A JENSEN–ROHRLICH TYPE FORMULA FOR THE HYPERBOLIC 3-SPACE

S. HERRERO, Ö. IMAMOḠLU, A.-M. VON PIPPICH, AND Á. TÓTH

ABSTRACT. In this article we give a Jensen–Rohrlich type formula for a certain class of automorphic functions on the hyperbolic 3-space for the group $\mathrm{PSL}_2(\mathcal{O}_K)$.

1. INTRODUCTION

1.1. **Rohrlich's formula.** The classical Jensen's formula is a well-known theorem of complex analysis which characterizes, for a meromorphic function f on the unit disc, the value of the integral of $\log|f(z)|$ on the unit circle in terms of the zeros and poles of f inside the unit disc. An important theorem of Rohrlich [11] establishes a version of Jensen's formula for modular functions f with respect to the full modular group $\mathrm{PSL}_2(\mathbb{Z})$ and expresses the integral of $\log|f(z)|$ over a fundamental domain in terms of special values of Dedekind's eta function.

To be more precise, let $\mathbb{H}^2 = \{\tau = x + iy \,|\, x, y \in \mathbb{R}, y > 0\}$, $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$, and $X = \mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}^2$. Let Γ_τ denote the stabilizer subgroup of τ in Γ and let $\nu(\tau)$ denote its order. The hyperbolic measure on X is given by $d\mu(\tau) = dxdy/y^2$ and the hyperbolic Laplacian on X is given by

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

The quotient space X has the structure of a hyperbolic Riemann surface of finite hyperbolic volume $\mathrm{vol}(X) = \pi/3$, admitting one cusp which we denote by ∞. The field of modular functions on X is given by $\mathbb{C}(j(\tau))$, with $j(\tau)$ denoting Klein's j-invariant [13] satisfying

$$j(\tau) = \frac{1}{q_\tau} + 744 + O(q_\tau),$$

as $\tau \to \infty$, where $q_\tau = e^{2\pi i\tau}$.

Consider now the class $\mathcal{M}$ of functions $F : \mathbb{H}^2 \to \mathbb{R} \cup \{\infty\}$ satisfying the following properties:

(M1) The function $F(\tau)$ is Γ-invariant and can therefore be considered as a function on X.

(M2) There exist distinct points $\tau_1, \ldots, \tau_m \in X$ together with constants $n_1, \ldots, n_m \in \mathbb{Z}$ satisfying $\sum_{\ell=1}^m n_\ell = 0$ such that, for $\ell \in \{1, \ldots, m\}$, the bound

$$F(\tau) = n_\ell\,\nu(\tau_\ell)\log|\tau - \tau_\ell|^{-1} + O(1),$$

as $\tau \to \tau_\ell$, holds and such that $F(\tau)$ is smooth at any point $\tau \in X$ with $\tau \neq \tau_\ell$ for $\ell \in \{1, \ldots, m\}$.

(M3) For $\tau \in X$ with $\tau \neq \tau_\ell$ for $\ell \in \{1, \ldots, m\}$, we have $\Delta F(\tau) = 0$.

(M4) The function $F(\tau)$ is square-integrable on X.

If $F : \mathbb{H}^2 \to \mathbb{R} \cup \{\infty\}$ satisfies the properties (M1)–(M4), then the limit $F(\infty) := \lim_{\tau \to \infty} F(\tau)$ exists and we have the equality

$$(1.1) \qquad F(\tau) = \log|f(\tau)|, \quad \text{with } f(\tau) = e^{F(\infty)}\prod_{\ell=1}^m (j(\tau) - j(\tau_\ell))^{-n_\ell}.$$

Now, Rohrlich's Theorem can be rephrased as follows

Theorem 1.1 (Rohrlich [11])**.** *Let $F : \mathbb{H}^2 \to \mathbb{R} \cup \{\infty\}$ be in $\mathcal{M}$, the class of functions satisfying the properties (M1)–(M4). Then, we have the equality*

$$\frac{3}{\pi} \int_X F(\tau) d\mu(\tau) = F(\infty) + 6 \sum_{\ell=1}^m n_\ell \log \left(|\eta(\tau_\ell)|^4 \operatorname{Im}(\tau_\ell) \right),$$

where $\eta(\tau) = q_\tau^{1/24} \prod_{n=1}^\infty (1 - q_\tau^n)$ is the classical Dedekind's eta function.

Observe that the function on the right hand side of the equality in Theorem 1.1 is given by the constant term in the Laurent expansion of the non-holomorphic Eisenstein series $E_\infty(\tau, s)$ at $s = 1$. For $\tau \in \mathbb{H}^2$ and $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1$, this series is defined by

$$E_\infty(\tau, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \operatorname{Im}(\gamma \tau)^s.$$

The Eisenstein series is Γ-invariant with respect to τ and holomorphic in s, and it admits a meromorphic continuation to the whole complex s-plane with a simple pole at $s = 1$ with residue

$$\operatorname{res}_{s=1} E_\infty(P, s) = \frac{1}{\operatorname{vol}(X)} = \frac{3}{\pi}.$$

In this context, the well-known Kronecker's limit formula for $\mathrm{PSL}_2(\mathbb{Z})$ (see, e.g., [14]) states

$$\lim_{s \to 1} \left(E_\infty(z, s) - \frac{3}{\pi(s-1)} \right) = -\frac{3}{\pi} \log \left(|\eta(z)|^4 \operatorname{Im}(z) \right) + C,$$

where $C = 6(1 - 12\,\zeta'(-1) - \log(4\pi))/\pi$ and $\zeta(s)$ denotes the Riemann zeta function. Note that the constant C does not appear in Theorem 1.1, since $\sum_{\ell=1}^m n_\ell = 0$.

The proof of Rohrlich's formula is an application of this Kronecker's limit formula. The formula admits several generalizations and has many applications in number theory, see, e.g. [5], [8]. There is also an extension of Rohrlich's formula which has applications to the computation of arithmetic intersection numbers in Arakelov theory, see, e.g., [9].

1.2. **Purpose of the article.** The goal of this paper is to give an analogue of Rohrlich's formula in $\mathbb{H}^3$, the hyperbolic 3-space. We write $\mathbb{H}^3 = \{P = z + rj \mid z \in \mathbb{C}, r \in \mathbb{R}_{>0}\}$, which is a subset of the usual quaternions $\mathbb{R}[i, j, k]$, and we will view z and r as coordinate functions on $\mathbb{H}^3$. The quaternionic norm on $\mathbb{R}[i, j, k]$ induces a norm on $\mathbb{H}^3$ given explicitly by $\|P\| = \sqrt{|z|^2 + r^2}$. We let K be an imaginary quadratic field, $\mathcal{O}_K$ its ring of integers, h_K its class number, and d_K its discriminant. From now on, we let $\Gamma = \mathrm{PSL}_2(\mathcal{O}_K) \subset \mathrm{PSL}_2(\mathbb{C})$, which is a discrete and cofinite subgroup, and we let $X = \Gamma \backslash \mathbb{H}^3$. By Γ_P we denote the stabilizer subgroup of P in Γ and by $\nu(P)$ its order. By $d\mu(P)$ we denote the hyperbolic measure on X and by Δ the hyperbolic Laplacian on X (see 2.1). The quotient space X has finite hyperbolic volume, which is explicitly given by

$$(1.2) \qquad\qquad \operatorname{vol}(X) = \frac{|d_K|^{3/2}}{4\pi^2} \zeta_K(2)$$

with $\zeta_K(s)$ denoting the Dedekind zeta function, and it admits h_K cusps (see Section 2).

For $P \in \mathbb{H}^3$ and $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1$, the Eisenstein series associated to the cusp ∞ is defined by

$$E_\infty(P, s) = \sum_{\gamma \in \Gamma'_\infty \backslash \Gamma} r(\gamma P)^{s+1},$$

2

where Γ'_∞ is the maximal unipotent subgroup of the stabilizer group Γ_∞ of ∞ in Γ. The Eisenstein series is Γ-invariant with respect to P and holomorphic in s, and it admits a meromorphic continuation to the whole complex s-plane with a simple pole at $s = 1$ with residue

$$(1.3) \qquad \operatorname{res}_{s=1} E_\infty(P, s) = \frac{\operatorname{covol}(\mathcal{O}_K)}{\operatorname{vol}(X)} = \frac{2\pi^2}{|d_K|\zeta_K(2)}.$$

Here, $\operatorname{covol}(\mathcal{O}_K)$ denotes the euclidean covolume of the lattice $\mathcal{O}_K$ in $\mathbb{C}$. In this case, Kronecker's limit formula states

$$(1.4) \qquad \lim_{s\to 1}\left(E_\infty(P,s) - \frac{2\pi^2}{|d_K|\zeta_K(2)(s-1)}\right) = -\frac{2\pi^2}{|d_K|\zeta_K(2)}\log\left(\eta_\infty(P)\,r(P)\right) + C_K,$$

where C_K is an explicit constant depending only on K. Here, the function $\eta_\infty : \mathbb{H}^3 \to \mathbb{R}$ satisfies $\eta_\infty(\gamma P) = \|cP + d\|^2 \eta_\infty(P)$ for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and can be considered as the analogue of the weight 2 real-analytic modular form $|\eta(z)|^4$. The function η_∞ is essentially the function defined by Asai in [1]. More precisely, we have

$$-\frac{2\pi^2 \log\left(\eta_\infty(P)\right)}{|d_K|\zeta_K(2)} = \frac{|\mathcal{O}_K^\times|}{2}r^2 + 4\pi \sum_{\substack{\mu \in \mathcal{D}^{-1} \\ \mu \neq 0}} |\mu|\varphi_{\infty,\infty}(\mu;1)\,rK_1(4\pi|\mu|r)e^{2\pi i\operatorname{tr}(\mu z)}.$$

Here, we employed the notation of Section 2. The value $\varphi_{\infty,\infty}(\mu;1)$ can be explicitly given in terms of special values of certain generalized divisors sums. For these results, we refer the reader to [3], Chapter 8, Sections 1–3.

Consider now the class $\mathcal{A}$ of functions $F : \mathbb{H}^3 \to \mathbb{R} \cup \{\infty\}$ satisfying the following properties:

($\mathcal{A}$1) The function $F(P)$ is Γ-invariant and can therefore be considered as a function on X.

($\mathcal{A}$2) There exist distinct points $Q_1, \ldots, Q_m \in X$ together with constants $c_1, \ldots, c_m \in \mathbb{R}$ satisfying $\sum_{\ell=1}^m c_\ell = 0$ such that, for $\ell \in \{1, \ldots, m\}$, the bound

$$F(P) = c_\ell\,\nu(Q_\ell)\frac{r_\ell}{\|P - Q_\ell\|} + O(1),$$

as $P \to Q_\ell = z_\ell + r_\ell j$, holds and $F(P)$ is smooth at any point $P \in X$ with $P \neq Q_\ell$ for $\ell \in \{1, \ldots, m\}$.

($\mathcal{A}$3) For $P \in X$ with $P \neq Q_\ell$ for $\ell \in \{1, \ldots, m\}$, we have $\Delta F(P) = 0$.

($\mathcal{A}$4) The function $F(P)$ is square-integrable on X.

We note that the bounds in ($\mathcal{M}$2) and ($\mathcal{A}$2) are the natural bounds that arise from the type of singularities of the corresponding Green's functions.

In Proposition 5.2 of Section 5, we will show that, if $F : \mathbb{H}^3 \to \mathbb{R} \cup \{\infty\}$ satisfies the properties ($\mathcal{A}$1)–($\mathcal{A}$4), then the limit $F(\infty) := \lim_{r\to\infty} F(P)$ exists, and we will prove the analogue of (1.1) in this case. Our main theorem is

Theorem 1.2. *Let $F : \mathbb{H}^3 \to \mathbb{R} \cup \{\infty\}$ be in $\mathcal{A}$, the class of functions satisfying the properties ($\mathcal{A}$1)–($\mathcal{A}$4). Then, we have the equality*

$$\frac{1}{\operatorname{vol}(X)}\int_X F(P)d\mu(P) = F(\infty) + \frac{2\pi}{\operatorname{vol}(X)}\sum_{\ell=1}^m c_\ell \log\left(\eta_\infty(Q_\ell)\,r_\ell\right).$$

Note that, in analogy with Rohrlich's Theorem, the constant C_K arising in (1.4) does not appear in Theorem 1.2, since $\sum_{\ell=1}^m c_\ell = 0$.

It is known to the experts that Rohrlich's formula can be proven using the theory of the resolvent kernel of the hyperbolic Laplacian and our proof of Theorem 1.2 is a generalization of this method to the hyperbolic 3-space. The advantage of this method is that it can be

3

generalized to other settings such as the case of the hyperbolic n-space. This method also naturally leads to an analogue of the function $\log|j(\tau_1) - j(\tau_2)|$ (see the function defined in (5.1)). The properties of this function play a central role in our proof of Theorem 1.2, and the proof of these follow from properties of the resolvent kernel and of Niebur type Poincaré series.

1.3. **Outline of the article.** The paper is organized as follows. In Section 2, we begin by collecting background information. In Section 3, we compute the Fourier expansion of the resolvent kernel associated to the hyperbolic Laplacian on X. In addition, we give the Fourier expansion of the Niebur type Poincaré series which appear as coefficients in the Fourier expansion of the resolvent kernel. To the best of the authors' knowledge these expansions have not been explicitly stated elsewhere in the literature and are of independent interest. In Section 4, we study some of the analytic properties of the Niebur type Poincaré series and we prove the meromorphic continuation of the resolvent kernel via its Fourier expansion. In Section 5, we construct the above mentioned analogue of $\log|j(\tau_1) - j(\tau_2)|$, prove its main properties, and give our proof of Theorem 1.2 using these properties. Identities involving special functions that are needed in the paper as well as some technical lemmas are given in the Appendix and in Section 6, respectively.

2. BACKGROUND MATERIAL

2.1. **The hyperbolic 3-space and the group** $\mathrm{PSL}_2(\mathcal{O}_K)$. Let $\mathbb{H}^3 := \{P = z + rj \,|\, z \in \mathbb{C}, r \in \mathbb{R}_{>0}\}$ denote the upper half-space model of the three-dimensional hyperbolic space, where $\{1, i, j, k\}$ is the standard basis for the quaternions $\mathbb{R}[i,j,k]$. The quaternionic norm on $\mathbb{R}[i,j,k]$ induces a norm on $\mathbb{H}^3$ given explicitly by $\|P\| = \|z + rj\| = \sqrt{|z|^2 + r^2}$. For $z \in \mathbb{C}$, we set $\mathrm{tr}(z) := z + \overline{z}$. The hyperbolic volume element, resp. the hyperbolic Laplacian are given as

$$(2.1) \qquad d\mu(P) := \frac{dx\,dy\,dr}{r^3}, \quad \text{resp.} \quad \Delta := -r^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial r^2}\right) + r\frac{\partial}{\partial r}.$$

Let $d(P, Q)$ denote the hyperbolic distance between the points P and Q. An explicit formula is given by

$$(2.2) \qquad \cosh\left(d(P,Q)\right) = \frac{|z_1 - z_2|^2 + r_1^2 + r_2^2}{2r_1 r_2},$$

where $P = z_1 + r_1 j$ and $Q = z_2 + r_2 j$. An element $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}_2(\mathbb{C})$ acts on $\mathbb{H}^3$ by

$$\gamma P = \frac{(az + b)\overline{(cz + d)} + a\overline{c}r^2}{|cz + d|^2 + |c|^2 r^2} + \frac{r}{|cz + d|^2 + |c|^2 r^2}\,j,$$

where $P = z + rj$. By abuse of notation, we represent an element of $\mathrm{PSL}_2(\mathbb{C})$ by a matrix.

As mentioned in the Introduction, we let K be an imaginary quadratic field, $\mathcal{O}_K$ its ring of integers, h_K its class number, and d_K its discriminant. We let $\Gamma = \mathrm{PSL}_2(\mathcal{O}_K) \subset \mathrm{PSL}_2(\mathbb{C})$ and we let $X := \Gamma\backslash\mathbb{H}^3$. By Γ_P we denote the stabilizer subgroup of P in Γ and by $\nu(P)$ its order.

In a slight abuse of notation, we will at times identify X with a fundamental domain in $\mathbb{H}^3$ and identify points on X with their preimages in such a fundamental domain. The hyperbolic volume $\mathrm{vol}(X)$ of X is given by formula (1.2) in terms of a special value of Dedekind's zeta function, which is defined by

$$\zeta_K(s) = \sum_{\substack{I \subseteq \mathcal{O}_K \text{ ideal} \\ I \neq (0)}} \mathcal{N}(I)^{-s},$$

where $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ and $\mathcal{N}(I)$ denotes the norm of I.

A cusp of X is the Γ-orbit of a parabolic fixed point of Γ, and X has h_K cusps. From now on we fix a complete set of representatives $C_\Gamma \subseteq \mathbb{P}^1(K)$ for the cusps of X. We write elements of C_Γ as $[a:b]$ for $a, b \in \mathcal{O}_K$, not both equal to 0, and we write $\infty := [1:0]$ and assume that $\infty \in C_\Gamma$. Furthermore, for any cusp $\kappa = [a:b] \in C_\Gamma$, we fix a scaling matrix $\sigma_\kappa = \left(\begin{smallmatrix} a & * \\ b & * \end{smallmatrix} \right) \in \mathrm{PSL}_2(K)$ such that $\sigma_\kappa \infty = \kappa$ and

$$(2.3) \qquad \sigma_\kappa^{-1} \Gamma_\kappa \sigma_\kappa = \left\{ \begin{pmatrix} u & \lambda \\ 0 & u^{-1} \end{pmatrix} \middle| u \in \mathcal{O}_K^\times, \lambda \in \Lambda_\kappa \right\}$$

with the full lattice $\Lambda_\kappa = (a\mathcal{O}_K + b\mathcal{O}_K)^{-2} \subseteq \mathbb{C}$ (see, e.g., [15]). For the cusp ∞, we choose σ_∞ to be the identity. Furthermore, for the maximal unipotent subgroup Γ_κ', which consists of all the parabolic elements of Γ_κ together with the identity, we have

$$(2.4) \qquad \sigma_\kappa^{-1} \Gamma_\kappa' \sigma_\kappa = \left\{ \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \middle| \lambda \in \Lambda_\kappa \right\}.$$

We let $\Lambda_\kappa^* = \{\nu \in \mathbb{C} : \mathrm{tr}(\nu\lambda) \in \mathbb{Z} \text{ for any } \lambda \in \Lambda_\kappa\}$ denote its dual lattice. In particular, we have $\Lambda_\infty = \mathcal{O}_K$ and $\Lambda_\infty^* = \mathcal{D}^{-1}$ with $\mathcal{D}^{-1} = \{\nu \in K \mid \mathrm{tr}(\nu\lambda) \in \mathbb{Z} \text{ for any } \lambda \in \mathcal{O}_K\}$ denoting the inverse different.

2.2. Fourier expansion of automorphic functions.
A function $f : \mathbb{H}^3 \to \mathbb{C}$ is called automorphic with respect to Γ if it is Γ-invariant, that is, $f(\gamma P) = f(P)$ for any $\gamma \in \Gamma$. An important tool to study the behavior of an automorphic function f at a cusp $\xi \in C_\Gamma$, with scaling matrix σ_ξ, is its Fourier expansion. More precisely, since the function $P \mapsto f(\sigma_\xi P)$ is $\sigma_\xi^{-1} \Gamma_\xi' \sigma_\xi$-invariant, employing (2.4), we have $f(\sigma_\xi(P + \lambda)) = f(\sigma_\xi P)$ for any $\lambda \in \Lambda_\xi$. If f is smooth, the Fourier expansion of f with respect to the cusp ξ is therefore of the form

$$(2.5) \qquad f(\sigma_\xi P) = \sum_{\mu \in \Lambda_\xi^*} a_\mu(r) e^{2\pi i \, \mathrm{tr}(\mu z)},$$

where $P = z + rj$ and with Fourier coefficients given by

$$a_\mu(r) = \frac{1}{\mathrm{covol}(\Lambda_\xi)} \int_{\mathbb{C}/\Lambda_\xi} f(\sigma_\xi P) e^{-2\pi i \, \mathrm{tr}(\mu z)} dz.$$

If we assume that f is an eigenfunction of the hyperbolic Laplacian, satisfying $\Delta f = (1 - s^2) f$ for some $s \in \mathbb{C}$ with $s \neq 0$, and that f is of polynomial growth as $r \to \infty$, that is $f(z + rj) = O(r^C)$ as $r \to \infty$ for some constant C, then the expansion (2.5) has the form (see, e.g., [3], Theorem 3.1, p. 105)

$$(2.6) \qquad f(\sigma_\xi P) = a_0 r^{1+s} + b_0 r^{1-s} + \sum_{\substack{\mu \in \Lambda_\xi^* \\ \mu \neq 0}} a_\mu r K_s(4\pi|\mu|r) e^{2\pi i \, \mathrm{tr}(\mu z)}$$

with $a_0, b_0, a_\mu \in \mathbb{C}$ and with $K_s(\cdot)$ denoting the modified Bessel function of the second kind.

5

2.3. **Poincaré series.** For later purposes, we define two families of eigenfunctions of the hyperbolic Laplacian, namely the Eisenstein series and the Niebur type Poincaré series, which from now one will be called Niebur–Poincaré series for simplicity. For this, let $\kappa \in C_\Gamma$ be a cusp with scaling matrix σ_κ.

For $P \in \mathbb{H}^3$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, the Eisenstein series associated to the cusp κ is given by

$$E_\kappa(P, s) = \sum_{\gamma \in \Gamma'_\kappa \backslash \Gamma} r(\sigma_\kappa^{-1} \gamma P)^{s+1}.$$

The Eisenstein series is an automorphic function for Γ and it is holomorphic in s in the region $\mathrm{Re}(s) > 1$. Moreover, it satisfies the differential equation

$$\left(\Delta - (1 - s^2) \right) E_\kappa(P, s) = 0,$$

i.e. it is an eigenfunction of Δ. For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, the Eisenstein series admits a Fourier expansion of the form (2.6) given by (see, e.g., [3], Theorem 4.1, p. 111)

(2.7)

$$E_\kappa(\sigma_\xi P, s) = \delta_{\kappa,\xi} [\Gamma_\kappa : \Gamma'_\kappa] \, r^{1+s} + \varphi_{\kappa,\xi}(0; s) \, r^{1-s} + \frac{2^{1+s} \pi^s}{\Gamma(s)} \sum_{\substack{\mu \in \Lambda_\xi^* \\ \mu \neq 0}} |\mu|^s \varphi_{\kappa,\xi}(\mu; s) \, r K_s(4\pi|\mu|r) e^{2\pi i \, \mathrm{tr}(\mu z)},$$

where $\delta_{\kappa,\xi}$ is Kronecker's delta symbol and, for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, we have set

(2.8)

$$\varphi_{\kappa,\xi}(\mu; s) := \frac{\pi}{\mathrm{covol}(\Lambda_\xi) s} \sum_{\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix} \right) \in \sigma_\kappa^{-1} \mathcal{R}_{\kappa,\xi} \sigma_\xi} \frac{e^{2\pi i \, \mathrm{tr}(\mu \frac{d}{c})}}{|c|^{2s+2}}$$

with

(2.9)

$$\mathcal{R}_{\kappa,\xi} := \Gamma'_\kappa \backslash \{ \gamma \in \Gamma : \gamma \xi \neq \kappa \} / \Gamma'_\xi.$$

Note that $\{ \gamma \in \Gamma : \gamma \xi \neq \kappa \} = \Gamma$ if $\xi \neq \kappa$. It is known (see, e.g., [3], [12]) that the function $\varphi_{\kappa,\xi}(\mu; s)$ admits a meromorphic continuation to all $s \in \mathbb{C}$, which is holomorphic at $s = 1$ if $\mu \neq 0$. It is also well-known that one can use the above Fourier expansion in order to prove that $E_\kappa(P, s)$ admits a meromorphic continuation to the whole complex s-plane. There is always a simple pole at $s = 1$ with residue given by

(2.10)

$$\mathrm{res}_{s=1} E_\kappa(P, s) = \frac{\mathrm{covol}(\Lambda_\kappa)}{\mathrm{vol}(X)} = \mathrm{res}_{s=1} \varphi_{\kappa,\kappa}(0; s).$$

In case that $\kappa = \infty$, the residue is explicitly given by (1.3) and we have

$$\varphi_{\infty,\infty}(0; s) = \frac{\pi |\mathcal{O}_K^\times|}{h_K |d_K|^{1/2} s} \sum_\chi \frac{L(s, \chi)}{L(s+1, \chi)},$$

where the sum runs over all characters χ of the class group of K and $L(s, \chi)$ denotes the associated L-function (see, e.g., [3], Chapter 8, Theorems 1.5 and 2.11). From these, a straightforward computation yields the Kronecker's limit formula (1.4) stated in the Introduction.

Finally, we recall the definition of the Niebur–Poincaré series. For $P \in \mathbb{H}^3$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, the Niebur–Poincaré series associated to the cusp κ and to $\nu \in \Lambda_\kappa^*$, $\nu \neq 0$, is given by

(2.11)

$$F_{\kappa,\nu}(P, s) = \sum_{\gamma \in \Gamma'_\kappa \backslash \Gamma} r(\sigma_\kappa^{-1} \gamma P) \, I_s \left(4\pi|\nu| r(\sigma_\kappa^{-1} \gamma P) \right) e^{2\pi i \, \mathrm{tr}\left(\nu z(\sigma_\kappa^{-1} \gamma P) \right)},$$

6

where $I_s(\cdot)$ denotes the modified Bessel function of the first kind. We recall that the Niebur–Poincaré series converges absolutely and defines an automorphic function, which is holomorphic for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ (see, e.g., [10]). Moreover, it satisfies the differential equation

$$\left(\Delta - (1 - s^2)\right) F_{\kappa,\nu}(\cdot, s) = 0,$$

i.e. it is an eigenfunction of Δ.

2.4. **The resolvent kernel.** The resolvent kernel for the hyperbolic Laplacian is given by the automorphic Green's function. For $P, Q \in \mathbb{H}^3$ with $P \neq \gamma Q$ for any $\gamma \in \Gamma$, and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, it is defined by

$$G_s(P, Q) = \frac{1}{2\pi} \sum_{\gamma \in \Gamma} \varphi_s\left(\cosh(d(P, \gamma Q))\right),$$

where $\varphi_s(t) = (t + \sqrt{t^2 - 1})^{-s}(t^2 - 1)^{-1/2}$. The series defining $G_s(P, Q)$ converges uniformly on compact subsets of $\{(P, Q) \in \mathbb{H}^3 \times \mathbb{H}^3 : P \neq \gamma Q \text{ for any } \gamma \in \Gamma\} \times \{s \in \mathbb{C} : \mathrm{Re}(s) > 1\}$. We recall the following well-known properties of $G_s(P, Q)$ (see, e.g., [3]):

(G1) The function $G_s(P, Q)$ is Γ-invariant in each variable and can therefore be considered as a function on $X \times X$, away from the diagonal. Moreover, we have $G_s(P, Q) = G_s(Q, P)$.

(G2) For fixed $Q \in X$, we have a singularity of the form

$$G_s(P, Q) = \frac{\nu(Q)}{2\pi} \frac{1}{d(P, Q)} + O_Q(1),$$

as $P \to Q$.

(G3) For $P, Q \in X$ with $P \neq Q$, we have $(\Delta_P - (1 - s^2))G_s(P, Q) = 0$.

The Green's function is holomorphic for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ and it admits a meromorphic continuation to the whole complex s-plane with a simple pole at $s = 1$ with residue

$$(2.12) \qquad\qquad \mathrm{res}_{s=1} G_s(P, Q) = \frac{1}{\mathrm{vol}(X)}.$$

Moreover, using the spectral expansion of $G_s(P, Q)$ given in [3] (Proposition 4.6, p. 285), it is easy to see that the function

$$P \mapsto \lim_{s \to 1} \left(G_s(P, Q) - \frac{2}{\mathrm{vol}(X)(s^2 - 1)} \right)$$

is square-integrable on X, for fixed $Q \in X$, and orthogonal to the constant functions, i.e.

$$(2.13) \qquad\qquad \int_X \lim_{s \to 1} \left(G_s(P, Q) - \frac{2}{\mathrm{vol}(X)(s^2 - 1)} \right) d\mu(P) = 0.$$

3. Fourier expansions

In this section, we compute the Fourier expansion of the Green's function and that of the Niebur–Poincaré series. Part of the computations involve explicit evaluations of certain integrals in terms of special functions. The proof of these technical identities is postponed to Section 6 in order to keep the exposition simple.

Proposition 3.1. *Let* $P = z + rj \in \mathbb{H}^3$ *with* $r > r(\sigma_\xi^{-1} \gamma Q)$ *for any* $\gamma \in \Gamma$, *and* $s \in \mathbb{C}$ *with* $\mathrm{Re}(s) > 1$. *Then, we have the following Fourier expansion*

$$G_s(\sigma_\xi P, Q) = \frac{1}{\mathrm{covol}(\Lambda_\xi)} \left(\frac{r^{1-s}}{s} E_\xi(Q, s) + 2 \sum_{\substack{\mu \in \Lambda_\xi^* \\ \mu \neq 0}} F_{\xi, -\mu}(Q, s) \, r K_s(4\pi|\mu|r) e^{2\pi i \, \mathrm{tr}(\mu z)} \right).$$

7

Proof. The Fourier coefficient $a_\mu(r) = a_{\mu,s}(r, Q)$ in the Fourier expansion (2.5) of the function $P \mapsto G_s(P, Q)$ with respect to the cusp ξ is given by

$$a_{\mu,s}(r, Q) = \frac{1}{\text{covol}(\Lambda_\xi)} \int_{\mathbb{C}/\Lambda_\xi} G_s(\sigma_\xi P, Q) e^{-2\pi i \operatorname{tr}(\mu z)} dz.$$

To compute this integral, we start by writing

$$G_s(\sigma_\xi P, Q) = \frac{1}{2\pi} \sum_{\gamma \in \Gamma'_\xi \backslash \Gamma} \sum_{\eta \in \Gamma'_\xi} \varphi_s \left(\cosh(d(\eta^{-1} \sigma_\xi P, \gamma Q)) \right)$$

$$= \frac{1}{2\pi} \sum_{\gamma \in \Gamma'_\xi \backslash \Gamma} \sum_{\lambda \in \Lambda_\xi} \varphi_s \left(\cosh(d(P + \lambda, \sigma_\xi^{-1} \gamma Q)) \right),$$

where for the last equality we employed (2.4), namely the identity $\sigma_\xi^{-1} \Gamma'_\xi \sigma_\xi = \{ \left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix} \right) | \lambda \in \Lambda_\xi \}$. Hence, we get

$$a_{\mu,s}(r, Q) = \frac{1}{2\pi \text{covol}(\Lambda_\xi)} \sum_{\gamma \in \Gamma'_\xi \backslash \Gamma} \int_{\mathbb{C}} \varphi_s \left(\cosh(d(P, \sigma_\xi^{-1} \gamma Q)) \right) e^{-2\pi i \operatorname{tr}(\mu z)} dz.$$

Now, we set $\tilde{z} := z(\sigma_\xi^{-1} \gamma Q)$ and $\tilde{r} := r(\sigma_\xi^{-1} \gamma Q)$. Using formula (2.2), namely

$$\cosh(d(P, \sigma_\xi^{-1} \gamma Q)) = \frac{|z - \tilde{z}|^2 + r^2 + \tilde{r}^2}{2r\tilde{r}},$$

we obtain by a change of variables ($z \mapsto z + \tilde{z}$),

$$a_{\mu,s}(r, Q) = \frac{1}{2\pi \text{covol}(\Lambda_\xi)} \sum_{\gamma \in \Gamma'_\xi \backslash \Gamma} e^{-2\pi i \operatorname{tr}(\mu \tilde{z})} I_{\mu,s}(r, \tilde{r}),$$

where we have set

$$I_{\mu,s}(r, \tilde{r}) := \int_{\mathbb{C}} \varphi_s \left(\frac{|z|^2 + r^2 + \tilde{r}^2}{2r\tilde{r}} \right) e^{-2\pi i \operatorname{tr}(\mu z)} dz.$$

By Lemma 6.1, we have

$$I_{\mu,s}(r, \tilde{r}) = \begin{cases} 2\pi s^{-1} r^{1-s} \tilde{r}^{s+1}, & \text{if } \mu = 0, \\ 4\pi r\tilde{r} K_s(4\pi |\nu| r) I_s(4\pi |\nu| \tilde{r}), & \text{if } \mu \neq 0. \end{cases}$$

Summing up and recalling that $\tilde{r} = r(\sigma_\xi^{-1} \gamma Q)$, we conclude

$$a_{0,s}(r, Q) = \frac{1}{\text{covol}(\Lambda_\xi)} \frac{r^{1-s}}{s} \sum_{\gamma \in \Gamma'_\xi \backslash \Gamma} r(\sigma_\xi^{-1} \gamma Q)^{s+1} = \frac{1}{\text{covol}(\Lambda_\xi)} \frac{r^{1-s}}{s} E_\xi(Q, s)$$

and, for $\mu \neq 0$, we derive

$$a_{\mu,s}(r, Q) = \frac{2}{\text{covol}(\Lambda_\xi)} r K_s(4\pi |\mu| r) \sum_{\gamma \in \Gamma'_\xi \backslash \Gamma} \tilde{r} I_s(4\pi |\mu| \tilde{r}) e^{-2\pi i \operatorname{tr}(\mu \tilde{z})}$$

$$= \frac{2}{\text{covol}(\Lambda_\xi)} r K_s(4\pi |\mu| r) F_{\xi, -\mu}(Q, s),$$

as asserted. This completes the proof. □

We proceed by computing the Fourier expansion of the Niebur–Poincaré series $F_{\kappa,\nu}(P,s)$, where $\kappa \in C_\Gamma$ and $\nu \in \Lambda_\kappa^*$, $\nu \neq 0$. For this, we define the function $\mathcal{J}_s : \mathbb{C}^\times \to \mathbb{C}$ by

$$(3.1) \qquad \mathcal{J}_s(z) := \begin{cases} J_s(4\pi\sqrt{z})J_s(4\pi\sqrt{\overline{z}}), & \text{if } \operatorname{Re}(z) \geq 0, \\ I_s(4\pi\sqrt{-z})I_s(4\pi\sqrt{-\overline{z}}), & \text{if } \operatorname{Re}(z) \leq 0. \end{cases}$$

Using the identity $I_s(z) = e^{\mp s\pi/2}J_s(ze^{\pm\pi i/2})$ for $z \in \mathbb{C}$ with $\operatorname{Re}(z) > 0$, it is easy to verify that $\mathcal{J}_s(z)$ is well-defined for $z \in \mathbb{C}$, $z \neq 0$, with $\operatorname{Re}(z) = 0$. With this, we have

Proposition 3.2. *Let $P = z + rj \in \mathbb{H}^3$ and $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1$. Then, we have the following Fourier expansion*

$$F_{\kappa,\nu}(\sigma_\xi P, s) = \delta_{\kappa,\xi}\, rI_s(4\pi|\nu|r) \sum_{u \in \mathcal{O}_K^\times/\{\pm 1\}} e^{2\pi i\, \operatorname{tr}(\nu u^2 z)} + \frac{\operatorname{covol}(\Lambda_\kappa)}{\operatorname{covol}(\Lambda_\xi)} \frac{(2\pi|\nu|)^s}{s\Gamma(s)} \varphi_{\xi,\kappa}(-\nu;s)\, r^{1-s}$$

$$+ \sum_{\substack{\mu \in \Lambda_\xi^* \\ \mu \neq 0}} \mathcal{B}_{\kappa,\xi}(\nu,\mu;s)\, rK_s(4\pi|\mu|r)e^{2\pi i\, \operatorname{tr}(\mu z)}.$$

Here,

$$\mathcal{B}_{\kappa,\xi}(\nu,\mu;s) := \frac{2\pi}{\operatorname{covol}(\Lambda_\xi)} \sum_{\left(\begin{smallmatrix} a & * \\ c & d \end{smallmatrix}\right) \in \sigma_\kappa^{-1}\mathcal{R}_{\kappa,\xi}\sigma_\xi} \frac{e^{2\pi i\, \operatorname{tr}((\nu a + \mu d)/c)}}{|c|^2} \mathcal{J}_s\left(\frac{\nu\mu}{c^2}\right),$$

and $\varphi_{\xi,\kappa}(-\nu;s)$, $\mathcal{R}_{\kappa,\xi}$, and $\mathcal{J}_s(\cdot)$ are given by (2.8), (2.9), (3.1), respectively.

Proof. To simplify the notation, we set $f(P) := r(P)I_s\left(4\pi|\nu|r(P)\right)e^{2\pi i\, \operatorname{tr}(\nu z(P))}$ and we define

$$\widehat{F}_{\kappa,\nu}(P,s) := \sum_{\gamma \in \mathcal{R}_{\kappa,\xi}} \sum_{\eta \in \Gamma_\xi'} f(\sigma_\kappa^{-1}\gamma\eta P).$$

Recalling the definition (2.11) of the Niebur-Poincaré series, we then deduce

$$F_{\kappa,\nu}(\sigma_\xi P, s) = \sum_{\gamma \in \Gamma_\kappa'\backslash\Gamma} f\left(\sigma_\kappa^{-1}\gamma\sigma_\xi P\right)$$

$$(3.2) \qquad\qquad = \delta_{\kappa,\xi} \sum_{\gamma \in \Gamma_\kappa'\backslash\Gamma_\kappa} f\left(\sigma_\kappa^{-1}\gamma\sigma_\xi P\right) + \widehat{F}_{\kappa,\nu}(\sigma_\xi P, s).$$

To treat the first term in (3.2), we assume that $\delta_{\kappa,\xi} = 1$, that is $\kappa = \xi$ and $\sigma_\kappa = \sigma_\xi$. Then

$$\delta_{\kappa,\xi} \sum_{\gamma \in \Gamma_\kappa'\backslash\Gamma_\kappa} f\left(\sigma_\kappa^{-1}\gamma\sigma_\xi P\right) = \sum_{\gamma \in \sigma_\kappa^{-1}(\Gamma_\kappa'\backslash\Gamma_\kappa)\sigma_\kappa} f\left(\gamma P\right) = rI_s(4\pi|\nu|r) \sum_{u \in \mathcal{O}_K^\times/\{\pm 1\}} e^{2\pi i\, \operatorname{tr}(\nu u^2 z)},$$

where for the second equality we note that $\sigma_\kappa^{-1}(\Gamma_\kappa'\backslash\Gamma_\kappa)\sigma_\kappa \cong \left\{ \left(\begin{smallmatrix} u & 0 \\ 0 & u^{-1} \end{smallmatrix}\right) \mid u \in \mathcal{O}_K^\times \right\}/\{\pm 1\}$, which is an immediate consequence of (2.3) and (2.4), and we used the identity $\left(\begin{smallmatrix} u & 0 \\ 0 & u^{-1} \end{smallmatrix}\right)P = u^2 z + rj$ for $u \in \mathcal{O}_K^\times$.

To treat the second term in (3.2), we note that the function $\widehat{F}_{\kappa,\nu}(\sigma_\xi\cdot,s)$ is $\sigma_\xi^{-1}\Gamma_\xi'\sigma_\xi$-invariant. The Fourier coefficient $b_\mu(r) = b_{\mu,\kappa,\nu}(r,s)$ in the Fourier expansion (2.5) of the function $\widehat{F}_{\kappa,\nu}(P,s)$ with respect to the cusp ξ is given by

$$b_{\mu,\kappa,\nu}(r,s) = \frac{1}{\operatorname{covol}(\Lambda_\xi)} \int_{\mathbb{C}/\Lambda_\xi} \widehat{F}_{\kappa,\nu}(\sigma_\xi P, s)e^{-2\pi i\, \operatorname{tr}(\mu z)}dz.$$

To compute this integral, we start by writing

$$\widehat{F}_{\kappa,\nu}(\sigma_\xi P, s) = \sum_{\gamma \in \mathcal{R}_{\kappa,\xi}} \sum_{\eta \in \Gamma'_\xi} f\left(\sigma_\kappa^{-1} \gamma \eta \sigma_\xi P\right)$$

$$= \sum_{\gamma \in \mathcal{R}_{\kappa,\xi}} \sum_{\lambda \in \Lambda_\xi} f\left(\sigma_\kappa^{-1} \gamma \sigma_\xi (P + \lambda)\right),$$

where for the last equality we employed (2.4), namely the identity $\sigma_\xi^{-1} \Gamma'_\xi \sigma_\xi = \{\left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right) | \lambda \in \Lambda_\xi\}$. Hence, we get

$$b_{\mu,\kappa,\nu}(r, s) = \frac{1}{\mathrm{covol}(\Lambda_\xi)} \sum_{\gamma \in \sigma_\kappa^{-1} \mathcal{R}_{\kappa,\xi} \sigma_\xi} \int_{\mathbb{C}} f\left(\gamma P\right) e^{-2\pi i \, \mathrm{tr}(\mu z)} dz.$$

Now, writing $z(\gamma P) = \frac{a}{c} - \frac{1}{c} \frac{\overline{cz+d}}{|cz+d|^2 + |c|^2 r^2}$ with $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and using

$$f\left(\gamma P\right) = \frac{r}{|cz+d|^2 + |c|^2 r^2} I_s\left(\frac{4\pi|\nu|r}{|cz+d|^2 + |c|^2 r^2}\right) e^{2\pi i \, \mathrm{tr}(\nu \frac{a}{c})} e^{-2\pi i \, \mathrm{tr}\left(\frac{\nu}{c} \frac{\overline{cz+d}}{|cz+d|^2 + |c|^2 r^2}\right)},$$

we obtain by a change of variables $(z \mapsto z - \frac{d}{c})$

$$b_{\mu,\kappa,\nu}(r, s) = \frac{1}{\mathrm{covol}(\Lambda_\xi)} \sum_{\left(\begin{smallmatrix} a & * \\ c & d \end{smallmatrix}\right) \in \sigma_\kappa^{-1} \mathcal{R}_{\kappa,\xi} \sigma_\xi} e^{2\pi i \, \mathrm{tr}\left(\nu \frac{a}{c} + \mu \frac{d}{c}\right)} \mathcal{I}(r, \nu, \mu, c),$$

where we have set

$$\mathcal{I}(r, \nu, \mu, c) := \int_{\mathbb{C}} \frac{r}{|c|^2(|z|^2 + r^2)} I_s\left(\frac{4\pi|\nu|r}{|c|^2(|z|^2 + r^2)}\right) e^{-2\pi i \, \mathrm{tr}\left(\nu \frac{\overline{z}}{c^2(|z|^2 + r^2)} + \mu z\right)} dz.$$

By Lemma 6.2, we have

$$\mathcal{I}(r, \nu, \mu, c) = \begin{cases} \dfrac{2^s \pi^{s+1} |\nu|^s}{|c|^{2(s+1)} s^2 \Gamma(s)} r^{1-s}, & \text{if } \mu = 0, \\[2ex] \dfrac{2\pi}{|c|^2} \mathcal{J}_s\left(\frac{\nu\mu}{c^2}\right) r K_s(4\pi|\mu|r), & \text{if } \mu \neq 0. \end{cases}$$

Summing up, we conclude

$$b_{0,\kappa,\nu}(r, s) = \frac{1}{\mathrm{covol}(\Lambda_\xi)} \frac{2^s \pi^{s+1} |\nu|^s}{s^2 \Gamma(s)} r^{1-s} \sum_{\left(\begin{smallmatrix} a & * \\ c & * \end{smallmatrix}\right) \in \sigma_\kappa^{-1} \mathcal{R}_{\kappa,\xi} \sigma_\xi} \frac{e^{2\pi i \, \mathrm{tr}(\nu \frac{a}{c})}}{|c|^{2s+2}}$$

$$= \frac{1}{\mathrm{covol}(\Lambda_\xi)} \frac{2^s \pi^{s+1} |\nu|^s}{s^2 \Gamma(s)} r^{1-s} \sum_{\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \sigma_\xi^{-1} \mathcal{R}_{\xi,\kappa} \sigma_\kappa} \frac{e^{2\pi i \, \mathrm{tr}(-\nu \frac{d}{c})}}{|c|^{2s+2}}.$$

Recalling definition (2.8), we get that

$$b_{0,\kappa,\nu}(r, s) = \frac{\mathrm{covol}(\Lambda_\kappa)}{\mathrm{covol}(\Lambda_\xi)} \frac{2^s \pi^s |\nu|^s}{s \Gamma(s)} r^{1-s} \varphi_{\xi,\kappa}(-\nu; s),$$

as asserted. Furthermore, for $\mu \neq 0$, we conclude

$$b_{\mu,\kappa,\nu}(r, s) = \frac{2\pi}{\mathrm{covol}(\Lambda_\xi)} r K_s(4\pi|\mu|r) \sum_{\left(\begin{smallmatrix} a & * \\ c & d \end{smallmatrix}\right) \in \sigma_\kappa^{-1} \mathcal{R}_{\kappa,\xi} \sigma_\xi} \frac{e^{2\pi i \, \mathrm{tr}(\nu \frac{a}{c} + \mu \frac{d}{c})}}{|c|^2} \mathcal{J}_s\left(\frac{\nu\mu}{c^2}\right)$$

$$= \mathcal{B}_{\kappa,\xi}(\nu, \mu; s) r K_s(4\pi|\mu|r).$$

10

This completes the proof. □

4. Analytic continuation

The main goal of this section is to prove the meromorphic continuation of the Green's function via its Fourier expansion. We remark here that the existence of this meromorphic continuation is well-known and follows from the spectral expansion of the Green's function (see, e.g., [3], Proposition 4.6, p. 285). Here we choose a different approach as we also need precise information about the growth at the cusp ∞ of this meromorphic continuation. In order to do this, we first analytically continue the Niebur–Poincaré series $F_{\infty,\nu}(P,s)$ by using the explicit Fourier expansion given in Proposition 3.2 with $\xi = \infty$. Before doing so, we need the following result.

Lemma 4.1. *We have the bounds*

$$|\mathcal{J}_s(z)| = \begin{cases} O\left(|z|^{\mathrm{Re}(s)}\right), & \text{for } 0 < |z| \leq 1, \\ O\left(e^{8\pi\sqrt{|z|}}|z|^{\mathrm{Re}(s)}\right), & \text{for } |z| > 1, \end{cases}$$

holding uniformly for s in any compact set contained in $\mathrm{Re}(s) > -1/2$.

Proof. Using the asymptotic formula (A.15), we conclude

$$\mathcal{J}_s(z) \sim \frac{|4\pi^2 z|^s}{\Gamma(s+1)^2},$$

for $z \to 0$. This implies the first bound. In order to obtain the second bound, we use (A.13) and get

$$I_s(4\pi\sqrt{-z})I_s(4\pi\sqrt{-\overline{z}}) = O\left(|z|^{\mathrm{Re}(s)}e^{8\pi\,\mathrm{Re}(\sqrt{-z})}\right),$$

for $z \to \infty$, $\mathrm{Re}(z) \leq 0$. On the other hand, formula (A.14) gives

$$J_s(4\pi\sqrt{z})J_s(4\pi\sqrt{\overline{z}}) = O\left(|z|^{\mathrm{Re}(s)}e^{8\pi|\mathrm{Im}(\sqrt{z})|}\right),$$

for $z \to \infty$, $\mathrm{Re}(z) \geq 0$. The second bound follows easily from these estimates. Since the used asymptotic formulas and bounds are uniform for s in any compact set contained in $\mathrm{Re}(s) > -1/2$, we conclude that these estimates are also uniform. This completes the proof of the Lemma. □

Given $\nu, \mu \in \mathcal{D}^{-1}$ both non zero and $s \in \mathbb{C}$ define

$$\mathcal{Z}(\nu, \mu; s) := \sum_{\substack{c \in \mathcal{O}_K/\{\pm 1\} \\ c \neq 0}} \frac{|\mathcal{S}(\nu, \mu, c)|}{|c|^{2+2s}},$$

where

$$\mathcal{S}(\nu, \mu, c) := \sum_{\substack{u,u^* \in \mathcal{O}_K/c\mathcal{O}_K \\ uu^* = 1}} e^{2\pi i\,\mathrm{tr}((u\nu + u^*\mu)/c)}.$$

By using the trivial bound for $|\mathcal{S}(\nu, \mu, c)|$, namely $|\mathcal{S}(\nu, \mu, c)| \leq \mathcal{N}(c) = |c|^2$, one sees that the series $\mathcal{Z}(\nu, \mu; s)$ converges absolutely for $\mathrm{Re}(s) > 1$.

Lemma 4.2. *The series $\mathcal{Z}(\nu, \mu; s)$ converges absolutely for $\mathrm{Re}(s) > 1/2$. Moreover, there exists $\alpha > 0$ such that the bound*

$$|\mathcal{Z}(\nu, \mu; s)| = O\left(\mathcal{N}(\nu\mu\mathcal{D}^2)^\alpha\right)$$

holds uniformly for s in any compact set contained in $\mathrm{Re}(s) > 1/2$.

Proof. This result is essentially due to Sarnak. Indeed, from the proof of Proposition 3.4 in [12] we have

$$\sum_{\substack{c\in\mathcal{O}_K/\{\pm1\}\\c\neq0}}\frac{|\mathcal{S}(\nu,\mu,c)|}{|c|^{2+2\sigma}}\leq\frac{|\mathcal{O}_K^\times|}{2}\prod_{\substack{P\subset\mathcal{O}_K\\\nu\mu\mathcal{D}^2\subseteq P}}\left(1-\mathcal{N}(P)^{-\sigma}\right)^{-1}\prod_{\substack{P\subset\mathcal{O}_K\\P\neq(0)}}\left(1+2\mathcal{N}(P)^{-\frac{1}{2}-\sigma}+\frac{\mathcal{N}(P)^{-2\sigma}}{1-\mathcal{N}(P)^{-\sigma}}\right),$$

where $\sigma=\mathrm{Re}(s)$ and the products run over prime ideals $P\subset\mathcal{O}_K$. The infinite product

$$\prod_{\substack{P\subset\mathcal{O}_K\\P\neq(0)}}\left(1+2\mathcal{N}(P)^{-\frac{1}{2}-\sigma}+\frac{\mathcal{N}(P)^{-2\sigma}}{1-\mathcal{N}(P)^{-\sigma}}\right)$$

converges for $\sigma>1/2$, proving the absolute convergence of $\mathcal{Z}(\nu,\mu,s)$ for $\mathrm{Re}(s)>1/2$. On the other hand, since the function $x\mapsto(1-x^{-\sigma})^{-1}$ is decreasing for $x>1$ and $\mathcal{N}(P)\geq2$ for any prime ideal P, we have

$$\prod_{\substack{P\subset\mathcal{O}_K\\\nu\mu\mathcal{D}^2\subseteq P}}\left(1-\mathcal{N}(P)^{-\sigma}\right)^{-1}\leq(1-2^{-\sigma})^{-\ell},$$

where ℓ is the number of prime ideals dividing $\nu\mu\mathcal{D}^2$. But $\ell\leq2\omega^\#(\mathcal{N}(\nu\mu\mathcal{D}^2))$, where $\omega^\#(n)$ is the number of prime divisors of $n\in\mathbb{N}$. It is known that $\omega^\#(n)=O(\log(n))$, which gives $(1-2^{-\sigma})^{-\ell}\leq\mathcal{N}(\nu\mu\mathcal{D}^2)^\alpha$ for some $\alpha>0$ depending on σ. Moreover, one can choose $\alpha>0$ such that this bound holds uniformly for σ in any fixed compact set contained in $\sigma>1/2$. This implies the desired bound for $|\mathcal{Z}(\nu,\mu,s)|$. □

Lemma 4.3. *Let $\nu,\mu\in\mathcal{D}^{-1}$ both non zero. Then, the series $\mathcal{B}_{\infty,\infty}(\nu,\mu;s)$ converges absolutely for $\mathrm{Re}(s)>1/2$ and the bound*

$$|\mathcal{B}_{\infty,\infty}(\nu,\mu;s)|=O\left(e^{8\pi\sqrt{|\nu\mu|}}|\nu\mu|^{\mathrm{Re}(s)+1}\right)$$

holds uniformly for s in any compact set contained in $\mathrm{Re}(s)>1/2$.

Proof. We start by writing

$$\mathcal{B}_{\infty,\infty}(\nu,\mu;s)=\frac{2\pi}{\mathrm{covol}(\mathcal{O}_K)}\sum_{\substack{c\in\mathcal{O}_K/\{\pm1\}\\c\neq0}}\frac{\mathcal{S}(\nu,\mu,c)}{|c|^2}\mathcal{J}_s\left(\frac{\nu\mu}{c^2}\right).$$

For fixed $s\in\mathbb{C}$ with $\mathrm{Re}(s)>1/2$, Lemma 4.1 gives

$$\sum_{\substack{c\in\mathcal{O}_K/\{\pm1\}\\c\neq0}}\left|\frac{\mathcal{S}(\nu,\mu,c)}{c^2}\mathcal{J}_s\left(\frac{\nu\mu}{c^2}\right)\right|=O\Bigg(|\nu\mu|^{\mathrm{Re}(s)}\sum_{|\frac{\nu\mu}{c^2}|\leq1}\frac{|\mathcal{S}(\nu,\mu,c)|}{|c|^{2+2\,\mathrm{Re}(s)}}$$

$$+|\nu\mu|^{\mathrm{Re}(s)}\sum_{|\frac{\nu\mu}{c^2}|>1}\frac{|\mathcal{S}(\nu,\mu,c)|}{|c|^{2+2\,\mathrm{Re}(s)}}e^{8\pi\sqrt{|\nu\mu/c^2|}}\Bigg).$$

Since

$$\sum_{|\frac{\nu\mu}{c^2}|>1}\frac{|\mathcal{S}(\nu,\mu,c)|}{|c|^{2+2\,\mathrm{Re}(s)}}e^{8\pi\sqrt{|\nu\mu/c^2|}}\leq e^{8\pi\sqrt{|\nu\mu|}}\cdot\#\left\{c\in\mathcal{O}_K:|c|^2<|\nu\mu|\right\}=O\left(e^{8\pi\sqrt{|\nu\mu|}}|\nu\mu|\right),$$

we have

$$\sum_{\substack{c\in\mathcal{O}_K/\{\pm1\}\\c\neq0}}\left|\frac{\mathcal{S}(\nu,\mu,c)}{c^2}\mathcal{J}_s\left(\frac{\nu\mu}{c^2}\right)\right|=O\left(|\nu\mu|^{\mathrm{Re}(s)}\mathcal{Z}(\nu,\mu,\mathrm{Re}(s))+e^{8\pi\sqrt{|\nu\mu|}}|\nu\mu|^{\mathrm{Re}(s)+1}\right).$$

This together with Lemma 4.2 implies the absolute convergence of $\mathcal{B}_{\infty,\infty}(\nu, \mu; s)$ and the desired bound for $|\mathcal{B}_{\infty,\infty}(\nu, \mu; s)|$. This completes the proof of this Lemma. $\qquad\square$

We now give the analytic continuation of the Niebur–Poincaré series.

Proposition 4.4. *The Niebur–Poincaré series $F_{\infty,\nu}(P, s)$ has an analytic continuation to $\mathrm{Re}(s) > 1/2$. Moreover, for fixed $P \in \mathbb{H}^3$ and $\delta > 1$, the bound*

$$F_{\infty,\nu}(P, s) = O_{P,\delta}\left(\max\left\{|\nu|^{\mathrm{Re}(s)}e^{4\pi|\nu|r}, |\nu|^{\mathrm{Re}(s)+1}e^{\frac{4\delta\pi|\nu|}{r}}\right\}\right)$$

holds uniformly for s in any compact set contained in $\mathrm{Re}(s) > 1/2$.

Proof. By Proposition 3.2, we have

$$(4.1) \qquad F_{\infty,\nu}(P, s) = rI_s(4\pi|\nu|r)\sum_{u \in \mathcal{O}_K^{\times}/\{\pm 1\}} e^{2\pi i \, \mathrm{tr}(\nu u^2 z)} + \frac{(2\pi|\nu|)^s}{s\Gamma(s)}r^{1-s}\varphi_{\infty,\infty}(-\nu; s)$$

$$+ \sum_{\substack{\mu \in \Lambda_\xi^* \\ \mu \neq 0}} \mathcal{B}_{\infty,\infty}(\nu, \mu; s)rK_s(4\pi|\mu|r)e^{2\pi i \, \mathrm{tr}(\mu z)}.$$

For s in a fixed compact set in $\mathrm{Re}(s) > 1/2$ we have, by Lemma 4.3 and (A.12), the bound

$$\sum_{\substack{\mu \in \Lambda_\xi^* \\ \mu \neq 0}} |\mathcal{B}_{\infty,\infty}(\nu, \mu; s)rK_s(4\pi|\mu|r)| = O\left(\sqrt{r}|\nu|^{\sigma+1}\sum_{\substack{\mu \in \mathcal{D}^{-1} \\ \mu \neq 0}} |\mu|^{\sigma+1/2}e^{8\pi\sqrt{|\nu||\mu|}}e^{-4\pi|\mu|r}\right),$$

where $\sigma = \mathrm{Re}(s)$. The inequality

$$e^{8\pi\sqrt{|\nu||\mu|}}e^{-4\pi|\mu|r} \leq e^{\frac{4\delta|\nu|\pi}{r}}e^{-\pi(1-\delta^{-1})|\mu|r},$$

which holds for $\delta > 1$, gives

$$(4.2) \qquad \sum_{\substack{\mu \in \Lambda_\xi^* \\ \mu \neq 0}} |\mathcal{B}_{\infty,\infty}(\nu, \mu; s)rK_s(4\pi|\mu|r)| = O_s\left(\sqrt{r}|\nu|^{\sigma+1}e^{\frac{4\delta|\nu|\pi}{r}}\sum_{\substack{\mu \in \mathcal{D}^{-1} \\ \mu \neq 0}} |\mu|^{\sigma+1/2}e^{-\pi(1-\delta^{-1})|\mu|r}\right).$$

In particular, the series on the left hand side converges. This, together with the Fourier expansion (4.1) and the analytic continuation of $\varphi_{\infty,\infty}(-\nu; s)$, give the analytic continuation of $F_{\infty,\nu}(P, s)$. Now, by the asymptotic bound (A.13), we have

$$(4.3) \qquad rI_s(4\pi|\nu|r) = O\left(r^{\mathrm{Re}(s)+1}|\nu|^{\mathrm{Re}(s)}e^{4\pi|\nu|r}\right).$$

On the other hand, as mentioned in the Introduction, the function $\varphi_{\infty,\infty}(-\nu; s)$ can be expressed in terms of certain generalized divisors sums and it therefore has at most polynomial growth with respect to $|\nu|$, uniformly for s in any fixed compact set contained in $\mathrm{Re}(s) > 0$. This, together with (4.2) and (4.3), gives the result on the growth of $|F_{\infty,\nu}(P, s)|$. This completes the proof of the Proposition. $\qquad\square$

We can now state the existence of the meromorphic continuation of $G_s(P, Q)$ together with precise information about its growth at the cusp ∞.

Theorem 4.5. *For fixed $P, Q \in \mathbb{H}^3$ with $r = r(P) > \max\{r(Q), r(Q)^{-1}\}$, the automorphic Green's function $G_s(P, Q)$ has an analytic continuation to $\mathrm{Re}(s) > 1/2$, $s \neq 1$, with a simple pole at $s = 1$. Moreover, we have*

$$(4.4) \qquad \lim_{s \to 1}\left(G_s(P, Q) - \frac{1}{\mathrm{covol}(\mathcal{O}_K)}E_{\infty}(Q, s)\right) = -\frac{1 + \log(r)}{\mathrm{vol}(X)} + o(1),$$

as $r \to \infty$.

13

Note that the analyticity of $G_s(P,Q)$ for $\mathrm{Re}(s) > \frac{1}{2}$, $s \neq 1$, is equivalent to Sarnak's lower bound for the first "exceptional" discrete eigenvalue of the Laplacian on X ([12], Theorem 3.1).

Proof. Let $P,Q \in \mathbb{H}^3$ and assume that $r = r(P) > \max\{r(Q), r(Q)^{-1}\}$. By Theorem 3.1, we have

$$G_s(P,Q) = \frac{1}{\mathrm{covol}(\mathcal{O}_K)} \left(\frac{r^{1-s}}{s} E_\infty(Q,s) + 2 \sum_{\substack{\mu \in \mathcal{D}^{-1} \\ \mu \neq 0}} F_{\infty,-\mu}(Q,s)\, r K_s(4\pi|\mu|r) e^{2\pi i \,\mathrm{tr}(\mu z)} \right).$$

By Proposition 4.4, together with the asymptotic bound (A.12), we have

$$\sum_{\substack{\mu \in \mathcal{D}^{-1} \\ \mu \neq 0}} |F_{\infty,-\mu}(Q,s)\, r K_s(4\pi|\mu|r)| = O_{Q,\delta} \left(\sqrt{r} \sum_{\substack{\mu \in \mathcal{D}^{-1} \\ \mu \neq 0}} |\mu|^{\mathrm{Re}(s)-\frac{1}{2}} e^{-4\pi|\mu|r} \max\left\{ e^{4\pi|\mu|r(Q)}, |\mu| e^{\frac{4\delta\pi|\mu|}{r(Q)}} \right\} \right)$$

for any $\delta > 1$, uniformly for s in any compact set contained in $\mathrm{Re}(s) > 1/2$. Choosing δ such that $r > \max\{r(Q), \delta r(Q)^{-1}\}$ we conclude that the series on the left hand side is convergent. This proves that $G_s(P,Q)$ has a meromorphic continuation to $\mathrm{Re}(s) > 1/2$ having poles only where $E_\infty(P,s)$ has poles, in which case the multiplicities also agree. Since $E_\infty(P,s)$ admits an analytic continuation to $\mathrm{Re}(s) > 0$, $s \neq 1$, with a simple pole at $s = 1$, we conclude the same for $G_s(P,Q)$. Now, we note that the above computations also give

$$(4.5) \qquad \lim_{s \to 1} \left(G_s(P,Q) - \frac{r^{1-s}}{\mathrm{covol}(\mathcal{O}_K)s} E_\infty(Q,s) \right) = o(1),$$

as $r \to \infty$. A straight-forward computation using (1.3) gives

$$(4.6) \qquad \lim_{s \to 1} \left(\frac{r^{1-s}}{\mathrm{covol}(\mathcal{O}_K)s} E_\infty(Q,s) - \frac{1}{\mathrm{covol}(\mathcal{O}_K)} E_\infty(Q,s) \right) = -\frac{1+\log(r)}{\mathrm{vol}(X)}.$$

Formula (4.4) follows by combining (4.5) with (4.6). This completes the proof of the Theorem. $\square$

5. Proof of the main theorem

To prove our main theorem, we first introduce a building block for the class of functions in $\mathcal{A}$. More precisely, for $P,Q \in \mathbb{H}^3$ with $P \neq \gamma Q$ for any $\gamma \in \Gamma$, we define

$$(5.1) \qquad \mathcal{L}(P,Q) := \lim_{s \to 1} \left(G_s(P,Q) - \frac{1}{\mathrm{covol}(\mathcal{O}_K)} \left(E_\infty(Q,s) + E_\infty(P,s) - \varphi_{\infty,\infty}(0;s) \right) \right).$$

Recalling (2.12) and (2.10), the above limit exists. This function can be seen as the analogue of $\log|j(\tau_1) - j(\tau_2)|$ (see Proposition 5.1 in [7]). The next lemma summarizes the properties of the function $\mathcal{L}(P,Q)$.

Lemma 5.1. *The function $\mathcal{L}(P,Q)$ satisfies the following properties:*

($\mathcal{L}$1) *The function $\mathcal{L}(P,Q)$ is Γ-invariant in each variable and can therefore be considered as a function on $X \times X$. Moreover, we have $\mathcal{L}(P,Q) = \mathcal{L}(Q,P)$.*

($\mathcal{L}$2) *For fixed $Q \in X$, we have a singularity of the form*

$$\mathcal{L}(P,Q) = \frac{\nu(Q)}{2\pi} \frac{r(Q)}{\|P - Q\|} + O_Q(1),$$

as $P \to Q$, and the function $P \mapsto \mathcal{L}(P,Q)$ is smooth at any point $P \in X$ with $P \neq Q$.

($\mathcal{L}$3) *For $P,Q \in X$ with $P \neq Q$, we have $\Delta_P \mathcal{L}(P,Q) = 0$.*

14

($\mathcal{L}4$) *For fixed $Q \in X$, we have*

$$\mathcal{L}(P,Q) = -\frac{1}{\text{vol}(X)} - \frac{|\mathcal{O}_K^\times|}{2\,\text{covol}(\mathcal{O}_K)} r^2 + o(1),$$

as $r = r(P) \to \infty$.

Proof. Properties ($\mathcal{L}1$), ($\mathcal{L}2$), and ($\mathcal{L}3$) follow from properties ($G1$), ($G2$), and ($G3$) of the Green's function $G_s(P,Q)$ together with the equality

$$\frac{1}{d(P,Q)} = \frac{r(Q)}{\|P - Q\|} + O_Q(1),$$

as $P \to Q$. In order to prove property ($\mathcal{L}4$), we consider the Fourier expansion (2.7) of $E_\infty(P, s)$, namely the equality

$$E_\infty(P,s) = \frac{|\mathcal{O}_K^\times|}{2} r^{1+s} + \varphi_{\infty,\infty}(0;s)\,r^{1-s} + \frac{2^{1+s}\pi^s}{\Gamma(s)} \sum_{\substack{\mu \in \mathcal{D}^{-1} \\ \mu \neq 0}} |\mu|^s \varphi_{\infty,\infty}(\mu;s)\,r K_s(4\pi|\mu|r) e^{2\pi i\,\text{tr}(\mu z)},$$

where we employed the identity $[\Gamma_\infty : \Gamma_\infty'] = |\mathcal{O}_K^\times|/2$. A straight-forward computation using (2.10) gives

$$\lim_{s \to 1} \varphi_{\infty,\infty}(0;s)(r^{1-s} - 1) = -\frac{\text{covol}(\mathcal{O}_K)}{\text{vol}(X)} \log(r).$$

Since $|\varphi_{\infty,\infty}(\mu;1)|$ is of at most polynomial growth in $|\mu|$ and $K_1(r)$ has exponential decay as $r \to \infty$, we therefore get

$$\lim_{s \to 1} \left(E_\infty(P,s) - \varphi_{\infty,\infty}(0;s) \right) = \frac{|\mathcal{O}_K^\times|}{2} r^2 - \frac{\text{covol}(\mathcal{O}_K)}{\text{vol}(X)} \log(r) + o(1),$$

as $r \to \infty$. Property ($\mathcal{L}4$) now follows from this together with (4.4). This completes the proof of the Lemma. $\qquad\square$

The function $\mathcal{L}(P,Q)$ is a building block for functions in $\mathcal{A}$. More precisely, we have the following proposition which can be seen as an analogue of (1.1).

Proposition 5.2. *Let $F : \mathbb{H}^3 \to \mathbb{R} \cup \{\infty\}$ be in $\mathcal{A}$, the class of functions satisfying ($\mathcal{A}1$)–($\mathcal{A}4$). Then, the limit $F(\infty) := \lim_{r \to \infty} F(P)$ exists and we have the equality*

$$F(P) = F(\infty) + 2\pi \sum_{\ell=1}^m c_\ell\, \mathcal{L}(P, Q_\ell),$$

for any $P \in X$ with $P \neq Q_\ell$, for $\ell = 1, \ldots, m$.

Proof. Let us define $\widetilde{F}(P)$, for $P \in X$ with $P \neq Q_\ell$, for $\ell = 1, \ldots, m$, by

$$\widetilde{F}(P) = F(P) - 2\pi \sum_{\ell=1}^m c_\ell\, \mathcal{L}(P, Q_\ell).$$

By properties ($\mathcal{A}3$) and ($\mathcal{L}3$), we have that $\Delta\widetilde{F}(P) = 0$ for $P \neq Q_\ell$, $\ell = 1, \ldots, m$. On the other hand, properties ($\mathcal{A}2$) and ($\mathcal{L}2$) imply that $\widetilde{F}(P)$ is locally bounded around any point in X. This implies that $\widetilde{F}(P)$ extends to a smooth function $\widetilde{F} : X \to \mathbb{R}$ satisfying $\Delta\widetilde{F}(P) = 0$ everywhere. Indeed, by taking geodesic normal coordinates around any point, one can reduce the problem to the case where $\widetilde{F}(P)$ is a harmonic function with respect to the euclidean

Laplacian, at least locally. The existence of the harmonic extension of $\widetilde{F}(P)$ then follows from Theorem 2.3 in [2]. Using $\sum_{\ell=1}^{m} c_\ell = 0$, we note that

$$
\begin{aligned}
\sum_{\ell=1}^{m} c_\ell \, \mathcal{L}(P, Q_\ell) &= \sum_{\ell=1}^{m} c_\ell \lim_{s \to 1} \left(G_s(P, Q_\ell) - \frac{1}{\text{covol}(\mathcal{O}_K)} E_\infty(Q_\ell, s) \right) \\
&= \sum_{\ell=1}^{m} c_\ell \lim_{s \to 1} \left(G_s(P, Q_\ell) - \frac{2}{\text{vol}(X)(s^2 - 1)} \right) \\
&\quad + \sum_{\ell=1}^{m} c_\ell \lim_{s \to 1} \left(\frac{2}{\text{vol}(X)(s^2 - 1)} - \frac{1}{\text{covol}(\mathcal{O}_K)} E_\infty(Q_\ell, s) \right) \\
&= \sum_{\ell=1}^{m} c_\ell \lim_{s \to 1} \left(G_s(P, Q_\ell) - \frac{2}{\text{vol}(X)(s^2 - 1)} \right) + \frac{1}{\text{vol}(X)} \sum_{\ell=1}^{m} c_\ell \log(\eta_\infty(Q_\ell) r_\ell).
\end{aligned}
$$

As mentioned in Section 2.4, the function

$$
P \mapsto \lim_{s \to 1} \left(G_s(P, Q_\ell) - \frac{2}{\text{vol}(X)(s^2 - 1)} \right)
$$

is square-integrable on X, for fixed Q_ℓ. This implies that the function

$$
P \mapsto \sum_{\ell=1}^{m} c_\ell \, \mathcal{L}(P, Q_\ell)
$$

is also square-integrable. By property $(\mathcal{A}4)$, we conclude that $\widetilde{F}(P)$ is square-integrable over X. By Theorem 4.1.8 in [3], p. 140, we know that any smooth, harmonic, square-integrable function on X is constant. We conclude that $\widetilde{F}(P)$ is constant. Finally, using $(\mathcal{L}4)$ together with $\sum_{\ell=1}^{m} c_\ell = 0$, we have

$$
\sum_{\ell=1}^{m} c_\ell \, \mathcal{L}(P, Q_\ell) = o(1),
$$

as $r \to \infty$. We conclude that $\widetilde{F}(P) = F(\infty)$. This proves the result. $\qquad \square$

We now prove our main theorem.

Proof of Theorem 1.2. Let $F : \mathbb{H}^3 \to \mathbb{R} \cup \{\infty\}$ be a function in the class $\mathcal{A}$ satisfying the properties $(\mathcal{A}1)$–$(\mathcal{A}4)$. By Proposition 5.2 we have

$$
\frac{1}{\text{vol}(X)} \int_X F(P) d\mu(P) = F(\infty) + \frac{2\pi}{\text{vol}(X)} \int_X \sum_{\ell=1}^{m} c_\ell \, \mathcal{L}(P, Q_\ell) d\mu(P).
$$

Since $\sum_\ell c_\ell = 0$, we have

$$
\begin{aligned}
\int_X \sum_{\ell=1}^{m} c_\ell \, \mathcal{L}(P, Q_\ell) d\mu(P) &= \int_X \lim_{s \to 1} \sum_{\ell=1}^{m} c_\ell \left(G_s(P, Q_\ell) - \frac{1}{\text{covol}(\mathcal{O}_K)} E_\infty(Q_\ell, s) \right) d\mu(P) \\
&= \int_X \sum_{\ell=1}^{m} c_\ell \lim_{s \to 1} \left(G_s(P, Q_\ell) - \frac{2}{\text{vol}(X)(s^2 - 1)} \right) d\mu(P) \\
&\quad + \int_X \sum_{\ell=1}^{m} c_\ell \lim_{s \to 1} \left(\frac{2}{\text{vol}(X)(s^2 - 1)} - \frac{1}{\text{covol}(\mathcal{O}_K)} E_\infty(Q_\ell, s) \right) d\mu(P).
\end{aligned}
$$

16

Using (2.13) and (1.4), we obtain

$$\int_X \sum_{\ell=1}^m c_\ell \, \mathcal{L}(P, Q_\ell) d\mu(P) = \lim_{s \to 1} \sum_{\ell=1}^m c_\ell \left(\frac{2}{s^2 - 1} - \frac{\text{vol}(X)}{\text{covol}(\mathcal{O}_K)} E_\infty(Q_\ell, s) \right)$$

$$= \sum_{\ell=1}^m c_\ell \log(\eta_\infty(Q_\ell) \, r_\ell).$$

This completes the proof of Theorem 1.2. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

6. Technical lemmas

In this section we prove two lemmas that were used in Section 3 for the computation of the Fourier coefficients of the Green's function and of the Niebur–Poincaré series.

Lemma 6.1. *For $\mu, s \in \mathbb{C}$ with $\text{Re}(s) > 0$ and $r > \tilde{r} > 0$, let*

$$I_{\mu,s}(r, \tilde{r}) := \int_{\mathbb{C}} \varphi_s \left(\frac{|z|^2 + r^2 + \tilde{r}^2}{2r\tilde{r}} \right) e^{-2\pi i \, \text{tr}(\mu z)} dz$$

with $\varphi_s(t) = (t + \sqrt{t^2 - 1})^{-s}(t^2 - 1)^{-1/2}$. Then, we have

$$I_{\mu,s}(r, \tilde{r}) = \begin{cases} 2\pi s^{-1} r^{1-s} \tilde{r}^{s+1}, & \text{if } \mu = 0, \\ 4\pi r\tilde{r} K_s(4\pi|\nu|r) I_s(4\pi|\nu|\tilde{r}), & \text{if } \mu \neq 0. \end{cases}$$

Proof. Using polar coordinates $z = \rho e^{i\theta}$, we get

$$I_{\mu,s}(r, \tilde{r}) = \int_0^{2\pi} \int_0^\infty \varphi_s \left(\frac{\rho^2 + r^2 + \tilde{r}^2}{2r\tilde{r}} \right) e^{-2\pi i \rho \, \text{tr}(\mu e^{i\theta})} \rho \, d\rho d\theta.$$

Letting $t = \rho^2$ and $f(t) := (t + r^2 + \tilde{r}^2)/2r\tilde{r}$, we have

$$I_{0,s}(r, \tilde{r}) = \pi \int_0^\infty \varphi_s\left(f(t)\right) dt = -\frac{2\pi r\tilde{r}}{s} \left[\left(f(t) + \sqrt{f(t)^2 - 1} \right)^{-s} \right]_{t=0}^{t=\infty} = \frac{2\pi}{s} r^{1-s} \tilde{r}^{s+1},$$

where we have used that $r > \tilde{r}$. This proves the first formula. For $\mu \neq 0$, we write $\mu = |\mu|e^{i\alpha}$ and get

$$\int_0^{2\pi} e^{-2\pi i \rho \, \text{tr}(\mu e^{i\theta})} d\theta = \int_0^{2\pi} e^{-4\pi i \rho |\mu| \sin(\theta)} d\theta = 2\pi J_0(4\pi|\mu|\rho),$$

by using formula (A.1). Replacing this in the above formula for $I_{\mu,s}(r, \tilde{r})$ and making the change of variables $t = \rho^2/r^2$, we get

$$I_{\mu,s}(r, \tilde{r}) = \pi r^2 \int_0^\infty \varphi_s \left(\frac{r}{2\tilde{r}} \left(t + 1 + \frac{\tilde{r}^2}{r^2} \right) \right) J_0(4\pi|\mu|r\sqrt{t}) dt.$$

Using formula (A.10), we have

$$\varphi_s(b/a) = a \int_0^\infty I_s(au) e^{-bu} du,$$

for $b > a > 0$. Using this identity with $a = 2\tilde{r}/r$ and $b = t + 1 + \frac{\tilde{r}^2}{r^2}$, we get

$$I_{\mu,s}(r, \tilde{r}) = 2\pi r\tilde{r} \int_0^\infty I_s \left(\frac{2\tilde{r}u}{r} \right) e^{-\left(1 + \frac{\tilde{r}^2}{r^2}\right)u} \int_0^\infty J_0(4\pi|\mu|r\sqrt{t}) e^{-tu} dt du.$$

17

Formula (A.2) with $a = u$ and $b = 4\pi|\mu|r$ yields

$$I_{\mu,s}(r, \tilde{r}) = 2\pi r \tilde{r} \int_0^\infty I_s\left(\frac{2\tilde{r}u}{r}\right) e^{-\left(1+\frac{\tilde{r}^2}{r^2}\right)u} e^{-4\pi^2|\mu|^2 r^2/u} \frac{du}{u}.$$

Next, we make the change of variables $t = 8\pi^2|\mu|^2 r^2/u$ and we get

$$I_{\mu,s}(r, \tilde{r}) = 2\pi r \tilde{r} \int_0^\infty I_s\left(\frac{ab}{t}\right) e^{-\frac{a^2+b^2}{2t}} e^{-t/2} \frac{dt}{t},$$

with $a = 4\pi|\mu|r$ and $b = 4\pi|\mu|\tilde{r}$. Observing that $a > b > 0$ and using (A.3) we conclude

$$I_{\mu,s}(r, \tilde{r}) = 4\pi r \tilde{r} K_s(a) I_s(b).$$

This completes the proof of the Lemma. $\qquad\qquad\square$

Lemma 6.2. *For $\nu, \mu, c \in \mathbb{C}$ with ν, c both non zero, $r > 0$, and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$, put*

$$\mathfrak{I}(r, \nu, \mu, c) = \int_\mathbb{C} \frac{r}{|c|^2(|z|^2 + r^2)} I_s\left(\frac{4\pi|\nu|r}{|c|^2(|z|^2 + r^2)}\right) e^{-2\pi i\,\mathrm{tr}\left(\nu\frac{\bar{z}}{c^2(|z|^2+r^2)} + \mu z\right)} dz.$$

Then, we have

$$\mathfrak{I}(r, \nu, \mu, c) = \begin{cases} \dfrac{\pi^{1+s}2^s|\nu|^s}{|c|^{2(1+s)}s\Gamma(1+s)} r^{1-s}, & \text{if } \mu = 0, \\[2ex] \dfrac{2\pi}{|c|^2} r K_s(4\pi|\mu|r) \mathfrak{J}_s\left(\frac{\nu\mu}{c^2}\right), & \text{if } \mu \neq 0, \end{cases}$$

where $\mathfrak{J}_s(z)$ is given in (3.1).

Proof. We start with the case $\mu = 0$. Using polar coordinates $z = \rho e^{i\theta}$, we get

$$\mathfrak{I}(r, \nu, 0, c) = \frac{r}{|c|^2} \int_0^\infty \int_0^{2\pi} \frac{\rho}{\rho^2 + r^2} I_s\left(\frac{4\pi|\nu|r}{|c|^2(\rho^2 + r^2)}\right) e^{-\frac{4\pi i|\nu|\rho}{|c|^2(\rho^2+r^2)}\cos(\theta)} d\theta d\rho.$$

Using formula (A.1) and making the change of variables $\xi = \rho/r$, we get

$$\mathfrak{I}(r, \nu, 0, c) = \frac{2\pi r}{|c|^2} \int_0^\infty \frac{\xi}{\xi^2 + 1} I_s\left(\frac{4\pi|\nu|}{|c|^2 r(\xi^2 + 1)}\right) J_0\left(\frac{4\pi|\nu|\xi}{|c|^2 r(\xi^2 + 1)}\right) d\xi.$$

Using Lemma A.1 with $a = \frac{4\pi|\nu|}{|c|^2 r}$, we get

$$\mathfrak{I}(r, \nu, 0, c) = \frac{\pi^{1+s}2^s|\nu|^s}{|c|^{2(1+s)}s\Gamma(1+s)} r^{1-s}.$$

This proves the first formula. For $\mu \neq 0$, we start by writing

$$|\nu|\mathfrak{I}\left(\frac{r}{|\mu|}, \nu, \mu, c\right) = \int_\mathbb{C} \frac{|\beta|r}{|\mu z|^2 + r^2} I_s\left(\frac{4\pi|\beta|r}{|\mu z|^2 + r^2}\right) e^{-2\pi i\,\mathrm{tr}\left(\beta\frac{\bar{z}\mu}{(|z\mu|^2+r^2)}\right) - 2\pi i\,\mathrm{tr}(\mu z)} dz$$

with $\beta = \frac{\nu\mu}{c^2}$. Making the change of variables $\xi = \mu z$ and using polar coordinates $\xi = \rho e^{i\theta}$, we obtain

$$|\nu|\mathfrak{I}\left(\frac{r}{|\mu|}, \nu, \mu, c\right) = \frac{|\beta|r}{|\mu|^2} \int_0^\infty \frac{\rho}{\rho^2 + r^2} I_s\left(\frac{4\pi|\beta|r}{\rho^2 + r^2}\right) \int_0^{2\pi} e^{-2\pi i\left(\frac{\beta\rho e^{-i\theta} + \bar{\beta}\rho e^{i\theta}}{\rho^2+r^2} + 2\rho\cos(\theta)\right)} d\theta d\rho.$$

18

Now, we compute

$$\int_0^{2\pi} e^{-2\pi i \left(\frac{\beta \rho e^{-i\theta} + \overline{\beta} \rho e^{i\theta}}{\rho^2 + r^2} + 2\rho \cos(\theta) \right)} d\theta = \int_0^{2\pi} e^{-4\pi i \frac{\rho}{\rho^2 + r^2} |\beta + \rho^2 + r^2| \sin(\theta)} d\theta$$

$$= 2\pi J_0 \left(\frac{4\pi \rho}{\rho^2 + r^2} |\beta + \rho^2 + r^2| \right),$$

by formula (A.1). We conclude

$$|\nu| \mathfrak{I} \left(\frac{r}{|\mu|}, \nu, \mu, c \right) = \frac{2\pi |\beta| r}{|\mu|^2} \int_0^\infty \frac{\rho}{\rho^2 + r^2} I_s \left(\frac{4\pi |\beta| r}{\rho^2 + r^2} \right) J_0 \left(\frac{4\pi \rho}{\rho^2 + r^2} |\beta + \rho^2 + r^2| \right) d\rho.$$

Making the change of variables $t = \rho/r$, we get

$$|\nu| \mathfrak{I} \left(\frac{r}{|\mu|}, \nu, \mu, c \right) = \frac{2\pi |\beta| r}{|\mu|^2} \int_0^\infty \frac{t}{t^2 + 1} I_s \left(\frac{4\pi |\beta|}{r(t^2 + 1)} \right) J_0 \left(\frac{4\pi r t}{t^2 + 1} \left| \frac{\beta}{r^2} + t^2 + 1 \right| \right) dt.$$

Now, by formula (A.11) with $\lambda = \left| \frac{\beta}{r^2(t^2+1)} + 1 \right|$, $z = 4\pi r t$, and $s = 0$, we get

$$J_0 \left(\frac{4\pi r t}{t^2 + 1} \left| \frac{\beta}{r^2} + t^2 + 1 \right| \right) = \sum_{k=0}^\infty \frac{(-1)^k}{k!} \frac{1}{(t^2 + 1)^{2k}} \left(\frac{|\beta|^2}{r^4} + \frac{\text{tr}(\beta)}{r^2}(t^2 + 1) \right)^k (2\pi r t)^k J_k(4\pi r t).$$

Therefore, using the binomial theorem, we have

$$|\nu| \mathfrak{I} \left(\frac{r}{|\mu|}, \nu, \mu, c \right) =$$

$$\frac{2\pi |\beta| r}{|\mu|^2} \sum_{k=0}^\infty \frac{(-1)^k (2\pi)^k}{k!} \sum_{j=0}^k \binom{k}{j} \frac{|\beta|^{2j} \text{tr}(\beta)^{k-j}}{r^{k+2j}} \int_0^\infty \frac{t^{k+1}}{(t^2 + 1)^{k+j+1}} I_s \left(\frac{4\pi |\beta|}{r(t^2 + 1)} \right) J_k(4\pi r t) dt.$$

Using the power expansion for $I_s(z)$ given in formula (A.6), we have

$$\int_0^\infty \frac{t^{k+1}}{(t^2 + 1)^{k+j+1}} I_s \left(\frac{4\pi |\beta|}{r(t^2 + 1)} \right) J_k(4\pi r t) dt$$

$$= \left(\frac{2\pi |\beta|}{r} \right)^s \sum_{\ell=0}^\infty \frac{(2\pi |\beta|)^{2\ell} r^{-2\ell}}{\ell! \, \Gamma(s + \ell + 1)} \int_0^\infty \frac{t^{k+1}}{(t^2 + 1)^{s+k+j+2\ell+1}} J_k(4\pi r t) dt$$

$$= \left(\frac{2\pi |\beta|}{r} \right)^s \sum_{\ell=0}^\infty \frac{(2\pi |\beta|)^{2\ell} r^{-2\ell}}{\ell! \, \Gamma(s + \ell + 1)} \frac{(2\pi r)^{s+k+j+2\ell} K_{s+j+2\ell}(4\pi r)}{\Gamma(s + k + j + 2\ell + 1)},$$

by formula (A.5) with $s = k$, $\mu = s + k + j + 2\ell$, and $a = 4\pi r$. This gives

$$\frac{|\nu||\mu|^2}{2\pi |\beta| r} \mathfrak{I} \left(\frac{r}{|\mu|}, \nu, \mu, c \right) = \sum_{k=0}^\infty \sum_{j=0}^k \sum_{\ell=0}^\infty \frac{(-1)^k (2\pi)^{2s+2k+j+4\ell} |\beta|^{s+2j+2\ell} \text{tr}(\beta)^{k-j} K_{s+j+2\ell}(4\pi r)}{(k-j)! j! \ell! \, \Gamma(s + \ell + 1) \Gamma(s + k + j + 2\ell + 1) r^j}$$

$$= \sum_{k=0}^\infty \sum_{j=0}^\infty \sum_{\ell=0}^\infty \frac{(-1)^{k+j} (2\pi)^{2s+2k+3j+4\ell} |\beta|^{s+2j+2\ell} \text{tr}(\beta)^k K_{s+j+2\ell}(4\pi r)}{k! j! \ell! \, \Gamma(s + \ell + 1) \Gamma(s + k + 2j + 2\ell + 1) r^j},$$

19

using a well-known identity for the double sum over k and j. Applying this identity again for the double sum over j and ℓ and using Lemma A.2 with $z = 4\pi r$, we get

$$\frac{|\nu||\mu|^2}{2\pi|\beta|r}\mathfrak{I}\left(\frac{r}{|\mu|}, \nu, \mu, c\right) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{k+j}(2\pi)^{2s+2k+3j}|\beta|^{s+2j}\operatorname{tr}(\beta)^k}{k!\,j!\,\Gamma(s+k+2j+1)r^j}\sum_{\ell=0}^{j}\binom{j}{\ell}\frac{(-2\pi r)^{\ell}K_{s+j+\ell}(4\pi r)}{\Gamma(s+\ell+1)}$$

$$= K_s(4\pi r)\sum_{j=0}^{\infty}\frac{(2\pi)^{2s+4j}|\beta|^{s+2j}}{j!\,\Gamma(s+j+1)}\sum_{k=0}^{\infty}\frac{(-1)^k(2\pi)^{2k}\operatorname{tr}(\beta)^k}{k!\,\Gamma(s+k+2j+1)}.$$

Assuming that $\operatorname{tr}(\beta) > 0$, using formula (A.7) and recalling that $\beta = \frac{\nu\mu}{c^2}$, we therefore obtain

$$\mathfrak{I}(r, \nu, \mu, c) = \frac{2\pi}{|c|^2}rK_s(4\pi|\mu|r)\sum_{j=0}^{\infty}\frac{\left((2\pi)|\beta|/\sqrt{\operatorname{tr}(\beta)}\right)^{s+2j}}{j!\,\Gamma(s+j+1)}J_{s+2j}\left(4\pi\sqrt{\operatorname{tr}(\beta)}\right).$$

Applying Lemma (A.3) with $x = 4\pi\sqrt{\operatorname{tr}(\beta)}$ and $A = \sqrt{\beta/\operatorname{tr}(\beta)}$, we get

$$\mathfrak{I}(r, \nu, \mu, c) = \frac{2\pi}{|c|^2}rK_s(4\pi|\mu|r)J_s\left(4\pi\sqrt{\beta}\right)J_s\left(4\pi\sqrt{\overline{\beta}}\right)$$

$$= \frac{2\pi}{|c|^2}rK_s(4\pi|\mu|r)\,\mathfrak{J}_s\left(\frac{\nu\mu}{c^2}\right).$$

This implies the second formula in the case $\mu \neq 0, \operatorname{Re}\left(\frac{\nu\mu}{c^2}\right) > 0$. The case $\operatorname{Re}\left(\frac{\nu\mu}{c^2}\right) < 0$ is completely analogous, so we omit the details. Finally, the case $\operatorname{Re}\left(\frac{\nu\mu}{c^2}\right) = 0$ follows from any of the two other cases by taking the limit $\beta \to it, t \in \mathbb{R}, t \neq 0$. This completes the proof of the Lemma. $\qquad\square$

APPENDIX A. IDENTITIES INVOLVING SPECIAL FUNCTIONS

In this appendix we recall some identities involving special functions that are used in the paper. Most of these identities are well-known and can be found in the literature and for these we just give a reference. For some of the less-known identities we give sketch of proofs.

We start with the well-known identities. These are

$$(A.1) \qquad \int_0^{2\pi} e^{-ia\sin(\theta)}d\theta = 2\pi J_0(a), \quad a \geq 0,$$

$$(A.2) \qquad \int_0^{\infty} e^{-at}J_0(b\sqrt{t})dt = \frac{1}{a}e^{-b^2/4a}, \quad a > 0, b \in \mathbb{R},$$

$$(A.3) \qquad \int_0^{\infty} I_s\left(\frac{ab}{t}\right)e^{-\frac{t}{2}-\frac{1}{2t}(a^2+b^2)}\frac{dt}{t} = 2K_s(a)I_s(b), \quad a > b > 0, \operatorname{Re}(s) > -1,$$

$$(A.4) \qquad \int_0^{\infty}\frac{x^{\mu-1}}{(x+1)^{\nu}}dx = \frac{\Gamma(\mu)\Gamma(\nu-\mu)}{\Gamma(\nu)}, \quad \operatorname{Re}(\nu) > \operatorname{Re}(\mu) > 0,$$

and

$$(A.5) \qquad \int_0^{\infty}\frac{t^{s+1}}{(t^2+1)^{\mu+1}}J_s(at)dt = \frac{(a/2)^{\mu}K_{\mu-s}(a)}{\Gamma(\mu+1)}, \quad a > 0, -1 < \operatorname{Re}(s) < \operatorname{Re}\left(2\mu+\frac{3}{2}\right),$$

which can be found in [6] (formulas 8.411-1, 6.614-1, 6.653-2, 6.565-4, and 3.194-3, respectively). We also have

$$(A.6) \qquad I_s(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{s+2k}}{k!\,\Gamma(s+k+1)}, \quad z \in \mathbb{C} \setminus (-\infty, 0],$$

$$(A.7) \qquad J_s(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{s+2k}}{k!\,\Gamma(s+k+1)}, \quad z \in \mathbb{C} \setminus (-\infty, 0],$$

and

$$(A.8) \qquad \sum_{\ell=0}^{n} \binom{n}{\ell}(-1)^\ell \frac{\Gamma(\ell+b)}{\Gamma(\ell+a)} = \frac{\Gamma(n+a-b)\Gamma(b)}{\Gamma(a-b)\Gamma(n+a)},$$

in *loc. cit.* (formulas 8.402, 8.445, and 0.160-2, respectively). For the Gauss hypergeometric series $_2F_1(a, b; c; z)$, we have the transformation property (formula 9.134-2 in *loc. cit.*)

$$(A.9) \qquad _2F_1(a, b; a-b+1; z) = (1+z)^{-a} \, _2F_1\left(\frac{a}{2}, \frac{1+a}{2}; a-b+1; \frac{4z}{(1+z)^2}\right).$$

Formulas 4.16-1 in [4] and 5·22-16 in [16] are

$$(A.10) \qquad \int_0^{\infty} I_s(at)e^{-bt}dt = \frac{a^s}{\sqrt{b^2-a^2}}(b+\sqrt{b^2-a^2})^{-s}, \quad \text{Re}(s) > -1, \text{Re}(b) > |\text{Re}(a)|,$$

$$(A.11) \qquad J_s(\lambda z) = \lambda^s \sum_{k=0}^{\infty} \frac{(-z/2)^k(\lambda^2-1)^k}{k!}J_{s+k}(z).$$

We also have the well-known asymptotic bounds, valid uniformly for s in a compact set contained in the half-plane $\text{Re}(s) > -1/2$,

$$(A.12) \qquad |K_s(x)| = O\left(x^{-1/2}e^{-x}\right), \quad \text{for } x \to \infty, x \in \mathbb{R},$$

and

$$(A.13) \qquad I_s(z) = O\left(|z|^{\text{Re}(s)}e^{\text{Re}(z)}\right),$$

$$(A.14) \qquad J_s(z) = O\left(|z|^{\text{Re}(s)}e^{|\text{Im}(z)|}\right),$$

for $z \to \infty, |\arg(z)| \le \frac{1}{2}\pi - \delta$ with fixed $\delta > 0$. The asymptotic formulas

$$(A.15) \qquad I_s(z) \sim J_s(z) \sim \frac{(z/2)^s}{\Gamma(s+1)}, \quad \text{for } z \to 0,$$

also hold uniformly for s in a fixed compact set.

We now give the less-known identities in the form of several lemmas.

Lemma A.1. *We have*

$$\int_0^{\infty} \frac{\xi}{\xi^2+1}I_s\left(\frac{a}{\xi^2+1}\right)J_0\left(\frac{a\xi}{\xi^2+1}\right)d\xi = \frac{a^s}{2^{s+1}s\Gamma(s+1)}$$

for any $a > 0$ and $s \in \mathbb{C}$ with $\text{Re}(s) > 0$.

Proof. Using formulas (A.6) and (A.7), we have

$$\int_0^{\infty} \frac{\xi}{\xi^2+1}I_s\left(\frac{a}{\xi^2+1}\right)J_0\left(\frac{a\xi}{\xi^2+1}\right)d\xi$$

$$= \left(\frac{a}{2}\right)^s \sum_{k,j=0}^{\infty} \frac{(-1)^j a^{2k+2j}}{4^{k+j}k!j!\,\Gamma(s+k+1)\Gamma(j+1)}\int_0^{\infty} \frac{\xi^{2j+1}}{(\xi^2+1)^{2k+2j+s+1}}d\xi.$$

21

By putting $x = \xi^2$, $n = k + j$, and using formula (A.4), we see that this equals

$$\frac{1}{2}\left(\frac{a}{2}\right)^s \sum_{n=0}^{\infty} \frac{a^{2n}}{4^n n!\,\Gamma(2n + s + 1)} \sum_{j=0}^{n} \binom{n}{j} \frac{(-1)^j\,\Gamma(s + 2n - j)}{\Gamma(s + n - j + 1)}.$$

By formula (A.8) we have

$$\sum_{j=0}^{n} \binom{n}{j} \frac{(-1)^j\,\Gamma(s + 2n - j)}{\Gamma(s + n - j + 1)} = (-1)^n \sum_{j=0}^{n} \binom{n}{j} \frac{(-1)^j\Gamma(s + n + j)}{\Gamma(s + j + 1)} = \begin{cases} s^{-1}, & \text{if } n = 0, \\ 0, & \text{if } n \geq 1. \end{cases}$$

Replacing this in the previous expression gives the desired formula. $\qquad\square$

Lemma A.2. *For $s, z \in \mathbb{C}$, we have*

$$\sum_{\ell=0}^{j} \binom{j}{\ell} \frac{(-z/2)^\ell K_{s+j+\ell}(z)}{\Gamma(s + \ell + 1)} = \frac{(-z/2)^j K_s(z)}{\Gamma(s + j + 1)}.$$

Proof. This identity can be proved by induction over j, the case $j = 0$ being obvious. For the inductive step one can use the identity

$$K_{s+1}(z) = K_{s-1}(z) + \frac{2s}{z}K_s(z)$$

(see formula 8.468-10 in [6]). We omit the details. $\qquad\square$

Lemma A.3. *Assume that $A \in \mathbb{C}\backslash]-\infty, 0]$, $\mathrm{Re}(A^2) = 1/2$, and $x > 0$. Then*

$$J_s(Ax)J_s(\overline{A}x) = \sum_{n=0}^{\infty} \frac{(x|A|^2/2)^{s+2n}}{n!\,\Gamma(s + n + 1)} J_{s+2n}(x)$$

and similarly

$$I_s(Ax)I_s(\overline{A}x) = \sum_{n=0}^{\infty} \frac{(x|A|^2/2)^{s+2n}}{n!\,\Gamma(s + n + 1)} I_{s+2n}(x).$$

Proof. By using formula (A.7) and collecting the powers of x, we have

$$J_s(Ax)J_s(\overline{A}x) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!}\left(-\frac{x^2}{4}\right)^\ell \sum_{n=0}^{\ell} \binom{\ell}{n} \frac{A^{2n}\overline{A}^{2\ell-2n}}{\Gamma(s + n + 1)\Gamma(s + \ell - n + 1)}$$

and

$$\sum_{n=0}^{\infty} \frac{(x|A|^2/2)^{s+2n}}{n!\,\Gamma(s + n + 1)} J_{s+2n}(x) = \sum_{\ell=0}^{\infty} \left(-\frac{x^2}{4}\right)^\ell \frac{1}{\Gamma(s + \ell + 1)} \sum_{n=0}^{[\ell/2]} \frac{|A|^{4n}}{n!(\ell - 2n)!\,\Gamma(s + n + 1)},$$

where $[\ell/2]$ denotes the integral part of $\ell/2$. Putting $a = A^2$ and using that $\overline{a} = 1 - a$, we get

$$\sum_{n=0}^{\ell} \binom{\ell}{n} \frac{A^{2n}\overline{A}^{2\ell-2n}}{\Gamma(s + n + 1)\Gamma(s + \ell - n + 1)} = \frac{(1 - a)^\ell}{\Gamma(s + 1)\Gamma(s + 1 + \ell)}\,{}_2F_1\left(-s - \ell, -\ell; s + 1; \frac{a}{1 - a}\right)$$

and

$$\sum_{n=0}^{[\ell/2]} \frac{|A|^{4n}}{n!(\ell - 2n)!\,\Gamma(s + n + 1)} = \frac{1}{\ell!\,\Gamma(s + 1)}\,{}_2F_1\left(\frac{1 - \ell}{2}, -\frac{\ell}{2}; s + 1; 4a(1 - a)\right).$$

Thus, we just have to prove the identity

$$(1-a)^\ell \, {}_2F_1\left(-s-\ell, -\ell; s+1; \frac{a}{1-a}\right) = \, {}_2F_1\left(\frac{1-\ell}{2}, -\frac{\ell}{2}; s+1; 4a(1-a)\right).$$

This follows from identity (A.9) applied to $a = -\ell$, $b = -s-\ell$, and $z = a/(1-a)$. This completes the proof of the first formula. The proof of the second formula is completely analogous, so we omit the details. $\qquad\square$

References

[1] Asai, T.: *On a certain function analogous to* $\log|\eta(z)|$. Nagoya Math. J. **40** (1970), 193–211.

[2] Axler, S., Bourdon, P., and Wade, R.: *Harmonic function theory.* Graduate Texts in Mathematics **137**, Springer-Verlag, New York, 2001.

[3] Elstrodt, J., Grunewald, F., and Mennicke, J.: *Groups acting on hyperbolic space. Harmonic analysis and number theory.* Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1998.

[4] Erdélyi, A., Magnus, W., Oberhettinger, F., and Tricomi, F. G.: *Tables of integral transforms. Vol. I.* McGraw-Hill, New York, 1954.

[5] Funke, J.: *CM points and weight 3/2 modular forms.* In: Analytic number theory. A tribute to Gauss and Dirichlet, 107–127, Clay Math. Proc. **7**, Amer. Math. Soc., Providence, RI, 2007.

[6] Gradshteyn, I. S. and Ryzhik, I. M.: *Table of integrals, series, and products.* Eighth edition. Elsevier/Academic Press, Amsterdam, 2015.

[7] Gross, B. H. and Zagier, D. B.: *On singular moduli.* J. Reine Angew. Math. **355** (1985), 191–220.

[8] Kudla, S. S.: *Integrals of Borcherds forms.* Compositio Math. **137** (2003), no. 3, 293–349.

[9] Kühn, U.: *Generalized arithmetic intersection numbers.* J. Reine Angew. Math. **534** (2001), 209–236.

[10] Matthes, R.: *Regularized theta lifts and Niebur-type Poincaré series on n-dimensional hyperbolic space.* J. Number Theory **133** (2013), no. 1, 20–47.

[11] Rohrlich, D. E.: *A modular version of Jensen's formula.* Math. Proc. Cambridge Philos. Soc. **95** (1984), no. 1, 15–20.

[12] Sarnak, P.: *The arithmetic and geometry of some hyperbolic three-manifolds.* Acta Math. **151** (1983), no. 3-4, 253–295.

[13] Serre, J.-P.: *A course in arithmetic.* Graduate Texts in Mathematics **7**, Springer-Verlag, New York, 1973.

[14] Siegel, C. L.: *Advanced analytic number theory.* Second edition. Tata Institute of Fundamental Research Studies in Mathematics **9**, Tata Institute of Fundamental Research, Bombay, 1980.

[15] Sorensen, C. M.: *Fourier expansion of Eisenstein series on the Hilbert modular group and Hilbert class fields.* Trans. Amer. Math. Soc. **354** (2002), no. 12, 4847–4869.

[16] Watson, G. N.: *A treatise on the theory of Bessel functions.* Cambridge University Press, Cambridge, 1995.

Sebastián Herrero, Department of Mathematical Sciences, Chalmers University of Technology and University of Gothenburg, SE-412 96 Gothenburg, Sweden.
E-mail address: sebastian.herrero.m@gmail.com

Özlem Imamoḡlu, Department of Mathematics, ETH Zürich, Rämistrasse 101, CH-8092 Zürich, Switzerland.
E-mail address: ozlem@math.ethz.ch

Anna-Maria von Pippich, Fachbereich Mathematik, Technische Universität Darmstadt, Schlossgartenstr. 7, D-64289 Darmstadt, Germany.
E-mail address: pippich@mathematik.tu-darmstadt.de

Árpád Tóth, Department of Analysis, Eötvös Loránd University and MTA Rényi Intézet Lendület Automorphic Research Group, South Building Room 3.207, Budapest, Hungary.
E-mail address: toth@cs.elte.hu