IS THERE ANY NONTRIVIAL COMPACT GENERALIZED SHIFT OPERATOR ON HILBERT SPACES?

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Abstract. In the following text for cardinal number $\tau > 0$, and self–map $\varphi : \tau \to \tau$ we show the generalized shift operator $\sigma_\varphi(\ell^2(\tau)) \subseteq \ell^2(\tau)$ (where $\sigma_\varphi((x_\alpha)_{\alpha<\tau}) = (x_{\varphi(\alpha)})_{\alpha<\tau}$ for $(x_\alpha)_{\alpha<\tau} \in \mathbb{C}^\tau$) if and only if $\varphi : \tau \to \tau$ is bounded and in this case $\sigma_\varphi \restriction_{\ell^2(\tau)}: \ell^2(\tau) \to \ell^2(\tau)$ is continuous, consequently $\sigma_\varphi \restriction_{\ell^2(\tau)}: \ell^2(\tau) \to \ell^2(\tau)$ is a compact operator if and only if τ is finite.

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1. Preliminaries

The concept of generalized shifts has been introduced for the first time in [2] as a generalization of one-sided shift $\underset{(a_1,a_2,\cdots)\mapsto(a_2,a_3,\cdots)}{\{1,\ldots,k\}^{\mathbb{N}} \to \{1,\ldots,k\}^{\mathbb{N}}}$ and two-sided shift $\underset{(a_n)_{n\in\mathbb{Z}}\mapsto(a_{n+1})_{n\in\mathbb{Z}}}{\{1,\ldots,k\}^{\mathbb{Z}} \to \{1,\ldots,k\}^{\mathbb{Z}}}$ [10, 9]. Suppose K is a nonempty set with at least two elements, Γ is a nonempty set, and $\varphi : \Gamma \to \Gamma$ is an arbitrary map, then we call $\sigma_\varphi : \underset{(x_\alpha)_{\alpha\in\Gamma}\mapsto(x_{\varphi(\alpha)})_{\alpha\in\Gamma}}{K^\Gamma \to K^\Gamma}$ a generalized shift (for one-sided and two-sided shifts consider $\varphi(n) = n + 1$). It's evident that for topological space K, $\sigma_\varphi : K^\Gamma \to K^\Gamma$ is continuous, where K^Γ is equipped by product topology.

For Hilbert space H there exists unique cardinal number τ such that H and $\ell^2(\tau)$ are isomorphic [4, 8]. All members of the collection $\{\ell^2(\tau) : \tau$ is a non–zero cardinal number$\}$ are Hilbert spaces, moreover for cardinal number τ and $(x_\alpha)_{\alpha<\tau} \in \mathbb{K}^\tau$ (where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ depending on our choice for real Hilbert spaces or Complex Hilbert spaces) we have $x = (x_\alpha)_{\alpha<\tau} \in \ell^2(\tau)$ if and only if $||x||^2 := \sum\limits_{\alpha<\tau} |x_\alpha|^2 < +\infty$. Moreover for $(x_\alpha)_{\alpha<\tau}, (y_\alpha)_{\alpha<\tau} \in \ell^2(\tau)$ let $< (x_\alpha)_{\alpha<\tau}, (y_\alpha)_{\alpha<\tau} >= \sum\limits_{\alpha<\tau} x_\alpha \overline{y_\alpha}$ (inner product). For $\varphi : \tau \to \tau$, one may consider $\sigma_\varphi : \mathbb{K}^\tau \to \mathbb{K}^\tau$ in particular we may study $\sigma_\varphi \restriction_{\ell^2(\tau)}: \ell^2(\tau) \to \mathbb{K}^\tau$.

Convention. In the following text suppose $\tau > 1$ is a cardinal number and $\varphi : \tau \to \tau$ is arbitrary, we denote $\sigma_\varphi \restriction_{\ell^2(\tau)}: \ell^2(\tau) \to \mathbb{K}^\tau$ simply by $\sigma_\varphi : \ell^2(\tau) \to \mathbb{K}^\tau$, and equip $\ell^2(\tau)$ with its usual inner product introduced in the above lines. Also for cardinal number ψ let (for properties of cardinal numbers and their arithmetic see [7]):

$$\psi^* := \begin{cases} \psi & \psi \text{ is finite ,} \\ +\infty & \text{otherwise .} \end{cases}$$

Moreover for $s \neq t$ let $\delta_s^t = 0$ and $\delta_s^s = 1$.

If X, Y are normed vector spaces, we say the linear map $S : X \to Y$ is an operator if it is continuous. We call (X, T) a linear dynamical system, if X is a normed

vector space and $T : X \to X$ is an operator [5].
Let's recall that $\mathbb{R}$ is the set of real numbers, $\mathbb{C}$ is the set of complex numbers, and $\mathbb{N} = \{1, 2, \ldots\}$ is the set of natural numbers.

2. On generalized shift operators

In this section we show $\sigma_\varphi(\ell^2(\tau)) \subseteq \ell^2(\tau)$ (and $\sigma_\varphi : \ell^2(\tau) \to \ell^2(\tau)$ is continuous) if and only if $\varphi : \tau \to \tau$ is bounded. Moreover $\sigma_\varphi(\ell^2(\tau)) = \ell^2(\tau)$ if and only if $\varphi : \tau \to \tau$ is one–to–one.

Remark 2.1. We say $f : A \to A$ is bounded if there exists finite $n \geq 1$ such that for all $a \in A$ we have $\mathrm{card}(\varphi^{-1}(a)) \leq n$ [6].

Theorem 2.2. The following statements are equivalent:

1. $\sigma_\varphi(\ell^2(\tau)) \subseteq \ell^2(\tau)$,
2. $\varphi : \tau \to \tau$ is bounded,
3. $\sigma_\varphi : \ell^2(\tau) \to \ell^2(\tau)$ is a linear continuous map.

Moreover in the above case we have $||\sigma_\varphi|| = \sqrt{\sup\{(\mathrm{card}(\varphi^{-1}(\alpha)))^* : \alpha \in \tau\}}$.

Proof. First note that for $x = (x_\alpha)_{\alpha<\tau}$ we have

$$(*) \qquad ||\sigma_\varphi(x)||^2 = \sum_{\alpha<\tau} |x_{\varphi(\alpha)}|^2 = \sum_{\alpha<\tau} \left((\mathrm{card}(\varphi^{-1}(\alpha)))^* |x_\alpha|^2 \right)$$

(where $0(+\infty) = (+\infty)0 = 0$).
"(1) $\Rightarrow$ (2)" Suppose $\sigma_\varphi(\ell^2(\tau)) \subseteq \ell^2(\tau)$, for $\theta < \tau$ we have $||(\delta_\alpha^\theta)_{\alpha<\tau}|| = 1$ and:

$$||\sigma_\varphi((\delta_\alpha^\theta)_{\alpha<\tau})||^2 = (\mathrm{card}(\varphi^{-1}(\theta)))^*$$

by (*). Hence $(\delta_\alpha^\theta)_{\alpha<\tau} \in \ell^2(\tau)$ and $\sigma_\varphi((\delta_\alpha^\theta)_{\alpha<\tau}) \in \sigma_\varphi(\ell^2(\tau)) \subseteq \ell^2(\tau)$, thus $\varphi^{-1}(\theta)$ is finite.
Thus $\{\mathrm{card}(\varphi^{-1}(\alpha)) : \alpha \in \tau\}$ is a collection of finite cardinal numbers. If $\sup\{(\mathrm{card}(\varphi^{-1}(\alpha)))^* : \alpha \in \tau\} = +\infty$, then there exists a strictly increasing sequence $\{n_k\}_{k\geq 1}$ in $\mathbb{N}$ and sequence $\{\alpha_k\}_{k\geq 1}$ in τ such that for all $k \geq 1$ we have $\mathrm{card}(\varphi^{-1}(\alpha_k)) = n_k$. Since $\{n_k\}_{k\geq 1}$ is a one–to–one sequence, $\{\alpha_k\}_{k\geq 1}$ is a one–to–one sequence too. Consider $(x_\alpha)_{\alpha<\tau}$ with:

$$x_\alpha := \begin{cases} \frac{1}{k} & \alpha = \alpha_k, k \geq 1\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Then $\sum_{\alpha<\tau} |x_\alpha|^2 = \sum_{k\geq 1} x_{\alpha_k}^2 = \sum_{k\geq 1} \frac{1}{k^2} < +\infty$ and $(x_\alpha)_{\alpha<\tau} \in \ell^2(\tau)$. On the other hand by (*) we have $||\sigma_\varphi((x_\alpha)_{\alpha<\tau})||^2 = \sum_{k\geq 1} \frac{n_k}{k^2} \geq \sum_{k\geq 1} \frac{1}{k} = +\infty$ (note that $n_k \geq k$ for all $k \geq 1$), in particular $\sigma_\varphi((x_\alpha)_{\alpha<\tau}) \notin \ell^2(\tau)$ which leads to the contradiction $\sigma_\varphi(\ell^2(\tau)) \not\subseteq \ell^2(\tau)$. Therefore $\sup\{\mathrm{card}(\varphi^{-1}(\alpha)) : \alpha \in \tau\}$ is finite and is a natural number.
"(2) $\Rightarrow$ (3)" Suppose $n := \sup\{\mathrm{card}(\varphi^{-1}(\alpha)) : \alpha \in \tau\}$ is finite. For all $x = (x_\alpha)_{\alpha<\tau} \in \ell^2(\tau)$ we have:

$$||\sigma_\varphi(x)|| = \sqrt{\sum_{\alpha<\tau} ((\mathrm{card}(\varphi^{-1}(\alpha)))^* |x_\alpha|^2)} \leq \sqrt{\sum_{\alpha<\tau} (n|x_\alpha|^2)} = \sqrt{n}||x||$$

which shows continuity of σ_φ and $||\sigma_\varphi|| \leq \sqrt{n}$. On the other hand, there exists $\theta < \tau$ with $\mathrm{card}(\varphi^{-1}(\theta) = n$. By $||(\delta_\alpha^\theta)_{\alpha<\tau}|| = 1$ and (*) we have $||\sigma_\varphi((\delta_\alpha^\theta)_{\alpha<\tau})|| = \sqrt{n}$ which leads to $||\sigma_\varphi|| \geq \sqrt{n}$. ☐

By [1, Lemma 4.1] and [2], $\varphi : \tau \to \tau$ is one–to–one (resp. onto) if and only if $\sigma_\varphi : \mathbb{K}^\tau \to \mathbb{K}^\tau$ is onto (resp. one–to–one), however the following lemma deal with $\sigma_\varphi : \ell^2(\tau) \to \ell^2(\tau)$.

Lemma 2.3. The following statements are equivalent:

1. $\sigma_\varphi(\ell^2(\tau)) = \ell^2(\tau)$,
2. $\sigma_\varphi(\ell^2(\tau))$ is a dense subset of $\ell^2(\tau)$,
3. $\varphi : \tau \to \tau$ is one–to–one.

In addition the following statements are equivalent too:

i. $\sigma_\varphi : \ell^2(\tau) \to \mathbb{K}^\tau$ is one–to–one,
ii. $\varphi : \tau \to \tau$ is onto.

Proof. "(2) ⇒ (3)" Suppose $\varphi : \tau \to \tau$ is not one–to–one, then there exists $\theta \neq \psi$ with $\mu := \varphi(\theta) = \varphi(\psi)$. There exists $(y_\alpha)_{\alpha<\tau} \in \ell^2(\tau)$ with $||\sigma_\varphi((y_\alpha)_{\alpha<\tau}) - (\delta^\theta_\alpha)_{\alpha<\tau}|| < \frac{1}{4}$, thus for all $\alpha < \tau$ we have $|y_{\varphi(\alpha)} - \delta^\theta_\alpha| < \frac{1}{4}$ in particular $|y_{\varphi(\psi)} - \delta^\theta_\psi| < \frac{1}{4}$ and $|y_{\varphi(\theta)} - \delta^\theta_\theta| < \frac{1}{4}$, thus $|y_\mu| < \frac{1}{4}$ and $|y_\mu - 1| < \frac{1}{4}$, which is a contradiction, therefore $\varphi : \tau \to \tau$ is one–to–one.
"(3) ⇒ (1)" Suppose $\varphi : \tau \to \tau$ is one–to–one, then by Theorem 2.2, $\sigma_\varphi(\ell^2(\tau)) \subseteq \ell^2(\tau)$. For $y = (y_\alpha)_{\alpha<\tau} \in \ell^2(\tau)$, define $x = (x_\alpha)_{\alpha<\tau}$ in the following way:

$$x_\alpha = \begin{cases} y_\beta & \alpha = \varphi(\beta), \beta < \tau \,, \\ 0 & \alpha \in \tau \setminus \varphi(\tau) \,, \end{cases}$$

then $||x|| = ||y||$ and $x \in \ell^2(\tau)$, moreover $\sigma_\varphi(x) = y$, which leads to $\sigma_\varphi(\ell^2(\tau)) = \ell^2(\tau)$.
In order to complete the proof we should prove that (i) and (ii) are equivalent however by [1, Lemma 4.1], (ii) implies (i), so we should just prove that (i) implies (ii).
"(i) ⇒ (ii)" Suppose $\varphi : \tau \to \tau$ is not onto and choose $\theta \in \tau \setminus \varphi(\tau)$. Then $(\delta^\theta_\alpha)_{\alpha<\tau}, (0)_{\alpha<\tau}$ are two distinct elements of $\ell^2(\tau)$, however

$$\sigma_\varphi((\delta^\theta_\alpha)_{\alpha<\tau}) = \sigma_\varphi((0)_{\alpha<\tau}) = (0)_{\alpha<\tau}$$

and $\sigma_\varphi : \ell^2(\tau) \to \mathbb{K}^\tau$ is not one–to–one. □

Corollary 2.4. The following statements are equivalent:

1. $\varphi : \tau \to \tau$ is bijective,
2. $\sigma_\varphi : \ell^2(\tau) \to \ell^2(\tau)$ is bijective,
3. $\sigma_\varphi : \ell^2(\tau) \to \ell^2(\tau)$ is an isomorphism,
4. $\sigma_\varphi : \ell^2(\tau) \to \ell^2(\tau)$ is an isometry.

Proof. Using Lemma 2.3, (1) and (2) are equivalent. It's evident that (3) implies (2), moreover if $\varphi : \tau \to \tau$ is bijective, then by Theorem 2.2 two linear maps $\sigma_\varphi : \ell^2(\tau) \to \ell^2(\tau)$ and its inverse $\sigma_{\varphi^{-1}} : \ell^2(\tau) \to \ell^2(\tau)$ are continuous, hence (1) implies (3).
(1) implies (4), is evident by (*) in Theorem 2.2. In order to complete the proof, we should just prove that (4) implies (1).
"(4) ⇒ (1)" Suppose $\sigma_\varphi : \ell^2(\tau) \to \ell^2(\tau)$ is an isometry, then $\sigma_\varphi : \ell^2(\tau) \to \ell^2(\tau)$ is one–to–one and by Lemma 2.3, $\varphi : \tau \to \tau$ is onto. Moreover, $||\sigma_\varphi|| = 1$ since $\sigma_\varphi : \ell^2(\tau) \to \ell^2(\tau)$ is an isometry. By Lemma 2.2 we have $1 = ||\sigma_\varphi||^2 = \sup\{\mathrm{card}(\varphi^{-1}(\alpha)) : \alpha \in \tau\}$, thus for all $\alpha < \tau$ we have $\mathrm{card}(\varphi^{-1}(\alpha)) \leq 1$ and $\varphi : \tau \to \tau$ is one–to–one. □

Lemma 2.5. Let $\mathcal{D} = \{z \in \ell^2(\tau) : \sigma_\varphi(z) \in \ell^2(\tau)\}$ (consider $\mathcal{D}$ with induced normed and topology of $\ell^2(\tau)$), then:

1. $\mathcal{D}$ is a subspace of $\ell^2(\tau)$,
2. $\{\theta < \tau : \exists (z_\alpha)_{\alpha<\tau} \in \mathcal{D}\ z_\theta \neq 0\} = \{\alpha < \tau : \varphi^{-1}(\alpha) \text{ is finite }\}$.

Proof. Since $\sigma_\varphi : \ell^2(\tau) \to \mathbb{K}^\tau$ is linear, we have immediately (1).
2) We have

$$(**) \qquad ||\sigma_\varphi((\delta^\theta_\alpha)_{\alpha<\tau})|| = \begin{cases} \sqrt{(\text{card}(\varphi^{-1}(\theta)))^*} & \varphi^{-1}(\theta) \text{ is finite,} \\ +\infty & \varphi^{-1}(\theta) \text{ is infinite.} \end{cases}$$

Thus if $\varphi^{-1}(\theta)$ is finite we have $\sigma_\varphi((\delta^\theta_\alpha)_{\alpha<\tau}) \in \ell^2(\tau)$ and $(\delta^\theta_\alpha)_{\alpha<\tau} \in \mathcal{D}$, which shows $\theta \in \{\beta < \tau : \exists (z_\alpha)_{\alpha<\tau} \in \mathcal{D}\ z_\beta \neq 0\}$. Therefore:

$$\{\alpha < \tau : \varphi^{-1}(\alpha) \text{ is finite }\} \subseteq \{\theta < \tau : \exists (z_\alpha)_{\alpha<\tau} \in \mathcal{D}\ z_\theta \neq 0\}$$

Now for $\theta < \tau$ suppose there exists $(z_\alpha)_{\alpha<\tau} \in \mathcal{D}$ with $z_\theta \neq 0$. Using the fact that $\mathcal{D}$ is a subspace of $\ell^2(\tau)$ we may suppose $z_\theta = 1$, now we have (since $\sigma_\varphi(z) \in \ell^2(\tau)$):

$$||\sigma_\varphi((\delta^\theta_\alpha)_{\alpha<\tau})|| = ||\sigma_\varphi((z_\alpha\delta^\theta_\alpha)_{\alpha<\tau})|| \leq ||\sigma_\varphi(z)|| < +\infty\,,$$

by (**), $\varphi^{-1}(\theta)$ is finite, which completes the proof of (2). □

Note 2.6. For $H \subseteq \tau$ the closure of subspace generated by $\{(\delta^\theta_\alpha)_{\alpha<\tau} : \theta \in H\}$ (in $\ell^2(\tau)$) is $\{(x_\alpha)_{\alpha<\tau} \in \ell^2(\tau) : \forall \alpha \notin H\ (x_\alpha = 0)\}$.

Theorem 2.7. For $\mathcal{D} = \{z \in \ell^2(\tau) : \sigma_\varphi(z) \in \ell^2(\tau)\}$ as in Lemma 2.5 and $M := \{\alpha < \tau : \varphi^{-1}(\alpha) \text{ is finite }\}$, the following statemnts are equivalent:

1. $\sigma_\varphi \restriction_{\mathcal{D}}: \mathcal{D} \to \ell^2(\tau)$ is continuous,
2. there exists finite $n \geq 1$ with $\text{card}(\varphi^{-1}(\alpha)) \leq n$ for all $\alpha \in M$,
3. $\mathcal{D} = \{(x_\alpha)_{\alpha<\tau} \in \ell^2(\tau) : \forall \theta \notin M\ \ x_\theta = 0\}$,
4. $\mathcal{D}$ is a closed subspace of $\ell^2(\tau)$,

Proof. "(1) ⇒ (2)" Suppose $\sigma_\varphi \restriction_{\mathcal{D}}: \mathcal{D} \to \ell^2(\tau)$ is continuous, consider $\theta < \tau$ with finite $\varphi^{-1}(\theta)$. By proof of item (2) in Lemma 2.5, $(\delta^\theta_\alpha)_{\alpha<\tau} \in \mathcal{D}$ and $||\sigma_\varphi((\delta^\theta_\alpha)_{\alpha<\tau})|| = \sqrt{(\text{card}(\varphi^{-1}(\theta)))^*}$, thus

$$+\infty > ||\sigma_\varphi|| \geq \sup\{\sqrt{(\text{card}(\varphi^{-1}(\theta)))^*} : \theta \in M\}\,.$$

"(2) ⇒ (1)" For $n := \sup\{\text{card}(\varphi^{-1}(\alpha)) : \alpha \in M\} < +\infty$ and $x = (x_\alpha)_{\alpha<\tau} \in \mathcal{D}$ we have $||\sigma_\varphi(x)|| = \sqrt{\sum\limits_{\alpha\in M}((\text{card}(\varphi^{-1}(\alpha)))^*|x_\alpha|^2)} \leq \sqrt{\sum\limits_{\alpha\in M}(n|x_\alpha|^2)} = \sqrt{n}||x||$ which shows continuity of $\sigma_\varphi \restriction_{\mathcal{D}}: \mathcal{D} \to \ell^2(\tau)$.
"(3) ⇒ (4)" Note that for nonempty M, $\{(x_\alpha)_{\alpha<\tau} \in \ell^2(\tau) : \forall \theta \notin M\ x_\theta = 0\}$ is just $\ell^2(M)$.
"(4) ⇒ (3)" By proof and notations in Lemma 2.5, we have $\{(\delta^\theta_\alpha)_{\alpha<\tau} : \theta \in M\} \subseteq \mathcal{D}$. Use Note 2.6 to complete the proof.
"(2) ⇒ (3)" For $n := \sup\{\text{card}(\varphi^{-1}(\alpha)) : \alpha \in M\} < +\infty$ and $x = (x_\alpha)_{\alpha<\tau} \in \ell^2(\tau)$ with $x_\alpha = 0$ for all $\alpha \notin M$ we have $||\sigma_\varphi(x)|| \leq \sqrt{n}||x||$ which shows $x \in \mathcal{D}$ and $\{(x_\alpha)_{\alpha<\tau} \in \ell^2(\tau) : \forall \theta \notin M\ \ x_\theta = 0\} \subseteq \mathcal{D}$. Using Lemma 2.5 we have $\mathcal{D} \subseteq \{(x_\alpha)_{\alpha<\tau} \in \ell^2(\tau) : \forall \theta \notin M\ \ x_\theta = 0\}$.
"(3) ⇒ (2)" If $\sup\{(\text{card}(\varphi^{-1}(\alpha)))^* : \alpha \in M\} = +\infty$, then there exists a strictly increasing sequence $\{n_k\}_{k\geq 1}$ in $\mathbb{N}$ and sequence $\{\alpha_k\}_{k\geq 1}$ in M such that for all $k \geq 1$

we have $\mathrm{card}(\varphi^{-1}(\alpha_k)) = n_k$. Using a similar method described in Lemma 2.2, consider $(x_\alpha)_{\alpha<\tau} \in \mathcal{D}$ with:

$$x_\alpha := \begin{cases} \frac{1}{k} & \alpha = \alpha_k, k \geq 1 \,, \\ 0 & \text{otherwise} \,. \end{cases}$$

Then $||\sigma_\varphi((x_\alpha)_{\alpha<\tau})||^2 = \sum\limits_{k\geq 1} \frac{n_k}{k^2} \geq \sum\limits_{k\geq 1} \frac{1}{k} = +\infty$, in particular $\sigma_\varphi((x_\alpha)_{\alpha<\tau}) \notin \ell^2(\tau)$ which is in contradiction with $(x_\alpha)_{\alpha<\tau} \in \mathcal{D}$ and completes the proof. □

2.1. **Compact generalized shift operators.** For normed vector spaces X, Y we say the operator $T : X \to Y$ is a compact operator, if $\overline{T(B^X(0,1))}$ is a compact subset of Y, where $B^X(0,1) = \{x \in X : ||x|| < 1\}$ [3].

Theorem 2.8. The generalized shift operator $\sigma_\varphi : \ell^2(\tau) \to \ell^2(\tau)$ is compact if and only if τ is finite.

Proof. If τ is infinite, then by Theorem 2.2, $\{\varphi^{-1}(\alpha) : \alpha < \tau\} \setminus \{\varnothing\}$ is a partition of τ to its finite subsets, thus there exists a one–to–one sequence $\{\alpha_n\}_{n\geq 1}$ in τ such that $\{\varphi^{-1}(\alpha_n\}_{n\geq 1}$ is a sequence of nonempty finite and disjoint subsets of τ. For all distinct $n, m \geq 1$ we have

$$||\sigma_\varphi((\delta_\alpha^{\alpha_n})_{\alpha<\tau}) - \sigma_\varphi((\delta_\alpha^{\alpha_m})_{\alpha<\tau})|| = \sqrt{2}$$

so $\{\sigma_\varphi(\frac{1}{2}(\delta_\alpha^{\alpha_n})_{\alpha<\tau})\}_{n\geq 1}$ (is a sequence in $\overline{\sigma_\varphi(B((0)_{\alpha<\tau}, 1))}$) without any converging subsequence. Therefore $\overline{\sigma_\varphi(B((0)_{\alpha<\tau}, 1))}$ is not compact and $\sigma_\varphi : \ell^2(\tau) \to \ell^2(\tau)$ is not a compact operator.

On the other hand, if τ is finite, then every linear operator on $\ell^2(\tau)$ is compact, hence $\sigma_\varphi : \ell^2(\tau) \to \ell^2(\tau)$ is a compact operator. □

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