla f|^2$, originating from p at the initial time $t = a$; i.e., $H(p, -)$ is the solution to the initial value problem

$$\frac{\partial H}{\partial t}(p, t) = \frac{\nabla f(H(p, t))}{|\nabla f(H(p, t))|^2} \quad \text{with} \quad H(p, a) = p. \quad (3)$$

Clearly, $f(H(p, t)) = t$, as $f(H(p, t)) - f(H(p, a)) = \int_a^t \nabla f \cdot (\partial H/\partial \tau) d\tau = \int_a^t 1 d\tau$. Due to existence, uniqueness, and smooth dependence on initial condition of solution to the initial value problem, the map $H(-, t) : f^{-1}(a) \rightarrow f^{-1}(t)$, and consequently the map $H : f^{-1}(a) \times [a, b] \rightarrow f^{-1}([a, b])$, are diffeomorphisms; see [14, p. 65] and the Remark that follows this proof.

Now assume that $f^{-1}(a)$ is connected and hence a simple closed curve; otherwise, treat each component of $f^{-1}(a)$ and its corresponding component of $f^{-1}([a, b])$ separately. Let $\gamma(-, a) : [0, 1] \rightarrow \mathbb{R}^2$ be a regular parametrization of

³The improper Riemann integral $\int_a^b \varphi(t) dt$ is a well-defined notion when φ is continuous at all but finitely many $t \in [a, b]$. In the present case, $\varphi(t) = \int_{f^{-1}(t)} (g/|\nabla f|) ds$.

$f^{-1}(a)$ that is one-to-one on $[0, 1)$ with $\gamma(0, a) = \gamma(1, a)$ and that induces the same orientation that \mathbf{T} does on $f^{-1}(a)$. The map

$$\gamma(u, t) := H(\gamma(u, a), t) \quad (4)$$

then parametrizes $f^{-1}([a, b])$, with

$$\left| \det \frac{\partial(x, y)}{\partial(u, t)} \right| = \left| \frac{\partial\gamma}{\partial t} \times \frac{\partial\gamma}{\partial u} \right| = \left| \frac{\partial\gamma}{\partial t} \right| \left| \frac{\partial\gamma}{\partial u} \right| = \frac{1}{|\nabla f|} \left| \frac{\partial\gamma}{\partial u} \right|.$$

Hence, by changing variables in integration,

$$\iint_{f^{-1}([a, b])} g \, dA = \int_a^b \left(\int_0^1 \frac{g}{|\nabla f|} \cdot \left| \frac{\partial\gamma}{\partial u} \right| du \right) dt = \int_a^b \left(\int_{f^{-1}(t)} \frac{g}{|\nabla f|} ds \right) dt.$$

Case 2. Suppose that $[a, b]$ contains critical values comprising the (finite) set S . Then, $(a, b) \setminus S$ is a disjoint union of finitely many intervals $I_j := (c_j, c_{j+1})$ of regular values attained by f . As $f^{-1}([a, b]) = \cup_j f^{-1}(I_j) \cup f^{-1}(S \cup \{a, b\})$ and $f^{-1}(S \cup \{a, b\})$ has zero area,

$$\iint_{f^{-1}([a, b])} g \, dA = \sum_j \iint_{f^{-1}(I_j)} g \, dA.$$

Applying Case 1 to $f^{-1}([c_j + \epsilon, c_{j+1} - \delta])$ and letting $\epsilon, \delta \rightarrow 0^+$, we have

$$\begin{aligned} \iint_{f^{-1}(I_j)} g \, dA &= \lim_{\epsilon, \delta \rightarrow 0} \iint_{f^{-1}([c_j + \epsilon, c_{j+1} - \delta])} g \, dA \\ &= \lim_{\epsilon, \delta \rightarrow 0} \int_{c_j + \epsilon}^{c_{j+1} - \delta} \left(\int_{f^{-1}(t)} \frac{g}{|\nabla f|} ds \right) dt \\ &= \int_{I_j} \left(\int_{f^{-1}(t)} \frac{g}{|\nabla f|} ds \right) dt. \end{aligned}$$

Summing these integrals over j proves the assertion. \blacksquare

Choices for g in Lemma 1 yield integral identities. For example, the choice $g \equiv 1$ gives the area of $f^{-1}([a, b])$ as $\int_a^b \left(\int_{f^{-1}(t)} 1/|\nabla f| ds \right) dt$. We next show that Lemma 1 can result in interesting relations among the length of level curves, their curvature, and the gradient field.

Theorem 2 *Assume Hypothesis (†).*

(a) $\int \int_{f^{-1}[a, b]} (h \circ f) \cdot |\nabla f| \, dA = \int_a^b h(t) L(t) \, dt$ for any function h such that both members of the equation are meaningful. In particular, for any $t \in [a, b]$,

$$\int_a^t L(\tau) \, d\tau = \iint_{f^{-1}([a, t])} |\nabla f| \, dA,$$

or equivalently,

$$L(t) = \frac{d}{dt} \iint_{f^{-1}([a, t])} |\nabla f| \, dA.$$

(b) For any regular value $t \in [a, b]$,

$$L'(t) = \int_{f^{-1}(t)} \frac{\kappa}{|\nabla f|} ds,$$

or equivalently, for any $t \in [a, b]$,

$$L(t) = L(a) + \iint_{f^{-1}([a, t])} \kappa dA.$$

As promised in §2.1, the integrability of κ will be proven in §5.

Proof. Part (a) follows from Lemma 1 by letting $g = (h \circ f) \cdot |\nabla f|$; the “particular” case results from letting $h = \chi_{[a, t]}$, the characteristic function of the interval $[a, t]$. (Continuity of L makes applicable the fundamental theorem of calculus, i.e., $\frac{d}{dt} \int_a^t L(\tau) d\tau = L(t)$.)

In Part (b), the equivalence between the two assertions is due to Lemma 1, as we now show. First, assume the formula for L' . Since L is continuous at the (finitely many) critical values in $[a, b]$, the fundamental theorem of calculus applies to give, for $t \in [a, b]$,

$$L(t) - L(a) = \int_a^t L'(\tau) d\tau = \int_a^t \left(\int_{f^{-1}(\tau)} \frac{\kappa}{|\nabla f|} ds \right) d\tau = \iint_{f^{-1}([a, t])} \kappa dA,$$

where the last equality follows from Lemma 1 (with t playing the role of b). Now assume the integral formula for L . Then, for a regular value $t \in (a, b)$,

$$L'(t) = \frac{d}{dt} \iint_{f^{-1}([a, t])} \kappa dA = \frac{d}{dt} \int_a^t \left(\int_{f^{-1}(\tau)} \frac{\kappa}{|\nabla f|} ds \right) d\tau = \int_{f^{-1}(t)} \frac{\kappa}{|\nabla f|} ds,$$

where, in the last equality, t being a regular value is essential, as $\frac{d}{dt} \int_a^t \varphi(\tau) d\tau = \varphi(t)$ iff φ is continuous at t .

We now give two alternative proofs for Part (b) by directly proving each of the two equivalent formulae.

To prove the formula for L' , let $t \in (a, b)$ be a regular value. There is an interval $[a', b']$ of regular values such that $t \in (a', b') \subset [a, b]$. Let a' and b' play the role of a and b in case 1 of the proof of Lemma 1 and let

$$\gamma : [0, 1] \times [a', b'] \rightarrow f^{-1}([a', b'])$$

be as in (4). For the speed $v := |\partial\gamma/\partial u|$, we have $v^2 = \langle \partial\gamma/\partial u, \partial\gamma/\partial u \rangle$ and

$$\begin{aligned} \frac{1}{2} \frac{\partial(v^2)}{\partial t} &= \left\langle \frac{\partial}{\partial u} \left(\frac{\partial\gamma}{\partial t} \right), \frac{\partial\gamma}{\partial u} \right\rangle = \left\langle \frac{\partial}{\partial u} \left(\frac{\nabla f}{|\nabla f|^2} \right), \frac{\partial\gamma}{\partial u} \right\rangle \\ &= \left\langle \frac{\partial}{\partial u} \left(\frac{-1}{|\nabla f|} \mathbf{N} \right), v\mathbf{T} \right\rangle = \left\langle \frac{-1}{|\nabla f|} \frac{\partial\mathbf{N}}{\partial u}, v\mathbf{T} \right\rangle. \end{aligned}$$

As $\partial\mathbf{N}/\partial u = v\partial\mathbf{N}/\partial s = -v\kappa\mathbf{T}$,

$$\frac{\partial v}{\partial t} = \frac{1}{2v} \frac{\partial(v^2)}{\partial t} = \frac{1}{v} \left\langle \frac{1}{|\nabla f|} v\kappa\mathbf{T}, v\mathbf{T} \right\rangle = \frac{\kappa v}{|\nabla f|}.$$

and

$$L'(t) = \frac{d}{dt} \int_0^1 v \, du = \int_0^1 \frac{\partial v}{\partial t} \, du = \int_0^1 \frac{\kappa}{|\nabla f|} v \, du = \int_{f^{-1}(t)} \frac{\kappa}{|\nabla f|} \, ds.$$

To prove the integral formula for L , first consider the case when $[a, b]$ is free of critical values. Let R denote $f^{-1}([a, b])$ in this argument. Let \mathbf{n} denote the unit outward normal (relative to R) on ∂R ; it should be clear that $\mathbf{n} = -\nabla f / |\nabla f|$ on $f^{-1}(a)$ and $\mathbf{n} = \nabla f / |\nabla f|$ on $f^{-1}(b)$. Then, by Green's theorem and (2),

$$L(b) - L(a) = \int_{\partial R} \left\langle \frac{\nabla f}{|\nabla f|}, \mathbf{n} \right\rangle \, ds = \iint_R \operatorname{div} \left(\frac{\nabla f}{|\nabla f|} \right) \, dA = \iint_R \kappa \, dA.$$

If $[a, b]$ has critical values only in its interior, this argument can be modified by excising from R small discs around the critical points (where $\nabla f / |\nabla f|$ ceases to be defined); if a (resp. b) is a critical value, replace it by $a + \epsilon$ (resp. $b - \epsilon$). We omit the details but note that the integrability of κ over R is essential. ■

Comparing the two proofs given for Theorem 2(b), the one deriving the formula for L' is perhaps more geometrically revealing, for, besides its independence on (2), it shows directly how curvature and gradient affect length of level curves. Indeed, varying metric properties of a family of curves belong to the broader field of evolution of curves and surfaces, which encompasses topics such as the curve-shortening flow and mean curvature flow. In §4.1, we show how our results on level curves may inform certain instances of curve evolution.

Remark Assume Hypothesis (†). By Theorem 2(a), we have the estimate

$$\min_{t \in [a, b]} L(t) < \frac{1}{b-a} \iint_{f^{-1}([a, b])} |\nabla f| \, dA < \max_{t \in [a, b]} L(t).$$

This inequality is optimal over all f satisfying the hypothesis, as evidenced by the functions r^n ($n \in \mathbb{N}$) on the unit disc. ■

Remark Suppose that $R \subset \Omega$ is a closed Jordan domain bounded by a component of a level set of f . Then, the length $\mathcal{L}(\partial R)$ of ∂R has a simple formula

$$\mathcal{L}(\partial R) = \pm \iint_R \kappa \, dA. \quad (5)$$

The proof for the integral formula in Theorem 2(b) will also prove this result, once we note that no critical points of f lie on ∂R (due to our assumption that f is a Morse function⁴); the integrability of κ is again essential, as R necessarily contains critical points in its interior. ■

3 Level Curves of Complex Analytic Functions

3.1 Level Curves near a Critical Point

We explain why level curves of a complex analytic function are *locally* so well-behaved that their intersections with a small disc around any point admit continuous length.

⁴While a level curve passing through a saddle point of a *Morse function* necessarily self-intersects, this may not be so in general, as evidenced by the level-0 curve of the function $g(x, y) = y^3 - x^6$.

For $z_0 \in \mathbb{C}$ and $r > 0$, let

$$D(z_0; r) = \{z : |z - z_0| < r\} \quad \text{and} \quad C(z_0; r) = \{z : |z - z_0| = r\} .$$

Let f denote a nonconstant analytic function on an open connected subset Ω of \mathbb{C} . By the *level- t set of f* , we mean the level- t set of $|f|$. As $|f|$ is not differentiable at any simple zero of f , we will consider $|f|^2$ when classifying level curves. By a *regular (resp. critical) level* of f , we mean a regular (resp. critical) level of $|f|^2$. Let $L(t)$ denote the length (whenever it can be defined) of the level- t set of f .

A simple calculation using the Cauchy-Riemann equations shows that

$$\left| \nabla \left(|f|^2 \right) \right| = 2 |f \cdot f'| .$$

The critical points of $|f|^2$ are then the zeros of f and of f' . If $|f(z_0)|^2 = t^2$ ($t > 0$) is a regular value of $|f|^2$, then, since $f'(z_0) \neq 0$, f conformally maps an arc $C \ni z_0$ on the level- t curve onto an arc on the circle $C(0; t)$. In other words, C is the conformal image (under the inverse of $f|_{D(z_0; \delta)}$) of a circular arc; hence a regular level of f is an analytic curve, the smoothest of all.

However, $|f|^2$ may not be a Morse function; its level curves near a saddle point are somewhat more complicated but still have a simple description. For simplicity, we will call z_0 a *saddle point* of f if it is a saddle point of $|f|^2$. If z_0 a saddle point of f , then $f(z_0) \neq 0$ but $f'(z_0) = 0$, and we show that level curves of f within $D(z_0; \epsilon)$ are images under a conformal map of level curves of $z^n - 1$ within $D(0; \delta)$, where n is the order of z_0 as a zero of $f(z) - f(z_0)$. (For contrast, recall that the behavior of a nondefinite quadratic form near the origin dictates that of a Morse function near a saddle point.)

In general, if f is analytic at z_0 , then there is a neighborhood U of z_0 and a conformal equivalence $w : U \rightarrow D(0; \delta)$ with $\delta < 1$ such that

$$f(z) = [w(z)]^n + f(z_0) \quad \text{for} \quad z \in U ;$$

necessarily, $w(z_0) = 0$ and $n = \min \{k : f^{(k)}(z_0) \neq 0\}$.

Suppose now that z_0 is a critical point of $|f|^2$, in which case either $f(z_0) = 0$ or $f'(z_0) = 0$.

If $f(z_0) = 0$, then the level-0 set of $f|_U$ is the singleton $\{z_0\}$, while the level- ϵ set of $f|_U$ (for small $\epsilon > 0$) is the level- $\sqrt[n]{\epsilon}$ set of w , which is the (conformal) image under w^{-1} of the origin-centered circle of radius $\sqrt[n]{\epsilon}$. So the length $L(t)$ of the level- t curves of $f|_U$ is continuous at $0 = |f(z_0)|$.

If $f(z_0) \neq 0$ but $f'(z_0) = 0$, then, by considering $-f(z)/f(z_0)$, we may assume that $f(z_0) = -1$, in which case $f = w^n - 1$ on U (with $n \geq 2$). For t near the critical value 1, the level- t set of $f|_U$ is the image under w^{-1} of the set $\{\zeta \in D(0; \delta) : |\zeta^n - 1| = t\}$. Hence, it suffices to consider the level- t curves (with t near 1) of the function $p_n : \zeta \mapsto (\zeta^n - 1)$ on $D(0; \delta)$, to which we now turn.

Denote the input for p_n by z (instead of ζ). With Γ_t denoting the level- t set of p_n in $D(0; \delta)$, i.e.,

$$\Gamma_t := \{z \in D(0; \delta) : |z^n - 1| = t\} ,$$

we have

$$z \in \Gamma_t \quad \text{iff} \quad z^n \in \Sigma_t := C(1; t) \cap D(0; \delta^n) ;$$

in other words, Γ_t is comprised of all the n th roots of every complex number on the circular arc Σ_t .

As t ranges over $[1 - \epsilon, 1]$, the circular arc Σ_t sweeps out the “annular strip” $\cup_{t \in [1 - \epsilon, 1]} \Sigma_t$, on which there are exactly n *continuous* n th roots, i.e., continuous functions g_1, \dots, g_n on $\cup_{t \in [1 - \epsilon, 1]} \Sigma_t$ such that $[g_j(w)]^n = w$. As a result, for each $t \in [1 - \epsilon, 1]$, Γ_t has exactly n branches, with the j th branch parametrized by

$$\theta \mapsto g_j(1 + te^{i\theta})$$

(where the range of θ depends continuously on t as well). It is then clear that the length of Γ_t varies continuously with t . For t ranging over $[1, 1 + \epsilon]$, similar consideration leads to the same conclusion concerning length but results in a different partition of Γ_1 into branches.

Remark Concerning the polynomial $p_n(z) := z^n - 1$, we mention two connections, one a simple fact while the other possibly (but improbably) fiction. First, $|p_n(z)| = t$ iff $|z|^{2n} - 2 \operatorname{Re}(z^n) + 1 = t^2$, and the latter is a real polynomial equation of degree $2n$ in $(\operatorname{Re} z, \operatorname{Im} z)$; therefore the level curves of p_n are real algebraic curves of degree $2n$. Second, for $n \geq 2$, the level-1 curve of p_n is conjectured around the mid-twentieth century by [9] to be no shorter than the level-1 curve of *any* monic n th-degree polynomial. This conjecture remains undecided except for the case $n = 2$, which is proven true by [10] near the end of the last century. The level-1 curve of $z^2 - 1$ is a Bernoulli’s lemniscate. ■

3.2 Length in Relation to Curvature and Derivative

With the same notation as in §3.1, let f be a *nonconstant* analytic function on an open connected subset Ω of \mathbb{C} .

The real function $|f|$ is *not* differentiable at and only at simple zeros of f , as shown in [1]. Where it is differentiable,

$$|\nabla |f|| = |f'| = |\nabla \operatorname{Re} f|$$

by the Cauchy-Riemann equations. The critical points of $|f|$, including points at which $|f|$ is not differentiable, are isolated and Lemma 1 is applicable to $|f|$.⁵ Given the good behavior of length of level curves of f (as seen in §3.1), the theory developed in §2.2 applies to $|f|$ *verbatim*, as long as the curvature κ of level curves is integrable over a disc around any critical point. While the integrability of κ is to be established in §5, we state brief versions of the results in Lemma 1 and Theorem 2.

Theorem 3 *Let a, b be values attained by $|f|$ such that $D := \{z : |f(z)| \in [a, b]\}$ is compact. Then,*

(a) *the area of D equals $\int_a^b \left(\int_{\{|f(z)|=t\}} (1/|f'(z)|) |dz| \right) dt$;*

(b) $\int_a^b L(t) dt = \iint_D |f'| dA$;

(c) $L(b) = L(a) + \iint_D \kappa dA$.

⁵If one wishes to avoid dealing with nondifferentiable functions, one may work with $|f|^2$ instead. A change of variable will result in the same formula as obtained by working with $|f|$.

Remark. Suppose that a component of the level- c curve of f bounds an open Jordan domain $D \subset \Omega$. In this Remark, let $L(t)$ denote the length of the level- t curve of $f|_D$.

This situation is markedly simpler than its counterpart for real functions discussed for (5) in §2.2. The extrema principle for analytic functions implies that $|f|$ never again attains the value c in D , that f vanishes somewhere in D , and that $\{|f(z)| : z \in D\} = [0, c)$.

If, for each $t \in (0, c)$, the level- t set of f in D is a simple closed convex curve (in which case $f^{-1}(0) \cap D$ is necessarily a singleton $\{z_0\}$ and f' does not vanish on $D \setminus \{z_0\}$), then

$$L'(t) = \int_{\{z \in D : |f(z)|=t\}} \frac{\kappa}{|f'(z)|} |dz| > 0$$

for $t \in (0, c)$.⁶ Hence, L is strictly increasing and we have the estimate

$$L(c) > \frac{1}{c} \iint_D |f'| dA,$$

which is optimal as exemplified by the functions z^n on $D(0; 1)$ for $n \in \mathbb{N}$. ■

In closing this section, we say a word about curvature. Since each regular level of f is an analytic curve, its Schwarz function in principle can yield its curvature; see [6, p. 45]. In practice, with machine-aided computation, the curvature formula (1) applied to $|f|^2$ suffices. However, we note a formula for κ purely in terms of f and f' (which, despite the best efforts of the author, has not been found in the literature), and we leave its derivation to the reader.

Proposition 4 *The level curves of an analytic function f has curvature*

$$\kappa = \frac{|f'|}{|f|} - \frac{\langle \nabla|f|, \nabla|f'| \rangle}{|f'|^2}.$$

4 Applications and Extensions

In §4.1, we give some geometric applications, among which is an analytic characterization of circles. In §4.2, we deduce some additional identities relating curvature and gradient and then extend our results to level surfaces.

4.1 Curve Evolution: Two Instances

We show that some simple instances of curve evolution in the plane can be framed in our elementary setting and addressed by our lines of inquiry. Although most results that we deduce are well-known, it is the simplicity of our method in obtaining them that we wish to illustrate. As a by-product, we obtain a characterization of circles in terms of existence of certain functions.

Consider a C^2 function f on an open planar domain. Suppose that the interval $[a, b]$ consists only of regular values attained by f and that $f^{-1}([a, b])$

⁶This positivity assertion is due to two reasons: our orientation stipulation subject to which the formula for L' holds, and semi-definiteness of the sign of κ because of the convexity assumption on the curve $|f(z)| = t$.

is connected and compact. For each $t \in [a, b]$, $f^{-1}(t)$ is then a simple closed curve. At $P \in f^{-1}([a, b])$, let $\mathbf{n}(P)$ be the outward unit normal at P of the (simple closed) curve $f^{-1}(f(P))$ and define

$$\sigma(P) = \left\langle \mathbf{n}(P), \frac{\nabla f(P)}{|\nabla f(P)|} \right\rangle .$$

Being continuous and integer-valued, σ is constant on (the connected) $f^{-1}([a, b])$. By our stipulation in §2.1, the level curves comprising $f^{-1}([a, b])$ all receive positive (resp. negative) orientation iff $\sigma = +1$ (resp. -1). It is in this context that we will apply our results to curve evolution.

Let $A(t)$ be the area enclosed by $f^{-1}(t)$. By Lemma 1 and Theorem 2,

$$L(b) = L(a) + \int_a^b \left(\int_{f^{-1}(t)} \frac{\kappa}{|\nabla f|} ds \right) dt; \quad A(b) = A(a) + \sigma \int_a^b \left(\int_{f^{-1}(t)} \frac{1}{|\nabla f|} ds \right) dt .$$

Example 1 Parallel closed curves and a characterization of circles

Let $\alpha : [0, 1] \rightarrow \mathbb{R}^2$ be a simple closed C^2 curve. For $t \geq 0$, define $\gamma_t : [0, 1] \rightarrow \mathbb{R}^2$ to be $\alpha + t\mathbf{n}$, where \mathbf{n} is the outward unit normal field along α ; i.e., the point $\gamma_t(u)$ is obtained by traveling from $\alpha(u)$ along $\mathbf{n}(u)$ for a distance of t units. As long as t is sufficiently small, γ_t defines a *simple* closed curve; if α is convex, then γ_t will be simple and convex for any $t > 0$. In any case, for any t such that γ_τ remains a simple closed curve for all $\tau \in [0, t]$, we seek the length $L(t)$ of γ_t and the area $A(t)$ bounded by γ_t .

Define a function f on the exterior of α by letting f assume the value t on the curve γ_t ; i.e., $f(\gamma_t(u)) := t$. Then, γ_t is the level- t curve of f , $\nabla f(\gamma_t(u))$ points in the outward normal direction $\mathbf{n}(u)$, and γ_t is given the positive orientation according to the stipulation in §2.1. Now, $|\nabla f|$, being the derivative of f with respect to distance along \mathbf{n} , clearly equals 1 (as f is the normal distance from α). Hence,

$$L(t) = L(0) + \int_0^t \left(\int_{\gamma_\tau} \kappa ds \right) d\tau = L(0) + \int_0^t 2\pi d\tau = L(0) + 2\pi t, \quad (6)$$

and

$$\begin{aligned} A(t) &= A(0) + \int_0^t \left(\int_{\gamma_\tau} 1 ds \right) d\tau = A(0) + \int_0^t L(\tau) d\tau \\ &= A(0) + L(0)t + \pi t^2, \end{aligned} \quad (7)$$

which are well known and stated for convex curves in [7, p. 47].

Note by the way an interesting consequence of (6) and (7). If a simple closed convex curve were to undergo *inward* parallel evolution, then $A(t)$ and $L(t)$ vanish at the same time iff $A(0) = L(0)^2/4\pi$, in which case the initial curve is the optimal curve for the isoperimetric inequality, i.e., a circle. For a more general notion of parallel curves in relation to the isoperimetric inequality, see [11, pp. 79-85].

Formula (6) shows that the evolving parallel curves lengthen at the rate of 2π per unit normal distance Δt . We can draw the same conclusion under a somewhat weaker hypothesis.

Proposition 5 *Suppose that $f^{-1}([a, b])$ is compact and that, for each $t \in [a, b]$, $f^{-1}(t)$ is a simple closed curve on which $|\nabla f|$ equals a positive constant $c(t)$. Then,*

(a) $|\nabla(L \circ f)| \equiv 2\pi$ on $f^{-1}([a, b])$;

(b) the area of $f^{-1}([a, b])$ equals $|L(b)^2 - L(a)^2|/4\pi$.

Proof. Letting ℓ denote $L \circ f$ and $P_0 \in f^{-1}([a, b])$ with $f(P_0) = t_0$, we calculate:

$$\begin{aligned} |\nabla \ell(P_0)| &= |L'(t_0)| \cdot |\nabla f(P_0)| = \left| \int_{f^{-1}(t_0)} \frac{\kappa}{|\nabla f|} ds \right| \cdot |\nabla f(P_0)| \\ &= \frac{1}{|\nabla f(P_0)|} \left| \int_{f^{-1}(t_0)} \kappa ds \right| \cdot |\nabla f(P_0)| = 2\pi, \end{aligned}$$

proving Part (a).

For Part (b), recall that $\sigma := \langle \mathbf{n}, \nabla f / |\nabla f| \rangle$ is constant on $f^{-1}([a, b])$. Noting our orientation stipulation (subject to which Theorem 2(b) holds), we have

$$L'(t) = \int_{f^{-1}(t)} \frac{\kappa}{|\nabla f|} ds = \frac{1}{c(t)} \int_{f^{-1}(t)} \kappa ds = \sigma \frac{2\pi}{c(t)}.$$

Thus, L' has a definite sign on $[a, b]$, i.e., that of σ , and so $L' \circ f = \sigma |L' \circ f|$. Hence

$$(L' \circ f) |\nabla f| = \sigma |L' \circ f| |\nabla f| = \sigma |\nabla(L \circ f)| = 2\pi\sigma.$$

Now consider

$$I := \iint_{f^{-1}[a, b]} (L' \circ f) |\nabla f| dA.$$

On one hand,

$$I = \iint_{f^{-1}[a, b]} 2\pi\sigma dA = 2\pi\sigma \iint_{f^{-1}[a, b]} 1 dA;$$

on the other hand, by Theorem 2(a) (with $h = L'$),

$$I = \int_a^b L'(t)L(t)dt = \frac{1}{2} [L(b)^2 - L(a)^2].$$

Comparing the two expressions for I proves the identity. ■

In light of this result, we raise and address the following question.

Problem *Given a simple closed C^2 curve C bounding a compact Jordan domain R and a point P interior to R , does there exist a C^2 Morse function $f : R \rightarrow [0, 1]$ such that $f^{-1}(0) = \{P\}$, $f^{-1}(1) = C$, and, for each $t \in (0, 1)$, $f^{-1}(t)$ is a simple closed curve on which $|\nabla f|$ equals a positive constant?*

Solution. It is perhaps plausible that R would have to be “highly symmetric” (i.e. a disc) about P . This is indeed the case, and can be deduced from Proposition 5(b). If there were such a function f , then the area of $f^{-1}([\epsilon, 1])$ would equal $[L^2(1) - L^2(\epsilon)]/4\pi$. Letting $\epsilon \rightarrow 0$, it follows that the area $\mathcal{A}(R)$ of R would equal $\mathcal{L}(C)^2/4\pi$, which, unless C is a circle, violates the isoperimetric inequality.

In conclusion, only when C is a circle can such a function exist; in other words, existence of such a function characterizes the circular domain that supports it. ■

Example 2 The curve-shortening flow

Begin with a simple closed convex C^2 curve $\gamma_0 : [0, 1] \rightarrow \mathbb{R}^2$ with nonvanishing curvature. Move each point P on γ_0 following the inward normal with initial speed equal to the curvature of γ_0 at P . Once motion begins (i.e., after an infinitesimal period of time), the moved points comprise a new curve and will then be moved with a new speed equal to the curvature of the new curve, so on and so forth. (As points with differing curvature moves with different speed, it is geometrically apparent that the evolution has the tendency to uniformize curvature.) Obviously, we are attempting a verbal description of a one-parameter family of evolving curves $\gamma_t : [0, 1] \rightarrow \mathbb{R}^2$ governed by a differential equation. The velocity of the point $\gamma_t(u)$ is on one hand $\partial\gamma_t(u)/\partial t$ by definition and on the other hand $\partial^2\gamma_t(u)/\partial s^2$ by prescription (where s is arc length along γ_t); i.e.,

$$\frac{\partial\gamma_t}{\partial t} = \frac{\partial^2\gamma_t}{\partial s^2}. \quad (8)$$

We assume the existence, uniqueness, and smoothness of solution (over a time interval $[0, t]$) to this equation⁷, as well as the simplicity and convexity of any future curve γ_t ; see [4] for a thorough account or [2, Appendix B] for a brief account. The solution is known as the *curve-shortening flow* for γ_0 .

Define a function f by letting f assume the value t on the curve γ_t . Then, γ_t is the level- t curve of f , along which ∇f points in the inward normal direction $-\mathbf{n}$. According to our stipulation in §2.1, γ_t is given the negative orientation and its curvature $\kappa \leq 0$. Now, $|\nabla f|$, being the derivative of time t with respect to distance along $-\mathbf{n}$, clearly equals the reciprocal of the speed of evolution, i.e.,

$$|\nabla f| = 1/|\kappa| = -1/\kappa.$$

Therefore,

$$\begin{aligned} L(t) &= L(0) - \int_0^t \left(\int_{\gamma_\tau} \kappa^2 ds \right) d\tau \\ \text{and } A(t) &= A(0) - \int_0^t \left(\int_{\gamma_\tau} \frac{1}{-1/\kappa} ds \right) d\tau = A(0) - 2\pi t, \end{aligned}$$

from which it follows that the evolution ceases by the time $A(0)/2\pi$ and that L is a strictly decreasing function because $\int_{\gamma_\tau} \kappa^2 ds > 0$.

Remark When γ_0 is not convex, the PDE (8) still well-defines the curve-shortening flow of γ_0 . (In this case, points on γ_t where the curve is not convex move outward.) Then the difficulty in applying our method lies in the construction of f , because γ_t may intersect $\gamma_{t'}$ whereas the level curves of f should not intersect. If we wish, we may disentangle the intersecting curves by constructing the surface

$$S := \{(\gamma_t(u), t) : u \in [0, 1], t \in [0, T]\} \subset \mathbb{R}^3$$

⁷This matter, belonging to the realm of partial differential equations, is highly nontrivial and will take us too far afield.

and then define f on S by $(\gamma_t(u), t) \mapsto t$. The level- t set of f is then the curve γ_t lifted to altitude t . Our results, once generalized to functions on surfaces, may conceivably address this situation. ■

4.2 Extensions

Before we mention extensions of our results to level (hyper)surfaces, we first note two additional relations between the curvature of level curves of a two-variable function and its gradient field.

Proposition 6 *Assume Hypothesis (†). Then,*

- (a) $\iint_{f^{-1}([a,b])} \kappa f_x dA = 0 = \iint_{f^{-1}([a,b])} \kappa f_y dA;$
- (b) *if, in addition, $[a, b]$ is free of critical values and $f^{-1}([a, b])$ is connected, then $\iint_{f^{-1}([a,b])} \kappa |\nabla f| dA = \pm 2\pi(b - a)$.*

Proof. For Part (a), let g in Lemma 1 be the vector-valued function $\kappa \nabla f$. Then,

$$\iint_{f^{-1}([a,b])} \kappa \nabla f dA = \int_a^b \left(\int_{f^{-1}(t)} \kappa \frac{\nabla f}{|\nabla f|} ds \right) dt.$$

It suffices to note that

$$\int_{f^{-1}(t)} \kappa \frac{\nabla f}{|\nabla f|} ds = - \int_{f^{-1}(t)} \kappa \mathbf{N} ds = - \int_{f^{-1}(t)} \frac{d\mathbf{T}}{ds} ds = \mathbf{0}$$

where the line integral is taken over all the components of $f^{-1}(t)$.

In Part (b), $f^{-1}([a, b])$ is diffeomorphic to $f^{-1}(a) \times [a, b]$ and $f^{-1}(a)$ is then also connected. Hence, for $t \in [a, b]$, $f^{-1}(t)$, being diffeomorphic to $f^{-1}(a)$, is a simple closed curve and $\int_{f^{-1}(t)} \kappa ds = \pm 2\pi$ (with the same sign in force for all t , as shown in §4.1). Letting $g = \kappa |\nabla f|$ in Lemma 1 proves the assertion. ■

To extend these and earlier results to level surfaces, we recall a few differential-geometric preliminaries. Let M be a connected, oriented, compact C^2 surface in \mathbb{R}^3 . An *orientation* on M is a choice of a continuous unit normal field \mathbf{N} on M ; the map

$$G : M \rightarrow S^2; P \mapsto \mathbf{N}(P)$$

is known as the *Gauss map*. For $P \in M$ and a unit vector $\mathbf{v} \in T_P M := \mathbf{N}(P)^\perp$, take the normal section $\Gamma_{\mathbf{v}}$ of M whose velocity at P is \mathbf{v} .⁸ The (signed) curvature $\kappa_{\mathbf{v}}$ of the curve $\Gamma_{\mathbf{v}}$ at P is defined by

$$\left. \frac{d^2 \Gamma_{\mathbf{v}}}{ds^2} \right|_P = \kappa_{\mathbf{v}} \mathbf{N}(P).$$

The *mean curvature* H at P is defined⁹ to be, as the term suggests, the mean of $\kappa_{\mathbf{v}}$ for \mathbf{v} varying over the unit circle in $T_P M$; in explicit terms,

$$H(P) := \frac{1}{2\pi} \int_{\{\mathbf{v} \in T_P M : |\mathbf{v}|=1\}} \kappa_{\mathbf{v}} ds.$$

⁸The curve $\Gamma_{\mathbf{v}}$ is the intersection of M with the plane determined by \mathbf{N} and \mathbf{v} .

⁹Depending on the approach taken, there are various ways to define H . We adopt the one that is the easiest to motivate.

(Euler's theorem on $\kappa_{\mathbf{v}}$ then implies that $H(P) = (\kappa_1 + \kappa_2)/2$, where κ_1 and κ_2 are the extrema of $\kappa_{\mathbf{v}}$, known as the *principal curvatures* at P .) The sign of $H(P)$ obviously depends on the orientation of M . The *Gaussian curvature* K is simpler to define:

$$K(P) := \det(dG_P : T_P M \rightarrow T_{\mathbf{N}(P)} S^2),$$

which can be interpreted as the local (signed) area expansion factor at P of the Gauss map. Clearly, $K(P)$ is independent of orientation on M . (As a matter of fact, the principal curvatures κ_1 and κ_2 are the eigenvalues of dG_P , $K(P) = \kappa_1 \kappa_2$, and $H(P) = (\text{Tr } dG_P)/2$.)

Several facts are relevant here. Concerning K , we have

$$\iint_M K \mathbf{N} d\sigma_M = \mathbf{0} \quad \text{and} \quad \iint_M K d\sigma_M = 2\pi\chi(M), \quad (9)$$

where $d\sigma_M$ is the surface area form on M and $\chi(M)$ is the Euler characteristic of M . The second identity is the well-known Gauss-Bonnet Theorem, while the first identity follows from a routine calculation with differential forms.¹⁰ Concerning H , it is a fact (as shown in [8, p. 142]) that

$$H = -\frac{1}{2} \text{div } \mathbf{N}.$$

Returning to our context, assume for simplicity that

f is a C^2 function on a connected open set $\Omega \subset \mathbb{R}^3$, $[a, b]$ is an interval of *regular* values attained by f , and $f^{-1}([a, b])$ is connected and compact.

Then, for $t_1, t_2 \in [a, b]$, $f^{-1}(t_1)$ and $f^{-1}(t_2)$ are diffeomorphic compact surfaces. We orient $f^{-1}(t)$ by letting $\mathbf{N} = -\nabla f / |\nabla f|$. Then, H and K , both meaningful on $f^{-1}(t)$, become functions on $f^{-1}([a, b])$ and have explicit formulae in terms of the partial derivatives of f , as shown in [12, p. 204]. With our choice of \mathbf{N} ,

$$H = -\frac{1}{2} \text{div } \mathbf{N} = \frac{1}{2} \text{div } \frac{\nabla f}{|\nabla f|}.$$

Lemma 1 in the present context takes the form

$$\iiint_{f^{-1}([a, b])} g dV = \int_a^b \left(\iint_{f^{-1}(t)} \frac{g}{|\nabla f|} d\sigma \right) dt, \quad (10)$$

which yields a formula for the volume of $f^{-1}([a, b])$ upon letting $g \equiv 1$.

Applying (10) with suitable choices of g , we have, as consequences of (9),

$$\iiint_{f^{-1}([a, b])} K \nabla f dV = \mathbf{0} \quad \text{and} \quad \iiint_{f^{-1}([a, b])} K |\nabla f| dV = 2\pi(b-a)\chi(f^{-1}(a)).$$

¹⁰For detail, define the vector-valued 2-form ω on S^2 by letting $\omega = \text{Id}_{S^2} d\sigma_{S^2}$. Then $G^*\omega = K \mathbf{N} d\sigma_M$, as can be verified pointwise. Hence,

$$\int_M K \mathbf{N} d\sigma_M = \int_M G^* \omega = \text{deg } G \cdot \int_{S^2} \omega.$$

But $\int_{S^2} \omega = \int_{S^2} \text{Id}_{S^2} d\sigma_{S^2} = \mathbf{0}$ due to cancellation of antipodal contributions.

Writing $A(t)$ for the surface area of $f^{-1}(t)$, we have, for $t \in (a, b)$,

$$\frac{d}{dt} \iiint_{f^{-1}([a,t])} |\nabla f| dV = A'(t) = A(a) + 2 \iiint_{f^{-1}([a,t])} H dV,$$

with the latter equality equivalent to

$$A'(t) = \iint_{f^{-1}(t)} \frac{2H}{|\nabla f|} d\sigma.$$

The proofs for these statements parallel those given for their counterparts on level curves; except for the integral identity concerning $K |\nabla f|$, we may as well allow $[a, b]$ to have critical values. Similar to the treatment of curve evolution in §4.1, certain instances of surface evolution, such as parallel expansion and mean curvature flow of a convex surface, can be treated by these results.

We conclude with a cursory mention of level hypersurfaces of multivariable functions. Given the background laid, it suffices to note that, for a connected oriented one-codimensional compact submanifold M of \mathbb{R}^{n+1} with Gauss map $G : M \ni P \mapsto \mathbf{N}(P) \in S^n$, we have the mean curvature¹¹ and the Gaussian curvature

$$H := \frac{1}{n} \operatorname{Tr} dG \quad \text{and} \quad K := \det dG$$

satisfying

$$H = -\frac{1}{n} \operatorname{div} \mathbf{N} \quad \text{and} \quad \int_M K d\sigma_M = \frac{1}{2} \chi(M) \sigma(S^n) = \frac{\pi^{(n+1)/2}}{\Gamma(\frac{n+1}{2})} \chi(M).$$

5 Integrability of Curvature and a Generalized Morse Condition

Finally, we prove that, if f is a Morse function or the square of the modulus of an analytic function, then the curvature κ of its level curves is integrable over a small neighborhood of any critical point. In fact, the proof is valid under a more general hypothesis.

We first review the definition of improper multiple integral in a limited context that meets our purpose; cf. [3, pp. 221-223] or [5, pp. 257-260]. Let D be a closed disc centered at P and let g be continuous on $D \setminus \{P\}$. Consider $I_j := \iint_{D_j} g dA$ where $\{D_j\}$ is an approximating sequence of $D \setminus \{P\}$, i.e., an expanding sequence of Jordan measurable closed subsets of $D \setminus \{P\}$ such that any interior point of $D \setminus \{P\}$ is eventually interior to some D_j . If I_j converges to a limit that is independent of the choice of the approximating sequence $\{D_j\}$, then $\lim I_j$ is taken to be the value of the improper Riemann integral $\iint_D g dA$.

It is shown in [3, pp. 221-223] and [5, pp. 257-260] that $\iint_D g dA$ converges iff $\iint_D |g| dA$ converges. If $|g(X)|$ is bounded by $1/|XP|$ for $X \in D$, a calculation in polar coordinates shows the convergence of $\iint_D |g| dA$ and hence of $\iint_D g dA$.

¹¹There is a more general notion of mean curvature $\mu_r := p_r(\kappa_1, \dots, \kappa_n) / \binom{n}{r}$ where p_r is the r th-degree elementary symmetric polynomial and κ_j 's are the principal curvatures, i.e., the eigenvalues of dG . Then, $\mu_1 = H$ and $\mu_n = K$. The integral of μ_r has geometric significance as well, but discussion of them will take us too far afield.

Proposition 7 *Suppose that, near the origin O , $f(x, y) = f(O) + p(x, y) + o(r^n)$ for some homogeneous polynomial p of degree $n \geq 2$. If O (necessarily a critical point of f) is the only critical point of p (and hence an isolated critical point of f), then the improper Riemann integral $\iint_D \kappa dA$ converges on a sufficiently small disc D centered at O .*

Proof. Take D to be so small that O is the only critical point of f in it, in which case κ is continuous on $D' := D \setminus \{O\}$. (Recall (1) in §2.1, a formula for κ .)

Under our hypothesis,

$$f_x = p_x + r^{n-1}\epsilon_1 \quad \text{and} \quad f_y = p_y + r^{n-1}\epsilon_2,$$

where $\epsilon_j \rightarrow 0$ as $r \rightarrow 0$. Using polar coordinates (r, θ) in D' , we may write $p_x(\mathbf{r}) = r^{n-1}\alpha_1(\theta)$ and $p_y(\mathbf{r}) = r^{n-1}\alpha_2(\theta)$, where, e.g., $\alpha_1(\theta) = p_x(\cos \theta, \sin \theta)$. Hence, on D' ,

$$|\nabla f(\mathbf{r})|^2 = r^{2n-2} \left[(\alpha_1(\theta) + \epsilon_1)^2 + (\alpha_2(\theta) + \epsilon_2)^2 \right].$$

As O is assumed to be the only critical point of p , $\alpha_1(\theta)^2 + \alpha_2(\theta)^2 \neq 0$ for $\theta \in [0, 2\pi]$. Letting $m = \min(\alpha_1(\theta)^2 + \alpha_2(\theta)^2)$, then $m > 0$ and we have, for sufficiently small r , that $|\nabla f(\mathbf{r})|^2 > \frac{1}{2}mr^{2n-2}$ and hence

$$|\nabla f(\mathbf{r})|^3 > Cr^{3n-3}$$

for some constant $C > 0$. By an entirely similar analysis, we can show that, for sufficiently small r , there is some constant M such that

$$|f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2|(\mathbf{r}) < Mr^{3n-4}.$$

By (1), we have

$$|\kappa(\mathbf{r})| = \frac{|f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2|(\mathbf{r})}{|\nabla f(\mathbf{r})|^3} < \frac{M}{C} \frac{1}{r}$$

for sufficiently small r . Hence, κ is integrable on D . ■

The hypothesis of Proposition 7 motivates the following definition.

Definition Let $n \geq 2$. A C^n function f on a planar domain is said to satisfy the *generalized Morse condition of degree n* if, at any critical point (x_0, y_0) , its n th-degree Taylor polynomial equals

$$f(x_0, y_0) + p(x - x_0, y - y_0)$$

for a *homogeneous* n th-degree polynomial p whose *only* critical point is the origin.

Lastly, we show that, for an analytic function f , the function $|f|^2$ satisfies the generalized Morse condition.

Lemma 8 *For an analytic function f , the function $|f|^2$ satisfies the generalized Morse condition.*

Proof. Let $Q = |f|^2$. Suppose, without loss of generality, that 0 is a critical point of Q , in which case either $f(0) = 0$ or $f'(0) = 0$. For z near 0,

$$f(z) = f(0) + z^n g(z)$$

for a unique integer $n \geq 1$ and a unique analytic function g with $g(0) \neq 0$.

If $f(0) = 0$,

$$Q(z) = |z^n g(z)|^2 = z^{2n} |g(z)|^2 = (x^2 + y^2)^n |g(z)|^2$$

and therefore $(x^2 + y^2)^n$ is the lowest-degree term in the Taylor expansion of $Q(x, y)$ (as a real function). Obviously, $(x^2 + y^2)^n$ has only one critical point at the origin, and hence Q satisfies the generalized Morse condition of degree $2n$.

If $f'(0) = 0$, then $n \geq 2$. Let $a = f(0)$ and $b = g(0)$. Then

$$\begin{aligned} Q(z) &= |a + z^n g(z)|^2 = (a + z^n g(z)) (\bar{a} + \bar{z}^n \overline{g(z)}) \\ &= |a|^2 + 2 \operatorname{Re}(\bar{a} z^n g(z)) + |z|^{2n} |g(z)|^2 \\ &= |a|^2 + \operatorname{Re}(2\bar{a} b z^n) + o(|z|^n). \end{aligned}$$

Let $p(x, y) = \operatorname{Re}(2\bar{a} b(x + iy)^n)$, a homogeneous real polynomial of degree $n \geq 2$. To see that $\nabla p(x, y) = \mathbf{0}$ iff $x = 0 = y$, it suffices to note that, for any analytic function w , $|\nabla \operatorname{Re} w| = |w'|$ (by Cauchy-Riemann equations). Hence, Q satisfies the generalized Morse condition of degree n . ■

At long last, Proposition 7 and Lemma 8 imply the following.

Corollary 9 *The curvature of level curves of an analytic function is integrable around each of its critical points.*

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