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Coloring of cozero-divisor graphs of commutative von Neumann regular rings*

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Abstract

Let R be a commutative ring with non-zero identity. The cozero-divisor graph of R, denoted by $\Gamma'(R)$, is a graph with vertices in $W^*(R)$, which is the set of all non-zero and non-unit elements of R, and two distinct vertices a and b in $W^*(R)$ are adjacent if and only if $a \notin Rb$ and $b \notin Ra$. In this paper, we show that the cozero-divisor graph of a von Neumann regular ring with finite clique number is not only weakly perfect but also perfect. Also, an explicit formula for the clique number is given.

1 Introduction

The cozero-divisor graphs associated with commutative rings, as the dual notion of zero-divisor graphs, was first introduced by Afkhami and Khashyarmanesh in [2], where they investigated some fundamental properties on the structure of this graph and the relation between cozero-divisor and zero-divisor graphs. Study of the complement of cozero-divisor graphs and characterization of commutative rings with forest, star, double-star or unicyclic cozero-divisor graphs were made bay the same authors in [3]. Planar, outerplanar and ring graph cozero-divisor graphs may be found in [4]. Akbari et al. gave

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further results on rings with forest cozero-divisor graphs and diameter of cozero-divisor graphs associated with $R[x]$ and $R[[x]]$ (see [6]). The cozero-divisor graph has also been studied in several other papers (e.g., [5, 7, 8, 12]). In this paper, we deal with the coloring cozero-divisor graphs problem. Interested readers may find some methods in coloring of graphs associated with rings in [1, 13]. First we recall some terminology and notation.

Throughout this paper, all rings are assumed to be commutative with identity. We denote by $\mathrm{Max}(R)$, $U(R)$, $W(R)$ and $\mathrm{Nil}(R)$, the set of all maximal ideals of R, the set of all invertible elements of R, the set of all non-unit elements of R and the set of all nilpotent elements of R, respectively. For a subset T of a ring R we let $T^* = T \setminus \{0\}$. The ring R is said to be *reduced* if it has no non-zero nilpotent element. The ring R is called *von Neumann regular* if for every $r \in R$, there exists an $s \in R$ such that $r = r^2 s$. The *krull dimension of* R, denoted by $\dim(R)$, is the supremum of the lengths of all chains of prime ideals. For any undefined notation or terminology in ring theory, we refer the reader to [9].

Let $G = (V, E)$ be a graph, where $V = V(G)$ is the set of vertices and $E = E(G)$ is the set of edges. By $\overline{G}$, we mean the complement graph of G. We write $u - v$, to denote an edge with ends u, v. If $U \subseteq V(G)$, then by $N(U)$ we mean the set of all neighbors of U in G. A graph $H = (V_0, E_0)$ is called a *subgraph of* G if $V_0 \subseteq V$ and $E_0 \subseteq E$. Moreover, H is called an *induced subgraph by* V_0, denoted by $G[V_0]$, if $V_0 \subseteq V$ and $E_0 = \{\{u, v\} \in E \,|\, u, v \in V_0\}$. Also G is called a *null graph* if it has no edge. A *clique* of G is a maximal complete subgraph of G and the number of vertices in the largest clique of G, denoted by $\omega(G)$, is called the *clique number* of G. For a graph G, let $\chi(G)$ denote the *vertex chromatic number* of G, i.e., the minimal number of colors which can be assigned to the vertices of G in such a way that every two adjacent vertices have different colors. A graph G is said to be *weakly perfect* if $\omega(G) = \chi(G)$. A *perfect graph* G is a graph in which every induced subgraph is weakly perfect. For any undefined notation or terminology in graph theory, we refer the reader to [14].

Let R be a commutative ring with nonzero identity. *The cozero-divisor graph of* R, denoted by $\Gamma'(R)$, is a graph with the vertex set $W^*(R)$ and two distinct vertices a and b in $W^*(R)$ are adjacent if and only if $a \notin Rb$ and $b \notin Ra$. In this paper, it is shown that the cozero-divisor graph of a von Neumann regular ring with finite clique number is weakly perfect. Moreover, an explicit formula for the clique number is given. Finally, we strengthen this result; Indeed it is proved that this graph is perfect.

2 Clique and Chromatic Number of $\Gamma'(R)$

Let R be a von Neumann regular ring and $\omega(\Gamma'(R)) < \infty$. The main of this section is to show that $\omega(\Gamma'(R)) = \chi(\Gamma'(R)) = \binom{n}{[n/2]}$, where $n = |\mathrm{Min}(R)|$. First, we need a series of lemma.

Lemma 2.1 *Let R be a ring. If $\omega(\Gamma'(R)) < \infty$, then R is a Noetherian ring.*

Proof. It is enough to show that every ideal of R is finitely generated. Suppose to the contrary, there exists an ideal I of R which is generate by the set $(x_i)_{i\in\Lambda}$, where $|\Lambda| = \infty$ and it is not generate by the set $(x_i)_{i\in\Upsilon}$, where $\Upsilon = \Lambda\setminus\{i\}$, for every $i \in \Lambda$. Thus $x_i \notin Rx_j$ and $x_j \notin Rx_i$, for every two distinct elements $i, j \in \Lambda$. Hence the set $(x_i)_{i\in\Lambda}$ is a clique of $\Gamma'(R)$ and so $\omega(\Gamma'(R)) = \infty$, which is a contradiction. Therefore, every ideal of R is finitely generated. □

Lemma 2.2 *Let R be a von Neumann regular ring. If $\omega(\Gamma'(R)) < \infty$, then $R \cong F_1 \times \cdots \times F_n$, where every F_i is a field and $|\mathrm{Min}(R)| = n$.*

Proof. By [11, Theorem 3.1], R is a reduced ring and $\dim(R) = 0$. Moreover, by Lemma 2.1, R is a Noetherian ring. Thus R is a reduced Artinian ring. The result now follows from [9, Theorem 8.7]. □

Lemma 2.3 *Let R be a ring. Then the following statements are equivalent.*

(1) *$a - b$ is an edge of $\Gamma'(R)$.*

(2) *$Ra \nsubseteq Rb$ and $Rb \nsubseteq Ra$.*

Proof. It is straightforward. □

Lemma 2.4 *Let R be a ring and $x, y \in V(\Gamma'(R))$ such that $Ra = Rb$. Then $N(a) = N(b)$.*

Proof. Suppose that $c \in N(a)$. By Lemma 2.3, $Ra \nsubseteq Rc$ and $Rc \nsubseteq Ra$. Since $Ra = Rb$, we deduce that $Rb \nsubseteq Rc$ and $Rc \nsubseteq Rb$ and thus by Lemma 2.3, $c \in N(b)$. Hence $N(a) \subseteq N(b)$. Similarly, $N(b) \subseteq N(a)$, as desired. □

Lemma 2.5 *Let $2 \leq n < \infty$ be an integer and $R = \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ (n times). Then*

$$\omega(\Gamma'(R)) = \chi(\Gamma'(R)) = \binom{n}{[n/2]}.$$

Proof. Let $x = (x_1, \ldots, x_n) \in V(\Gamma'(R))$. Obviously, $x_i = 0$ for some $i \in \{1, \ldots, n\}$. Let $NZC(x)$ be the number zero x_i's in x, for every $x = (x_1, \ldots, x_n) \in V(\Gamma'(R))$. Clearly, $1 \leq NZC(x) \leq n-1$, for every $x = (x_1, \ldots, x_n) \in V(\Gamma'(R))$. For every $1 \leq i \leq n-1$, let

$$A_i = \{x = (x_1, \ldots, x_n) \in V(\Gamma'(R)) |\ NZC(x) = i\}.$$

It is easily seen that $V(\Gamma'(R)) = \cup_{i=1}^{n-1} A_i$ and $A_i \cap A_j = \varnothing$, for every $i \neq j$ and so $\{A_1, \ldots, A_{n-1}\}$ is a partition of $V(\Gamma'(R))$. We show that $\Gamma'(R)[A_i]$ is a complete (induced) subgraph of $\Gamma'(R)$, for every $1 \leq i \leq n-1$. Let $x = (x_1 \ldots, x_n), y = (y_1, \ldots, y_n) \in A_i$, for some $1 \leq i \leq n-1$ and $x \neq y$. Since $NZC(x) = NZC(y)$, there exist $1 \leq i \neq j \leq n$ such that $x_i = 0$, $y_i = 1$ and $x_j = 1$, $y_j = 0$. This implies that $x \notin Ry$ and $y \notin Rx$ and so x and y are adjacent. Hence $\Gamma'(R)[A_i]$ is a complete (induced) subgraph of $\Gamma'(R)$, for every $1 \leq i \leq n-1$. Furthermore, $|A_i| = \binom{n}{i}$, for every $1 \leq i \leq n$ and $|A_t| \geq |A_i|$, for every $1 \leq i \leq n-1$, where $t = [n/2]$. Let $i \neq j$ and $i < j < t$. Then $|A_i| \leq |A_j|$ and for every $x \in A_i$ there exists a vertex $y \in A_j$ such that $Ry \subseteq Rx$. Thus by Lemma 2.3, x is not adjacent to y (by replacing one of the zero components of $y \in A_j$ by 1, we have $x \in A_i$). Hence

$$\omega(\Gamma'(R)[\cup_{i=1}^{t} A_i]) = \chi(\Gamma'(R)[\cup_{i=1}^{t} A_i]) = \binom{n}{t}.$$

Similarly,

$$\omega(\Gamma'(R)[\cup_{i=t}^{n-1} A_i]) = \chi(\Gamma'(R)[\cup_{i=t}^{n-1} A_i]) = \binom{n}{t}.$$

Indeed, there are enough colors in $\Gamma'(R)[A_t]$ to color $\Gamma'(R)$. Thus

$$\omega(\Gamma'(R) = \chi(\Gamma'(R)) = \binom{n}{t}.$$

□

Remark 2.1 *Let G be a graph and $x \in V(G)$. If there exists a vertex $y \in V(G)$ which is not adjacent to x and $N(x) = N(y)$, then $\omega(G) = \omega(G \setminus \{x\})$ and $\chi(G) = \chi(G \setminus \{x\})$.*

We are now in a position to state our main result of this section.

Theorem 2.1 *Let R be a von Neumann regular ring and $|\mathrm{Min}(R)| = n$. If $|\omega(\Gamma'(R))| < \infty$, then*

$$\omega(\Gamma'(R)) = \chi(\Gamma'(R)) = \binom{n}{[n/2]}.$$

Proof. By Lemma 2.2, $R \cong F_1 \times \cdots \times F_n$, where F_i is a field, for every $1 \leq i \leq n < \infty$. Let

$$A = \{(x_1, \ldots, x_n) \in V(\Gamma'(R)) | \ \ x_i \in \{0, 1\} \text{ for every } 1 \leq i \leq n\}.$$

Consider the following claims:

Claim 1. $\omega(\Gamma'(R)[A]) = \omega(\Gamma'(R))$ and $\chi(\Gamma'(R)[A]) = \chi(\Gamma'(R))$.

Suppose that $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ are vertices of $\Gamma'(R)$. Define the relation $\sim$ on $V(\Gamma'(R))$ as follows: $x \sim y$, whenever "$x_i = 0$ if and only if $y_i = 0$", for every $1 \leq i \leq n$. Obviously, $\sim$ is an equivalence relation on $V(\Gamma'(R))$. Thus $V(\Gamma'(R)) = \cup_{i=1}^{2^n-2}[x]_i$, where $[x]_i$ is the equivalence class of x_i (We note that the number of equivalence classes is $2^n - 2$). Let $[x]$ be a equivalence class of x. Then $|[x] \cap A| = 1$ and so one may choose $a \in [x] \cap A$ and $b \in [x] \setminus \{a\}$. Since $Ra = Rb$, by Lemma 2.4, $N(a) = N(b)$. By Remark 3.1, $\omega(\Gamma'(R)) = \omega(\Gamma'(R) \setminus \{b\})$ and $\chi(\Gamma'(R)) = \chi(\Gamma'(R) \setminus \{b\})$. If we continue this procedure for $|V(\Gamma'(R)) \setminus A|$ times, then we get $\omega(\Gamma'(R)[A]) = \omega(\Gamma'(R))$ and $\chi(\Gamma'(R)[A]) = \chi(\Gamma'(R))$.

Claim 2. $\omega(\Gamma'(R)[A]) = \omega(\Gamma'(S))$ and $\chi(\Gamma'(R)[A]) = \chi(\Gamma'(S))$, where $S = \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ (n times).

Let $x = (x_1, \ldots, x_n) \in S \setminus \{0, 1\}$ and $y = (y_1, \ldots, y_n) \in A$. Consider the map $\varphi : S \setminus \{0, 1\} \longrightarrow A$ defined by the rule $\varphi(x) = y$, whenever $x_i = 0$ if and only if $y_i = 0$. It is not hard to check that φ is well-defined, bijective and if $x, y \in S \setminus \{0, 1\}$ such that x is adjacent y, then $\varphi(x)$ is adjacent $\varphi(y)$. This implies that $\Gamma'(S) \cong \Gamma'(R)[A]$ and thus $\omega(\Gamma'(R)[A]) = \omega(\Gamma'(S))$ and $\chi(\Gamma'(R)[A]) = \chi(\Gamma'(S))$.

By Claims 1,2 and Lemma 2.5,

$$\omega(\Gamma'(R)) = \chi(\Gamma'(R)) = \omega(\Gamma'(R)[A]) = \chi(\Gamma'(R)[A]) = \omega(\Gamma'(S)) = \chi(\Gamma'(S)) = \binom{n}{[n/2]}.$$

□

We close this section with the following proposition.

Proposition 2.1 *Let R be a ring which is not an integral domain. If $|\omega(\Gamma'(R))| < \infty$, then $\Gamma'(R)$ is a null graph if and only if $(R, \mathfrak{m})$ is local ring and $\mathfrak{m}$ is principal.*

Proof. First, suppose that $\Gamma'(R)$ is a null graph. If R is not local, then one may choose $x \in \mathfrak{m}_1 \setminus \mathfrak{m}_2$ and $y \in \mathfrak{m}_2 \setminus \mathfrak{m}_1$, where $\mathfrak{m}_1, \mathfrak{m}_2 \in \mathrm{Max}(R)$. Since x is not adjacent to y, we find a contradiction. Thus $(R, \mathfrak{m})$ is local ring. Also, by a similar argument to the proof of Lemma 2.1, one may show that $\mathfrak{m}$ is principal.

To prove the converse, suppose that $(R, \mathfrak{m})$ is a local ring and $\mathfrak{m}$ is principal. We show that $\dim(R) = 0$. It is enough to show that $\mathfrak{m} \in \mathrm{Min}(R)$. Assume that $\mathfrak{p} \subseteq \mathfrak{m}$, for some $\mathfrak{p} \in \mathrm{Min}(R)$. Since R is not an integral domain, $\mathfrak{p} \neq (0)$ and so we may pick $0 \neq a \in \mathfrak{p}$. Since $\mathfrak{m}$ is principal, $\mathfrak{m} = Rx$, for some $x \in R$. If $x \in \mathfrak{p}$, then $\mathfrak{p} = \mathfrak{m}$ and thus $\dim(R) = 0$. So let $x \notin \mathfrak{p}$. Since $\mathfrak{p} \subseteq \mathfrak{m}$, $a = r_1 x$ for some $r_1 \in R$. Also $x \notin \mathfrak{p}$ implies that $r_1 \in \mathfrak{p}$ and thus $r_1 = r_2 x$, for some $r_2 \in R$. Hence $a = r_2 x^2$ and so $a \in \mathfrak{m}^2$. If we continue this procedure, then $a \in \mathfrak{m}^n$, for every positive integer n. Therefore $a \in \cap_{n=1}^{\infty} \mathfrak{m}^n$. This, together with [9, Corolary 10.19] imply that $a = 0$, a contradiction. Hence $\mathfrak{p} = \mathfrak{m}$ and so $\dim(R) = 0$. Since R is Noetherian with $\dim(R) = 0$, R is an Artinian local ring. Finally, by [9, Proposition 8.8], every ideal of R is principal and hence $\Gamma'(R)$ is a null graph. □

3 Perfectness of $\Gamma'(R)$

Let R be a von Neumann regular ring and $\omega(\Gamma'(R)) < \infty$. In this section, we show that $\Gamma'(R)$ is a perfect graph. We begin with the following celebrate result.

Lemma 3.1 ([10] The Strong Perfect Graph Theorem) *A graph G is perfect if and only if neither G nor $\overline{G}$ contains an induced odd cycle of length at least 5.*

Theorem 3.1 *Let $R = \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ (n times). Then $\Gamma'(R)$ is perfect.*

Proof. By Lemma 3.1, it is enough to prove the following claims.

Claim 1. $\Gamma'(R)$ contains no induced odd cycle of length at least 5. Assume to the contrary,

$$a_1 - a_2 - \cdots - a_n - a_1$$

is an induced odd cycle of length at least 5 in $\Gamma'(R)$.

By Lemma 2.3, either $Ra_1 \subseteq Ra_3$ or $Ra_3 \subseteq Ra_1$. We show that these two cases lead to contradictions. First assume that the case $Ra_1 \subseteq Ra_3$ happens. We continue the proof by proving the following subclaims.

Subclaim 1. $Ra_1 \subseteq Ra_i$, for every $3 \leq i \leq n-1$.

Clearly, $Ra_1 \subseteq Ra_3$. By Lemma 2.3, $Ra_1 \subseteq Ra_4$ or $Ra_4 \subseteq Ra_1$. If $Ra_4 \subseteq Ra_1$, then $Ra_4 \subseteq Ra_3$, a contradiction, by Lemma 2.3. So $Ra_1 \subseteq Ra_4$. Again, by Lemma

2.3, $Ra_1 \subseteq Ra_5$ or $Ra_5 \subseteq Ra_1$. If $Ra_5 \subseteq Ra_1$, then since $Ra_1 \subseteq Ra_4$, $Ra_5 \subseteq Ra_4$, a contradiction. Thus $Ra_1 \subseteq Ra_5$. Similarly, $Ra_1 \subseteq Ra_i$, for every $6 \leq i \leq n-1$.

Subclaim 2. $Ra_2 \subseteq Ra_i$, for every $4 \leq i \leq n$. Obviously, $Ra_1 \subseteq Ra_4$, by the Subclaim 1. By Lemma 2.3, $Ra_2 \subseteq Ra_4$ or $Ra_4 \subseteq Ra_2$. If $Ra_4 \subseteq Ra_2$, then $Ra_1 \subseteq Ra_2$, a contradiction. So $Ra_2 \subseteq Ra_4$. Next, we show that $Ra_2 \subseteq Ra_5$. If $Ra_5 \subseteq Ra_2$, then since $Ra_2 \subseteq Ra_4$, we deduce that $Ra_5 \subseteq Ra_4$, a contradiction. Therefore $Ra_2 \subseteq Ra_5$. Similarly, $Ra_2 \subseteq Ra_i$, for every $6 \leq i \leq n$.

Now, using Subclaims 1 and 2, we show that $Ra_3 \subseteq Ra_1$. By Lemma 2.3, $Ra_3 \subseteq Ra_5$ or $Ra_5 \subseteq Ra_3$. If $Ra_5 \subseteq Ra_3$, then by Subclaim 2, $Ra_2 \subseteq Ra_3$, a contradiction. Thus $Ra_3 \subseteq Ra_5$. We show that $Ra_3 \subseteq Ra_6$. If $Ra_6 \subseteq Ra_3$, then by Subcase 2, $Ra_2 \subseteq Ra_3$, a contradiction. So $Ra_3 \subseteq Ra_6$. Similarly, $Ra_3 \subseteq Ra_i$, for every $7 \leq i \leq n$. Since $Ra_1 \subseteq Ra_3$, $Ra_1 \subseteq Ra_i$, for every $5 \leq i \leq n$, i.e., $Ra_1 \subseteq Ra_n$, a contradiction. Thus $Ra_3 \subseteq Ra_1$ and this contradicts Subclaim 1. Therefore, $\Gamma'(R)$ contains no induced odd cycle of length at least 5.

Claim 2. $\overline{\Gamma'(R)}$ contains no induced odd cycle of length at least 5. Assume to the contrary,

$$a_1 - a_2 - \cdots - a_n - a_1$$

is an induced odd cycle of length at least 5 in $\overline{\Gamma'(R)}$. By Lemma 2.3, we may assume that $Ra_1 \subseteq Ra_2$. If $Ra_2 \subseteq Ra_3$, then $Ra_1 \subseteq Ra_3$, a contradiction. Thus

$$Ra_1 \subseteq Ra_2,$$
$$Ra_3 \subseteq Ra_2.$$

If $Ra_4 \subseteq Ra_3$, then $Ra_4 \subseteq Ra_2$, a contradiction. Hence $Ra_3 \subseteq Ra_4$. If $Ra_4 \subseteq Ra_5$, then $Ra_3 \subseteq Ra_4$ implies that $Ra_3 \subseteq Ra_5$, a contradiction. Thus

$$Ra_3 \subseteq Ra_4,$$
$$Ra_5 \subseteq Ra_4.$$

Since n is odd, by continuing this procedure, we find

$$Ra_{n-2} \subseteq Ra_{n-1},$$
$$Ra_n \subseteq Ra_{n+1} = Ra_1.$$

This implies that $Ra_n \subseteq Ra_1$ and since $Ra_1 \subseteq Ra_2$, $Ra_n \subseteq Ra_2$, a contradiction. Therefore, $\overline{\Gamma'(R)}$ contains no induced odd cycle of length at least 5.

The proof now is complete. $\square$

Remark 3.1 *Let G be a graph and $x \in V(G)$. If there exists a vertex $y \in V(G)$ which is not adjacent to x and $N(x) = N(y)$, then G is perfect if and only if $G \setminus \{x\}$ is perfect.*

We close this paper with the following result.

Theorem 3.2 *Let R be a von Neumann regular ring and $\omega(\Gamma'(R)) < \infty$. Then $\Gamma'(R)$ is a perfect graph.*

Proof. Since $|\omega(\Gamma'(R))| < \infty$, it follows from Lemma 2.2 that $R \cong F_1 \times \cdots \times F_n$, where F_i is a field, for every $1 \leq i \leq n < \infty$. Let

$$A = \{(x_1, \ldots, x_n) \in V(\Gamma'(R))| \;\; x_i \in \{0, 1\} \text{ for every } 1 \leq i \leq n\}.$$

By Lemma 2.4 and Remark 3.1, it is not hard to check that $\Gamma'(R)$ is perfect graph if and only if $\Gamma'(R)[A]$ is perfect. In fact if

$$a_1 - a_2 - \cdots - a_n - a_1$$

is an induced odd cycle of length at least 5 in $\overline{\Gamma'(R)}$ or $\Gamma'(R)$, then $Ra_i \neq Ra_j$, for every $1 \leq i, j \leq n$, $i \neq j$. By the proof of Theorem 2.1, we find that $\Gamma'(R)[A] \cong \Gamma'(S)$, where $S = \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ (n times). Thus $\Gamma'(R)$ is perfect if and only if $\Gamma'(S)$ is perfect. The result now follows from Lemma 3.1. $\square$

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