p-ADIC L-FUNCTIONS AND CLASSICAL CONGRUENCES

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ABSTRACT. In this paper, using p-adic analysis and p-adic L-functions, we show how to extend classical congruences (due to Wilson, Gauss, Dirichlet, Jacobi, Wolstenholme, Glaisher, Morley, Lemher and other people) to modulo p^k for any $k > 0$.

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1. INTRODUCTION

Let p be an odd prime. A famous congruence due to Wilson (according to Waring) states that [13, p.68]:

$$(p-1)! \equiv -1 \ (mod\ p). \tag{1}$$

This congruence was first formulated by Waring in 1770. The first proof was given by Lagrange in 1771. In 1828, Dirichlet proved another related congruence:

$$(\frac{p-1}{2})! \equiv (-1)^N \ (mod\ p), \tag{2}$$

where $4|p-3$ and N is the number of quadratic nonresidues less than $p/2$. In 1900 Glaisher [10] extended Wilson's theorem as follow:

$$(p-1)! \equiv pB_{p-1} - p \ (mod\ p^2), \tag{3}$$

where B_n is the Bernoulli number. In 2000, Sun [26] went one step further by showing

$$(p-1)! \equiv \frac{pB_{2p-2}}{2p-2} - \frac{pB_{p-1}}{p-1} - \frac{1}{2}(\frac{pB_{p-1}}{p-1})^2 \ (mod\ p^3). \tag{4}$$

Assume that $p \equiv 1(mod\ 4)$, then by Fermat's two square theorem, we have $p = a^2+b^2$, where a can be uniquely determined by requiring $a \equiv 1\ (mod\ 4)$. Another famous congruence, due to Gauss(1828) states that

$$\binom{(p-1)/2}{(p-1)/4} \equiv 2a \ (mod\ p). \tag{5}$$

Date:

The following extension of Gauss's congruence was first conjectured by Beukers [3] and proved by Chowla, Dwork, Evans[6]:

$$\binom{(p-1)/2}{(p-1)/4} \equiv \Big(1+\frac{1}{2}pq_p(2)\Big)\Big(2a-\frac{p}{2a}\Big)\ (mod\ p^2). \tag{6}$$

In 1837, Jacobi proved a congruence analogous to (5):

$$\binom{2(p-1)/3}{(p-1)/3} \equiv -r\ (mod\ p), \tag{7}$$

where $p\equiv 1(mod\ 6)$, $4p = r^2 + 27s^2$, and $r \equiv 1(mod\ 3)$. Evans and Yeung [33] independently extended Jacobi' congruence to modulo p^2 as follow:

$$\binom{2(p-1)/3}{(p-1)/3} \equiv -r+\frac{p}{r}\ (mod\ p^2). \tag{8}$$

In 2010 Cosgrave and Dilcher further extended Gauss' and Jacobi' congruences to modulo p^3 [7] as follows:

$$\begin{aligned}\binom{(p-1)/2}{(p-1)/4} &\equiv \Big(2a-\frac{p}{2a}-\frac{p^2}{8a^2}\Big)\cdot \\ &\Big(1+\frac{1}{2}q_p(2)p-\frac{1}{8}q_p(2)^2p^2+\frac{1}{4}E_{p-3}p^2\Big)\ (mod\ p^3)\end{aligned} \tag{9}$$

$$\binom{2(p-1)/3}{(p-1)/3} \equiv \Big(-r+\frac{p}{r}+\frac{p^2}{r^3}\Big)\Big(1+\frac{1}{6}B_{p-2}(\frac{1}{3})p^2\Big)\ (mod\ p^3). \tag{10}$$

Later, (9) and (10) were extended to cover similar binomial coefficients by Al-Shaghay and Dilcher[1].

In 1852, Wolstenholme proved his famous theorem[32] which states that if $p \geq 5$ is a prime, then

$$\sum_{k=1}^{p-1}\frac{1}{k} \equiv 0\ (mod\ p^2). \tag{11}$$

In the same paper, Wolstenholme also proved the following congruence:

$$\sum_{k=1}^{p-1}\frac{1}{k^2} \equiv 0\ (mod\ p). \tag{12}$$

In 1900, Glaisher gave the following generalizations of Wolstenholme's theorem [10, 11]:

$$\sum_{k=1}^{p-1}\frac{1}{k^m} \equiv \begin{cases}\frac{m}{m+1}pB_{p-m-1} & (mod\ p^2)\quad \text{if } m \text{ is even}\\[2ex] \frac{-m(m+1)}{2(m+2)}p^2B_{p-m-2} & (mod\ p^3)\quad \text{if } m \text{ is odd}\end{cases} \tag{13}$$

where $m > 0$ and $p \geq m+3$,

Since then, the above congruences have been extended by several authors to modulo p^k [10, 11, 18, 26, 29], and the multiple harmonic sums[14, 24, 25, 35, 36, 37].

Another form of Wolstenholme's theorem, which can be easily deduced from (11) is that:

$$\binom{2p-1}{p-1} \equiv 1 \ (mod\ p^3). \tag{14}$$

Many extensions of (14) to modulo p^k in terms of Bernoulli numbers or harmonic sums have been obtained in [9, 10, 12, 21, 25, 29]. We refer the readers to [20] for more references about various generalizations of Wolstenholme's theorem.

Another classical congruence for binomial coefficient due to Morley (1895) [22] is:

$$\binom{p-1}{(p-1)/2} \equiv (-1)^{(p-1)/2} 4^{p-1} \ (mod\ p^3). \tag{15}$$

For integer a with $(a,p)=1$, set $q_p(a) = \frac{a^{p-1}-1}{p}$. In 1938, Lemher [18] proved the following four congruences:

$$\sum_{k=1}^{[p/2]} \frac{1}{k} \equiv -2q_p(2) + pq_p(2)^2 \ (mod\ p^2); \tag{16}$$

$$\sum_{k=1}^{[p/3]} \frac{1}{p-3k} \equiv \frac{1}{2}q_p(3) - \frac{1}{4}pq_p(3)^2 \ (mod\ p^2); \tag{17}$$

$$\sum_{k=1}^{[p/4]} \frac{1}{p-4k} \equiv \frac{3}{4}q_2 - \frac{3}{8}pq_p(2)^2 \ (mod\ p^2); \tag{18}$$

$$\sum_{k=1}^{[p/6]} \frac{1}{p-6k} \equiv \frac{1}{4}q_p(3) + \frac{1}{3}q_p(2) - \frac{1}{8}pq_p(3)^2 - \frac{1}{6}pq_p(2)^2) \ (mod\ p^2), \tag{19}$$

and used them to derive congruences about $\binom{p-1}{[p/m]}$ for $m = 2, 3, 4$ or 6. In[5, 23, 26, 27], Morley's and Lemher's congruences are extended to congruences for $\binom{p-1}{[p/m]}$ and $\sum_{k=1}^{[p/m]} \frac{1}{k^n}$ modulo p^k.

Remark 1.1. *Congruences (16) and (18) modulo p were given by Glaisher* [11]*, while congruence (16) and (17) modulo p were given proved by Lerch* [17].

There arises the following question:

Question 1.2. *Can we extend the above congruences to modulo p^k for arbitrarily large k ?*

The first breakthrough in this direction is due to Washington [31], who gave an explicit p-adic expansion of the sums

$$\sum_{\substack{k=1 \\ (k,np)=1}}^{np} \frac{1}{k^m} \tag{20}$$

in terms of p-adic L-functions. Washington' expansions, together with Kummer's congruences for p-adic L-functions, immediately imply mod p^k evaluations of $\sum_{k=1}^{p-1} \frac{1}{k^m}$ for arbitrarily large k.

In this paper, we give p-adic expansions for the sums of Lemher's type

$$\sum_{\substack{k=1 \\ (k,np)=1}}^{[np/r]} \frac{1}{k^m}, \tag{21}$$

where $(r, np) = 1$. It turn outs that many binomial coefficients also admit nice expansions in terms of p-adic L-functions. As applications, we can extend all the congruences mentioned above to modulo p^k for arbitrarily large k.

This paper is structured as follows: In Section 2, we give preliminaries that will be used throughout this paper. In Section 3, we give a review of Washington's p-adic expansion of the power sums and its applications. In Section 4, we give a similar p-adic expansion for the sums of Lemher's type and derive many corollaries. Sections 5 and 6 are devoted to extensions of Gauss's and Jacobi's congruences, and Wilson's theorem respectively.

2. PRELIMINARIES

The Bernoulli numbers B_n and the Bernoulli polynomials $B_n(x)$ are defined respectively by

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n z^n / n!; \tag{22}$$

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(x) z^n / n!. \tag{23}$$

Thus $B_0(x) = 1$, $B_1(x) = x - \frac{1}{2}$, $B_2(x) = x^2 - x + \frac{1}{6}$, $B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$, $B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}$, etc.

From the above definitions, we have

$$B_n(x) = \sum_{r=0}^{n} \binom{n}{r} B_r x^{n-r}. \tag{24}$$

In particular, $B_n(0) = B_n$. Note that $B_n = 0$ whenever $n > 1$ is odd.

For a Dirichlet character χ modulo m, the generalized Bernoulli numbers $B_{n,\chi}$ are defined by

$$\sum_{a=1}^{m} \frac{\chi(a) z e^{az}}{e^{mz}-1} = \sum_{n=0}^{\infty} B_{n,\chi} z^n / n!. \tag{25}$$

From the definitions, we have

$$B_{n,\chi} = m^{n-1} \sum_{a=1}^{m} \chi(a) B_n(\tfrac{a}{m}). \tag{26}$$

For a Dirichlet character χ modulo m and a positive integer d, let χ' be the character modulo md induced by χ . Then we have

$$B_{n,\chi'} = B_{n,\chi} \prod_{p|d, p\ prime} (1-\chi(p)p^{n-1}). \tag{27}$$

We need the following identity of power sums due to Szmidt, Urbanowicz and Zagier [28]

Lemma 2.1. *Let χ be a Dirichlet character modulo d and N be a multiple of d. Let m and r be positive integers, with $(r,N)=1$. Then*

$$mr^{m-1} \sum_{n=1}^{[N/r]} \chi(n) n^{m-1} = -B_{m,\chi} + \frac{\overline{\chi}(r)}{\varphi(r)} \sum_{\psi} \overline{\psi}(-N) B_{m,\chi\psi}(N).$$

Now we recall definition and basic properties of p-adic L-functions and refer the readers to [30] for more details.

Throughout this paper, p denotes an odd prime, and $\mathbb{Z}_p$ and $\mathbb{Z}_p^*$ denote the ring of p-adic integers and the group of invertible p-adic integers respectively. The p-adic-valued Teichmüller character ω is defined as follows:

For an integer a with $(a,p)=1$, $\omega(a) \in \mathbb{Z}_p$ is the $p-1$-st root of unit satisfying $\omega(a) \equiv a (mod\ p)$. Set $\langle a \rangle = \omega(a)^{-1} a$

The p-adic exponential and logarithm functions are defined respectively by

$$exp(s) = \sum_{n=0}^{\infty} s^n / n!, \tag{28}$$

$$log_p(1+s) = \sum_{n=0}^{\infty} (-1)^{n+1} s^n / n, \tag{29}$$

for $s \in p\mathbb{Z}_p$. As usual, we have $exp(log_p(1+s)) = 1+s$ and $log_p(exp(s)) = s$, and

$$log_p(1+s) + log_p(1+t) = log_p((1+s)(1+t)), \tag{30}$$

for $s, t \in p\mathbb{Z}_p$.

Let χ be a primitive Dirichlet character modulo d and let D be any multiple of p and d. The p-adic L-function χ is defined by:

$$L_p(s,\chi) = \frac{1}{D}\frac{1}{s-1}\sum_{\substack{a=1\\(a,p)=1}}^{D}\chi(a)\langle a\rangle^{1-s}\sum_{n=0}^{\infty}\binom{1-s}{n}(B_n)(\frac{D}{a})^n, \tag{31}$$

where $s \in \mathbb{Z}_p$ and

$$\langle a\rangle^{1-s} = exp((1-s)log_p(\langle a\rangle)) = \sum_{n=0}^{\infty}\binom{1-s}{n}(\langle a\rangle - 1)^n. \tag{32}$$

From the definition, we have, for $n \geq 1$,

$$L_p(1-n,\chi) = -(1-\chi\omega^{-n}(p)p^{n-1})\frac{B_{n,\chi\omega^{-n}}}{n}. \tag{33}$$

By (33) it is easy to see that $L_p(s,\chi)$ is identically zero if $\chi(-1) = -1$. We note that $L_p(s,\chi)$ is analytic if $\chi \neq \mathbf{1}$, and $L_p(s,\mathbf{1})$ is analytic except for a pole at $s = 1$ with residue $(1-1/p)$.

Lemma 2.2. *$p(1-s)L_p(s,\mathbf{1}) \in \mathbb{Z}_p$ for $0 \neq s \in \mathbb{Z}_p$. If χ is a nontrivial primitive Dirichlet character modulo d, with $p^2 \nmid d$, then $L_p(s,\chi) \in \mathbb{Z}_p$ for $s \in \mathbb{Z}_p$.*

Proof. The first assertion follows directly from the definition. For the second assertion, see [30, Corollary 5.13]. □

The following congruences generalize the Kummer's congruences for generalized Bernoulli numbers [30]:

Lemma 2.3. *Let χ be a nontrivial primitive character modulo d, $p^2 \nmid d$. Then for integers k, s and t, with $0 < k < p-2$, we have*

$$L_p(s,\chi) \equiv L_p(s+p^{k-1}t,\chi)\ (mod\ p^k), \tag{34}$$

and

$$\Delta_t^k L_p(s,\chi) = \sum_{i=0}^{k}(-1)^i\binom{k}{i}L_p(s+it,\chi) \equiv 0\ (mod\ p^k), \tag{35}$$

where Δ_t is the forward difference operator with increment t.

Proof. The first congruence follows from [30, Theorem 5.12]. Thus it suffices to prove the second. We choose D in (31) such that $p^2 \nmid D$. Then by definition, $L_p(s,\chi)$ is an infinite sum of the terms

$$g(s,m,n) = \chi(a)\frac{1}{D}\frac{1}{s-1}\binom{1-s}{m}\binom{1-s}{n}(B_n)(\langle a\rangle - 1)^m(\frac{D}{a})^n$$

where $m+n > 0$. If $m+n > k$, we have $p^k | g(s,m,n)$ for $s \in \mathbb{Z}_p$, hence $p^k | \Delta_t^k g(s,m,n)$. If $m+n \leq k$, then $g(s,m,n)$ is a polynomial in s of degree less than k, hence $\Delta_t^k g(s,m,n) = 0$. □

For a primitive character χ, and two positive integers m, k, with $m < p-1$, set

$$B_p(m,k;\chi) := \sum_{i=1}^{k}(-1)^i\binom{k}{i}\frac{B_{i(p-1)+1-m,\chi}}{i(p-1)+1-m},$$

and set $B_p(m,k) = B_p(m,k;\mathbf{1})$. Then by Lemma 2.3 and (33), if $m+k \leq p-1$ and p^2 does not divide the conductor of $\chi\omega^{1-m}$, we have

$$L_p(m,\chi\omega^{1-m}) \equiv B_p(m,k;\chi) \equiv -\frac{B_{p^{k-1}(p-1)+1-m,\chi}}{p^{k-1}(p-1)+1-m} \ (mod\ p^k). \tag{36}$$

3. Washington's p-adic expansions of sums of powers

In[31], Washington gave the following p-adic expansions of harmonic sums

Theorem 3.1. *Let p be an odd prime and let d, m be positive integers. Then*

$$\sum_{\substack{k=1\\(k,p)=1}}^{dp}\frac{1}{k^m} = -\sum_{n=1}^{\infty}\binom{-m}{n}L_p(m+n,\omega^{1-m-n})(dp)^n. \tag{37}$$

Theorem 3.1 together with (36), immediately implies the following generalization of Wolstenholme's Theorem:

Corollary 3.2. *Let p, d, and m be as before. Let j be another positive integer, with $j+m \leq p-1$. Then we have*

$$\sum_{\substack{k=1\\(k,p)=1}}^{dp}\frac{1}{k^m} \equiv -\sum_{n=1}^{j-1}\binom{-m}{n}B_p(m+n,j-n)(dp)^n \ (mod\ p^j), \tag{38}$$

and

$$\sum_{\substack{k=1\\(k,p)=1}}^{dp}\frac{1}{k^m} \equiv \sum_{n=1}^{j-1}-\binom{-m}{n}\frac{B_{p^{j-n-1}(p-1)+1-m-n}}{p^{j-n-1}+m+n-1}(dp)^n \ (mod\ p^j). \tag{39}$$

Now we give p-adic expansions of $\binom{cp}{dp}/\binom{c}{d}$ for $c > d > 0$. Set

$$H_p(d;m) = \sum_{\substack{k=1\\(k,p)=1}}^{dp}\frac{1}{k^m}.$$

Theorem 3.3. *For $c > d > 0$, we have*

$$\binom{cp}{dp}\Big/\binom{c}{d} = exp\Big(-\sum_{k=3}^{\infty}(c^k-(c-d)^k-d^k)L_p(k,\omega^{1-k})p^k/k\Big). \tag{40}$$

Proof. By Corollary 3.2,

$$\binom{cp}{dp}\Big/\binom{c}{d} = \prod_{\substack{k=1\\(k,p)=1}}^{dp} (1+(c-d)p/k)$$

$$= exp\Big(\sum_{\substack{k=1\\(k,p)=1}}^{dp} log_p(1+\frac{(c-d)p}{k})\Big)$$

$$= exp\Big(\sum_{\substack{k=1\\(k,p)=1}}^{dp} \sum_{m=1}^{\infty}(-1)^{m-1}\frac{(c-d)^m p^m}{mk^m}\Big)$$

$$= exp\Big(\sum_{m=1}^{\infty}(-1)^{m-1}H_p(d;m)(c-d)^m p^m/m\Big)$$

$$= exp\Big(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{m+n}\binom{m+n-1}{n}L_p(m+n,\omega^{1-m-n})(c-d)^m d^n p^{m+n}/m\Big)$$

$$= exp\Big(-\sum_{k=3}^{\infty}\sum_{m=1}^{k-1}\binom{k}{m}L_p(k,\omega^{1-k})(c-d)^m d^{k-m} p^k/k\Big)$$

$$= exp\Big(-\sum_{k=3}^{\infty}(c^k-(c-d)^k-d^k)L_p(k,\omega^{1-k})p^k/k\Big),$$

where the sixth = follows from the fact that $L_p(k,\omega^{1-k})=0$ whenever k is even. □

The above expansion in terms of $H_p(d;m)$ or p-adic L-functions, together with (36), covers many known congruence about binomial coefficients[2, 5, 9, 10, 12, 15, 21, 25, 29, 34]. By Theorem 3.3, we can easily write down a *mod* p^8 evaluation of $\binom{cp}{dp}/\binom{c}{d}$ in terms of Bernoulli numbers:

Corollary 3.4. *For $p>7$, we have*

$$\binom{cp}{dp}\Big/\binom{c}{d} \equiv 1-(c^2d-cd^2)\frac{B_{p^5-p^4-2}}{p^4+2}p^3 \tag{41}$$
$$-(c^4d-2c^3d^2+2c^2d^3-cd^4)\frac{B_{p^3-p^2-4}}{p^2+4}p^5$$
$$-(c^6d-3c^5d^2+5c^4d^3-5c^3d^4+3c^2d^5-cd^6)\frac{B_{p-7}}{7}p^7$$
$$+(c^2d-cd^2)^2\frac{B^2_{p^2-p-2}}{2(p+2)^2}p^6 \quad (mod\ p^8)$$

Next, we show that the *mod* p^k evaluations of the homogeneous multiple harmonic sums(HMHS)

$$M_p(d,m;n)=\sum_{\substack{1\le k_1<\cdots<k_n<dp\\(k_i,p)=1}}\frac{1}{k_1^m\cdots k_n^m}, \tag{42}$$

and

$$\overline{M}_p(d,m;n)=\sum_{1\le k_1\le\cdots\le k_n<dp}\frac{1}{k_1^m\cdots k_n^m}, \tag{43}$$

can also be reduced to that of $H_p(d;m)$. Let t be an indeterminate, then formally we have

$$\begin{aligned}1+\sum_{n=1}^{dp}M_p(d,m;n)t^n &= \prod_{\substack{k=1\\(k,p)=1}}^{dp}(1+\frac{t}{k^m})\\ &= exp\Big(\sum_{\substack{k=1\\(k,p)=1}}^{dp}log(1+\frac{t}{k^m})\Big)\\ &= exp\Big(\sum_{\substack{k=1\\(k,p)=1}}^{dp}\sum_{j=1}^{\infty}(-1)^{j-1}\frac{t^j}{jk^{jm}}\Big)\\ &= exp\Big(\sum_{j=1}^{\infty}(-1)^{j-1}H_p(d;mj)\frac{t^j}{j}\Big).\end{aligned} \tag{44}$$

Similarly we have

$$1+\sum_{n=1}^{\infty}\overline{M}_p(d,m;n)t^n=\prod_{\substack{k=1\\(k,p)=1}}^{dp}(1-\frac{t}{k^m})^{-1}=exp\Big(\sum_{j=1}^{\infty}H_p(d;mj)\frac{t^j}{j}\Big). \tag{45}$$

Applying Corollary 3.2 to (44) and (45), we get the following congruences which improve the previous results about HMHS [37].

Corollary 3.5. *For $n > 1$, $p > mn + 4$, we have*

$$M_p(d,m;n) \equiv \begin{cases} \sum_{j=1}^{n-1} \frac{m^2(mj+1)B_{p-mj-2}B_{p-m(n-j)-1}}{2(mj+2)(mn-mj+1)} d^3p^3 \\ -\frac{m(mn+1)B_{p^2-p-mn-1}}{2(p+mn+1)} d^2p^2 \quad (mod\ p^4) \quad \text{if } mn \text{ is odd} \\ (-1)^n \sum_{j=1}^{n-1} \frac{m^2 B_{p-mj-1}B_{p-m(n-j)-1}}{2(mj+1)(mn-mj+1)} d^2p^2 \\ -(-1)^n \frac{mB_{p^2-p-mn}}{p+mn} dp \quad (mod\ p^3) \quad \text{if } m \text{ is even} \\ \sum_{j=1}^{n-1} \frac{m^2 B_{p-mj-1}B_{p-m(n-j)-1}}{2(mj+1)(mn-mj+1)} d^2p^2 \\ -\frac{mB_{p^2-p-mn}}{p+mn} dp \quad (mod\ p^3) \quad \text{if } m \text{ is odd and } n \text{ is even} \end{cases}$$

and

$$\overline{M}_p(d,m;n) \equiv \begin{cases} -\sum_{j=1}^{n-1} \frac{m^2(mj+1)B_{p-mj-2}B_{p-m(n-j)-1}}{2(mj+2)(mn-mj+1)} d^3p^3 \\ -\frac{m(mn+1)B_{p^2-p-mn-1}}{2(p+mn+1)} d^2p^2 \quad (mod\ p^4) \quad \text{if } mn \text{ is odd} \\ \sum_{j=1}^{n-1} \frac{m^2 B_{p-mj-1}B_{p-m(n-j)-1}}{2(mj+1)(mn-mj+1)} d^2p^2 \\ +\frac{mB_{p^2-p-mn}}{p+mn} dp \quad (mod\ p^3) \quad \text{if } m \text{ is even} \\ \sum_{j=1}^{n-1} \frac{m^2 B_{p-mj-1}B_{p-m(n-j)-1}}{2(mj+1)(mn-mj+1)} d^2p^2 \\ -\frac{mB_{p^2-p-mn}}{p+mn} dp \quad (mod\ p^3) \quad \text{if } m \text{ is odd and } n \text{ is even} \end{cases}$$

4. p-adic expansions of sums of Lemher's type

This section is parallel to the previous section. First we generalize Washington's p-adic expansions to cover the sums of Lemher's type.

Theorem 4.1. *Let d, r, m be positive integers, with $(r, dp) = 1$. Then, when $m = 1$,*

$$(46) \sum_{\substack{k=1 \\ (k,p)=1}}^{[dp/r]} \frac{1}{k} = -\frac{1}{p}\big(log_p r^{p-1} + \sum_{\substack{q|r \\ q \ prime}} \frac{log_p q^{p-1}}{q-1}\big) - \frac{r}{\varphi(r)} \sum_{\substack{n=0 \\ f_\psi | r (\psi,n)\neq(1,0)}}^{\infty} \overline{\psi}(-dp)(-1)^n d(\psi, r, -1-n) L_p(1+n, \psi\omega^{-n})(dp)^n,$$

and, when $m > 1$,

$$(47) \sum_{\substack{k=1 \\ (k,p)=1}}^{[dp/r]} \frac{1}{k^m} = L_p(m, \omega^{1-m}) - \frac{r^m}{\varphi(r)} \sum_{f_\psi | r} \overline{\psi}(-dp) \sum_{n=0}^{\infty} \binom{-m}{n} d(\psi, r, -m-n) L_p(m+n, \psi\omega^{1-m-n})(dp)^n,$$

where f_ψ is the conductor of primitive character ψ, and

$$d(\psi, r, n) = \prod_{\substack{q | \frac{r}{f_\psi} \\ q \ prime}} (1 - \psi(q) q^n).$$

Proof. Note that the term $-\frac{1}{p}\big(log_p r^{p-1} + \sum_{\substack{q|r \\ q \ prime}} \frac{log_p q^{p-1}}{q-1}\big)$ in (46) is just the value of

$$\Big(1 - r^{s(p-1)} \prod_{\substack{q|r \\ q \ prime}} \frac{1 - q^{-1-s(p-1)}}{1 - 1/q}\Big)/(s(p-1))$$

at $s=0$. Hence it suffices to prove the following identity for $m\geq 1$ and $s\in\mathbb{Z}_p$:

$$\sum_{\substack{k=1\\(k,p)=1}}^{[dp/r]}\frac{1}{k^{m+s(p-1)}}=L_p(m+s(p-1),\omega^{1-m})$$

$$-\frac{r^{m+s(p-1)}}{\varphi(r)}\sum_{f_\psi|r}\overline{\psi}(-dp)\sum_{n=0}^{\infty}\binom{-m-s(p-1)}{n}d(\psi,r,-m-s(p-1)-n)$$

$$L_p(m+s(p-1)+n,\psi\omega^{1-m-n})(dp)^n,$$

$$=(m+s(p-1)-1)L_p(m+s(p-1),\omega^{1-m})\frac{\left(1-r^{m+s(p-1)-1}\prod_{\substack{q|r\\ q\ prime}}\frac{1-q^{-m-s(p-1)}}{1-1/q}\right)}{(m+s(p-1)-1)}$$

$$-\frac{r^{m+s(p-1)}}{\varphi(r)}\sum_{\substack{n=0\\ f_\psi|r(\psi,n)\neq(\mathbf{1},0)}}^{\infty}\overline{\psi}(-dp)\binom{-m-s(p-1)}{n}d(\psi,r,-m-s(p-1)-n)$$

$$L_p(m+s(p-1)+n,\psi\omega^{1-m-n})(dp)^n,$$

where $a^{s(p-1)}=(a^{p-1})^s$. It is easy to see that both sides are analytic on $\mathbb{Z}_p$. We first prove the case $m+s(p-1)$ is negative integer, and the theorem follows by continuity. We note that for $n>0$,

$$\binom{n-1-s(p-1)}{n}L_p(1+s(p-1),\mathbf{1})|_{s=0}=-\frac{1}{n}(1-\frac{1}{p}).$$

By Lemma 2.1, when $m<0$,

$$\sum_{\substack{k=1\\(k,p)=1}}^{[dp/r]}\frac{1}{k^m}$$

$$=\sum_{k=1}^{[dp/r]}\frac{1}{k^m}-\sum_{k=1}^{[d/r]}\frac{1}{p^mk^m}$$

$$=-(1-p^{-m})\frac{B_{-m+1}}{-m+1}+\frac{r^m}{\varphi(r)}d(\mathbf{1},r,-1)\big(\frac{1-\frac{1}{p}}{-m+1}\big)(dp)^{-m+1}$$

$$+\frac{r^m}{\varphi(r)}\sum_{\substack{\psi\neq\mathbf{1}\\ f_\psi|r}}\overline{\psi}(-dp)\sum_{n=0}^{-m}\binom{-m}{n}d(\psi,r,-m-n)(1-\psi(p)p^{-m-n})\frac{B_{-m-n+1,\psi}}{-m-n+1}c$$

$$=L_p(m,\omega^{1-m})-\frac{r^m}{\varphi(r)}d(\mathbf{1},r,-1)(dp)^{-m+1}\binom{-m-s(p-1)}{-m+1}L_p(1+s(p-1),\mathbf{1})|_{s=0}$$

$$-\frac{r^m}{\varphi(r)}\sum_{f_\psi|r}\overline{\psi}(-dp)\sum_{n=0}^{-m}\binom{-m}{n}d(\psi,r,-m-n)L_p(m+n,\psi\omega^{1-m-n})(dp)^n.$$

Hence the theorem follows. ∎

Combining Theorem 4.1 with (36), implies the following generalization of Lemher's congruences:

Corollary 4.2. *Let p, d, r and m be as before. Assume that $(\varphi(r),p)=1$. Let j be another positive integer, with $j+m\le p-1$. Then, when $m=1$,*

$$\sum_{\substack{k=1\\(k,p)=1}}^{[dp/r]}\frac{1}{k}\equiv-\frac{r}{\varphi(r)}\sum_{\substack{n=0,\\ f_\psi|r,(\psi,n)\neq(\mathit{1},0)}}^{j-1}\overline{\psi}(-dp)d(\psi,r,-1-n)B_p(1+n,j-n;\psi)(-dp)^n$$
$$-\frac{1}{p}\big(log_p r^{p-1}+\sum_{\substack{q|r\\ q\ \ prime}}\frac{log_p q^{p-1}}{q-1}\big)\ \ (mod\ p^j),$$

and when $m>1$,

$$\sum_{\substack{k=1\\(k,p)=1}}^{[dp/r]}\frac{1}{k^m}\equiv-\frac{r^m}{\varphi(r)}\sum_{f_\psi|r}\overline{\psi}(-dp)\sum_{n=0}^{j-1}\binom{-m}{n}d(\psi,r,-m-n)B_p(m+n,j-n;\psi)(dp)^n$$
$$+B_p(m,j;\mathit{1})\ \ (mod\ p^j).$$

The above congruences remain valid if we replace some $B_p(m,k;\chi)$ with $-\frac{B_{p^{k-1}(p-1)+1-m,\chi}}{p^{k-1}(p-1)+1-m}$.

Let E_n(Euler numbers) be defined by

$$\frac{2e^z}{e^{2z}+1}=\sum_{n=0}^{\infty}E_nz^n/n!. \tag{48}$$

Then it is easy to see that

$$\frac{B_{n+1,\eta}}{n+1}=-\frac{E_n}{2}=\begin{cases}2^{2n+1}\frac{B_{n+1}(1/4)}{n+1} & \text{if } n \text{ is even}\\ 0 & \text{if } n \text{ is odd}\end{cases} \tag{49}$$

where η is the unique quadratic character module 4. Let $r=2$, or 4 in Corollary 4.2, we get

Corollary 4.3. *Assume that $(2,d)=1$ and $j+m\le p-1$. Then, when $m>1$, we have*

$$\sum_{\substack{k=1\\(k,p)=1}}^{[dp/2]}\frac{1}{k^m}\equiv\frac{B_{p^{j-1}(p-1)+1-m}}{p^{j-1}+m-1}$$
$$-\sum_{n=0}^{j-1}(2^m-2^{-n})\binom{-m}{n}\frac{B_{p^{j-n-1}(p-1)+1-m-n}}{p^{j-n-1}+m+n-1}(dp)^n\ (mod\ p^j),$$

and

$$\sum_{\substack{k=1\\(k,p)=1}}^{[dp/4]}\frac{1}{k^m}\equiv\frac{B_{p^{j-1}(p-1)+1-m}}{p^{j-1}+m-1}$$
$$-\sum_{n=0}^{j-1}(2^{2m-1}-2^{m-n-1})\binom{-m}{n}\frac{B_{p^{j-n-1}(p-1)+1-m-n}}{p^{j-n-1}+m+n-1}(dp)^n$$
$$+(-1)^{\frac{dp-1}{2}}2^{2m-2}\sum_{n=0}^{j-1}\binom{-m}{n}E_{p^{j-n-1}(p-1)-m-n}(dp)^n \pmod{p^j}.$$

When $m=1$, *we have*

$$\sum_{\substack{k=1\\(k,p)=1}}^{[dp/2]}\frac{1}{k}\equiv-\frac{2log_p2^{p-1}}{p}-\sum_{n=1}^{j-1}(2-2^{-n})\frac{B_{p^{j-n-1}(p-1)-n}}{p^{j-n-1}+n}(-dp)^n$$
$$\equiv 2\sum_{n=1}^{j}(-1)^n\frac{q_p(2)^np^{n-1}}{n}$$
$$-\sum_{n=1}^{j-1}(2-2^{-n})\frac{B_{p^{j-n-1}(p-1)-n}}{p^{j-n-1}+n}(-dp)^n \pmod{p^j},$$

and

$$\sum_{\substack{k=1\\(k,p)=1}}^{[dp/4]}\frac{1}{k}\equiv-\frac{3log_p2^{p-1}}{p}-\sum_{n=1}^{j-1}(2-2^{-n})\frac{B_{p^{j-n-1}(p-1)-n}}{p^{j-n-1}+n}(-dp)^n$$
$$-(-1)^{\frac{dp-1}{2}}\sum_{n=0}^{j-1}E_{p^{j-n-1}(p-1)-n-1}(dp)^n$$
$$\equiv 3\sum_{n=1}^{j}(-1)^n\frac{q_p(2)^np^{n-1}}{n}-\sum_{n=1}^{j-1}(2-2^{-n})\frac{B_{p^{j-n-1}(p-1)-n}}{p^{j-n-1}+n}(-dp)^n$$
$$-(-1)^{\frac{dp-1}{2}}\sum_{n=0}^{j-1}E_{p^{j-n-1}(p-1)-n-1}(dp)^n \pmod{p^j};$$

Let p, d, r be as before and let c be a nonzero integer. Set

$$H_p(d,r;m)=\sum_{\substack{k=1\\(k,p)=1}}^{[dp/r]}\frac{1}{k^m}.$$

As in Section 3, we have the following p-adic expansion of $\binom{cp+[dp/r]}{[dp/r]}/\binom{c+[d/r]}{[d/r]}$

Theorem 4.4.

$$\binom{cp+[dp/r]}{[dp/r]}\Big/\binom{c+[d/r]}{[d/r]} == exp\Big(\sum_{m=1}^{\infty}(-1)^{m-1}H_p(d,r;m)c^m p^m/m\Big)$$

$$= exp\Big(-c log_p r^{p-1} - c\sum_{\substack{q|r\\ q\ \ prime}}\frac{log_p q^{p-1}}{q-1}\Big)\cdot$$

$$exp\big(\sum_{k=3}^{\infty}L_p(k,\omega^{1-k})c^k p^k/k\big)\cdot$$

$$exp\big(\frac{-1}{\varphi(r)}\sum_{k=3}^{\infty}((cr+d)^k-d^k)d(\mathbf{1},r,-k)L_p(k,\omega^{1-k})p^k/k)\big)\cdot$$

$$exp\big(\frac{-1}{\varphi(r)}\sum_{\substack{\psi\neq \mathbf{1}\\ f_\psi|r}}\overline{\psi}(dp)\sum_{k=1}^{\infty}((cr+d)^k-d^k)d(\psi,r,-k)L_p(k,\psi\omega^{1-k})p^k/k)\big)$$

The above expansion, together with (36), immediately implies many known congruence about $\binom{p-1}{[p/r]}$ $(r>1)$ [5, 23, 26, 27]. For example, setting $c=-1$, $d=1$, $r=2$, or 4 in Theorem 4.4, we get:

Corollary 4.5. *For $p>7$,*

$$(-1)^{\frac{p-1}{2}}4^{-p+1}\binom{p-1}{\frac{p-1}{2}} \equiv 1+\frac{1}{4}\frac{B_{p^5-p^4-2}}{p^4+2}p^3+\frac{3}{16}\frac{B_{p^3-p^2-4}}{p^2+4}p^5$$
$$+\frac{1}{16}\frac{B^2_{p^2-p-2}}{2(p+2)^2}p^6+\frac{9}{64}\frac{B_{p-7}}{7}p^7 \pmod{p^8}.$$

Corollary 4.6. *For $p>5$,*

$$(-1)^{[\frac{p}{4}]}2^{-3p+3}\binom{p-1}{[\frac{p}{4}]} \equiv 1-(-1)^{\frac{p-1}{2}}E_{p^4-p^3-2}p^2+\frac{15}{4}\frac{B_{p^3-p^2-2}}{p^2+2}p^3$$
$$-(-1)^{\frac{p-1}{2}}5E_{p^2-p-4}p^4+\frac{75}{16}B_{p-5}p^5$$
$$\frac{1}{2}(2E_{p-3}-E_{2p-4})^2p^4-(-1)^{\frac{p-1}{2}}\frac{5}{4}E_{p-3}B_{p-3}p^5 \pmod{p^6}.$$

Finally, we consider the *mod* p^k evaluations of the homogeneous multiple harmonic sums of Lemher's type

$$M_p(d,r,m;n)=\sum_{\substack{1\le k_1<\cdots<k_n<dp/r\\ (k_i,p)=1}}\frac{1}{k_1^m\cdots k_n^m},$$

and

$$\overline{M}_p(d,r,m;n)=\sum_{\substack{1\le k_1\le\cdots\le k_n<dp/r\\ (k_i,p)=1}}\frac{1}{k_1^m\cdots k_n^m}.$$

As before we have

$$1+\sum_{n=1}^{[dp/r]} M_p(d,r,m;n)t^n = exp(\sum_{j=1}^{\infty}(-1)^{j-1}H_p(d,r;mj)\frac{t^j}{j}), \tag{50}$$

and

$$1+\sum_{n=1}^{\infty} \overline{M}_p(d,r,m;n)t^n = exp(\sum_{j=1}^{\infty}H_p(d,r;mj)\frac{t^j}{j}). \tag{51}$$

Applying Corollary 4.2 to (50) and (51) we can deduce $mod\ p^k$ evaluations of $M_p(d,r,m;n)$ and $\overline{M}_p(d,r,m;n)$ for any k. The following are two examples:

Corollary 4.7. *Assume that $p > mn+4$, and $(d,2)=1$, then when $m>1$, we have*

$$M_p(d,2,m;n) \equiv \begin{cases} \sum_{j=1}^{n-1} \frac{(2^{mj}-2)(2^{m(n-j)}-\frac{1}{2})B_{p-mj}B_{p-m(n-j)-1}}{j^2(mn-mj+1)}dp \\ -\frac{(2^{mn}-2)B_{p^2-p-mn+1}}{n(p+mn-1)} \qquad (mod\ p^2) & \text{if } mn \text{ is odd} \\ (-1)^n \sum_{j=1}^{n-1} \frac{m^2(2^{mj}-\frac{1}{2})(2^{m(n-j)}-\frac{1}{2})B_{p-mj-1}B_{p-m(n-j)-1}}{2(mj+1)(mn-mj+1)}d^2p^2 \\ -(-1)^n\frac{m(2^{mn}-\frac{1}{2})B_{p^2-p-mn}}{p+mn}dp \qquad (mod\ p^3) & \text{if } m \text{ is even} \end{cases}$$

and, when $m=1$, and $n>1$ is odd we have

$$M_p(d,2,1;n) \equiv \sum_{j=2}^{n-2}\frac{(2^j-2)(2^{n-j}-\frac{1}{2})B_{p-j}B_{p-n+j-1}}{j^2(n-j+1)}dp$$
$$+\frac{q_p(2)(2^n-1)B_{p-n}}{n}dp - \frac{(2^n-2)B_{p^2-p-n+1}}{n(p+n-1)} \qquad (mod\ p^2).$$

5. Congruences of Gauss and Jacobi

In this section, we will give p-adic expansions of $\binom{(p-1)/2}{(p-1)/4}$ (when $4|p-1$) and $\binom{2(p-1)/3}{(p-1)/3}$ (when $3|p-1$), and hence full generalizations of congruences of Gauss and Jacobi. First we introduce Morita's p-adic gamma function. For a positive integer k, set

$$\Gamma_p(k) = (-1)^k \prod_{\substack{j=1 \\ (j,p)=1}}^{k-1} j. \tag{52}$$

Then Γ_p extends uniquely to a continuous function from $\mathbb{Z}_p$ to $\mathbb{Z}_p^*$[16]. Let a, b, and m be positive integers satisfying $p \equiv 1\ (mod\ m)$, and $a+b \leq m$.

Then we have

$$(53)\binom{(a+b)(p-1)/m}{a(p-1)/m} = \frac{\binom{(a+b)(p^k-1)/m}{a(p^k-1)/m}\binom{a(p^k-1)/m}{a(p-1)/m}\binom{b(p^k-1)/m}{b(p-1)/m}}{\binom{(a+b)(p^k-1)/m}{(a+b)(p-1)/m}\binom{(a+b)(p^k-p)/m}{a(p^k-p)/m}}$$

$$= \frac{-\Gamma_p(1-\frac{a+b}{m})}{\Gamma_p(1-\frac{a}{m})\Gamma_p(1-\frac{b}{m})}\lim_{k\to\infty}\frac{\binom{a(p^k-1)/m}{a(p-1)/m}\binom{b(p^k-1)/m}{b(p-1)/m}\binom{(a+b)(p^{k-1}-1)/m}{a(p^{k-1}-1)/m}}{\binom{(a+b)(p^k-1)/m}{(a+b)(p-1)/m}\binom{(a+b)(p^k-p)/m}{a(p^k-p)/m}}$$

By (41) and (50), the above limit has a p-adic expansion. Now assume that $4|p-1$ and let a be as in congruence (5).

Theorem 5.1.

$$\binom{(p-1)/2}{(p-1)/4} = \Big(2a - 2a\sum_{j=1}^{\infty}\frac{1}{j}\binom{2j-2}{j-1}(\frac{p}{4a^2})^j\Big)$$

$$exp\Big(\frac{1}{2}log_p 2^{p-1} + \sum_{k=2}^{\infty}L_p(k,\eta\omega^{1-k})p^k/k)\Big),$$

where η is the unique quadratic character module 4.

Proof. We need the following expansion from [7, Corollary 1]

$$(54) \qquad \frac{\Gamma_p(1-\frac{1}{2})}{\Gamma_p(1-\frac{1}{4})^2} = -2a + 2a\sum_{j=1}^{\infty}\frac{1}{j}\binom{2j-2}{j-1}(\frac{p}{4a^2})^j.$$

By Theorem 3.3, we have

$$(55) \quad \lim_{k\to\infty}\frac{\binom{(p^k-p)/2}{(p^k-p)/4}}{\binom{(p^{k-1}-1)/2}{(p^{k-1}-1)/4}} = exp\Big(\sum_{k=3}^{\infty}(2^{-k}-2^{-2k+1})L_p(k,\omega^{1-k})p^k/k)\Big),$$

and by Theorem 4.4,

$$\lim_{k\to\infty}\frac{\binom{(p^k-1)/4}{(p-1)/4}^2}{\binom{(p^k-1)/2}{(p-1)/2}} = exp\Big(\frac{1}{2}log_p 2^{p-1} + \sum_{k=3}^{\infty}(2^{-k}-2^{-2k+1})L_p(k,\omega^{1-k})p^k/k\Big)\cdot$$

$$exp\Big(\sum_{k=2}^{\infty}L_p(k,\eta\omega^{1-k})p^k/k)\Big)$$

Hence

$$\lim_{k\to\infty}\frac{\binom{(p^k-1)/4}{(p-1)/4}^2\binom{(p^{k-1}-1)/2}{(p^{k-1}-p)/4}}{\binom{(p^k-1)/2}{(p-1)/2}\binom{(p^k-p)/2}{(p^k-p)/4}} = exp\Big(\frac{1}{2}log_p 2^{p-1} + \sum_{k=2}^{\infty}L_p(k,\eta\omega^{1-k})p^k/k)\Big),$$

and the theorem follows. □

Now assume that $3|p-1$ and let a be as in congruence (7). Using exactly the same proof, we get

Theorem 5.2.

$$\binom{2(p-1)/3}{(p-1)/3} = \Big(-r + r\sum_{j=1}^{\infty}\frac{1}{j}\binom{2j-2}{j-1}(\frac{p}{r^2})^j\Big)\cdot$$
$$exp\Big(\sum_{k=3}^{\infty}(1-2^{k-1})(1-3^{-k})L_p(k,\omega^{1-k})p^k/k\Big)\cdot$$
$$exp\Big(\sum_{k=2}^{\infty}(1+2^{k-1})L_p(k,\phi\omega^{1-k})p^k/k)\Big),$$

where ϕ is the unique quadratic character module 3.

Remark 5.3. *Using the p-adic expansions of $\frac{-\Gamma_p(1-\frac{a+b}{m})}{\Gamma_p(1-\frac{a}{m})\Gamma_p(1-\frac{b}{m})}$ (m=4,6,8) obtained in* [1]*, we can give similar expansions for $\binom{(a+b)(p-1)/m}{a(p-1)/m}$ (m=4,6,8).*

6. Wilson's theorem and related congruences

In this section, we show how to get the *mod* p^k evaluations of $(p-1)!$, and $(\frac{p-1}{2})!$ (when $4|p-3$) and $(\frac{p-1}{4})!^4$ (when $4|p-1$), and hence full generalizations of Wilson's theorem and related congruences. By (30), we have for $m < p$,

$$(56)\quad log_p(-(p-1)!) = \frac{1}{p-1}log_p(p-1)!^{p-1}$$
$$= \frac{1}{p-1}\sum_{k=1}^{p-1}log_p(1+(k^{p-1}-1))$$
$$= \frac{1}{p-1}\sum_{k=1}^{p-1}\sum_{n=1}^{\infty}(-1)^{n-1}(k^{p-1}-1)^n/n$$
$$\equiv \frac{1}{p-1}\sum_{n=1}^{m-1}\frac{-1}{n}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\sum_{k=1}^{p-1}k^{j(p-1)}$$
$$\equiv \frac{1}{p-1}\sum_{n=1}^{m-1}\frac{1}{n}\Big(\sum_{j=1}^{n}\frac{(-1)^{j-1}\binom{n}{j}}{j(p-1)+1}\sum_{k=0}^{j(p-1)}\binom{j(p-1)+1}{k+1}B_{j(p-1)-k}p^{k+1}\Big)$$
$$+\sum_{n=1}^{m-1}\frac{-1}{n}\ (mod\ p^m)$$

From the above expansion, we can deduce *mod* p^m evaluations of $(p-1)!$ for any $m > 0$. But when $m > 4$, the congruence would be too complicated to be written down. Thus we will only work out the case $m = 4$. Set $V_{p,i} = \frac{pB_{i(p-1)}}{p-1} - 1$.

Lemma 6.1.

$$V_{p,i} \equiv 0 \ (mod\ p), \tag{57}$$

$$2V_{p,1} - V_{p,2} \equiv 3V_{p,2} - 2V_{p,3} \equiv 0 \ (mod\ p^2), \tag{58}$$

and

$$3V_{p,1} - 3V_{p,2} + V_{p,3} \equiv 0 \ (mod\ p^3). \tag{59}$$

Proof. The first congruence follows from the von Staudt-Clausen theorem. The second and third follow by evaluating $\sum_{k=1}^{p-1}(k^{p-1}-1)^2$ modulo p^2, and $\sum_{k=1}^{p-1}(k^{p-1}-1)^3$ modulo p^3. □

Theorem 6.2. *For $p > 3$,*

$$\begin{aligned}(p-1)! \equiv& -1 - 3V_{p,1} + \frac{3}{2}V_{p,2} - \frac{1}{3}V_{p,3} - \frac{1}{2}(2V_{p,1} - \frac{1}{2}V_{p,2})^2 - \frac{1}{6}V_{p,1}^3 \\ &+ (B_{p-3} - \frac{3}{2}B_{2p-4} + \frac{2}{3}B_{3p-5})p^3 \ (mod\ p^4),\end{aligned}$$

$$\begin{aligned}(-1)^{\frac{p-1}{2}}4^{p-1}(\frac{p-1}{2})!^2 \equiv& -1 - 3V_{p,1} + \frac{3}{2}V_{p,2} - \frac{1}{3}V_{p,3} - \frac{1}{2}(2V_{p,1} - \frac{1}{2}V_{p,2})^2 - \frac{1}{6}V_{p,1}^3 \\ &+ (\frac{13}{12}B_{p-3} - \frac{3}{2}B_{2p-4} + \frac{2}{3}B_{3p-5})p^3 \ (mod\ p^4),\end{aligned}$$

Proof. by (56) we have

$$\begin{aligned}(p-1)! \equiv& -exp(3V_{p,1} - \frac{3}{2}V_{p,2} + \frac{1}{3}V_{p,3}) \cdot \\ & exp\Big((\frac{p-2}{2}B_{p-3} - \frac{2p-3}{2}B_{2p-4} + \frac{3p-4}{6}B_{3p-5})p^3\Big) \ (mod\ p^4).\end{aligned}$$

By (58) and (59),

$$(3V_{p,1} - \frac{3}{2}V_{p,2} + \frac{1}{3}V_{p,3})^2 \equiv (2V_{p,1} - \frac{1}{2}V_{p,2})^2 \ (mod\ p^4), \tag{60}$$

and

$$(3V_{p,1} - \frac{3}{2}V_{p,2} + \frac{1}{3}V_{p,3})^3 \equiv V_{p,1}^3 \ (mod\ p^4). \tag{61}$$

Hence the first assertion follows. The second follows from the first and Corollary 4.5. □

Corollary 6.3. *If $4|p-3$,*

$$\begin{aligned}2^{p-1}(\frac{p-1}{2})! \equiv& (-1)^N\Big(1 + \frac{3}{2}V_{p,1} - \frac{3}{4}V_{p,2} + \frac{1}{6}V_{p,3} + \frac{1}{8}(2V_{p,1} - \frac{1}{2}V_{p,2})^2 + \frac{1}{48}V_{p,1}^3 \\ &- (\frac{13}{24}B_{p-3} - \frac{3}{4}B_{2p-4} + \frac{1}{3}B_{3p-5})p^3\Big) \ (mod\ p^4)\end{aligned}$$

where N is the number of quadratic nonresidues less than $p/2$.

We close the paper by noting that, when $4|p-1$, we can deduce a congruence for $(\frac{p-1}{4})!^4$ from Theorems 5.1 and 6.2.

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