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Spatial-Homogeneity of Stable Solutions of Almost-Periodic Parabolic Equations with Concave Nonlinearity

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Abstract

We study the spatial-homogeneity of stable solutions of almost-periodic parabolic equations. It is shown that if the nonlinearity satisfies a concave or convex condition, then any linearly stable almost automorphic solution is spatially-homogeneous; and moreover, the frequency module of the solution is contained in that of the nonlinearity.

1 Introduction

We consider the semilinear parabolic equation with Neumann boundary condition

$$\begin{aligned} &u_t = \Delta u + f(t,u,\nabla u), \quad t>0, x\in\Omega \\ &\frac{\partial u}{\partial n}\,|_{\partial\Omega}= 0, \quad t>0 \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain and $f : \mathbb{R}\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}; (t, u, p) \mapsto f(t, u, p)$ together with its first and second derivatives are almost periodic in t uniformly for (u, p) in any compact

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subset of $\mathbb{R} \times \mathbb{R}^n$. Such equation is ubiquitous throughout the modeling of population dynamics and population ecology. The almost periodicity of the nonlinearity f captures the growth rate influenced by external effects which are roughly but not exactly periodic, or environmental forcing which exhibits different, non-commensurate periods.

In cases where f is independent of t (i.e., the autonomous case) or f is time-periodic with period $T > 0$ (i.e., the time T-periodic case), it has been known that stable equilibria or T-periodic solutions are not supposed to possess spatial variations on a convex domain. For instance, in terms of an autonomous equation on a convex domain Ω with f being independent of ∇u, Casten and Holland [2] and Matano [10] proved that any stable equilibrium is spatially-homogeneous (i.e., without any spatial structure). In other words, any spatially-inhomogeneous equilibrium on a convex domain must be unstable. Later, Hess [6] considered the time T-periodic equation and showed that all stable T-periodic solutions are spatially-homogeneous on a convex domain Ω.

When the system (1.1) is driven by a time almost periodic forcing, there usually exist almost automorphic solutions rather than almost periodic ones. As a matter of fact, the appearance of almost automorphic dynamics is a fundamental phenomenon in almost periodically forced parabolic equations [16–20]. We also refer to [7–9,12–14,23] on the study of almost automorphic dynamics in different types of almost-periodic differential systems. Among many others, Shen and Yi [20] showed that any stable almost automorphic solution of (1.1) is spatially-homogeneous on a convex domain Ω.

Besides the convexity of the domain, the convexity or concavity of the nonlinearity f in (1.1) (i.e., the function $f(t, \cdot, \cdot) : \mathbb{R}^{N+1} \to \mathbb{R}$ is convex or concave for all $t \in \mathbb{R}$) can be thought as an alternative condition which guarantees that any spatially-inhomogeneous equilibrium and time T-periodic solution are unstable in the autonomous case (Casten and Holland [2]) and the time T-periodic case (Hess [6]), respectively.

The present paper is mainly focusing on the almost periodically forced equation (1.1). We will show that, if $f(t, \cdot, \cdot) : \mathbb{R}^{N+1} \to \mathbb{R}$ is convex or concave for all $t \in \mathbb{R}$, then any linearly stable almost automorphic solution $u(t, x)$ (see Definition 2.2) of (1.1) is spatially-homogeneous; and moreover, the frequency module of $u(t, x)$ is contained in that of f (see Theorem 3.1).

Our result can be viewed as an effective supplement of the above-mentioned result in [20]; for the concavity or convexity of f, instead of for convex domains. It also generalizes to multi-frequency driven systems from that in the autonomous cases [2] and time-periodic cases [6].

The paper is organized as follows. In Section 2, we review the basic notations and concepts involving skew-product semiflows, linearly stable and almost periodic (automorphic) functions which will be useful in our discussions. In Section 3, we prove the spatial-homogeneity of linearly stable almost automorphic solutions to (1.1) under the assumption that the nonlinearity f is concave or convex.

2 Notations and Preliminary Results

2.1 Skew-product Semiflows and Linearly Stable Solutions

Let Y be a compact metric space with metric d_Y and $\mathbb{R}$ be the additive group of reals. A real *flow* $(Y, \mathbb{R})$ (or (Y, σ)) is a continuous mapping $\sigma : Y \times \mathbb{R} \to Y, (y, t) \mapsto y \cdot t$ satisfying: (i) $\sigma(y, 0) = y$; (ii) $\sigma(\sigma(y, s), t) = \sigma(y, s+t)$ for all $y \in Y$ and $s, t \in \mathbb{R}$. A subset $E \subset Y$ is *invariant* if $\sigma(y, t) \in E$ for each $y \in E$ and $t \in \mathbb{R}$, and is called *minimal* or *recurrent* if it is compact and the only non-empty compact invariant subset of it is itself. By Zorn's Lemma, every compact and σ-invariant set contains a minimal subset. Moreover, a subset E is minimal if and only if every trajectory is dense in E.

Let X, Y be metric spaces and (Y, σ) be a compact flow (called the base flow). Let also $\mathbb{R}^+ = \{t \in \mathbb{R} : t \geq 0\}$. A skew-product semiflow $\Pi^t : X \times Y \to X \times Y$ is a semiflow of the following form

$$\Pi^t(u, y) = (\varphi(t, u, y), y \cdot t), \quad t \geq 0,\, (u, y) \in X \times Y, \tag{2.1}$$

satisfying (i) $\Pi^0 = \mathrm{Id}_X$ and (ii) the co-cycle property: $\varphi(t + s, u, y) = \varphi(s, \varphi(t, u, y), y \cdot t)$ for each $(u, y) \in X \times Y$ and $s, t \in \mathbb{R}^+$. A subset $E \subset X \times Y$ is *positively invariant* if $\Pi^t(E) \subset E$ for all $t \in \mathbb{R}^+$. The *forward orbit* of any $(u, y) \in X \times Y$ is defined by $\mathcal{O}^+(u, y) = \{\Pi^t(u, y) : t \geq 0\}$, and the *$\omega$-limit set* of (u, y) is defined by $\omega(u, y) = \{(\widehat{u}, \widehat{y}) \in X \times Y : \Pi^{t_n}(u, y) \to (\widehat{u}, \widehat{y}) (n \to \infty)$ for some sequence $t_n \to \infty\}$.

A *flow extension* of a skew-product semiflow Π^t is a continuous skew-product flow $\widehat{\Pi}^t$ such that $\widehat{\Pi}^t(u, y) = \Pi^t(u, y)$ for each $(u, y) \in X \times Y$ and $t \in \mathbb{R}^+$. A compact positively invariant subset is said to admit *a flow extension* if the semiflow restricted to it does. Actually, a compact positively invariant set $K \subset X \times Y$ admits a flow extension if every point in K admits a unique backward orbit which remains inside the set K (see [20, part II]). A compact positively invariant set $K \subset X \times Y$ for Π^t is called *minimal* if it does not contain any other nonempty compact positively invariant set than itself.

Let X be a Banach space and the cocycle φ in (2.1) be C^1 for $u \in X$, that is, φ is C^1 in u, and the derivative φ_u is continuous in $u \in X$, $y \in Y$, $t > 0$; and moreover, for any $v \in X$,

$$\varphi_u(t, u, y)v \to v \quad \text{as} \quad t \to 0^+,$$

uniformly for (u, y) in compact subsets of $X \times Y$. Let $K \subset X \times Y$ be a compact, positively invariant set which admits a flow extension. Define $\Phi(t, u, y) = \varphi_u(t, u, y)$ for $(u, y) \in K$, $t \geq 0$. Then the operator Φ generates a linear skew-product semiflow Ψ on $(X \times K, \mathbb{R}^+)$ associated with (2.1) over K as follows:

$$\Psi(t, v, (u, y)) = (\Phi(t, u, y)v, \Pi^t(u, y)),\ t \geq 0,\ (u, y) \in K,\ v \in X. \tag{2.2}$$

For each $(u, y) \in K$, define the Lyapunov exponent $\lambda(u, y) = \limsup\limits_{t\to\infty} \frac{\ln ||\Phi(t,u,y)||}{t}$, where $|| \cdot ||$ is the operator norm of $\Phi(t, u, y)$. We call the number $\lambda_K = \sup_{(u,y)\in K} \lambda(u, y)$ the *upper Lyapunov exponent* on K.

Definition 2.1. K is said to be *linearly stable* if $\lambda_K \le 0$.

To carry out our study for the non-autonomous system (1.1), we embed it into a skew-product semiflow. Let $f_\tau(t,u,p) = f(t+\tau,u,p)(\tau \in \mathbb{R})$ be the time-translation of f, then the function f generates a family $\{f_\tau|\tau \in \mathbb{R}\}$ in the space of continuous functions $C(\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n,\mathbb{R})$ equipped with the compact open topology. Moreover, $H(f)$ (the closure of $\{f_\tau|\tau \in \mathbb{R}\}$ in the compact open topology) called the hull of f is a compact metric space and every $g \in H(f)$ has the same regularity as f. Hence, the time-translation $g\cdot t \equiv g_t\,(g \in H(f))$ naturally defines a compact minimal flow on $H(f)$ and equation (1.1) induces a family of equations associated to each $g \in H(f)$,

$$\begin{aligned} &u_t = \Delta u + g(t,u,\nabla u), \qquad t>0, \quad x\in\Omega,\\ &\frac{\partial u}{\partial n} = 0 \quad \text{on } \mathbb{R}^+ \times \partial\Omega. \end{aligned} \tag{2.3}$$

It follows from the standard theory of parabolic equations (see, e.g. [3]), for each $u_0 \in C^1(\overline{\Omega})$ satisfying $\frac{\partial u_0}{\partial n}$ on $\partial\Omega$, (2.3) admits a unique classical locally solution $\varphi(t,\cdot;u_0,g)$ with $\varphi(0,\cdot;u_0,g) = u_0$.

Hereafter, we always assume that X is a fractional power space (see [5]) associated with the operator $u \to -\Delta u$, $\mathcal{D} \to L^p(\Omega)$ such that $X \hookrightarrow C^1(\overline{\Omega})$ (X is compact embedded in $C^1(\overline{\Omega})$), where $\mathcal{D} = \{u|u \in W^{2,p}(\Omega)$ and $\frac{\partial u}{\partial n}|_{\partial\Omega} = 0\}$, $p>n$. For any $u\in X$ and $g \in H(f)$, (2.3) defines (locally) a unique solution $\varphi(t,\cdot;u,g)$ in X is C^2 in u and is continuous in g and t within its (time) interval of existence. In the language of dynamic systems, there is a well defined (local) skew-product semiflow $\Pi^t : X\times H(f) \to X\times H(f)$:

$$\Pi^t(u,g) = (\varphi(t,\cdot;u,g), g\cdot t), \quad t>0 \tag{2.4}$$

associated with (2.3). By the standard a *priori* estimates for parabolic equations (see [3, 5]), if $\varphi(t,\cdot;u,g)(u\in X)$ is bounded in X in the existence interval of the solution, then it is a globally defined classical solution. For any $\delta>0$, $\{\varphi(t,\cdot;u,g)\}$ is relatively compact, hence the ω-limit set $\omega(u,g)$ is a nonempty connected compact subset of $X\times H(f)$. Moreover, by [4, 5], Π^t restricted to $\omega(u,g)$ is a (global) semiflow which admits a flow extension.

Let $X^+ = \{u \in X | u(x) \ge 0, x \in \overline{\Omega}\}$. Denote by $\mathrm{Int}X^+$ the interior of X^+. Clearly, $\mathrm{Int}X^+ \neq \emptyset$, since $\{u\in X|u(x)>0$ for $x\in\Omega$, $\frac{\partial u}{\partial n}<0$ for $x \in \partial\Omega\} \subset \mathrm{Int}X^+$. Thus, X^+ defines a strong ordering on X as follows:

$$\begin{aligned} u_1 \le u_2 &\Longleftrightarrow u_2 - u_1 \in X^+,\\ u_1 < u_2 &\Longleftrightarrow u_2 - u_1 \in X^+,\ u_2 \neq u_1,\\ u_1 \ll u_2 &\Longleftrightarrow u_2 - u_1 \in \mathrm{Int}X^+. \end{aligned}$$

Immediately, we have the following lemma from [20, Lemma III. 5.1].

Lemma 2.1. *The skew-product semiflow Π^t in (2.4) is strongly monotone, in the sense that: for any $(u,g) \in X\times H(f)$, $v\in X$ with $v>0$, one has $\Phi(t,u,g)v \gg 0$ for $t>0$.*

Definition 2.2. A bounded solution $u(t,x) = \varphi(t,x;u_0,g)$ of (2.3)($u_0 \in X$) is *linearly stable* if it satisfies the following conditions:

(i) $\omega(u_0,g)$ is linearly stable.

(ii) Let $\Phi(t,s)$ $(t \geq s \geq 0)$ be the solution operator of the following linearized equation along $u(t,x)$:

$$\begin{aligned} &v_t = \Delta v + g_u(t,u,\nabla u)v + g_p(t,u,\nabla u)\nabla v \quad \text{in } \mathbb{R}^+ \times \Omega, \\ &\frac{\partial v}{\partial n} = 0 \quad \text{on } \mathbb{R}^+ \times \partial\Omega. \end{aligned} \tag{2.5}$$

Then $\sup_{t\geq 0} \|\Phi(t,0)v_0\| < \infty$ for all $v_0 \in X$.

2.2 Almost Periodic and Almost Automorphic Functions

In this subsection, we always assume D is a non-empty subset of $\mathbb{R}^m$.

Definition 2.3. *A continuous function $f : \mathbb{R}\times D \to \mathbb{R}$ is said to be admissible if for any compact subset $K \subset D$, f is bounded and uniformly continuous on $\mathbb{R} \times K$. f is C^r ($r \geq 1$) admissible if f is C^r in $w \in D$ and Lipschitz in t, and f as well as its partial derivatives to order r are admissible.*

Let $f \in C(\mathbb{R} \times D, \mathbb{R})(D \subset \mathbb{R}^m)$ be admissible. Then $H(f) = \text{cl}\{f \cdot \tau : \tau \in \mathbb{R}\}$ (called the *hull of f*) is compact and metrizable under the compact open topology (see [15, 20]), where $f \cdot \tau(t,\cdot) = f(t+\tau,\cdot)$. Moreover, the time translation $g \cdot t$ of $g \in H(f)$ induces a natural flow on $H(f)$ (cf. [15]).

Definition 2.4. A function $f \in C(\mathbb{R},\mathbb{R})$ is *almost automorphic* if for every $\{t'_k\} \subset \mathbb{R}$ there is a subsequence $\{t_k\}$ and a function $g : \mathbb{R} \to \mathbb{R}$ such that $f(t+t_k) \to g(t)$ and $g(t-t_k) \to f(t)$ pointwise. f is *almost periodic* if for any sequence $\{t'_n\}$ there is a subsequence $\{t_n\}$ such that $\{f(t+t_n)\}$ converges uniformly. A function $f \in C(\mathbb{R} \times D,\mathbb{R})(D \subset \mathbb{R}^m)$ is *uniformly almost periodic (automorphic) in t*, if f is both admissible and almost periodic (automorphic) in $t \in \mathbb{R}$.

Remark 2.1. If f is a uniformly almost automorphic function in t, then $H(f)$ is always *minimal*, and there is a residual set $Y' \subset H(f)$, such that all $g \in Y'$ is a uniformly almost automorphic function in t. If f is a uniformly almost periodic function in t, then $H(f)$ is always *minimal*, and every $g \in H(f)$ is uniformly almost periodic function (see, e.g. [20]).

Let $f \in C(\mathbb{R} \times D, \mathbb{R})$ be uniformly almost periodic (almost automorphic) and

$$f(t,w) \sim \sum_{\lambda\in\mathbb{R}} a_\lambda(w)e^{i\lambda t} \tag{2.6}$$

be a Fourier series of f (see [20,22] for the definition and the existence of a Fourier series). Then $\mathcal{S} = \{\alpha_\lambda(w) \not\equiv 0\}$ is called the Fourier spectrum of f associated with Fourier series (2.6) and $\mathcal{M}$ be the smallest additive subgroup of $\mathbb{R}$ containing $\mathcal{S}(f)$ is called the frequency module of f. Moreover, $\mathcal{M}(f)$ is a countable subset of $\mathbb{R}$ (see, e.g. [20]).

Lemma 2.2. *Assume $f \in C(\mathbb{R} \times D, \mathbb{R})$ is a uniformly almost automorphic function, then for any uniformly almost automorphic function $g \in H(f)$, $\mathcal{M}(g) = \mathcal{M}(f)$.*

Proof. See [20, Corollary I.3.7]. □

3 Spatial-homogeneity of Linearly Stable Solutions

In this section, we always assume that the function $(u, p) \mapsto f(t, u, p)$ in (1.1) is concave (resp. convex) for each $t \in \mathbb{R}$, that is, $f(t, \lambda u_1 + (1-\lambda)u_2, \lambda p_1 + (1-\lambda)p_2) \geq$ (resp. $\leq$) $\lambda f(t, u_1, p_1) + (1-\lambda)f(t, u_2, p_2)$ for any $\lambda \in [0, 1]$, $t \in \mathbb{R}$ and $(u_i, p_i) \in \mathbb{R} \times \mathbb{R}^n, i = 1, 2$. Clearly, $g(t, u, p)$ is also concave (resp. convex) for any $g \in H(f)$. We further assume that $f(t, \cdot, \cdot)$ is C^2 uniformly almost periodic. Our main result is the following theorem

Theorem 3.1. *Assume that $f : (t, \cdot, \cdot) \mapsto f(t, \cdot, \cdot)$ is concave (or convex). Let $\varphi(t, \cdot, u_0, g) \in C^{1+\frac{\mu}{2}, 2+\mu}(\mathbb{R} \times \overline{\Omega})$ ($\mu \in (0, 1]$) be a linearly stable almost automorphic (almost periodic) solution of (2.3), then $\varphi(t, \cdot; u_0, g)$ is spatially-homogeneous and is a solution of*

$$u' = g(t, u, 0). \tag{3.1}$$

Moreover, $\mathcal{M}(\varphi) \subset \mathcal{M}(f)$.

Hereafter, we only consider the case when f is concave, because by a transformation from u to $-u$, the convexity of nonlinearity g can be changed into concavity.

Let $\varphi(t, \cdot; u_0, g) \in C^{1+\frac{\mu}{2}, 2+\mu}(\mathbb{R} \times \overline{\Omega})$ be an almost automorphic solution of (2.3) with $u(0) = u_0$. Then, $\omega(u_0, g)$ is an almost automorphic minimal set; and hence, $\varphi(t, x; u_0, g)$ is well defined for all $t \in \mathbb{R}$. For brevity, we write $u(t, x) = \varphi(t, x; u_0, g)$ and define the following function $c : \mathbb{R} \to \mathbb{R}$ by

$$c(t) := \max_{x \in \overline{\Omega}} u(t, x), \ \ t \in \mathbb{R}.$$

Let $M(t) = \{x \in \overline{\Omega} : u(t, x) = c(t)\}$. Then, similar as the arguments in [6, p.327], $c(t)$ is a Lipchitz continuous function and hence differentiable for a.e. $t \in \mathbb{R}$; define $\widetilde{\mathbb{R}} = \{t \in \mathbb{R} | c(t) \text{ is differentiable}\}$, then $\mathbb{R} \setminus \widetilde{\mathbb{R}}$ is a set of zero measure and $c'(t)$ is continuous on $\widetilde{\mathbb{R}}$; and moreover, $c'(t) = u_t(t, x)$ for any $t \in \widetilde{\mathbb{R}}$ and $x \in M(t)$. Since $u \in C^{1+\frac{\mu}{2}, 2+\mu}(\mathbb{R} \times \overline{\Omega})$ is an almost automporhic solution of (2.3), $c'(t) \in L^\infty(\mathbb{R})$. Moreover, we have the following

Lemma 3.2. *$c(t)$ is an almost automorphic function.*

Proof. Note that $u(t, x)$ is a uniformly almost automorphic function on $\mathbb{R} \times \overline{\Omega}$. Then, for any sequence $t_n \to \infty$, there are $v(t, x) \in H(u)$ (the hull of u) and a subsequence $\{t_{n_k}\} \subset \{t_n\}$, such that $u(t + t_{n_k}, x) \to v(t, x)$ and $v(t - t_{n_k}, x) \to u(t, x)$, uniformly for $(t, x) \in I \times \overline{\Omega}$, where I is any compact set contained in $\mathbb{R}$. In other words, for any $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that

$$\begin{cases} u(t + t_{n_k}, x) - \epsilon < v(t, x) < u(t + t_{n_k}, x) + \epsilon \\ v(t - t_{n_k}, x) - \epsilon < u(t, x) < v(t - t_{n_k}, x) + \epsilon, \end{cases}$$

for any $k > N$ and $(t,x) \in I \times \overline{\Omega}$. Therefore,

$$\begin{cases} \max_{x\in\overline{\Omega}} u(t+t_{n_k},x) - \epsilon < \max_{x\in\overline{\Omega}} v(t,x) < \max_{x\in\overline{\Omega}} u(t+t_{n_k},x) + \epsilon \\ \max_{x\in\overline{\Omega}} v(t-t_{n_k},x) - \epsilon < \max_{x\in\overline{\Omega}} u(t,x) < \max_{x\in\overline{\Omega}} v(t-t_{n_k},x) + \epsilon, \end{cases}$$

that is,

$$|c(t+t_{n_k}) - \max_{\overline{\Omega}} v(t,x)| < \epsilon \quad \text{and} \quad |c(t) - \max_{\overline{\Omega}} v(t-t_{n_k},x)| < \epsilon,$$

for any $k > N$ and $t \in \mathbb{R}$. This implies that $c(t)$ is an almost automorphic function. $\square$

Now, we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. Let $w(t,x) = c(t) - u(t,x)$. Then, it is clear that $w(t,x)$ is a uniformly almost automorphic function and $w(t,x) \geq 0$ on $\mathbb{R} \times \overline{\Omega}$. Since $u(t,x)$ is a solution of (2.3), denote $-\Delta$ by A, we have

$$w_t + Aw = c'(t) - u_t + \Delta u = c'(t) - g(t,u,\nabla u) \tag{3.2}$$

for all $t \in \widetilde{\mathbb{R}}$. Since g is concave,

$$g(t,c,0) \leq g(t,u,\nabla u) + \frac{\partial g}{\partial u}(t,u,\nabla u)w + \sum_{i=1}^{n} \frac{\partial g}{\partial p_i}(t,u,\nabla u)w_{x_j}. \tag{3.3}$$

Let

$$A(t) = A - \sum_{i=1}^{n} \frac{\partial g}{\partial p_i}(t,u,\nabla u)\frac{\partial}{\partial x_i} - \frac{\partial g}{\partial u}(t,u,\nabla u).$$

Together with (3.2)-(3.3), one has

$$w_t + A(t)w \leq c'(t) - g(t,c,0) := q(t)$$

for all $t \in \widetilde{\mathbb{R}}$.

Since $c'(t) \in L^\infty(\mathbb{R})$, one has $q \in L^\infty(\mathbb{R})$. We now divide our proof into the following two cases: (i) $q(t) \leq 0$ for a.e. $t \in \mathbb{R}$; (ii) $q(t) > 0$ on a set of positive measure.

Case (i). $q(t) \leq 0$ for a.e. $t \in \mathbb{R}$. Let $h \in L^\infty(\mathbb{R}, L^p(\Omega))$ be defined from (3.2) by

$$w_t + A(t)w =: h(t) \tag{3.4}$$

and $\Phi(t,s)$ be the fundamental solution associated with (3.4)(see Definition 2.2). If $h \in C(\mathbb{R}, L^p(\Omega))$, one can use the method of variation of constant to obtain

$$w(t) = \Phi(t,0)w(0) + \int_0^t \Phi(t,\tau)h(\tau)d\tau \tag{3.5}$$

in $L^p(\Omega)$ (see, e.g. [21, Theorem 5.2.2]). For general $h \in L^\infty(\mathbb{R}, L^p(\Omega))$ in (3.4), similarly as the argument in [6, p.328-329], by using the strong continuity of $\Phi(t,s)$ in s and [21, p.125, (5.33)], the following equation:

$$\frac{\partial}{\partial s}(\Phi(t,s)w(s)) = \Phi(t,s)w_t(s) + \Phi(t,s)A(s)w(s) = \Phi(t,s)h(s) \tag{3.6}$$

can be established for any $t \in \widetilde{\mathbb{R}}$. Furthermore, $\Phi(t,s)w(s)$ is in fact a Lipschitz continuous function of s from $\mathbb{R}$ to $L^p(\Omega)$ (hence, $\Phi(t,s)w(s)$ is an absolutely continuous function of s in $L^p(\Omega)$). By using [1, Corollary A] and integrating s in (3.6) from 0 to t, one can obtain (3.5).

Since $h(\tau) \le 0$ for a.e $\tau \in \mathbb{R}$, by strong positivity of Φ, one has $\Phi(t,\tau)h(\tau) \le 0$ for a.e. $\tau \in [0,t]$ $(t > 0)$; and hence

$$\int_0^t \Phi(t,\tau)h(\tau)d\tau \le 0, \quad \forall t > 0.$$

Therefore,

$$w(t) \le \Phi(t,0)w(0). \tag{3.7}$$

Suppose that $u(t,x)$ is not spatially-homogeneous. Then, $w(0) > 0$ in $C(\overline{\Omega})$ (i.e. $w(0,x) \ge 0$ for all $x \in \overline{\Omega}$, and $w(0,\cdot) \ne 0$). Noticing that the skew-product semiflow Π^t on $X \times H(f)$ is strongly monotone (see Lemma 2.1), $\omega(u_0, g)$ admits a continuous separation (see [20, Theorem II.4.4] or [11, Sec 3.5]) as follows: There exists continuous invariant splitting $X = X_1(v,\widetilde{g}) \oplus X_2(v,\widetilde{g})$ $((v,\widetilde{g}) \in \omega(u_0,g))$ with $X_1(v,\widetilde{g}) = \mathrm{span}\{\phi(v,\widetilde{g})\}$, $\phi(v,\widetilde{g}) \in \mathrm{Int}X^+$ and $X_2(v,\widetilde{g}) \cap X^+ = \{0\}$ such that

$$\Phi(t,v,\widetilde{g})X_1(v,\widetilde{g}) = X_1(\Pi^t(v,\widetilde{g})), \quad \Phi(t,v,\widetilde{g})X_2(v,\widetilde{g}) \subset X_2(\Pi^t(v,\widetilde{g})). \tag{3.8}$$

Moreover, there are $K, \gamma > 0$ satisfying

$$\|\Phi(t,v,\widetilde{g})|_{X_2(v,\widetilde{g})}\| \le Ke^{-\gamma t}\|\Phi(t,v,\widetilde{g})|_{X_1(v,\widetilde{g})}\| \tag{3.9}$$

for any $t \ge 0$ and $(v,\widetilde{g}) \in \omega(u_0,g)$. Write $w(0) = av_1 + v_2$ with $v_1 \in X_1(u_0,g)$, $\|v_1\| = 1$ and $v_2 \in X_2(u_0,g)$. Since $u(t,x)$ is linearly stable, $\sup_{t\ge 0}\|\Phi(t,0)v_1\|$ is bounded by Definition 2.2.

Case (ia): $\|\Phi(t,0)v_1\|$ is bounded away from zero. In this case, there exist $M \ge m > 0$ such that $m \le \inf_{t\ge 0}\|\Phi(t,0)v_1\| \le \sup_{t\ge 0}\|\Phi(t,0)v_1\| \le M$. Let $\Gamma = \{\overline{v} | \Phi(t_n,0)v_1 \to \overline{v}$ in X for some $t_n \to \infty\}$. Since $\sup_{t\ge 0}\|\Phi(t,0)v_1\| \le M$, by the regularity of $\Phi(t,0)$, one has $\Gamma \ne \emptyset$. *We further claim that* $\Gamma \subset \mathrm{Int}X^+$ *and* Γ *is a closed subset of* X. In fact, for any $\overline{v} \in \Gamma$, one can find a sequence $\tau_n \to \infty$, such that $\Phi(\tau_n,0)v_1 \to \overline{v}$. By virtue of (3.8), $\Phi(\tau_n,0)v_1 \in X_1(\Pi^{\tau_n}(u_0,g))$. Without loss of generality, one may assume that $\Pi^{\tau_n}(u_0,g) \to (\overline{u},\overline{g}) \in \omega(u_0,g)$. This implies that $\overline{v} \in X_1(\overline{u},\overline{g}) \subset \mathrm{Int}X^+ \cup \{0\}$. Note also that $\|v\| \ge m > 0$. Then $\overline{v} \in \mathrm{Int}X^+$. Next, we prove that Γ is closed in X. It suffices to prove that: if the sequence $v_n \in \Gamma$ converges to some $v^* \in X$, then $v^* \in \Gamma$. Indeed, for any positive integer $k \in \mathbb{N}$, there is $n_k > 0$ such that $\|v_n - v^*\| < \frac{1}{2k}$ for any $n \ge n_k$, particularly, $\|v_{n_k} - v^*\| < \frac{1}{2k}$. Noticing that $v_{n_k} \in \Gamma$, there exists $t_{n_k} \in \mathbb{R}^+$ such that $\|\Phi(t_{n_k},0)v_1 - v_{n_k}\| < \frac{1}{2k}$; and hence, $\|\Phi(t_{n_k},0)v_1 - v^*\| < \frac{1}{k}$. Without loss of generality, one may assume $t_{n_k} \to \infty$ as $k \to \infty$, by letting $k \to \infty$, one has $\Phi(t_{n_k}, 0)v_1 \to v^*$ as $t_{n_k} \to \infty$, which means $v^* \in \Gamma$. Thus we have proved the claim.

Recall that $\omega(u_0, g)$ is an almost automorphic minimal set, there is a sequence $t_n \to \infty$ such that $\Pi^{t_n}(u_0, g) \to (u_0, g)$. By choosing a subsequence, still denoted by t_n, one has that $\Phi(t_n, 0)v_1 \to v^* \in X_1(u_0, g) \cap \mathrm{Int}X^+$; in other words, there is a positive constant a^* such that $v^* = a^* v_1$. Moreover, $\Phi(t, 0)a^* v_1 \in \Gamma$ for any fixed $t \in \mathbb{R}^+$. Therefore, $\Phi(t_n, 0)a^* v_1 \in \Gamma$. Observing that $\Phi(t, 0)$ is a linear operator and Γ is a closed set, $\Phi(t_n, 0)a^* v_1 = a^* \Phi(t_n, 0)v_1 \to (a^*)^2 v_1 \in \Gamma$. Similarly, by repeating this argument, we have $(a^*)^n v_1 \in \Gamma$ for any $n \in \mathbb{N}$. Furthermore, by virtue of the boundedness of Γ, $a^* \le 1$. If $0 < a^* < 1$, then it is not hard to see $0 \in \Gamma$, a contradiction to $\Gamma \subset \mathrm{Int}X^+$. Therefore, $a^* = 1$. Note that $\sup_{t \ge 0} \|\Phi(t, 0)v_1\| \le M$, by (3.9), $\|\Phi(t, 0)v_2\| \to 0$ as $t \to \infty$. By letting $t = t_n$ and $n \to \infty$ in (3.7), one has

$$w(0) \le a v_1.$$

Therefore, $v_2 \le 0$. Observing that $X_2(u_0, g) \cap X^+ = \{0\}$, $v_2 = 0$. Hence, $w(0) = a v_1$ with $a \ge 0$. If $a > 0$, then $w(0) = a v_1 \in \mathrm{Int}X^+$, a contradiction to that $w(0) \notin \mathrm{Int}X^+$. Thus, $a = 0$ and $u(t, x)$ is spatially-homogeneous.

Case (ib): $\inf_{t \ge 0} \|\Phi(t, 0)v_1\| = 0$. There is a sequence $\{t_n\} \subset \mathbb{R}^+$ such that $\|\Phi(t_n, 0)v_1\| < \frac{1}{n}$. When the sequence $\{t_n\}$ is bounded, there exist $t^* \in \mathbb{R}^+$ and a subsequence t_{n_k} such that $t_{n_k} \to t^*$ as $k \to \infty$. Due to $\Phi(t, 0)v_1$ is continuous with respect to t, $\Phi(t^*, 0)v_1 = 0$, which contradicts to the strong positivity of $\Phi(t, 0)$. Thus, $\{t_n\}$ is unbounded. For simplicity, we assume $t_n \to \infty$ as $n \to \infty$. Again by (3.7), we have

$$0 \le w(t_n) \le a\Phi(t_n, 0)v_1 + \Phi(t_n, 0)v_2. \tag{3.10}$$

For such t_n, by choosing a subsequence if necessary, one may assume that $\Pi^{t_n}(u_0, g) \to (u^*, g^*) \in \omega(u_0, g)$ and $c(t_n) \to c^*$. Let $t_n \to \infty$ in (3.10), one has $0 \le w^* \le 0$ where $w^* = c^* - u^*$. So, $w_0^* = 0$, that is, $u^*(x) \equiv c^*$ on $\overline{\Omega}$ is spatially-homogeneous. By the minimality of $\omega(u_0, g)$, every point in $\omega(u_0, g)$ is spatially-homogeneous, thus, $u_0(x) = c(0)$ on $\overline{\Omega}$, a contradiction.

Thus, we have proved that $u(t, x)$ is spatially-homogeneous when $q(t) \le 0$ a.e. in $\mathbb{R}$.

Case (ii). There is a positive measure subset E in $\mathbb{R}$ such that $q(t) > 0$ for all $t \in E$. In the following, we will show that this case cannot occur. Actually, this can be proved by the same arguments in [6, p.329-330]. For the sake of completeness, we give a detailed proof below.

Suppose that there exists such subset $E \subset \mathbb{R}$. Then one can find some $t_0 \in \widetilde{\mathbb{R}}$ such that $q(t_0) > 0$. Recall that $c'(t)$ is continuous on $\widetilde{\mathbb{R}}$, there are nontrivial interval $[t_1, t_2] \subset \mathbb{R}$ and $\epsilon_0 > 0$ satisfying $q(t) \ge \epsilon_0$ for a.e. $t \in [t_1, t_2]$. By the concavity of $g(t, \cdot, \cdot)$, we have

$$g(t, u, \nabla u) \le g(t, c, 0) - \frac{\partial g}{\partial u}(t, c, 0)(c - u) - \sum_{i=1}^{n} \frac{\partial g}{\partial p_i}(t, c, 0)(c - u)_{x_i}. \tag{3.11}$$

Let

$$\overline{A}(t) = A - \sum_{i=1}^{n} \frac{\partial g}{\partial p_i}(t, c, 0)\frac{\partial}{\partial x_i} - \frac{\partial g}{\partial u}(t, c, 0)$$

and

$$\overline{h}(t) = \frac{d}{dt}(c-u)(t) + \overline{A}(t)(c-u)(t).$$

Combing with (3.2) and (3.11), one can obtain $\overline{h}(t) \geq q(t) \geq \epsilon_0$ for a.e. t in $[t_1, t_2]$. On the other hand, similarly as in (3.5), we have

$$(c-u)(t_2) = \overline{\Phi}(t_2, t_1)(c-u)(t_1) + \int_{t_1}^{t_2} \overline{\Phi}(t_2, s)\overline{h}(s)ds,$$

where $\overline{\Phi}(\cdot, \cdot)$ is the fundamental solution of $u_t = \overline{A}(t)u$. Note that

$$\int_{t_1}^{t_2} \overline{\Phi}(t_2, s)\overline{h}(s)ds \geq \epsilon_0 \int_{t_1}^{t_2} \overline{\Phi}(t_2, s)\mathbf{1}ds \gg 0 \quad \text{in } C(\overline{\Omega}),$$

where $\mathbf{1}$ is the unit constant-function. Together with $\overline{\Phi}(t_2, t_1)(c-u)(t_1) \geq 0$, it follows that $(c-u)(t_2) \gg 0$ in $C(\overline{\Omega})$, a contradiction to the definition of c. So, Case (ii) cannot happen.

Therefore, we have proved that $u(t, x) \equiv \varphi(t)$ is a spatially-homogeneous solution of (2.3); and moreover, it is an almost automorphic solution of (3.1). Finally, it follows from Lemma 2.2 and [20, Theorem III.3.4(c)] that $\mathcal{M}(\varphi) \subset \mathcal{M}(g) = \mathcal{M}(f)$. Thus, we have completed the proof. □

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