A SUB-SUPERSOLUTION APPROACH FOR SOME CLASSES OF NONLOCAL PROBLEMS INVOLVING ORLICZ SPACES

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Abstract. In the present paper we study the existence of solutions for some nonlocal problems involving Orlicz-Sobolev spaces. The approach is based on sub-supersolutions.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with C^2 boundary $\partial\Omega$. In the present paper we focus on the problems of quasilinear elliptic nonlocal equations

$$\begin{cases} -\Delta_\Phi u &= f(u)|u|^{\alpha}_{L^\Psi} + g(u)|u|^{\gamma}_{L^\Lambda} \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega \end{cases} \qquad (P_1)$$

and

$$\begin{cases} -\Delta_{\Phi_1} u &= f_1(v)|v|^{\alpha_1}_{L^{\Psi_1}} + g_1(v)|v|^{\gamma_1}_{L^{\Lambda_1}} \text{ in } \Omega, \\ -\Delta_{\Phi_2} v &= f_2(u)|u|^{\alpha_2}_{L^{\Psi_2}} + g_2(u)|u|^{\gamma_2}_{L^{\Lambda_2}} \text{ in } \Omega, \\ u = v &= 0 \text{ on } \partial\Omega, \end{cases} \qquad (P_2)$$

where α_i, γ_i, $i = 0, 1, 2$, with $\alpha_0 := \alpha$ and $\gamma_0 := \gamma$, are positive constants, $|.|_{L^\Psi}$ (resp. $|.|_{L^\Lambda}$) denotes the norm in the Orlicz space $L^\Psi(\Omega)$ (resp. $L^\Lambda(\Omega)$) and the nonlinearities $f_i, g_i : [0, +\infty) \to [0, +\infty)$, $i = 0, 1, 2$, with $f_0 := f$ and $g_0 := g$, are continuous and nondecreasing functions. Here, Δ_{Φ_i} stands for the Φ_i-Laplacian operator, that is, $\Delta_{\Phi_i} w = \operatorname{div}(\phi_i(|\nabla w|)\nabla w)$, for $i = 0, 1, 2$, where $\Phi_i : \mathbb{R} \to \mathbb{R}$ are N-functions of the form

$$\Phi_i(t) := \int_0^{|t|} \phi_i(s) s ds, \qquad (1.1)$$

with $\phi_i : [0, +\infty) \to [0, +\infty)$ being C^1 functions satisfying

(ϕ_1) $$(t\phi_i(t))'; \quad \forall t > 0$$

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$$(\phi_2) \qquad \lim_{t\to 0^+} t\phi_i(t) = 0, \quad \lim_{t\to +\infty} t\phi_i(t) = +\infty$$

and that there exist $l_i, m_i \in (1, N)$, $i = 0, 1, 2$ such that

$$(\phi_3) \qquad l_i - 1 \le \frac{(\phi_i(t)t)'}{\Phi_i(t)} \le m_i - 1, \quad \forall t > 0,$$

where $\phi_0 := \phi$, $l_0 := l$ and $m_0 := m$. Note that the condition (ϕ_3) implies that

$$(\phi_3)' \qquad l_i \le \frac{\phi_i(t)t^2}{\Phi_i(t)} \le m_i, \quad \forall t > 0,$$

for $i = 0, 1, 2$. In addition, Ψ_i and Λ_i, for $i = 0, 1, 2$, with $\Psi_0 := \Psi$ and $\Lambda_0 = \Lambda$, are N-functions satisfying the Δ_2 condition.

According to hypotheses $(\phi_1)-(\phi_3)$, a wide class of operators can be incorporated in problems (P_1) and (P_2), for instance:

- $\phi(t) = p|t|^{p-2}$, $t > 0$, with $p > 1$. The operator Δ_Φ is the p-Laplacian operator.
- $\phi(t) = p|t|^{p-2} + q|t|^{q-2}$, $t > 0, 1 < p < q$. Here Δ_Φ is the (p, q)-Laplacian operator applied in quantum physics (see [7]).
- $\phi(t) = 2\gamma(1 + t^2)^{\gamma-1}$, $t > 0$ and $\gamma > 1$. Δ_Φ appears in nonlinear elasticity problems [23].
- $\phi(t) = \gamma\frac{(\sqrt{1+t^2}-1)^{\gamma-1}}{\sqrt{1+t^2}}$, $t > 0$ and $\gamma \ge 1$. The operator Δ_Φ arises in minimal surfaces theory for $\gamma = 1$ (see [19, page 128]) and nonlinear elasticity for $\gamma > 1$ (see [22]).
- $\phi(t) = \frac{pt^{p-2}(1+t)\ln(1+t)+t^{p-1}}{1+t}$, $t > 0$. The operator Δ_Φ appears in plasticity problems (see [23]).

However, the lack of properties such as homogeneity complicates handling the nonlinear Φ_i-Laplacian operator which, therefore, constitutes a serious obstacle in the study of the problems (P_1) and (P_2). Thereby, it requires relevant topics of nonlinear functional analysis, especially theory of Orlicz and Orlicz-Sobolev spaces (see, e.g. [1, 30] and their abundant reference). Another mathematical difficulty encountered comes out from the nonlocal caracter of (P_1) and (P_2). It is due to the presence of terms $|\cdot|_{L^{\Psi_i}}$ and $|\cdot|_{L^{\Lambda_i}}$ that make the equations in (P_1) and (P_2) no longer a pointwise identities. For more inquiries on nonlocal problems we refer to [13, 6] where systems of elliptic equations are examined. With regard to the scalar case, we quote the papers [3, 5, 11, 25, 27, 28, 36]. Such problems are important for applications in view of the significant number of physical phenomena formulated into nonlocal mathematical models. For instance, they appear in the study of the flow of a fluid through a homogeneous isotropic rigid porous medium, as well as in the study of population dynamics (see, e.g., [15, 33]).

Relevant contributions regarding nonlocal problems fit the setting of (P_1) and (P_2). In particular, Alves & Covei [4] applied the sub-supersolution method to show the existence results for problem involving Kirchhoff-type operator

$$\begin{cases} -a\left(\int_\Omega u\right)\Delta u &= h_1(x,u)f\left(\int_\Omega |u|^p\right) + h_2(x,u)g\left(\int_\Omega |u|^r\right) \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \end{cases}$$

where a, f, g, and h_i $(i = 1, 2)$ are given functions. The case of nonlocal problems driven by the p-Laplacian differential operator is investigated by [12]. Combining sub-supersolutions method with the classical theorem due to Rabinowitz [29], the authors proved the the existence of solutions for quasilinear problem of the form

$$\begin{cases} -\Delta_p u &= |u|_{L^q}^{\alpha(x)} \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \end{cases} \tag{P}$$

where α is a nonnegative function defined in $\overline{\Omega}$. Then, they extend the results to the nonlocal quasilinear elliptic system

$$\begin{cases} -\Delta_{p_1} u &= |v|_{L^{q_1}}^{\alpha_1(x)} \text{ in } \Omega, \\ -\Delta_{p_2} v &= |u|_{L^{q_2}}^{\alpha_2(x)} \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega. \end{cases} \tag{S}$$

The semilinear case, that is when $p_1 = p_2 = 2$ is investigated by Corrêa-Lopes [13] and Chen-Gao [6] for systems of the form

$$\begin{cases} -\Delta u^m &= f(x,u)|v|_{L^p}^{\alpha} \text{ in } \Omega, \\ -\Delta v^n &= g(x,v)|u|_{L^q}^{\beta} \text{ in } \Omega, \\ u &= v = 0 \text{ on } \partial\Omega \end{cases} \tag{1.2}$$

with $m, n \geq 1$ and $\alpha, \beta > 0$. The existence of solutions is obtained by means of topological methods, namely, Galerkin method, fixed point theory as well as sub-supersolutions techniques.

Motivated by the aforementioned papers, our goal is to establish the existence of (positive) solutions for problems (P_1) and (P_2) involving sublinear and concave-convex terms. The approach relies on the method of sub-supersolution. However, besides the nonlocal nature of the problems, this method cannot be easily implemented due to the presence of Φ_i-Laplacian operator in the principle part of the equations. At this point, to the best of our knowledge, it is for the first time when nonlocal problems involving Φ_i-Laplacian operator are studied. A significant feature of our result lies in the obtaining of the sub- and supersolution in the Orlicz-Sobolev spaces setting and, involving nonlocal terms. This is achieved by the choice of suitable explicit functions with an adjustment of adequate constants.

The rest of the paper is organized as follows: Section 2 is devoted to the needed properties in Orlicz and Orlicz-Sobolev spaces. Section 3 (resp. Section 4) contains existence results for problem (P_1) (resp. (P_2)) involving sublinear and concave-convex structures.

2. Preliminaries

In this section we recall some results on Orlicz-Sobolev spaces. We say that a continuous function $\Phi : \mathbb{R} \to [0, +\infty)$ is a N-function if:

(i) Φ is convex,
(ii) $\Phi(t) = 0 \Leftrightarrow t = 0$,
(iii) $\displaystyle\lim_{t\to 0} \frac{\Phi(t)}{t} = 0$ and $\displaystyle\lim_{t\to+\infty} \frac{\Phi(t)}{t} = +\infty$,
(iv) Φ is even.

We say that a N-function Φ verifies the Δ_2-condition, and we denote by $\Phi \in \Delta_2$, if

$$\Phi(2t) \leq K\Phi(t), \quad \forall t \geq t_0,$$

for some constants $K, t_0 > 0$. Regarding the condition Δ_2 it is important to note that such property is satisfied under the condition $(\phi_3)'$ in the case of the $N-$function is given by (1.1).

We fix an open set $\Omega \subset \mathbb{R}^N$ and a N-function Φ. We define the Orlicz space associated with Φ as follows

$$L^{\Phi}(\Omega) = \left\{ u \in L^1(\Omega) \colon \int_\Omega \Phi\Big(\frac{|u|}{\lambda}\Big) dx < +\infty \text{ for some } \lambda > 0 \right\}.$$

The space $L^{\Phi}(\Omega)$ is a Banach space endowed with the Luxemburg norm given by

$$|u|_{L^\Phi} = \inf\left\{ \lambda > 0 : \int_\Omega \Phi\Big(\frac{|u|}{\lambda}\Big) dx \le 1 \right\}.$$

Lemma 1. [24] *Consider Φ a N-function of the form* (1.1) *and satisfying* $(\phi_1), (\phi_2)$ *and* (ϕ_3). *Set*

$$\zeta_0(t) = \min\{t^\ell, t^m\} \text{ and } \zeta_1(t) = \max\{t^\ell, t^m\},\ t \ge 0.$$

Then Φ satisfies

$$\zeta_0(t)\Phi(\rho) \le \Phi(\rho t) \le \zeta_1(t)\Phi(\rho),\ \rho, t > 0,$$
$$\zeta_0(|u|_\Phi) \le \int_\Omega \Phi(u) dx \le \zeta_1(|u|_\Phi),\ u \in L_\Phi(\Omega).$$

For a $N-$function Φ, the corresponding Orlicz-Sobolev space is defined as the Banach space

$$W^{1,\Phi}(\Omega) = \left\{ u \in L^{\Phi}(\Omega) \ : \ \frac{\partial u}{\partial x_i} \in L^{\Phi}(\Omega), \quad i = 1, ..., N \right\},$$

endowed with the norm

$$\|u\|_{1,\Phi} = |\nabla u|_{L^\Phi} + |u|_{L^\Phi}.$$

The Δ_2-condition also implies that

$$u_n \to u \text{ in } L_\Phi(\Omega) \Longleftrightarrow \int_\Omega \Phi(|u_n - u|) \to 0$$

and

$$u_n \to u \text{ in } W^{1,\Phi}(\Omega) \Longleftrightarrow \int_\Omega \Phi(|u_n - u|) \to 0 \text{ and } \int_\Omega \Phi(|\nabla u_n - \nabla u|) \to 0.$$

Consider $u, v \in W^{1,\Phi}(\Omega)$ we will say that $-\Delta_\Phi u \le -\Delta_\Phi v$ in Ω if

$$\int_\Omega \phi(|\nabla u|)\nabla u \nabla \varphi \le \int_\Omega \phi(|\nabla v|)\nabla v \nabla \varphi,$$

for all $\varphi \in W_0^{1,\Phi}(\Omega)$ with $\varphi \ge 0$.

The following results will be often used.

Lemma 2. [34, Lemma 4.1] *Let $u, v \in W^{1,\Phi}(\Omega)$ with $-\Delta_\Phi u \le -\Delta_\Phi v$ in Ω and $u \le v$ in $\partial\Omega$ (i.e $(u - v)^+ \in W_0^{1,\Phi}(\Omega)$), then $u(x) \le v(x)$ a.e in Ω.*

Lemma 3. [34, Lemma 4.5] *Let $\lambda > 0$ and consider z_λ the unique solution of the problem*

$$\begin{cases} -\Delta_\Phi z_\lambda &= \lambda \text{ in } \Omega, \\ z_\lambda &= 0 \text{ on } \partial\Omega, \end{cases} \tag{2.1}$$

where Φ is given by (1.1) and $\Omega \subset \mathbb{R}^N$ is an admissible domain. Define $\rho_0 = \frac{1}{2|\Omega|^{\frac{1}{N}} C_0}$. If $\lambda \geq \rho_0$, then $|z_\lambda|_{L^\infty} \leq C^* \lambda^{\frac{1}{l-1}}$ and $|z_\lambda|_{L^\infty} \leq C_* \lambda^{\frac{1}{m-1}}$ if $\lambda < \rho_0$. Here C^* and C_* are positive constants dependending only on l, m, N and Ω.

Regarding to the function z_λ of the previous result, it follows from [26, page 320] and [34, Lemma 4.2] that $z_\lambda \in C^1(\overline{\Omega})$ with $z_\lambda > 0$ in Ω.

3. THE SCALAR CASE

We say that $u \in W_0^{1,\Phi}(\Omega) \cap L^\infty(\Omega)$ is a (weak) solution of (P_1) if

$$\int_\Omega \phi(|\nabla u|)\nabla u \nabla \varphi = \int_\Omega (f(u)|u|_\Psi^\alpha + g(u)|u|_\Lambda^\gamma)\varphi,$$

for all $\varphi \in W_0^{1,\Phi}(\Omega)$.

Given $u, v \in \mathcal{S}(\Omega) := \{u : \Omega \to \mathbb{R} : u \text{ is measurable}\}$, we write $u \leq v$ if $u(x) \leq v(x)$ a.e in Ω. We denote by $[u, v]$ the set

$$[u, v] := \{w \in \mathcal{S}(\Omega) : u(x) \leq w(x) \leq v(x) \text{ a.e in } \Omega\}.$$

We say that $(\underline{u}, \overline{u})$ is a sub-super solution pair for (P) if $\underline{u}, \overline{u} \in W_0^{1,\Phi}(\Omega) \cap L^\infty(\Omega)$ are nonnegative functions that satisfy the inequality $0 < \underline{u} \leq \overline{u}$ in Ω and if for all $\varphi \in W_0^{1,\Phi}(\Omega)$ with $\varphi \geq 0$ the following inequalities hold

$$\int_\Omega \phi(|\nabla \underline{u}|)\nabla \underline{u} \nabla \varphi \leq \int_\Omega (f(\underline{u})|\underline{u}|_{L^\Psi}^\alpha + g(\underline{u})|\underline{u}|_{L^\Lambda}^\gamma)\varphi$$

and

$$\int_\Omega \phi(|\nabla \overline{u}|)\nabla \overline{u} \nabla \varphi \geq \int_\Omega (f(\overline{u})|\overline{u}|_{L^\Psi}^\alpha + g(\overline{u})|\overline{u}|_{L^\Lambda}^\gamma)\varphi.$$

The following result will play an important role in our arguments.

Lemma 4. *Suppose that $f, g : [0, +\infty) \to \mathbb{R}$ are nondecreasing, continuous and nonnegative functions. Consider also that $\alpha, \gamma \geq 0$ and that there exists a sub-supersolution pair $(\underline{u}, \overline{u})$ for problem (P_1). Then there exists a nontrivial solution u for (P_1) with $u \in [\underline{u}, \overline{u}]$.*

Proof. We have that

$$\begin{cases} -\Delta_\Phi \underline{u} & \leq & f(\underline{u})|\underline{u}|_{L^\Psi}^\alpha + g(\underline{u})|\underline{u}|_{L^\Lambda}^\gamma \text{ in } \Omega, \\ \underline{u} & = & 0 \text{ on } \partial\Omega. \end{cases}$$

and

$$\begin{cases} -\Delta_\Phi \overline{u} & \geq & f(\overline{u})|\overline{u}|_{L^\Psi}^\alpha + g(\overline{u})|\overline{u}|_{L^\Lambda}^\gamma \text{ in } \Omega, \\ \overline{u} & = & 0 \text{ on } \partial\Omega. \end{cases}$$

Denote by u_1 the unique solution of the problem

$$\begin{cases} -\Delta_\Phi u_1 & = & f(\underline{u})|\underline{u}|_{L^\Psi}^\alpha + g(\underline{u})|\underline{u}|_{L^\Lambda}^\gamma \text{ in } \Omega, \\ u_1 & = & 0 \text{ on } \partial\Omega. \end{cases}$$

Note that the mentioned solution exist because the term $f(\underline{u})|\underline{u}|_{L^\Psi}^\alpha$ is bounded. Since $\underline{u} \leq \overline{u}$ in Ω, f is nondecreasing and $\alpha, \gamma \geq 0$, we have that $f(\underline{u})|\underline{u}|_{L^\Psi}^\alpha \leq f(\overline{u})|\overline{u}|_{L^\Psi}^\alpha$ and $g(\underline{u})|\underline{u}|_{L^\Psi}^\gamma \leq g(\overline{u})|\overline{u}|_{L^\Lambda}^\gamma$, then it follows from Lemma 2 that $\underline{u} \leq u_1 \leq \overline{u}$ in Ω.

Let u_2 be the solution of the problem

$$\begin{cases} -\Delta_\Phi u_2 &= f(u_1)|u_1|^\alpha_{L^\Psi} + g(u_1)|u_1|^\gamma_{L^\Lambda} \text{ in } \Omega, \\ u_2 &= 0 \text{ on } \partial\Omega. \end{cases}$$

Since $\underline{u} \le u_1 \le \overline{u}$ in Ω, we have that

$$f(\underline{u})|\underline{u}|^\alpha_{L^\Psi} + g(\underline{u})|\underline{u}|^\alpha_{L^\Lambda} \le f(u_1)|u_1|^\alpha_{L^\Psi} + g(u_1)|u_1|^\gamma_{L^\Lambda} \le f(\overline{u})|\overline{u}|^\alpha_{L^\Psi} + g(\overline{u})|\overline{u}|^\gamma_{L^\Lambda} \text{ in } \Omega.$$

Thus from Lemma 2 we get,

$$\underline{u} \le u_1 \le u_2 \le \overline{u} \text{ in } \Omega.$$

Note also that $-\Delta_\Phi u_i \le f(\overline{u})|\overline{u}|^\alpha_{L^\Psi} + g(\overline{u})|\overline{u}|^\gamma_{L^\Lambda}, i = 1, 2$. Thus we can construct a sequence u_n such that

$$\begin{cases} -\Delta_\Phi u_n &= f(u_{n-1})|u_{n-1}|^\alpha_{L^\Psi} + g(u_{n-1})|u_{n-1}|^\gamma_{L^\Lambda} \text{ in } \Omega, \\ u_n &= 0 \text{ on } \partial\Omega. \end{cases} \tag{P_n}$$

with $-\Delta_\Phi u_n \le f(\overline{u})|\overline{u}|^\alpha_{L^\Psi} + g(\overline{u})|\overline{u}|^\gamma_{L^\Lambda}$ in Ω and $\underline{u} \le u_n \le \overline{u}$ in Ω for all $n \in \mathbb{N}$. Using the $C^{1,\alpha}$ estimates up to the boundary (see [26]), we have that u_n is a bounded sequence in $C^{1,\theta}(\overline{\Omega})$ for some $\theta \in (0, 1]$. Since the embedding $C^{1,\theta}(\overline{\Omega}) \hookrightarrow C^1(\overline{\Omega})$ is compact, we can extract a subsequence with $u_n \to u$ in $C^1(\overline{\Omega})$ for some $u \in C^1(\overline{\Omega})$. Passing to the limit in (P_n), we have that u is a nontrivial solution for problem (P_1). □

3.1. **A sublinear scalar problem.** In this section we use Lemma 4 and a suitable sub-supersolution pair to prove the existence of solution for a nonlocal problem of the type

$$\begin{cases} -\Delta_\Phi u &= u^\beta |u|^\alpha_{L^\Psi} \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \end{cases} \tag{P_S}$$

where $\alpha, \beta \ge 0$ are constants saisfying certain conditions. The above problem is considered in [12] for the $p-$Laplacian case and with $\beta = 0$. We complete the study done in [12, Theorem 4.1] by considering constants exponents and a more general operator.

Theorem 1. *Suppose that $\alpha, \beta \ge 0$ with $0 < \alpha + \beta < l - 1$, where l is given in (ϕ_3). Then (P_S) has a positive solution.*

Proof. We will start by constructing $\overline{u}$. Let $\lambda > 0$ and consider $z_\lambda \in W_0^{1,\Psi}(\Omega) \cap L^\infty(\Omega)$ the unique solution of (2.1) where λ will be chosen later.

For $\lambda > 0$ large by Lemma 3 there is a constant $K > 1$ that does not depend on λ such that

$$0 < z_\lambda(x) \le K\lambda^{\frac{1}{l-1}} \text{ in } \Omega. \tag{3.1}$$

Since $0 < \alpha + \beta < l - 1$ we can choose $\lambda > 1$ such that (3.1) occurs and

$$K^\beta \lambda^{\frac{\alpha+\beta}{l-1}} |K|^\alpha_{L^\Psi} \le \lambda. \tag{3.2}$$

By (3.1) and (3.2) we get

$$z_\lambda^\beta |z_\lambda|^\alpha_{L^\Psi} \le \lambda.$$

Therefore

$$\begin{cases} -\Delta_\Phi z_\lambda &\ge z_\lambda^\beta |z_\lambda|^\alpha_\Psi \text{ in } \Omega, \\ z_\lambda &= 0 \text{ on } \partial\Omega. \end{cases}$$

Now we will construct $\underline{u}$. Since $\partial\Omega$ is C^2 there is a constant $\delta > 0$ such that $d \in C^2(\overline{\Omega_{3\delta}})$ and $|\nabla d(x)| \equiv 1$ where $d(x) := dist(x, \partial\Omega)$ and $\overline{\Omega_{3\delta}} := \{x \in \overline{\Omega}; d(x) \leq 3\delta\}$(see [20, Lemma 14.16] and its proof). Let $\sigma \in (0, \delta)$. A direct computation implies that the function $\phi = \phi(k, \sigma)$ defined by

$$\eta(x) = \begin{cases} e^{kd(x)} - 1 & \text{if} \quad d(x) < \sigma, \\ e^{k\sigma} - 1 + \int_\sigma^{d(x)} ke^{k\sigma}\left(\frac{2\delta - t}{2\delta - \sigma}\right)^{\frac{m}{l-1}} dt & \text{if} \quad \sigma \leq d(x) < 2\delta, \\ e^{k\sigma} - 1 + \int_\sigma^{2\delta} ke^{k\sigma}\left(\frac{2\delta - t}{2\delta - \sigma}\right)^{\frac{m}{l-1}} dt & \text{if} \quad 2\delta \leq d(x) \end{cases}$$

belongs to $C_0^1(\overline{\Omega})$ where $k > 0$ is an arbitrary number. Direct computations implies that

$$-\Delta_\Phi(\mu\eta) = \begin{cases} -\mu k^2 e^{kd(x)} \frac{d}{dt}\left(\phi(t)t\right)\Big|_{t=\mu ke^{kd(x)}} - \phi(\mu ke^{kd(x)})\mu ke^{kd(x)}\Delta d \quad \text{if} \quad d(x) < \sigma, \\ \mu ke^{k\sigma}\left(\frac{m}{l-1}\right)\left(\frac{2\delta - d(x)}{2\delta - \sigma}\right)^{\frac{m}{l-1}-1}\left(\frac{1}{2\delta - \sigma}\right)\frac{d}{dt}\left(\phi(t)t\right)\Big|_{t=\mu ke^{k\sigma}\left(\frac{2\delta - d(x)}{2\delta - \sigma}\right)} \\ -\phi\left(\mu ke^{k\sigma}\left(\frac{2\delta - d(x)}{2\delta - \sigma}\right)^{\frac{m}{l-1}}\right)\mu ke^{k\sigma}\left(\frac{2\delta - d(x)}{2\delta - \sigma}\right)^{\frac{m}{l-1}}\Delta d \quad \text{if} \quad \sigma < d(x) < 2\delta, \\ 0 \quad \text{if} \quad 2\delta < d(x) \end{cases}$$

for all $\mu > 0$.

If k is large and $d(x) < \sigma$, we have that $-\Delta_\Phi(\mu\phi) \leq 0$. In fact, note that by (ϕ_3) we have for k large that

$$\begin{aligned} -\Delta_\Phi(\mu\eta) &= -\mu k^2 e^{kd(x)} \frac{d}{dt}\left(\phi(t)t\right)\Big|_{t=\mu ke^{kd(x)}} - \phi(\mu ke^{kd(x)})\mu ke^{kd(x)}\Delta d \\ &\leq -k^2\mu e^{kd(x)}(l-1)\phi(\mu ke^{kd(x)}) - \phi(\mu ke^{kd(x)})\mu ke^{kd(x)}\Delta d \\ &= \mu ke^{kd(x)}\phi(\mu ke^{kd(x)})(-k(l-1) - \Delta d) \\ &\leq 0, \end{aligned} \tag{3.3}$$

because Δd is bounded near the boundary and $l > 1$.

Now we will estimate $-\Delta_\Phi(\mu\eta)$ in the case $\sigma < d(x) < 2\delta$. Note that from (ϕ_3) and Lemma 1 we get

$$\begin{aligned} &\mu ke^{k\sigma}\left(\frac{m}{l-1}\right)\left(\frac{2\delta - d(x)}{2\delta - \sigma}\right)^{\frac{m}{l-1}-1}\left(\frac{1}{2\delta - \sigma}\right)\frac{d}{dt}\left(\phi(t)t\right)\Big|_{t=\mu ke^{k\sigma}\left(\frac{2\delta - d(x)}{2\delta - \sigma}\right)} \\ &\leq \mu ke^{k\sigma}\left(\frac{m}{l-1}\right)\left(\frac{2\delta - d(x)}{2\delta - \sigma}\right)^{\frac{m}{l-1}-1}\left(\frac{m-1}{2\delta - \sigma}\right)\phi\left(\mu ke^{k\sigma}\left(\frac{2\delta - d(x)}{2\delta - \sigma}\right)^{\frac{m}{l-1}}\right) \\ &\leq \left(\frac{m-1}{2\delta - \sigma}\right)\left(\frac{m}{l-1}\right)\frac{\Phi\left(\mu ke^{k\sigma}\left(\frac{2\delta - d(x)}{2\delta - \sigma}\right)^{\frac{m}{l-1}}\right)}{\mu ke^{k\sigma}\left(\frac{2\delta - d(x)}{2\delta - \sigma}\right)^{\frac{m}{l-1}}}\frac{1}{\left(\frac{2\delta - d(x)}{2\delta - \sigma}\right)} \\ &\leq \max\left\{(\mu ke^{k\sigma})^{m-1}\left(\frac{2\delta - d(x)}{2\delta - \sigma}\right)^{m\left(\frac{m}{l-1}\right) - \left(\frac{m}{l-1}+1\right)}, (\mu ke^{k\sigma})^{l-1}\left(\frac{2\delta - d(x)}{2\delta - \sigma}\right)^{l\left(\frac{m}{l-1}\right) - \left(\frac{m}{l-1}+1\right)}\right\} \\ &\times\left(\frac{m-1}{2\delta - \sigma}\right)\left(\frac{m}{l-1}\right). \end{aligned} \tag{3.4}$$

Since $m, l > 1$, we get $l\left(\frac{m}{l-1}\right) - m\left(\frac{m}{l-1}+1\right), m\left(\frac{m}{l-1}\right) - m\left(\frac{m}{l-1}+1\right) > 0$. Note that $0 \le \left(\frac{2\delta-d(x)}{2\delta-\sigma}\right) \le 1$. Thus by (3.4) we get

$$\begin{aligned}
&\mu k e^{k\sigma}\left(\frac{m}{l-1}\right)\left(\frac{2\delta-d(x)}{2\delta-\sigma}\right)^{\frac{m}{l-1}-1}\left(\frac{1}{2\delta-\sigma}\right)\frac{d}{dt}\left(\phi(t)t\right)\Big|_{t=\mu k e^{k\sigma}\left(\frac{2\delta-d(x)}{2\delta-\sigma}\right)} \\
\le&\left(\frac{m-1}{2\delta-\sigma}\right)\left(\frac{m}{l-1}\right)\max\{(\mu k e^{k\sigma})^{m-1},(\mu k e^{k\sigma})^{l-1}\} \\
=&C_1\left(\frac{1}{2\delta-\sigma}\right)\max\{(\mu k e^{k\sigma})^{m-1},(\mu k e^{k\sigma})^{l-1}\},
\end{aligned} \tag{3.5}$$

where C_1 is a constant that does not depend on μ and k. On other hand, we have by Lemma 1 that

$$\begin{aligned}
&\left|\phi\left(\mu k e^{k\sigma}\left(\frac{2\delta-d(x)}{2\delta-\sigma}\right)^{\frac{m}{l-1}}\right)\mu k e^{k\sigma}\left(\frac{2\delta-d(x)}{2\delta-\sigma}\right)^{\frac{m}{l-1}}\Delta d\right| \\
\le&\phi\left(\mu k e^{k\sigma}\left(\frac{2\delta-d(x)}{2\delta-\sigma}\right)^{\frac{m}{l-1}}\right)\mu k e^{k\sigma}\left(\frac{2\delta-d(x)}{2\delta-\sigma}\right)^{\frac{m}{l-1}}\sup_{\Omega_{3\delta}}|\Delta d| \\
\le&C\frac{\Phi\left(\mu k e^{k\sigma}\left(\frac{2\delta-d(x)}{2\delta-\sigma}\right)^{\frac{m}{l-1}}\right)}{\mu k e^{k\sigma}\left(\frac{2\delta-d(x)}{2\delta-\sigma}\right)^{\frac{m}{l-1}}} \\
\le&C\max\left\{(\mu k e^{k\sigma})^{m-1}\left(\frac{2\delta-d(x)}{2\delta-\sigma}\right)^{m\left(\frac{m}{l-1}\right)-\left(\frac{m}{l-1}+1\right)},(\mu k e^{k\sigma})^{l-1}\left(\frac{2\delta-d(x)}{2\delta-\sigma}\right)^{l\left(\frac{m}{l-1}\right)-\left(\frac{m}{l-1}+1\right)}\right\} \\
\le&C_2\max\{(\mu k e^{k\sigma})^{m-1},(\mu k e^{k\sigma})^{l-1}\},
\end{aligned} \tag{3.6}$$

where C_2 is a constant that does not depend on σ, k and μ. Thus from (3.5) and (3.6) we have that

$$-\Delta_\Phi u \le \max\left\{\frac{C_1}{2\delta-\sigma}, C_2\right\}\max\{(\mu k e^{k\sigma})^{m-1},(\mu k e^{k\sigma})^{l-1}\}, \tag{3.7}$$

if $\sigma < d(x) < 2\delta$.

Consider the function η and the numbers μ, σ and $k > 0$ described before. Let $\sigma = \frac{\ln 2}{k}$ and $\mu = e^{-k}$. Then $e^{k\sigma} = 2$.

If $k > 0$ is large, we have from (3.3) that

$$-\Delta_\Phi(\mu\eta) \le 0 \le (\mu\eta)^\beta|\mu\eta|^\alpha_{L^\Psi} \tag{3.8}$$

in the case $d(x) < \sigma$.

For any $k > 0$ we have $\eta(x) \ge e^{k\sigma} - 1 = 2 - 1 = 1$ in Ω. Thus there is a constant $C_3 > 0$ that does not depend on $k > 0$ such that

$$(\mu\eta)^\beta|\mu\eta|^\alpha_{L^\Psi} \ge \mu^{\alpha+\beta}C_3$$

Since $0 < \alpha + \beta < l - 1$, the L'Hospital's rule implies that

$$\lim_{k\to+\infty}\frac{k^{l-1}}{e^{k(l-1-(\alpha+\beta))}} = 0.$$

Thus, it is possible to consider a large $k_0 > 0$ such that

$$C_3 \geq \max\left\{C_1 \frac{1}{2\delta - \frac{\ln 2}{k}}, C_2\right\} \max\{2^{m-1}, 2^{l-1}\} \frac{k^{l-1}}{e^{k(l-1-(\alpha+\beta))}},$$

for all $k \geq k_0$. From (3.7), we have that

$$-\Delta_\Phi(\mu\eta) \leq (\mu\eta)^\beta |\mu\eta|^\alpha_{L^\Psi} \tag{3.9}$$

in the region $\sigma < d(x) < 2\delta$ for $k > 0$ is large enough.

If $d(x) > 2\delta$ we have

$$-\Delta_\Phi(\mu\eta) = 0 \leq (\mu\eta)^\beta |\mu\eta|^\alpha_{L^\Psi}. \tag{3.10}$$

Thus from (3.8), (3.9) and (3.10) we have that $\mu\eta$ is a subsolution for (Ps). Note that from (3.7), (3.8) and (3.10) we have for $k, \lambda > 0$ large enough that $-\Delta_\Phi(\mu\eta) \leq -\Delta_\Phi z_\lambda$. Thus from Lemma 3 we have $\mu\eta \leq z_\lambda$ in Ω. From Lemma 4 we have the result. □

Remark 1. *An interesting question for problem (P_S) is the existence of solution in the case $l - 1 < \alpha + \beta$.*

3.2. **A concave-convex scalar problem:** In this section we will consider a concave-convex problem of the type

$$\begin{cases} -\Delta_\Phi u = \lambda u^\beta |u|^\alpha_{L^\Psi} + \theta u^\xi |u|^\gamma_{L^\Lambda} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag*{$(P)_{\lambda,\theta}$}$$

where $\alpha, \beta, \xi, \gamma \geq 0$ are constants satisfying certain conditions and $\lambda, \theta > 0$ are positive numbers. The local version of $(P)_{\lambda,\theta}$ for the Laplacian operator was considered in the famous paper by Ambrosetti-Brezis-Cerami [2] in which a sub-supersolution argument is used. Our result is the following one.

Theorem 2. *Suppose that $\alpha, \beta, \xi, \gamma \geq 0$ and consider also that $0 < \alpha + \beta < l - 1$. The following assertions hold.*

(i) If $m - 1 < \xi + \gamma$, then given $\theta > 0$ there exists $\lambda_0 > 0$ such that for each $\lambda \in (0, \lambda_0)$ the problem $(P)_{\lambda,\theta}$ has a positive solution $u_{\lambda,\theta}$.

(ii) If $l-1 < \xi+\gamma$, then given $\lambda > 0$ there exists $\theta_0 > 0$ such that for each $\theta \in (0, \theta_0)$ the problem $(P)_{\lambda,\theta}$ has a positive solution $u_{\lambda,\theta}$.

Proof. Suppose that (i) occurs and fix $\theta > 0$. Let $z_\lambda \in W_0^{1,\Phi}(\Omega) \cap L^\infty(\Omega)$ be the unique solution of (2.1) where $\lambda \in (0, 1)$ will be chosen before.

Lemma 3 implies that for $\lambda > 0$ small enough there exists a constant $K > 1$ that does not depend on λ such that

$$0 < z_\lambda(x) \leq K\lambda^{\frac{1}{m-1}} \text{ in } \Omega. \tag{3.11}$$

Let $\overline{K} := \max\left\{K^\beta |K|^\alpha_{L^\Psi}, K^\xi |K|^\gamma_{L^\Lambda}\right\}$. For each $\theta > 0$ we can choose $0 < \lambda_0 < 1$ small enough, depending on θ, such that the inequalities

$$\lambda \geq \left(\lambda^{\frac{\alpha+\beta+m-1}{m-1}} \overline{K} + \theta \overline{K} \lambda^{\frac{\xi+\gamma}{m-1}}\right), \text{ for all } \lambda \in (0, \lambda_0)$$

and (3.11) hold because $\alpha+\beta>0$ and $m-1<\xi+\gamma$. Thus, there is a small $\lambda_0>0$ such that

$$\begin{aligned}(\lambda z_\lambda^\beta|z_\lambda|^\alpha_{L^\Psi}+\theta z_\lambda^\xi|z_\lambda|^\gamma_{L^\Lambda}) &\le \lambda(K\lambda^{\frac{1}{m-1}})^\beta|K\lambda^{\frac{1}{m-1}}|^\alpha_{L^\Psi}\\ &\quad+\theta(K\lambda^{\frac{1}{m-1}})^\xi|K\lambda^{\frac{1}{m-1}}|^\gamma_{L^\Lambda}\\ &\le \lambda.\end{aligned}$$

for all $\lambda\in(0,\lambda_0)$. Thus for $\lambda\in(0,\lambda_0)$ we get

$$\lambda z_\lambda^\beta|z_\lambda|^\alpha_{L^\Psi}+\theta z_\lambda^\xi|z_\lambda|^\gamma_{L^\Lambda}\le\lambda.$$

Now consider η,δ,σ,μ and as in the proof of Theorem 1. Fix $\lambda\in(0,\lambda_0)$.

Since $\alpha+\beta<l-1$ the arguments of the proof of Theorem 1 implies that if $\mu=\mu(\lambda)>0$ is small enough then

$$-\Delta_\Phi(\mu\eta)\le\lambda \text{ in } \Omega$$

and

$$\begin{aligned}-\Delta_\Phi(\mu\eta)&\le\lambda(\mu\eta)^\beta|\mu\eta|^\alpha_{L^\Psi}\\ &\le\lambda(\mu\eta)^\beta|\mu\eta|^\alpha_{L^\Psi}+\lambda(\mu\eta)^\xi|\mu\eta|^\gamma_{L^\Lambda}.\end{aligned}$$

The weak comparison principle implies that $\mu\eta\le z_\lambda$ for $\mu=\mu(\lambda)>0$ small enough. Therefore $(\mu\eta,z_\lambda)$ is a sub-super solution pair for $(P)_{\lambda,\theta}$.

Now we will prove the theorem in the second case. Consider again η,δ,σ and μ as in the proof of Theorem 1. Let $\lambda\in(0,\infty)$. Since $\alpha+\beta<l-1$ we can repeat the arguments of Theorem 1 to obtain $\mu=\mu(\lambda)>0$ small depending only on λ such that

$$-\Delta_\Phi(\mu\eta)\le 1 \ \text{ and } \ -\Delta_\Phi(\mu\eta)\le\lambda(\mu\eta)^\beta|\mu\eta|^\alpha_{L^\Psi} \ \text{ in } \Omega.$$

Let $z_M\in W_0^{1,\Phi}(\Omega)\cap L^\infty(\Omega)$ the unique solution of (2.1) where $M>0$ will be chosen later.

For $M\ge1$ large enough there is a constant $K>1$ that does not depend on M such that

$$0<z_M(x)\le KM^{\frac{1}{l-1}} \text{ in } \Omega. \tag{3.12}$$

We want to obtain $M>1$ such that

$$M\ge\left(\lambda z_M^\beta|z_M|^\alpha_{L^\Psi}+\theta z_M^\xi|z_M|^\gamma_{L^\Lambda}\right) \text{ in } \Omega \tag{3.13}$$

occurs.

Denoting by I the right-hand side of (3.13), we have from (3.12) that $I\le M$ if

$$1\ge\lambda\overline{K}M^{\frac{\alpha+\beta}{l-1}-1}+\theta\overline{K}M^{\frac{\xi+\gamma}{l-1}-1}, \tag{3.14}$$

where $\overline{K}:=\max\{K^\beta|K|^\alpha_{L^\Psi},K^\xi|K|^\gamma_{L^\Psi}\}$. Since $0<\alpha+\beta<l-1<\xi+\gamma$, the function

$$\Psi(t)=\lambda\overline{K}t^{\rho-1}+\theta\overline{K}t^{\tau-1}, t>0,$$

where $\rho:=\frac{\alpha+\beta}{l-1}$ and $\tau:=\frac{\xi+\gamma}{l-1}$, belongs to $C^1\big((0,\infty),\mathbb{R}\big)$ and attains a global minimum at

$$M_{\lambda,\theta}:=M(\lambda,\theta)=L\left(\frac{\lambda}{\theta}\right)^{\frac{1}{\tau-\rho}} \tag{3.15}$$

where $L := (\frac{1-\rho}{\tau-1})^{\frac{1}{\tau-\rho}}$. The inequality (3.14) is equivalent to find $M_{\lambda,\theta} > 0$ such that $\Psi(M_{\lambda,\theta}) \leq 1$. By (3.15) we have $\Psi(M_{\lambda,\theta}) \leq 1$ if and only if

$$\lambda\overline{K}(1-\rho)^{\frac{\rho-1}{\tau-\rho}}\left(\frac{\lambda}{\theta}\right)^{\frac{\rho-1}{\tau-\rho}} + \theta\overline{K}(1-\rho)^{\frac{\tau-1}{\tau-\rho}}\left(\frac{\lambda}{\theta}\right)^{\frac{\tau-1}{\tau-\rho}} \leq 1$$

Notice that the above inequality holds if $\theta > 0$ is small enough because $\alpha + \beta < l - 1 < \xi + \gamma$. Thus for $\lambda > 0$ fixed there exists $\theta_0 = \theta_0(\lambda)$ such that for each $\theta \in (0, \theta_0)$ there is a number $M = M_{\lambda,\theta} > 0$ such that (3.14) occurs. Consequently we have (3.13). Therefore

$$-\Delta_\Phi z_M \geq \lambda z_M^\beta |z_M|_{L^\Psi}^\alpha + \theta z_M^\xi |z_M|_{L^\Lambda}^\gamma \text{ in } \Omega.$$

Considering if necessary a smaller $\theta_0 > 0$, we get $M \geq 1$. Therefore $-\Delta_\Phi(\mu\eta) \leq -\Delta_\Phi z_M$ in Ω. The weak comparison principle implies that $\mu\eta \leq z_M$. Then $(\mu\eta, z_M)$ is a sub-supersolution pair for $(P)_{\lambda,\theta}$. The proof is finished. □

4. The system case

We say that $(u_1, u_2) \in (W_0^{1,\Phi_1}(\Omega) \cap L^\infty(\Omega)) \times (W_0^{1,\Phi_2}(\Omega) \cap L^\infty(\Omega))$ is a (weak) solution of (P_2) if

$$\int_\Omega \phi(|\nabla u_i|)\nabla u_i \nabla\varphi = \int_\Omega (f_i(u_j)|u_j|_{L^{\Psi_i}}^{\alpha_i} + g_i(u_j)|u_j|_{L^{\Lambda_i}}^{\alpha_i})\varphi_i,$$

for all $\varphi_i \in W_0^{1,\Phi_i}(\Omega)$ with $i, j = 1, 2$ and $i \neq j$.

We say that the pairs $(\underline{u}_i, \overline{u}_i), i = 1, 2$ are sub-supersolution pairs for (P_2) if $\underline{u}_i, \overline{u}_i \in W_0^{1,\Phi_i}(\Omega) \cap L^\infty(\Omega)$ are nonnegative functions with $0 < \underline{u}_i \leq \overline{u}_i$ in Ω and if for all $\varphi_i \in W_0^{1,\Phi_i}(\Omega)$ with $\varphi_i \geq 0$ the following inequalities are verified

$$\begin{cases} \int_\Omega \phi_i(|\nabla\underline{u}_i|)\nabla\underline{u}_i\nabla\varphi_i \leq \int_\Omega \left(f_i(\underline{u}_j)|\underline{u}_j|_{L^{\Psi_i}}^{\alpha_i} + g_i(\underline{u}_j)|\underline{u}_j|_{L^{\Lambda_i}}^{\gamma_i}\right)\varphi_i, \\ \int_\Omega \phi_i(|\nabla\overline{u}_i|)\nabla\overline{u}_i\nabla\varphi_i \geq \int_\Omega \left(f_i(\overline{u}_j)|\overline{u}_j|_{L^{\Psi_i}}^{\alpha_i} + g_i(\overline{u}_j)|\overline{u}_j|_{L^{\Lambda_i}}^{\gamma_i}\right)\varphi_i, \end{cases} \tag{4.1}$$

for all $\varphi_i \in W_0^{1,\Phi_i}(\Omega)$ with $i, j = 1, 2$ and $i \neq j$.

The following lemma is needed to obtain a solution for system (P_2).

Lemma 5. *Suppose that $f_i, g_i : [0, +\infty) \to \mathbb{R}, i = 1, 2$ are nondecreasing, continuous and nonnegative functions. Consider also that $\alpha_i, \gamma_i \geq 0, i = 1, 2$ and that there exist sub-supersolution pairs $(\underline{u}_i, \overline{u}_i), i = 1, 2$ for (P_2). Then there exists a solution $(u, \widetilde{u})$ for (P_2) with $u \in [\underline{u}_1, \overline{u}_1]$ and $\widetilde{u} \in [\underline{u}_2, \overline{u}_2]$.*

Proof. Consider u_1 the solution of the problem

$$\begin{cases} -\Delta_{\Phi_1} u_1 &= f_1(\underline{u}_2)|\underline{u}_2|_{L^{\Psi_1}}^{\alpha_1} + g_1(\underline{u}_2)|\underline{u}_2|_{L^{\Lambda_1}}^{\gamma_1} \text{ in } \Omega, \\ u_1 &= 0 \text{ on } \partial\Omega. \end{cases}$$

Using the monotonicity of f_1, g_1 and the fact that $\underline{u}_2 \leq \overline{u}_2$ a.e in Ω we get

$$-\Delta_{\Phi_1}\overline{u}_1 \geq f_1(\overline{u}_2)|\overline{u}_2|_{L^{\Psi_1}}^{\alpha_1} + g_1(\overline{u}_2)|\overline{u}_2|_{L^{\Lambda_1}}^{\gamma_1} \geq -\Delta_{\Phi_1}u_1 \text{ in } \Omega,$$

therefore $u_1 \leq \overline{u}_1$. Note also that

$$-\Delta_{\Phi_1}u_1 = f_1(\underline{u}_2)|\underline{u}_2|_{L^{\Psi_1}}^{\alpha_1} + g_1(\underline{u}_2)|\underline{u}_2|_{L^{\Lambda_1}}^{\gamma_1} \geq -\Delta_{\Phi_1}\underline{u}_1 \text{ in } \Omega.$$

Therefore $\underline{u}_1 \le u_1 \le \overline{u}_1$ a.e in Ω. Denote by $\widetilde{u}_1$ the weak solution of the problem

$$\begin{cases} -\Delta_{\Phi_1}\widetilde{u}_1 &= f_2(\underline{u}_1)|\underline{u}_1|^{\alpha_2}_{L^{\Psi_2}} + g_2(\underline{u}_1)|\underline{u}_1|^{\gamma_2}_{L^{\Lambda_2}} \text{ in } \Omega, \\ \widetilde{u}_1 &= 0 \text{ on } \partial\Omega. \end{cases}$$

From the definition of $\underline{u}_2$ and $\overline{u}_2$ we have that $-\Delta_{\Phi_2}\underline{u}_2 \le -\Delta_{\Phi_2}\widetilde{u}_1 \le -\Delta_{\Phi_2}\overline{u}_2$ in Ω. Therefore $\underline{u}_2 \le \widetilde{u}_1 \le \overline{u}_2$ in Ω.

Consider u_2 the solution of the problem

$$\begin{cases} -\Delta_{\Phi_1}u_2 &= f_1(\widetilde{u}_1)|\widetilde{u}_1|^{\alpha_1}_{L^{\Psi_1}} + g_1(\widetilde{u}_1)|\widetilde{u}_1|^{\gamma_1}_{L^{\Lambda_1}} \text{ in } \Omega, \\ u_2 &= 0 \text{ on } \partial\Omega. \end{cases}$$

Using the fact that $\underline{u}_2 \le \widetilde{u}_1 \le \overline{u}_2$ in Ω and the monotonicity of the functions f_1 and g_1, we have $-\Delta_{\Phi_1}u_1 \le -\Delta_{\Phi_1}u_2 \le -\Delta_{\Phi_1}\overline{u}_1$ in Ω. Therefore $\underline{u}_1 \le u_1 \le u_2 \le \overline{u}_1$ in Ω.

Consider $\widetilde{u}_2$ the solution of the problem

$$\begin{cases} -\Delta_{\Phi_2}\widetilde{u}_2 &= f_2(u_1)|u_1|^{\alpha_2}_{L^{\Psi_2}} + g_2(\widetilde{u}_1)|u_1|^{\gamma_2}_{L^{\Lambda_2}} \text{ in } \Omega, \\ \widetilde{u}_2 &= 0 \text{ on } \partial\Omega. \end{cases}$$

A direct computation imply that $\underline{u}_2 \le \widetilde{u}_1 \le \widetilde{u}_2 \le \overline{u}_2$ in Ω. Proceeding with the previous reasonings we construct sequences u_n and $\widetilde{u}_n$ satisfying

$$\begin{cases} -\Delta_{\Phi_1}u_n &= f_1(\widetilde{u}_{n-1})|\widetilde{u}_{n-1}|^{\alpha_1}_{L^{\Psi_1}} + g_1(\widetilde{u}_{n-1})|\widetilde{u}_{n-1}|^{\gamma_1}_{L^{\Lambda_1}} \text{ in } \Omega, \\ \widetilde{u}_n &= 0 \text{ on } \partial\Omega. \end{cases}$$

and

$$\begin{cases} -\Delta_{\Phi_2}\widetilde{u}_n &= f_2(u_{n-1})|u_{n-1}|^{\alpha_2}_{L^{\Psi_2}} + g_2(u_{n-1})|u_{n-1}|^{\gamma_2}_{L^{\Lambda_2}} \text{ in } \Omega, \\ \widetilde{u}_n &= 0 \text{ on } \partial\Omega. \end{cases}$$

where $\widetilde{u}_0 := \overline{u}_2$ and $u_0 := \overline{u}_1$. Arguing as in Lemma 4 we obtain the result.

A sublinear system

In this section we use Lemma 5 and suitable sub-supersolution pairs to prove the existence of solution for the the nonlocal system

$$\begin{cases} -\Delta_{\Phi_1}u &= v^{\beta_1}|v|^{\alpha_1}_{\Psi_1} \text{ in } \Omega, \\ -\Delta_{\Phi_2}v &= u^{\beta_2}|u|^{\alpha_2}_{\Psi_2} \text{ in } \Omega, \\ u = v &= 0 \text{ on } \partial\Omega, \end{cases} \tag{P_S'}$$

where α_i and $\beta_i, i = 1, 2$ are constants saisfying certain conditions. It is interesting to note that the set of hipothesis of the next result is different from the system version of (P) considered in [12, Theorem 5.2] in the constant exponent case. □

Theorem 3. *Suppose that $\alpha_i, \beta_i \ge 0$ with $0 < \alpha_1 + \beta_1 < l_i - 1, 0 < \alpha_2 + \beta_2 < l_i - 1, i = 1, 2$. Then (P_S') has a positive solution.*

Proof. Let $\lambda > 0$ and consider $z_\lambda \in W_0^{1,\Phi_1}(\Omega) \cap L^\infty(\Omega)$ and $y_\lambda \in W_0^{1,\Phi_2}(\Omega) \cap L^\infty(\Omega)$ the unique solutions of (2.1) where λ will be chosen later.

For $\lambda > 0$ sufficiently large, by Lemma 3 there is a constant $K > 0$ that does not depend on λ such that

$$0 < z_\lambda(x) \le K\lambda^{\frac{1}{l_1 - 1}} \text{ in } \Omega, \tag{4.2}$$

and

$$0 < y_\lambda(x) \le K\lambda^{\frac{1}{l_2 - 1}} \text{ in } \Omega. \tag{4.3}$$

Since $0 < \alpha_1 + \beta_1 < l_2 - 1$, we can choose $\lambda > 0$ large enough satisfying $K^{\beta_1}|K|^{\alpha_1}_{L^{\Psi_1}}\lambda^{\frac{\alpha_1+\beta_1}{l_2-1}} \leq \lambda$. Thus from (4.2) we have $y_\lambda^{\beta_1}|y_\lambda|^{\alpha_1}_{L^{\Psi_1}} \leq \lambda$ in Ω. Therefore

$$\begin{cases} -\Delta_{\Phi_1} z_\lambda \geq y_\lambda^{\beta_1}|y_\lambda|^{\alpha_1}_{L^{\Psi_1}} & \text{in } \Omega, \\ z_\lambda = 0 & \text{on } \partial\Omega. \end{cases}$$

From (4.3) and the fact that $0 < \alpha_2 + \beta_2 < l_1 - 1$ we also have that

$$\begin{cases} -\Delta_{\Phi_2} y_\lambda \geq z_\lambda^{\beta_2}|z_\lambda|^{\alpha_2}_{L^{\Psi_2}} & \text{in } \Omega, \\ y_\lambda = 0 & \text{on } \partial\Omega, \end{cases}$$

for $\lambda > 0$ large enough.

Since $\partial\Omega$ is C^2, there is a constant $\delta > 0$ such that $d \in C^2(\overline{\Omega_{3\delta}})$ and $|\nabla d(x)| \equiv 1$, where $d(x) := dist(x, \partial\Omega)$ and $\overline{\Omega_{3\delta}} := \{x \in \overline{\Omega}; d(x) \leq 3\delta\}$. For $\sigma \in (0, \delta)$ the function $\eta_i = \eta_i(k, \sigma), i = 1, 2$ defined by

$$\eta_i(x) = \begin{cases} e^{kd(x)} - 1 & \text{if} \quad d(x) < \sigma, \\ e^{k\sigma} - 1 + \int_\sigma^{d(x)} ke^{k\sigma}\left(\frac{2\delta - t}{2\delta - \sigma}\right)^{\frac{m_i}{l_i - 1}} dt & \text{if} \quad \sigma \leq d(x) < 2\delta, \\ e^{k\sigma} - 1 + \int_\sigma^{2\delta} ke^{k\sigma}\left(\frac{2\delta - t}{2\delta - \sigma}\right)^{\frac{m_i}{l_i - 1}} dt & \text{if} \quad 2\delta \leq d(x) \end{cases}$$

belongs to $C_0^1(\overline{\Omega})$ for $i = 1, 2$, where $k > 0$ is an arbitrary constant. Note that

$$-\Delta_\Phi(\mu\eta_i) = \begin{cases} -\mu k^2 e^{kd(x)} \frac{d}{dt}\left(\phi_i(t)t\right)\Big|_{t=\mu ke^{kd(x)}} - \phi_i(\mu ke^{kd(x)})\mu ke^{kd(x)}\Delta d \quad \text{if} \quad d(x) < \sigma, \\ \mu ke^{k\sigma}\left(\frac{m_i}{l_i - 1}\right)\left(\frac{2\delta - d(x)}{2\delta - \sigma}\right)^{\frac{m_i}{l_i-1} - 1}\left(\frac{1}{2\delta - \sigma}\right)\frac{d}{dt}\left(\phi_i(t)t\right)\Big|_{t=\mu ke^{k\sigma}\left(\frac{2\delta - d(x)}{2\delta - \sigma}\right)} \\ -\phi_i\left(\mu ke^{k\sigma}\left(\frac{2\delta - d(x)}{2\delta - \sigma}\right)^{\frac{m_i}{l_i - 1}}\right)\mu ke^{k\sigma}\left(\frac{2\delta - d(x)}{2\delta - \sigma}\right)^{\frac{m_i}{l_i - 1}}\Delta d \quad \text{if} \quad \sigma < d(x) < 2\delta, \\ 0 \quad \text{if} \quad 2\delta < d(x) \end{cases}$$

for all $\mu > 0$ and $i = 1, 2$. Arguing as in (3.3) we have $-\Delta_{\Phi_i}(\mu\eta_i) \leq 0, i = 1, 2$ for $k > 0$ large enough when $0 < d(x) < \sigma$.

Reasoning as in (3.7) we get

$$-\Delta_{\Phi_1}(\mu\eta_1) \leq \max\left\{\frac{K_1}{2\delta - \sigma}, K_2\right\}\max\{(\mu ke^{k\sigma})^{m_1 - 1}, (\mu ke^{k\sigma})^{l_1 - 1}\}, \tag{4.4}$$

and

$$-\Delta_{\Phi_2}(\mu\eta_2) \leq \max\left\{\frac{K_3}{2\delta - \sigma}, K_4\right\}\max\{(\mu ke^{k\sigma})^{m_2 - 1}, (\mu ke^{k\sigma})^{l_2 - 1}\}, \tag{4.5}$$

for $\sigma < d(x) < 2\delta$, where $K_i, i = 1, 2, 3, 4$ are positive constants that does not depend on $k > 0$.

Consider $\sigma = \frac{\ln 2}{k}$ and $\mu = e^{-k}$. We have $\eta_i(x) \geq e^{k\sigma} - 1 \geq 1$ for all $x \in \Omega$ and $i = 1, 2$. Thus there is a constant $K_5 > 0$ such that

$$(\mu\eta_j)^{\beta_i}|\mu\eta_j|^{\alpha_i}_{L^{\Psi_i}} \geq \mu^{\alpha_i + \beta_i}K_5, i, j = 1, 2, i \neq j$$

for $\sigma < d(x) < 2\delta$.

Since $0 < \alpha_i + \beta_i < l_i - 1$, the L'Hospital's rule implies that

$$\lim_{k \to +\infty}\frac{k^{l_i - 1}}{e^{k(l_i - 1 - (\alpha_i + \beta_i))}} = 0, i = 1, 2.$$

Thus, it is possible to consider $k_0 > 0$ large enough such that

$$K_5 \geq \max\left\{K_1 \frac{1}{2\delta - \frac{\ln 2}{k}}, K_2\right\} \max\{2^{m_1-1}, 2^{l_1-1}\} \frac{k^{l_1-1}}{e^{k(l_1-1-(\alpha_1+\beta_1))}}$$

and

$$K_5 \geq \max\left\{K_3 \frac{1}{2\delta - \frac{\ln 2}{k}}, K_4\right\} \max\{2^{m_2-1}, 2^{l_2-1}\} \frac{k^{l_2-1}}{e^{k(l_2-1-(\alpha_2+\beta_2))}},$$

for all $k \geq k_0$. Thus for $k > 0$ large enough we have $-\Delta_{\Phi_i}(\mu\eta_i) \leq (\mu\eta_j)^{\beta_i} |\mu\eta_j|^{\alpha_i}_{L^{\Psi_i}}, i, j = 1, 2, i \neq j$. for $\sigma < d(x) < 2\delta$. If $d(x) > 2\delta$ we have $-\Delta_{\Phi_i}(\mu\eta_j) = 0 \leq (\mu\eta_j)^{\beta_i} |\mu\eta_i|^{\alpha_i}_{L^{\Psi_i}}, i, j = 1, 2$ with $i \neq j$. For $k > 0$ large enough we also have that $-\Delta_{\Phi_1}(\mu\eta_1) \leq -\Delta_{\Phi_1} z_\lambda, -\Delta_{\Phi_2}(\mu\eta_2) \leq -\Delta_{\Phi_2} y_\lambda$ in Ω. Therefore $\mu\eta_1 \leq z_\lambda, \mu\eta_2 \leq y_\lambda$ in Ω. The result follows. □

4.1. **A concave-convex system.** In this section we prove the existence of solution for a concave-convex system of type

$$\begin{cases} -\Delta_{\Phi_1} u &= \lambda v^{\beta_1} |v|^{\alpha_1}_{\Psi_1} + \theta v^{\xi_1} |v|^{\gamma_1}_{L^{\Lambda_1}} \text{ in } \Omega, \\ -\Delta_{\Phi_2} v &= \lambda u^{\beta_2} |u|^{\alpha_2}_{\Psi_2} + \theta u^{\xi_2} |u|^{\gamma_2}_{L^{\Lambda_2}} \text{ in } \Omega, \\ u = v &= 0 \text{ on } \partial\Omega, \end{cases} \qquad (P')_{\lambda,\theta}$$

where $\alpha_i, \beta_i, \gamma_i, \xi_i, i = 1, 2$ are constants satisfying certain conditions.

Theorem 4. *Suppose that $\alpha_i, \beta_i, \gamma_i, \xi_i, i = 1, 2$ are nonnegative constants and suppose that $0 < \alpha_i + \beta_i < l_i - 1, i = 1, 2$. The following assertions hold*

(i) If $m_2 - 1 < \xi_1 + \gamma_1$ and $m_1 - 1 < \xi_2 + \gamma_2$, then for each $\theta > 0$ there exists $\lambda_0 > 0$ such that for each $\lambda \in (0, \lambda_0)$ the problem $(P')_{\lambda,\theta}$ has a positive solution $u_{\lambda,\theta}$.

(ii) If $0 < \alpha_1 + \beta_1 < l_2 - 1, 0 < \alpha_2 + \beta_2 < l_1 - 1 < \xi_1 + \gamma_1 < l_2 - 1$ and $\xi_2 + \gamma_2 < l_1 - 1$ then for each $\lambda > 0$ there exists $\theta_0 > 0$ such that for each $\theta \in (0, \theta_0)$ the problem $(P')_{\lambda,\theta}$ has a positive solution $u_{\lambda,\theta}$.

Proof. Suppose that (i) occurs. Consider $z_\lambda \in W_0^{1,\Phi_1}(\Omega) \cap L^\infty(\Omega)$ and $y_\lambda \in W_0^{1,\Phi_2}(\Omega) \cap L^\infty(\Omega)$ the unique solutions of (2.1), where $\lambda \in (0, 1)$ will be chosen before. Lemma 3 imply that for $\lambda > 0$ small enough there exists a constant $K > 0$ that does not depend on λ such that

$$0 < z_\lambda(x) \leq K\lambda^{\frac{1}{m_1-1}} \text{ in } \Omega, \qquad (4.6)$$

$$0 < y_\lambda(x) \leq K\lambda^{\frac{1}{m_2-1}} \text{ in } \Omega. \qquad (4.7)$$

We will prove, for each $\theta > 0$, that there exists $\lambda_0 > 0$ such that

$$\lambda y_\lambda^{\beta_1} |y_\lambda|^{\alpha_1}_{L^{\Psi_1}} + \theta y_\lambda^{\xi_1} |y_\lambda|^{\gamma_1}_{L^{\Lambda_1}} \leq \lambda \qquad (4.8)$$

and

$$\lambda z_\lambda^{\beta_2} |z_\lambda|^{\alpha_2}_{L^{\Psi_2}} + \theta z_\lambda^{\xi_2} |z_\lambda|^{\gamma_2}_{L^{\Lambda_2}} \leq \lambda \qquad (4.9)$$

in Ω. Since $0 < \alpha_i + \beta_i, i = 1, 2$, $m_2 - 1 < \xi_1 + \gamma_1$ and $m_1 - 1 < \xi_2 + \gamma_2$ there exists $\lambda_0 > 0$ such that

$$\lambda^{\frac{m_2-1+\alpha_1+\beta_1}{m_2-1}} K^{\beta_1} |K|^{\alpha_1}_{L^{\Psi_1}} + \theta \lambda^{\frac{\xi_1+\gamma_1}{m_2-1}} K^{\xi_1} |K|^{\gamma_1}_{L^{\Lambda_1}} \leq \lambda \qquad (4.10)$$

and

$$\lambda^{\frac{m_1-1+\alpha_2+\beta_2}{m_1-1}} K^{\beta_2} |K|^{\alpha_2}_{L^{\Psi_2}} + \theta \lambda^{\frac{\xi_2+\gamma_2}{m_1-1}} K^{\xi_2} |K|^{\gamma_2}_{L^{\Lambda_2}} \leq \lambda \qquad (4.11)$$

for all $\lambda \in (0, \lambda_0)$. From (4.6), (4.7), (4.10) and (4.11) we obtain (4.8) and (4.9). Therefore

$$-\Delta_{\Phi_1} z_\lambda \geq \lambda y_\lambda^{\beta_1} |y_\lambda|_{L^{\Psi_1}}^{\alpha_1} + \theta y_\lambda^{\xi_1} |y_\lambda|_{L^{\Lambda_1}}^{\gamma_1}$$

and

$$-\Delta_{\Phi_2} y_\lambda \geq \lambda z_\lambda^{\beta_2} |z_\lambda|_{L^{\Psi_2}}^{\alpha_2} + \theta z_\lambda^{\xi_2} |z_\lambda|_{L^{\Lambda_2}}^{\gamma_2}$$

in Ω for all $\lambda \in (0, \lambda_0)$.

Consider η_i, δ, σ and μ as in the proof of Theorem 3. Since $0 < \alpha_i + \beta_i < l_i - 1, i = 1, 2$ we have that exists $\mu > 0$ with $\mu\eta_1 \leq z_\lambda$, $\mu\eta_2 \leq y_\lambda$ and the inequalities

$$-\Delta_{\Phi_1}(\mu\eta_1) \leq \lambda, -\Delta_{\Phi_1}(\mu\eta_1) \leq \lambda(\mu\eta_2)^{\beta_1} |\mu\eta_2|_{L^{\Psi_1}}^{\alpha_1} + \theta(\mu\eta_2)^{\xi_1} |\mu\eta_2|_{L^{\Lambda_1}}^{\gamma_1}$$

and

$$-\Delta_{\Phi_2}(\mu\eta_2) \leq \lambda, -\Delta_{\Phi_2}(\mu\eta_2) \leq \lambda(\mu\eta_1)^{\beta_2} |\mu\eta_1|_{L^{\Psi_2}}^{\alpha_2} + \theta(\mu\eta_1)^{\xi_2} |\mu\eta_1|_{L^{\Lambda_2}}^{\gamma_2}$$

in Ω. Thus by Lemma 5 we have the first part of the result.

In order to prove the second part of the result consider η_i, δ and $\sigma_i, i = 1, 2$ as in the first part of the result and let $\lambda > 0$ fixed. Since $0 < \alpha_i + \beta_i < l_i - 1, i = 1, 2$ there exists $\mu > 0$ depending only on λ such that

$$-\Delta_{\Phi_i}(\mu\eta_i) \leq 1 \text{ and } -\Delta_{\Phi_i}(\mu\eta_i) \leq \lambda(\mu\eta_j)^{\beta_i} |\mu\eta_j|_{L^{\Psi_i}}^{\alpha_i}$$

in Ω with $i, j = 1, 2$ and $i \neq j$.

Let $M > 0$ which will be chosen before and consider $z_M \in W_0^{1,\Phi_1}(\Omega) \cap L^\infty(\Omega)$ and $y_M \in W_0^{1,\Phi_2}(\Omega) \cap L^\infty(\Omega)$ solutions of

$$\begin{cases} -\Delta_{\Phi_1} z_M = M \text{ in } \Omega, \\ z_M = 0 \text{ on } \partial\Omega. \end{cases} \qquad \begin{cases} -\Delta_{\Phi_2} y_M = M \text{ in } \Omega, \\ y_M = 0 \text{ on } \partial\Omega. \end{cases}$$

If $M > 0$ is large enough, then by Lemma 3 there exists a constant $K > 0$ that does not depend on M such that

$$0 < z_M(x) \leq K M^{\frac{1}{l_1 - 1}} \text{ in } \Omega, \tag{4.12}$$

$$0 < y_M(x) \leq K M^{\frac{1}{l_2 - 1}} \text{ in } \Omega. \tag{4.13}$$

In order to construct $\overline{u}_i, \overline{u}_i, i = 1, 2$ we will show that exist $\theta_0 > 0$ depending on λ with the following property: if we consider $\theta \in (0, \theta_0)$ then there will be a constant M depending only on λ and θ satisfying

$$M \geq \lambda {y_M}^{\beta_1} |y_M|_{L^{\Psi_1}}^{\alpha_1} + \theta {y_M}^{\xi_1} |y_M|_{L^{\Lambda_1}}^{\gamma_1} \tag{4.14}$$

and

$$M \geq \lambda {z_M}^{\beta_2} |z_M|_{L^{\Psi_2}}^{\alpha_2} + \theta {z_M}^{\xi_2} |z_M|_{L^{\Lambda_2}}^{\gamma_2} \tag{4.15}$$

in Ω. From (4.12) and (4.13) we have that (4.14) and (4.15) occur if $M \geq 1$ and

$$\lambda \overline{K} M^{\rho - 1} + \theta \overline{K} M^{\tau - 1} \leq 1 \tag{4.16}$$

where $\overline{K} := \max\{K^{\beta_1} |K|_{L^{\Psi_1}}^{\alpha_1}, K^{\beta_2} |K|_{L^{\Psi_2}}^{\alpha_2}, K^{\xi_1} |K|_{L^{\Lambda_1}}^{\gamma_1}, K^{\xi_2} |K|_{L^{\Lambda_2}}^{\gamma_2}\}$,

$$\rho := \max\left\{\frac{\alpha_1 + \beta_1}{l_2 - 1}, \frac{\alpha_2 + \beta_2}{l_1 - 1}\right\} \text{ and } \tau := \max\left\{\frac{\gamma_1 + \xi_1}{l_2 - 1}, \frac{\gamma_2 + \xi_2}{l_1 - 1}\right\}.$$

Since $0 < \rho < 1$ and $\tau > 1$ the function

$$\Psi(t) = \lambda \overline{K} t^{\rho - 1} + \theta \overline{K} t^{\tau - 1}, t > 0,$$

belongs to $C^1\big((0,\infty),\mathbb{R}\big)$ and attains a global minimum at

$$M_{\lambda,\theta} := M(\lambda,\theta) = L\left(\frac{\lambda}{\theta}\right)^{\frac{1}{\tau-\rho}} \tag{4.17}$$

where $L := (\frac{1-\rho}{\tau-1})^{\frac{1}{\tau-\rho}}$. The inequality (4.16) is equivalent to find $M_{\lambda,\theta} > 0$ such that $\Psi(M_{\lambda,\theta}) \leq 1$. By (4.17) we have $\Psi(M_{\lambda,\theta}) \leq 1$ if and only if

$$\lambda\overline{K}(1-\rho)^{\frac{\rho-1}{\tau-\rho}}\left(\frac{\lambda}{\theta}\right)^{\frac{\rho-1}{\tau-\rho}} + \theta\overline{K}(1-\rho)^{\frac{\tau-1}{\tau-\rho}}\left(\frac{\lambda}{\theta}\right)^{\frac{\tau-1}{\tau-\rho}} \leq 1$$

Notice that the above inequality holds if $\theta > 0$ is small enough because $0 < \rho < 1$ and $\tau > 1$. Thus for $\lambda > 0$ fixed there exists $\theta_0 = \theta_0(\lambda)$ such that for each $\theta \in (0,\theta_0)$ there is a number $M = M_{\lambda,\theta} > 0$ such that (4.16) occurs. Thus we can consider $M_{\lambda,\theta}$ large enough such that (4.14) and (4.15) occur. Therefore

$$-\Delta_{\Phi_1} z_M \geq \lambda y_M^{\beta_1}|y_M|^{\alpha_1}_{L^{\Psi_1}} + \theta y_M^{\xi_1}|y_M|^{\gamma_1}_{L^{\Lambda_1}}$$

and

$$-\Delta_{\Phi_2} y_M \geq \lambda z_M^{\beta_2}|z_M|^{\alpha_2}_{L^{\Psi_2}} + \theta z_M^{\xi_2}|z_M|^{\gamma_2}_{L^{\Lambda_2}}$$

Considering if necessary a smaller $\theta_0 > 0$, we get

$$-\Delta_{\Phi_1}(\mu\eta_1) \leq 1 \leq M_{\lambda,\theta_0} \leq M_{\lambda,\theta}$$

and

$$-\Delta_{\Phi_2}(\mu\eta_2) \leq 1 \leq M_{\lambda,\theta_0} \leq M_{\lambda,\theta}$$

in Ω for all $\theta \in (0,\theta_0)$ because $M_{\lambda,\theta} \to +\infty$ as $\theta \to 0^+$ and $\theta \longmapsto M_{\lambda,\theta}$ is non-increasing. Therefore $-\Delta_{\Phi_1}(\mu\eta_1) \leq -\Delta_{\Phi_1} z_M$, $-\Delta_{\Phi_2}(\mu\eta_2) \leq -\Delta_{\Phi_2} y_M$ in Ω. The weak comparison principle implies that $\mu\eta_1 \leq z_M$ and $\mu\eta_2 \leq y_M$ in Ω. The proof is finished. □

5. Final comments

A slightly modification in the arguments of Lemma 5 allow us to study a more general class of systems given by

$$\begin{cases} -\Delta_{\Phi_1} u &= f_1(u,v)|v|^{\alpha_1}_{L^{\Psi_1}} + g_1(u,v)|v|^{\gamma_1}_{L^{\Lambda_1}} \text{ in } \Omega, \\ -\Delta_{\Phi_2} v &= f_2(u,v)|u|^{\alpha_2}_{L^{\Psi_2}} + g_2(u,v)|u|^{\gamma_2}_{L^{\Lambda_2}} \text{ in } \Omega, \\ u = v &= 0 \text{ on } \partial\Omega, \end{cases} \tag{$\widetilde{P}$}$$

with $f_i, g_i : [0,+\infty) \times 0,+\infty) \to 0,+\infty), i = 1,2$ nondecreasing continuous functions in the variables u and v. The arguments used in this work allow us to consider results in the case for example when the functions f_i and g_i are power functions with convenient exponents. In order to avoid of a more technical exposition we choose to not prove results related with the case mentioned before, that is, systems involving the variables u and v in the local the terms of each equation of $(\widetilde{P})$.

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