Certain Metric Properties of Level Hypersurfaces

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ABSTRACT. This note establishes several integral identities relating certain metric properties of level hypersurfaces of Morse functions.

1 Introduction

Let f be a C^2 Morse function on an open connected subset Ω of \mathbb{R}^{n+1} where $n \geq 2$. Suppose that a and b are values of f such that $f^{-1}([a, b])$ is compact. For $t \in [a, b]$, let $\nu(t)$ be the (*n*-dimensional) volume of the level-t set $f^{-1}(t)$. Note that, since f is a Morse function, $\nu(t)$ is well-defined even if t is a critical value and that $\nu : [a, b] \to \mathbb{R}$ is continuous. At each regular point (i.e., noncritical point) on $f^{-1}(t)$, let $\mathbf{N} = -\nabla f / |\nabla f|$. This choice of unit normal induces a Gauss map G on the set of regular points on $f^{-1}(t)$, with $G(p) = \mathbf{N}(p) \in S^n$. The mean curvature H and the Gaussian curvature K are defined on the set of regular points on $f^{-1}(t)$ by the standard definitions

$$H = \frac{1}{n} \operatorname{Tr} dG$$
 and $K = \det dG$.

We henceforth view H and K as functions on the set of regular points of $f^{-1}([a,b])$; i.e., H(p) and K(p) are the mean curvature and Gaussian curvature of $f^{-1}(f(p))$ at p.

We now state our main results, in which $d\mu$ is the Lebesgue measure on \mathbb{R}^{n+1} and ∂_i denotes the *i*-th partial derivative.

Theorem Assume the preceding assumptions and notation.

- (a) $\nu(b) \nu(a) = n \int_{f^{-1}([a,b])} H \, d\mu$.
- (b) $\int_{a}^{b} \nu(t) dt = \int_{f^{-1}([a,b])} |\nabla f| d\mu$.
- (c) $\int_{f^{-1}([a,b])} K \partial_i f \, d\mu = 0$ for each $i \in \{1, \cdots, n+1\}$.

(Implicit in these results is the assertion that the functions H and $K\partial_i f$ are integrable on $f^{-1}([a, b])$. This is a consequence of f being a Morse function, as we shall demonstrate.)

2 Two Preparatory Results

In many results, we assume the following hypothesis.

Hypothesis \dagger : f is a C^2 Morse function on an open connected subset Ω of \mathbb{R}^{n+1} where $n+1 \geq 3$; a and b are values of f such that $f^{-1}([a,b])$ is compact.

Lemma 1 Assume Hypothesis \dagger . Suppose that g is a function that is continuous on the set of regular points in $f^{-1}([a,b])$ and integrable on $f^{-1}([a,b])$. Then

$$\int_{f^{-1}([a,b])} g \, d\mu = \int_a^b \left(\int_{f^{-1}(t)} \frac{g}{|\nabla f|} d\sigma \right) dt \,,$$

where $d\sigma$ is the (n-dimensional) volume form on $f^{-1}(t)$ and $\int_{f^{-1}(t)} (g/|\nabla f|) d\sigma$ is only defined for t a regular value.¹

Proof. There are two cases, according as whether [a, b] contains a critical value.

Case 1: [a, b] is free of critical values. For each $p \in f^{-1}(a)$, let $t \mapsto F(p, t)$ be the integral curve for the field $\nabla f / |\nabla f|^2$. The map $F : f^{-1}(a) \times [a, b] \to f^{-1}([a, b])$ is then a diffeomorphism, providing the transformation of variables that results in the claimed formula. (In detail, take a coordinate patch U on $f^{-1}(a)$ and apply Fubini's theorem to $U \times [a, b] \xrightarrow{F|_{U \times [a, b]}} F(U \times [a, b])$.)

Case 2: [a, b] contains a critical value. Let S be the (finite) set of critical values in [a, b]. Then, $(a, b) \\ S$ is a disjoint union of finitely many intervals $I_j := (c_j, c_{j+1})$ of regular values. As $f^{-1}([a, b]) = \cup_j f^{-1}(I_j) \cup f^{-1}(S \cup \{a, b\})$ and $f^{-1}(S \cup \{a, b\})$ has Lebesgue measure zero (as a subset of \mathbb{R}^{n+1}),

$$\int_{f^{-1}([a,b])} g \, d\mu = \sum_{j} \int_{f^{-1}(I_j)} g \, d\mu$$

Applying Case 1 to $f^{-1}([c_j + \epsilon, c_{j+1} - \delta])$ and letting $\epsilon, \delta \to 0^+$, we have

$$\int_{f^{-1}(I_j)} g \, d\mu = \lim_{\epsilon, \delta \to 0^+} \int_{f^{-1}([c_j + \epsilon, c_{j+1} - \delta])} g \, d\mu$$
$$= \lim_{\epsilon, \delta \to 0^+} \int_{c_j + \epsilon}^{c_{j+1} - \delta} \left(\int_{f^{-1}(t)} \frac{g}{|\nabla f|} d\sigma \right) dt$$
$$= \int_{c_j}^{c_{j+1}} \left(\int_{f^{-1}(t)} \frac{g}{|\nabla f|} d\sigma \right) dt \, .$$

Summing these integrals over j proves the assertion.

Recall from §1 the mean curvature H and Gaussian curvature K, both regarded as functions on the set of regular points of f. Explicit formulae are known for H and K. To state them, let Q be the Hessian quadratic form associated with f and define the quadratic form Q^* to be the one whose standard matrix is the adjugate (or "classical adjoint") of the standard matrix for Q; we shall regard the two quadratic forms Q and Q^* as real-valued functions of one vector variable. Then,

$$H = \frac{\left|\nabla f\right|^2 \operatorname{Tr} Q - Q(\nabla f)}{n \left|\nabla f\right|^3} \quad \text{and} \quad K = \frac{Q^*(\nabla f)}{\left|\nabla f\right|^{n+2}}.$$

These are implicit in [3, p. 204] and made explicit in [2]. (In both of these references, $f^{-1}(t)$ is oriented by $\nabla f / |\nabla f|$, the opposite of our choice of **N**.)

Lemma 2 For a C^2 Morse function f on an open set $\Omega \subset \mathbb{R}^{n+1}$, the functions $H, K\partial_i f$, and $K |\nabla f|$ are all integrable on any compact subset of Ω .

¹The "outer" integral $\int_{a}^{b} \cdots dt$ on the right may first be interpreted as an improper Riemann integral. Once the formula is proven, applying it to |g| shows that the one-variable function $\varphi(t) := \int_{f^{-1}(t)} (g/|\nabla f|) d\sigma$ is absolutely integrable over [a, b], since $|\varphi(t)| \le h(t) := \int_{f^{-1}(t)} (|g|/|\nabla f|) d\sigma$ and $\int_{a}^{b} h(t) dt = \int_{f^{-1}([a, b])} |g| d\mu$. Hence, $\int_{a}^{b} \varphi(t) dt$ may also be interpreted as a Lebesgue integral.

Proof. It suffices to show that they are integrable "near" each critical point p, i.e., on a closed ball D centered at p in which p is the only critical point. Without loss of generality, assume that p is the origin $\mathbf{0} \in \mathbb{R}^{n+1}$. We notate a typical point in \mathbb{R}^{n+1} by writing its position vector \mathbf{r} and we let $r = \|\mathbf{r}\|$. Then, for \mathbf{r} near $\mathbf{0}$,

$$f(\mathbf{r}) = f(\mathbf{0}) + P(\mathbf{r}) + o(r^2)$$

where $P(\mathbf{r})$ is the quadratic polynomial $\frac{1}{2}Q(\mathbf{r})$. For each $i \in \{1, \dots, n+1\}$,

$$\partial_i f = \partial_i P + r\epsilon_i \,,$$

where $\epsilon_i \to 0$ as $r \to 0$, and for $\mathbf{r} \in D' := D \setminus \{\mathbf{0}\},\$

$$\partial_i P(\mathbf{r}) = r\alpha_i(\mathbf{r}/r)$$

where α_i is a function on S^n . Hence, on D',

$$\left|\nabla f(\mathbf{r})\right|^2 = r^2 \sum_{i=1}^{n+1} \left(\alpha_i(\mathbf{r}/r) + \epsilon_i\right)^2 \,.$$

As f is a Morse function, **0** is the only critical point of P and thus $\sum_i \alpha_i(\mathbf{r})^2 > 0$ for $\mathbf{r} \in S^n$. Letting

$$m = \min_{\mathbf{r}\in S^n} \sum_{i=1}^{n+1} \alpha_i(\mathbf{r})^2 \,,$$

we have, for sufficiently small r, $\frac{1}{2}mr^2 \leq |\nabla f(\mathbf{r})|^2 \leq 2mr^2$. Hence, there are positive numbers C, M_1, M_2, δ such that, whenever $r \leq \delta$,

$$|\nabla f(\mathbf{r})| \ge Cr$$

as well as

$$\left|\left|\nabla f\right|^{2} \operatorname{Tr} Q - Q(\nabla f)\right|(\mathbf{r}) \leq M_{1}r^{2} \text{ and } \left|Q^{*}(\nabla f)\right|(\mathbf{r}) \leq M_{2}r^{2}.$$

Therefore, for $r \leq \delta$,

$$|H(\mathbf{r})| = \frac{\left|\left|\nabla f\right|^{2} \operatorname{Tr} Q - Q(\nabla f)\right|(\mathbf{r})}{n \left|\nabla f(\mathbf{r})\right|^{3}} \le \frac{M_{1}}{nC^{3}} \frac{1}{r}$$

and

$$|K(\mathbf{r})\partial_i f(\mathbf{r})| \le |K(\mathbf{r})\nabla f(\mathbf{r})| = \frac{|Q^*(\nabla f)|(\mathbf{r})|}{|\nabla f|^{n+1}} \le \frac{M_2}{C^{n+1}} \frac{1}{r^{n-1}}$$

It is a standard fact that, for any c > 0, $1/r^{n+1-c}$ is integrable on any origincentered ball in \mathbb{R}^{n+1} . Hence, H, $K\partial_i f$, and $K |\nabla f|$ are all integrable on D.

3 Main Results

We establish the main results of the article.

Theorem 3 Under Hypothesis \dagger , $\nu(b) - \nu(a) = n \int_{f^{-1}([a,b])} H d\mu$.

Proof. First recall (from [1, p. 142]) that $H = -\frac{1}{n} \operatorname{div} \mathbf{N}$. With $\mathbf{N} := -\nabla f / |\nabla f|$,

$$H = \frac{1}{n} \operatorname{div} \frac{\nabla f}{|\nabla f|} \,.$$

In the following, let $R = f^{-1}([a, b])$. There are two cases according as whether [a, b] contains a critical value.

Case 1: [a, b] is free of critical values. Then, R is an (n + 1)-manifold with boundary $f^{-1}(a) \cup f^{-1}(b)$. Let **n** denote the unit outward normal (relative to R) on ∂R ; then $\mathbf{n} = -\nabla f / |\nabla f|$ on $f^{-1}(a)$ and $\mathbf{n} = \nabla f / |\nabla f|$ on $f^{-1}(b)$. Now,

$$\nu(b) - \nu(a) = \int_{\partial R} \left\langle \frac{\nabla f}{|\nabla f|}, \mathbf{n} \right\rangle d\sigma = \int_{R} \operatorname{div} \frac{\nabla f}{|\nabla f|} d\mu = \int_{R} nH \, d\mu.$$

Case 2: [a, b] contains a critical value. Let S be the (finite) set of critical values in [a, b]. Then, $(a, b) \\ S$ is a disjoint union of finitely many intervals $I_j = (c_j, c_{j+1})$ of regular values. As $R = \bigcup_j f^{-1}(I_j) \cup f^{-1}(S \cup \{a, b\})$ and $f^{-1}(S \cup \{a, b\})$ has Lebesgue measure zero,

$$\int_R H \, d\mu = \sum_j \int_{f^{-1}(I_j)} H \, d\mu$$

It remains to note that, for each j,

$$\int_{f^{-1}(I_j)} H \, d\mu = \lim_{\epsilon \to 0^+} \int_{f^{-1}([c_j + \epsilon, c_{j+1} - \epsilon])} H \, d\mu \quad \text{(by integrability of } H\text{)}$$
$$= \lim_{\epsilon \to 0^+} \frac{1}{n} \left(\nu(c_{j+1} - \epsilon) - \nu(c_j + \epsilon)\right) \quad \text{(by Case 1)}$$
$$= \frac{1}{n} \left(\nu(c_{j+1}) - \nu(c_j)\right) \quad \text{(by continuity of } \nu\text{)}.$$

With the aid of Lemma 1, Theorem 3 easily yields a formula for ν' , which would take considerable effort to obtain otherwise.

Corollary 4 Assume Hypothesis \dagger . For any regular value $t_0 \in [a, b]$,

$$\nu'(t_0) = n \int_{f^{-1}(t_0)} \frac{H}{|\nabla f|} d\sigma$$

Proof. For a regular value $t_0 \in (a, b)$,

$$\nu'(t_0) = \left. \frac{d}{dt} \right|_{t_0} \int_{f^{-1}([a,t])} nH \, d\mu \quad \text{(by Theorem 3)}$$
$$= \left. \frac{d}{dt} \right|_{t_0} \int_a^t \left(\int_{f^{-1}(\tau)} \frac{nH}{|\nabla f|} d\sigma \right) d\tau \quad \text{(by Lemma 1)}$$
$$= \left. \int_{f^{-1}(t_0)} \frac{nH}{|\nabla f|} d\sigma \quad \text{(by fundamental theorem of calculus)} \right.$$

We show more applications of Lemma 1 with a certain choice of g.

Theorem 5 Under Hypothesis \dagger , $\int_{f^{-1}[a,b]} (h \circ f) \cdot |\nabla f| d\mu = \int_a^b h(t)\nu(t) dt$ for any integrable function h on [a,b]. In particular, for any $t_0 \in [a,b]$,

$$\int_{a}^{t_{0}} \nu(t) \, dt = \int_{f^{-1}([a,t_{0}])} |\nabla f| \, d\mu \,,$$

or equivalently,

$$\nu(t_0) = \left. \frac{d}{dt} \right|_{t_0} \int_{f^{-1}([a,t])} |\nabla f| \, d\mu \, .$$

Proof. The first assertion follows from Lemma 1 by letting $g = (h \circ f) \cdot |\nabla f|$. The second assertion results from letting h be the indicator function for $[a, t_0]$. Continuity of ν makes applicable the fundamental theorem of calculus, yielding the last assertion.

Proposition 6 Assume Hypothesis †.

(a)
$$\int_{f^{-1}([a,b])} K \partial_i f \, d\mu = 0 \text{ for } i \in \{1, \cdots, n+1\}.$$

(b) If, in addition, n is even and [a, b] is free of critical values, then

$$\int_{f^{-1}([a,b])} K |\nabla f| \, d\mu = \frac{1}{2} (b-a) \chi(f^{-1}(a)) \nu(S^n) \,,$$

where $\nu(S^n)$ is the (n-dimensional) volume of the unit sphere S^n and $\chi(f^{-1}(a))$ is the Euler characteristic of $f^{-1}(a)$.

Proof. For Part (a), let g in Lemma 1 be the vector-valued function $K\nabla f$. Then,

$$\int_{f^{-1}([a,b])} K\nabla f \, d\mu = \int_a^b \left(\int_{f^{-1}(t)} K \frac{\nabla f}{|\nabla f|} d\sigma \right) dt \, .$$

Now, note that

$$\int_{f^{-1}(t)} K \frac{\nabla f}{|\nabla f|} d\sigma = -\int_{f^{-1}(t)} K \mathbf{N} \, d\sigma = \mathbf{0} \, .$$

For detail of the last equality, let M denote $f^{-1}(t)$ and define the vector-valued n-form ω on S^n by letting $\omega = \operatorname{Id}_{S^n} d\sigma_{S^n}$, where $d\sigma_{S^n}$ is the volume form on S^n . Then, with G being the Gauss map $p \mapsto \mathbf{N}(p)$, $G^*\omega = K\mathbf{N} d\sigma$ as can be verified pointwise. Hence,

$$\int_M K\mathbf{N} \, d\sigma = \int_M G^* \omega = \deg G \cdot \int_{S^n} \omega \, .$$

 But

$$\int_{S^n} \omega = \int_{S^n} \operatorname{Id}_{S^n} d\sigma_{S^n} = \mathbf{0}$$

due to cancellation of antipodal contributions.

Under the hypothesis of Part (b), $f^{-1}(t)$ is diffeomorphic to $f^{-1}(a)$ for $t \in [a, b]$. By Gauss-Bonnet theorem,

$$\int_{f^{-1}(t)} K \, d\sigma = \frac{1}{2} \chi(f^{-1}(t)) \nu(S^n) = \frac{1}{2} \chi(f^{-1}(a)) \nu(S^n)$$

Letting $g = K |\nabla f|$ in Lemma 1 then proves Part (b).

References

- [1] M. P. do Carmo, Riemannian Geometry, Birkhäuser, Boston, 1992.
- [2] R. Goldman, Curvature formulas for implicit curves and surfaces, Computer Aided Geometric Design 22 (2005), pp. 632-658.
- [3] M. Spivak, A Comprehensive Introduction to Differential Geometry, Vol. 3, 2nd ed., Publish or Perish, Houston, 1979.

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