

Explicit estimates on positive supersolutions of nonlinear elliptic equations and applications

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Abstract

In this paper we consider positive supersolutions of the nonlinear elliptic equation

$$-\Delta u = \rho(x)f(u)|\nabla u|^p, \quad \text{in } \Omega,$$

where $0 \leq p < 1$, Ω is an arbitrary domain (bounded or unbounded) in \mathbb{R}^N ($N \geq 2$), $f : [0, a_f) \rightarrow \mathbb{R}_+$ ($0 < a_f \leq +\infty$) is a non-decreasing continuous function and $\rho : \Omega \rightarrow \mathbb{R}$ is a positive function. Using the maximum principle we give explicit estimates on positive supersolutions u at each point $x \in \Omega$ where $\nabla u \neq 0$ in a neighborhood of x . As consequences, we discuss the dead core set of supersolutions on bounded domains, and also obtain Liouville type results in unbounded domains Ω with the property that $\sup_{x \in \Omega} \text{dist}(x, \partial\Omega) = \infty$.

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1 Introduction and main estimates

The aim of this paper is to give explicit estimates on positive classical supersolutions of the following elliptic equation

$$-\Delta u = \rho(x)f(u)|\nabla u|^p, \quad \text{in } \Omega, \tag{1}$$

where Ω is an arbitrary domain (bounded or unbounded) in \mathbb{R}^N , $0 \leq p < 1$ and f, ρ satisfy

(C) $f : D_f = [0, a_f) \rightarrow [0, \infty)$ ($0 < a_f \leq +\infty$) is a non-decreasing continuous function and $\rho : \Omega \rightarrow \mathbb{R}$ is a positive function. Also we assume that $f(u) > 0$ for $u > 0$.

By a positive classical supersolution we mean a positive function $u \in C^2(\Omega)$ such that $-\Delta u \geq \rho(x)f(u)|\nabla u|^p$, for all $x \in \Omega$. Note in the case when f in (1) is not monotone we can still use our results if we additionally have $\inf_{s > s_0} f(s) > 0$ for every $s_0 > 0$. Indeed in this case one can take $g(t) := \inf_{s \geq t} f(s)$ then g is non-decreasing and every supersolution u of (1) is also a supersolution of $-\Delta u = \rho(x)g(u)|\nabla u|^p$ in Ω .

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In this paper, we give explicit estimates on positive classical supersolutions u of (1) at each point $x \in \Omega$ where $\nabla u \neq 0$ in a neighborhood of x . As we shall see, the simplicity and robustness of our maximum principle-based estimates provide for their applicability to many quasi-linear elliptic inequalities on arbitrary domains in \mathbb{R}^N , bounded or unbounded. The applications we are interested in applying the pointwise estimate to are: Liouville type theorems related to (1) in unbounded domains such as \mathbb{R}^N , \mathbb{R}_+^N , exterior domains or generally unbounded domains with the property that $\sup_{x \in \Omega} \text{dist}(x, \partial\Omega) = \infty$, and also we discuss issues related supersolutions of (1) on bounded domains which have dead cores.

In particular we apply our results to the equation

$$-\Delta u = |x|^\beta u^q |\nabla u|^p, \quad x \in \Omega, \quad (2)$$

where $\beta \in \mathbb{R}$, $q > 0$, $0 \leq p < 1$ and Ω is an arbitrary domain in \mathbb{R}^N ; note importantly that we allow $p = 0$, and hence we obtain results regarding semilinear equations. The existence and nonexistence of positive supersolutions of (2) when Ω is an exterior domain in \mathbb{R}^N , in particular in the case $\beta = 0$ and some similar equations have been studied extensively in recent years, see [1, 2, 3, 4, 5, 6, 7, 10, 11, 12, 14, 15, 18, 19, 20, 23].

In particular, recently Burgos-Perez, Garcia-Melian and Quaas in [14] considered positive supersolutions of the equation $-\Delta u = f(u)|\nabla u|^q$ posed in exterior domains of \mathbb{R}^N , where f is continuous in $[0, \infty)$ and positive in $(0, \infty)$ and $q > 0$. They classified supersolutions u into four types depending on the function $m(R) = \inf_{|x|=R} u(x)$ for large R , and give necessary and sufficient conditions in order to have supersolutions of each of these types. As consequences, they obtained many interesting Liouville theorems for supersolutions depending on the values of N, q and on some integrability properties on f at zero or infinity.

Also, very recently Bidaut-Veron, Garcia-Huidobro and Veron in [10] obtained several important results on positive supersolutions of equation (2) (with $\beta = 0$) in $\Omega \setminus \{0\}$ where Ω is an open subset of \mathbb{R}^N containing 0, p and q are real exponents. It worth mentioning that problem (1) has been studied in [13, 18] for some more general operators when $0 < p < 1$ and $f(u)$ is essentially like u^q .

As a simple application of our explicit estimates on supersolutions of (1), we see that if $\frac{q}{1-p} > 1$ then every positive supersolution of (2) is eventually constant if

$$(N-2)q + p(N-1) < N + \beta.$$

As some other applications of our main estimates we also examine equation (1) for nonlinearities like

$$f(u) = u^q + u^r \quad \text{or} \quad f(u) = \max\{u^q, u^r\}, \quad 0 < q < 1 - p < r,$$

or singular nonlinearity $f(u) = \frac{1}{(1-u)^q}$, ($q > 1$), and discuss the dead core set of positive supersolutions in bounded domains. In particular we show that there exists a $\beta > 0$ (we give the explicit value of β) such that if a domain Ω satisfies

$$\sup_{x \in \Omega} d_\Omega(x) > \beta,$$

then every positive solution u of $-\Delta u \geq f(u)|\nabla u|^p$, with the above nonlinearities f , must be a dead core solution.

Definition 1. (*Dead Core Solutions*). We call a non-negative nonzero solution u of (1) a dead core solution provided K_u^0 is nonempty; here K_u^0 denotes the interior of $K_u := \{x \in \Omega : \nabla u(x) = 0\}$.

Define, for a given positive supersolution u of (1),

$$m_x(r) = \inf_{y \in B_r(x)} u(y) \text{ and } \rho_x(r) = \inf_{y \in B_r(x)} \rho(y) \text{ for } 0 < r < d_\Omega(x) := \text{dist}(x, \partial\Omega).$$

Note when $\Omega = \mathbb{R}^N$ we set $d_\Omega(x) = \infty$. Also we set

$$\alpha_{N,p} = \frac{1-p}{2-p} \left(N + \frac{p}{1-p} \right)^{\frac{-1}{1-p}} \text{ for } 0 \leq p < 1.$$

Theorem 1. *Suppose f and ρ satisfy (C), Ω is an arbitrary domain in \mathbb{R}^N and u is a positive classical supersolution of (1).*

i) *If $0 < p < 1$ then for all $x \in \Omega \setminus K_u^0$ we have*

$$\int_{m_x(r)}^{u(x)} \frac{ds}{f(s)^{\frac{1}{1-p}}} \geq \frac{2-p}{1-p} \alpha_{N,p} \int_0^r (s\rho_x(s))^{\frac{1}{1-p}} ds, \text{ for } 0 < r < d_\Omega(x). \quad (3)$$

In particular, when $\rho \equiv 1$ we get

$$\int_{m_x(r)}^{u(x)} \frac{ds}{f(s)^{\frac{1}{1-p}}} \geq \alpha_{N,p} r^{\frac{2-p}{1-p}}, \quad 0 < r < d_\Omega(x). \quad (4)$$

ii) *When $p = 0$ the above estimates are true for all $x \in \Omega$.*

The above theorem leads us to the following explicit estimates on supersolutions of (1) depending on whether or not $f^{\frac{-1}{1-p}} \in L^1(0, a)$ for $0 < a < a_f$.

Proposition 1. *Suppose Ω is an arbitrary domain in \mathbb{R}^N and u is a positive classical supersolution of (1) with $0 \leq p < 1$ in Ω . If $f^{\frac{-1}{1-p}} \in L^1(0, a)$ for $0 < a < a_f$ then*

$$u(x) \geq F^{-1} \left(\alpha_{N,p} \frac{2-p}{1-p} \int_0^r (s\rho_x(s))^{\frac{1}{1-p}} ds \right) \text{ for } x \notin K_u^0, \quad (5)$$

where $F(t) := \int_0^t \frac{ds}{f(s)^{\frac{1}{1-p}}}$, $0 < t < a_f$.

In particular, when $\rho \equiv 1$ we have

$$u(x) \geq F^{-1} \left(\alpha_{N,p} d_\Omega(x)^{\frac{2-p}{1-p}} \right) \text{ for } x \notin K_u^0. \quad (6)$$

As a consequence, if

$$\alpha_{N,p} \frac{2-p}{1-p} \int_0^{d_\Omega(x)} (s\rho_x(s))^{\frac{1}{1-p}} ds > \|F\|_\infty = \int_0^{a_f} \frac{ds}{f(s)^{\frac{1}{1-p}}}, \quad (7)$$

then in case $0 < p < 1$ we have $\nabla u \equiv 0$ in a neighborhood of x , means that u is a deadcore solution, and in case $p = 0$ equation (1) has no any positive supersolution. When $\rho \equiv 1$ then (7) reads as

$$\alpha_{N,p} d_\Omega(x)^{\frac{2-p}{1-p}} > \|F\|_\infty. \quad (8)$$

Moreover, if $\|F\|_\infty < \infty$ then in case $\Omega = \mathbb{R}^N$ every positive supersolution u is constant when $0 < p < 1$, and there is no positive supersolution when $p = 0$. Also, when Ω is an unbounded domain with the property that

$$\sup_{x \in \Omega} d_\Omega(x) = \infty$$

then every positive supersolution u is constant in the region Ω_M^c for some $M > 0$ where

$$\Omega_M = \{x \in \Omega; \text{dist}(x, \partial\Omega) < M\},$$

while there is no positive supersolution when $p = 0$.

Remark 1. Note that using the above estimates of Proposition 1, where we assumed $f^{\frac{-1}{1-p}} \in L^1(0, a)$ for $0 < a < a_f$, we can formulate several general Liouville-type results on supersolutions of (1) in unbounded domains, depending on f and ρ , but we prefer to do this in concrete examples in Section 3.

For the case when $f^{\frac{-1}{1-p}} \notin L^1(0, a)$ for $a > 0$ we have

Proposition 2. Suppose Ω is an arbitrary domain in \mathbb{R}^N and u is a positive classical supersolution of (1) in Ω . If $f^{\frac{-1}{1-p}} \notin L^1(0, a)$ for $a > 0$, and $x \notin K_u^0$, then

$$\inf_{y \in B_r(x)} u(y) \leq G^{-1} \left(\alpha_{N,p} \frac{2-p}{1-p} \int_0^r (s\rho_x(s))^{\frac{1}{1-p}} ds \right). \quad (9)$$

where in this case G is a positive primitive of the function $\frac{-1}{f^{\frac{1}{1-p}}}$ on $(0, a_f)$ with $F(0) = \infty$.

In particular, when $\rho \equiv 1$ we have

$$\inf_{y \in B_r(x)} u(y) \leq G^{-1} \left(\alpha_{N,p} r^{\frac{2-p}{1-p}} \right) \quad \text{for } x \notin K_u^0, \quad 0 < r < d_\Omega(x). \quad (10)$$

As a consequence, if Ω is an exterior domain in \mathbb{R}^N and $0 < p < 1$ then u is eventually constant provided the following

$$N = 2 \quad \text{and} \quad \liminf_{r \rightarrow \infty} G^{-1} \left(\alpha_{N,p} \frac{2-p}{1-p} \int_0^r (s\rho_x(s))^{\frac{1}{1-p}} ds \right) = 0, \quad (11)$$

or

$$N > 2 \quad \text{and} \quad \liminf_{r \rightarrow \infty} r^{N-2} G^{-1} \left(\alpha \frac{2-p}{1-p} \int_0^r (s\rho_x(s))^{\frac{1}{1-p}} ds \right) = 0. \quad (12)$$

Also when Ω is an exterior domain in \mathbb{R}^N and $p = 0$ then there is no positive supersolution provided (11) or (12) hold.

2 Proof of the main results

Proof of Theorem 1. Assume $0 \leq p < 1$ and let u be a positive supersolution of (1). Fix an $x_0 \in \Omega$ with $\nabla u(x_0) \neq 0$ and $0 < r < d_\Omega(x_0)$. Then we have

$$-\Delta u \geq \rho_{x_0}(r) f(m_{x_0}(r)) |\nabla u|^p \quad \text{in } B_r(x_0). \quad (13)$$

Now set

$$w_r(y) = \alpha \rho_{x_0}(r)^{\frac{1}{1-p}} f(m_{x_0}(r))^{\frac{1}{1-p}} (r^q - |y - x_0|^q),$$

where $\alpha := \alpha_{N,p}$ and $q := \frac{2-p}{1-p}$. Then we have

$$-\Delta w_r = \rho_{x_0}(r) f(m_{x_0}(r)) |\nabla w_r|^p \quad \text{in } B_r(x_0) \quad \text{and} \quad w_r \equiv 0 \quad \text{on } \partial B_r(x_0). \quad (14)$$

Here we show that the function $u - w_r$ takes its minimum on $\partial B_r(x_0)$. When $p = 0$ this is obvious by the maximum principle and that we have $-\Delta(u - w_r) \geq 0$ in $B_r(x_0)$. When $0 < p < 1$, take an $s \in (0, 1)$ and set $v_s = u - sw_r$. We show that v_s takes its minimum on $\partial B_r(x_0)$. Assume not and suppose v_s takes its minimum at some $y \in B_r(x_0)$. First note that $y \neq x_0$ because $\nabla v_s(x_0) \neq 0$ by the above assumption that $\nabla u(x_0) \neq 0$. Now using $\nabla v_s(y) = 0$, that implies $\nabla u(y) = s\nabla w_r(y)$, we compute, using (13) and (14),

$$\Delta v_s(y) \leq (s - s^p)\rho_{x_0}(r)f(m_{x_0}(r))|\nabla w_r(y)|^p,$$

and since $s - s^p < 0$ and $\nabla w_r(y) \neq 0$ we get $\Delta v_s(y) < 0$, a contradiction. Hence v_s takes its minimum on $\partial B_r(x_0)$. And since $v_s|_{\partial B_r(x_0)} \geq m_{x_0}(r)$, then

$$u(y) - sw_r(y) \geq m_{x_0}(r), \quad y \in B_r(x_0)$$

and since $s \in (0, 1)$ was arbitrary we get

$$u(y) - m_{x_0}(r) \geq w_r(y), \quad y \in B_r(x_0).$$

Now let $0 < h < r$ and $y \in B_{r-h}(x_0) \subset B_r(x_0)$. Then from the above inequality we also have

$$u(y) - m_{x_0}(r) \geq w_r(y) \geq \alpha\rho_{x_0}(r)^{\frac{1}{1-p}}f(m_{x_0}(r))^{\frac{1}{1-p}}(r^q - (r-h)^q), \quad y \in B_{r-h}(x_0),$$

and taking infimum over $B_{r-h}(x_0)$ we obtain

$$m_{x_0}(r-h) - m_{x_0}(r) \geq \alpha\rho_{x_0}(r)^{\frac{1}{1-p}}f(m_{x_0}(r))^{\frac{1}{1-p}}(r^q - (r-h)^q),$$

or

$$\frac{m_{x_0}(r-h) - m_{x_0}(r)}{h} \geq \alpha\rho_{x_0}(r)^{\frac{1}{1-p}}f(m_{x_0}(r))^{\frac{1}{1-p}} \frac{(r^q - (r-h)^q)}{h}.$$

Letting $h \rightarrow 0$ in the above we arrive at the following ordinary differential inequality with initial value condition

$$\begin{cases} -m'_{x_0}(r) \geq q\alpha r^{q-1}\rho_{x_0}(r)^{\frac{1}{1-p}}f(m_{x_0}(r))^{\frac{1}{1-p}}, & r \in (0, d_\Omega(x_0)), \\ m_{x_0}(0) = u(x_0) \end{cases} \quad (15)$$

where $' = \frac{d}{dr}$. Dividing inequality (15) by $f(m_{x_0}(r))^{\frac{1}{1-p}}$ and integrate from 0 to r we get

$$\int_{m_{x_0}(r)}^{u(x_0)} \frac{ds}{f(s)^{\frac{1}{1-p}}} \geq q\alpha \int_0^r (s\rho_{x_0}(s))^{\frac{1}{1-p}} ds,$$

that proves the estimate (3) when $\nabla u(x_0) \neq 0$. To prove (3) in the case when $\nabla u(x) = 0$ but $x \notin K_u^0$ it suffices to take a sequence $x_n \in \Omega$ such that $\nabla u(x_n) \neq 0$ and $x_n \rightarrow x$, then write (3) for x_n and let $n \rightarrow \infty$. Also note that when $p = 0$ then $K_u^0 = \emptyset$ by the assumption that $f(t) > 0$ for $t > 0$, hence (3) is true for all $x \in \Omega$. \square

Remark 2. *The proof of Theorem 1 can be simplified when f is a C^1 increasing function. Indeed in this case taking $w(y) = F(u(y))$ in $B_r(x)$ for a fixed $x \in \Omega \setminus K_u^0$, where*

$$F(t) := \int_{m_x(r)}^t \frac{ds}{f(s)^{\frac{1}{1-p}}}, \quad t > m_x(r),$$

then by the formula $\Delta F(u) = F''(u)|\nabla u|^2 + f'(u)(\Delta u)$ and the fact that $F''(t) < 0$ we get

$$-\Delta w \geq \rho(y)|\nabla w|^p, \quad y \in B_r(x)$$

and then proceed as above we arrive at the desired estimate.

Now we give a short proof for Propositions 1 and 2. For the proof we also need the following lemma proved by J. Serrin and H. Zou in [22], see also [1].

Lemma 1. *Suppose $\{|x| > R > 0\} \subset \Omega$. Let u be a positive weak solution of the inequality*

$$-\Delta u \geq 0, \quad x \in \Omega. \quad (16)$$

Then there exist a constant $C = C(N, u, R) > 0$ such that

$$u(x) \geq C|x|^{2-N}, \quad (17)$$

provided $N > 2$, while

$$\liminf_{x \rightarrow \infty} u(x) > 0, \quad (18)$$

if $N \leq 2$.

Proof of Propositions 1 and 2. Let $\frac{1}{f^{1-p}} \in L^1(0, a)$ for $a > 0$ and $F(t) := \int_0^t \frac{ds}{f(s)^{\frac{1}{\alpha}}}$. Then F is an increasing function on D_f . Now if u is a positive supersolution of (1) in Ω and $x \notin K_u^0$ then from (3) we get

$$F(u(x)) \geq F(u(x)) - F(m_x(r)) \geq \frac{2-p}{1-p} \alpha_{N,p} \int_0^r (s\rho_x(s))^{\frac{1}{1-p}} ds, \quad \text{for } 0 < r < d_\Omega(x). \quad (19)$$

Now the above inequality easily gives the desired results in Proposition 1 (note also that when $p = 0$ then $K_u^0 = \emptyset$ for a positive supersolution u because of the assumption $f(u) > 0$ for $u > 0$). Similarly we get the estimates (9) and (10) in Proposition 2. To prove the rest we can simply use Lemma 1. Indeed, in an exterior domain Ω if there exists $x \notin K_u^0$ with $|x|$ sufficiently large then we take $r = \frac{|x|}{2}$ in (10) to get

$$\inf_{y \in B_r(x)} u(y) \leq G^{-1} \left(\alpha_{N,p} r^{\frac{2-p}{1-p}} \right).$$

By Lemma 1 there exists a constant C depends only on u, Ω and N such that $\inf_{y \in B_r(x)} u(y) \geq Cr^{2-N}$ when $N > 2$, and $\inf_{y \in B_r(x)} u(y) \geq C$ when $N = 2$. Using these in the above inequality and letting $r \rightarrow \infty$ taking into account (11) and (12) we arrive a contradiction. Hence there not exists $x \notin K_u^0$ with $|x|$ sufficiently large means that u is eventually constant. \square

3 Applications

In this section, as applications of the results in Section 2, we consider several concrete examples. We will frequently use the following lemma (in particular part (ii) of the lemma). We give here a sketch of the proof, for the detailed proof one can see Lemmas 2.1 and 2.2 in [14].

Lemma 2. *Let u be a positive, not eventually constant solution of $-\Delta u \geq 0$ in an exterior domain Ω . Denote $I(R) := \inf_{|x|=R} u(x)$, then we have*

(i) there exists R_1 large such that the function $I(R)$ is either strictly increasing, or strictly decreasing or constant in $(R_1, +\infty)$.

(ii) $I(R)$ is bounded when $N \geq 3$. Also when $N = 2$ we have

$$I(R) \leq C \log R.$$

Proof. We give just a proof for part(ii), to prove (i) see lemma 2,1 in [14].

Define for $x \in A(R_1, R_2) := \{x \in \mathbb{R}^N; R_1 < |x| < R_2\}$

$$\Phi_1(x) = \frac{I(R_1) - I(R_2)}{R_1^{2-N} - R_2^{2-N}} (|x|^{2-N} - R_2^{2-N}) + I(R_2),$$

when $N \geq 3$. We then see that Φ_1 is a harmonic function vanishing on the $\partial A(R_1, R_2)$, hence by the maximum principle we have $u \geq \Phi_1$ in $A(R_1, R_2)$. Now assume $I(R)$ is not bounded and fix an $R \in (R_1, R_2)$ then

$$I(R) \geq \frac{I(R_1) - I(R_2)}{R_1^{2-N} - R_2^{2-N}} (R^{2-N} - R_2^{2-N}) + I(R_2) \rightarrow \infty \text{ as } R_2 \rightarrow \infty,$$

a contradiction.

In the case $N = 2$, taking

$$\Phi_2(x) := \frac{I(R_1) - I(R_2)}{\log R_1 - \log R_2} (\log |x| - \log R_2) + I(R_2),$$

we see that Φ_2 is harmonic functions vanishing on the $\partial A(R_1, R_2)$ and similar as above, $u \geq \Phi_2$ in $A(R_1, R_2)$ hence for $R \in (R_1, R_2)$ we get

$$\begin{aligned} I(R) &\geq \frac{I(R_1) - I(R_2)}{\log R_1 - \log R_2} (\log R - \log R_2) + I(R_2) \\ &= \left(\frac{\log R_1 - \log R_2}{\log R - \log R_2} \right) I(R_1) + \left(\frac{\log R_1 - \log R}{\log R_1 - \log R_2} \right) I(R_2) \\ &\geq \left(\frac{\log R_1 - \log R}{\log R_1 - \log R_2} \right) I(R_2). \end{aligned}$$

Now fix R_1 and $R = 2R_1$, then for R_2 large we see from the above that

$$I(R_2) \leq C \log R_2, \text{ for } R_2 \text{ sufficiently large.}$$

□

3.1 Liouville-type results

Consider the equation

$$-\Delta u = u^q |\nabla u|^p, \quad x \in \Omega, \tag{20}$$

where Ω is an arbitrary domain in \mathbb{R}^N .

Theorem 2. Let u be a positive supersolution of (20), where $q \geq 0$ and $0 \leq p < 1$. Then
i) if $\frac{q}{1-p} < 1$ then

$$u(x) \geq \left(\frac{\alpha(1-p-q)}{1-p} \right)^{\frac{1-p}{1-p-q}} r^{\frac{2-p}{1-p-q}}, \quad 0 < r < d_\Omega(x) \quad x \notin K_u^0. \quad (21)$$

In particular, when $\Omega = \mathbb{R}^N$ then u is constant. Also, when $\Omega = B_R^c$ is an exterior domain then u is constant in $B_{R'}^c$ for some $R' > R$. Moreover, if Ω is unbounded with the property that

$$\sup_{x \in \Omega} d_\Omega(x) = \infty, \quad (22)$$

and u is bounded, then there exists an $M > 0$ so that u is constant in the region Ω_M^c where

$$\Omega_M = \{x \in \Omega; \text{dist}(x, \partial\Omega) < M\}.$$

See Example 1 to see (at least in the case of $p = 0$) that one does indeed need to assume u is bounded.

ii) If $\frac{q}{1-p} > 1$ then for all $x \notin K_u^0$ we have

$$\inf_{y \in B_r(x)} u(y) \leq Cr^{\frac{p-2}{p+q-1}}, \quad 0 < r < d_\Omega(x). \quad (23)$$

In particular if Ω is an exterior domain then u is eventually constant if

$$(N-2)q + (N-1)p < N \quad (24)$$

iii) If $\frac{q}{1-p} = 1$ then for all $x \notin K_u^0$ we have

$$u(x) \geq m_x(r) e^{\alpha_{N,p} r^{\frac{2-p}{1-p}}}, \quad 0 < r < d_\Omega(x). \quad (25)$$

In particular, when $\Omega = \mathbb{R}^N$ then u is constant. Also, when $\Omega = B_R^c$ is an exterior domain then u is constant in $B_{R'}^c$ for some $R' > R$.

Proof. i) Note equation (20) is of the form of equation (1) with $f(u) = u^q$ and $\rho(x) \equiv 1$. Hence if $\frac{q}{1-p} \neq 1$ then by Theorem 1 we get

$$\frac{1-p}{1-p-q} \left(u(x)^{\frac{1-p-q}{1-p}} - m_x(r)^{\frac{1-p-q}{1-p}} \right) \geq \alpha_{N,p} r^{\frac{2-p}{1-p}}, \quad 0 < r < d_\Omega(x), \quad (26)$$

for all $x \notin K_u^0$.

Now if $\frac{q}{1-p} < 1$ then we get the following pointwise estimate

$$u(x) \geq \left(\frac{\alpha(1-p-q)}{1-p} \right)^{\frac{1-p}{1-p-q}} r^{\frac{2-p}{1-p-q}}, \quad 0 < r < d_\Omega(x) \quad x \notin K_u^0. \quad (27)$$

Now if $\Omega = \mathbb{R}^N$ then the assertion of theorem is obvious. Because in this case we have $d_\Omega(x) = \infty$ for every $x \in \Omega$ and then from the estimate (27) we must have $K_u^0 = \Omega$ means that u is constant. To prove the second part of (i), for simplicity take $\Omega = B_1^c$ and let $I(R) := \inf_{|x|=R} u(x)$. Assume that u is not eventually constant then for any $R > 1$ there exists $x \notin K_u^0$ with $|x| \geq R$, and then from (27) we have $u(x) \geq CR^{\frac{2-p}{1-p-q}}$, where C is a constant independent of R for all large R . Now

by the continuity of u and the fact that u is constant in every connected component of K_u^0 we see that we have $u(x) \geq CR^{\frac{2-p}{1-p-q}}$ for all x with $|x| \geq R$, thus $I(R) \geq CR^{\frac{2-p}{1-p-q}} \rightarrow \infty$ as $R \rightarrow \infty$ which contradicts Lemma 2.

Now let Ω satisfy (22). For an $x \in \Omega$ with $\nabla u(x) \neq 0$ and $R < d_\Omega(x)$ we get, from (21), $u(x) \geq Cd_\Omega(x)^{\frac{2-p}{1-p-q}}$. Now assume u is a bounded solution but the assertion is not true, then there exists a sequence $\{x_n\} \subset \Omega$ such that $d_\Omega(x_n) \rightarrow \infty$ with $\nabla u(x_n) \neq 0$, then from the above we get $u(x_n) \geq Cd_\Omega(x_n)^{\frac{2-p}{1-p-q}}$ implies that u is unbounded, a contradiction.

ii) Now assume $\frac{q}{1-p} > 1$. Then by Proposition 2, or from (26), we get

$$\inf_{y \in B_r(x)} u(y) \leq Cr^{\frac{p-2}{p+q-1}}, \quad x \notin K_u^0. \quad (28)$$

Now let Ω be an exterior domain in \mathbb{R}^N and assume that u is not eventually constant then for any $R > 0$ large there exists $x \notin K_u^0$ with $|x| > R$. If $N = 2$ then (28) contradicts (18), hence u is eventually constant. Also when $N \geq 3$ we have $u(x) \geq C|x|^{2-N}$ by (17), hence we need $N - 2 \geq \frac{2-p}{p+q-1}$, or equivalently $(N - 2)q + p(N - 1) \geq N$. Thus u is eventually constant if we have

$$(N - 2)q + p(N - 1) < N.$$

(iii) Now assume $\frac{q}{1-p} = 1$, then by Theorem 1 and that $\int_{m_x(r)}^u \frac{ds}{f(s)^{\frac{1}{1-p}}} = \ln \frac{u}{m_x(r)}$ we get

$$u(x) \geq m_x(r)e^{\alpha_{N,p}r^{\frac{2-p}{1-p}}}, \quad 0 < r < d_\Omega(x), \quad x \notin K_u^0.$$

Now if $\Omega = \mathbb{R}^N$ then by Lemma 1 we get $m_x(r) > C > 0$ when $N = 2$, and $m_x(r) > C|r|^{2-N}$ when $N \geq 3$. Now using the fact that for arbitrary $\beta > 0$ we have $e^{\alpha_{N,p}r^{\frac{2-p}{1-p}}} \geq r^\beta$ for large r , then letting $r \rightarrow \infty$ in the above estimates easily gives that u is constant. A similar argument as in part (i) for the case when Ω is an exterior domain shows that u is eventually constant. Note in this case we also use the fact that by Lemma 1 we have, for $r = \frac{|x|}{2}$ with $|x|$ sufficiently large, $m_x(r) > C > 0$ when $N = 2$, and $m_x(r) > C|r|^{2-N}$ when $N \geq 3$. □

Example 1. Let $0 < q < 1$ and $S \subset S^{N-1}$ a smooth subset with nonempty boundary. For $r = |x|$ and $\theta = \frac{x}{|x|}$ we consider the cone $\Omega := \{x \in \mathbb{R}^N : r > 0, \theta \in S\}$. Then $u(x) = u(r, \theta) = r^{\frac{2}{1-q}}w(\theta)$ is a bounded positive classical solution of $-\Delta u = u^q$ in Ω with $u = 0$ on $\partial\Omega$ exactly when $w > 0$ is a bounded classical solution of

$$-\Delta_\theta w - \beta_q w = w^q \quad \text{in } S, \quad w = 0 \quad \text{on } \partial S, \quad (29)$$

where $\beta_q = \frac{2}{1-q} \left(N - 2 + 2\frac{1}{1-q} \right)$ and Δ_θ is the Laplace-Beltrami operator on S^{N-1} . Consider the energy

$$E(w) := \int_S \frac{|\nabla_\theta w|^2}{2} - \frac{\beta_q}{2} w^2 - \frac{|w|^{q+1}}{q+1} d\theta.$$

Provided $\beta_q < \lambda_1(S)$ (the first eigenvalue of $-\Delta_\theta$ in $H_0^1(S)$) and since $q < 1$ one sees there is a nonzero minimizer w of E over $H_0^1(S)$ and one can take $w > 0$. After standard arguments one sees this w is a positive bounded classical solution of (29).

Now consider the more general inequality

$$-\Delta u = |x|^\beta u^q |\nabla u|^p, \quad x \in \Omega, \quad (30)$$

where $\beta \in \mathbb{R}$ and Ω is an exterior domain in \mathbb{R}^N . For simplicity let $\Omega = \mathbb{R}^N \setminus B_1$.

Theorem 3. *Let u be a positive supersolution of (30) in $\Omega = \mathbb{R}^N \setminus B_1$, $q \geq 0$, $0 \leq p < 1$ and $\beta \in \mathbb{R}$.*

- i) If $\frac{q}{1-p} \leq 1$ then every positive supersolution is eventually constant if $\beta \geq p - 2$.
ii) If $\frac{q}{1-p} > 1$ then every positive supersolution is eventually constant if*

$$(N - 2)q + p(N - 1) < N + \beta.$$

Proof. (i) First note that when $\frac{q}{1-p} < 1$ then the case $\beta > 0$ is not interesting as in this case we have $|x|^\beta \geq 1$ and then u is also a supersolution of (20). So assume $\beta < 0$, then taking $\rho(x) := |x|^\beta$ we have

$$\rho_x(r) = \inf_{B_r(x)} \rho(y) = (|x| + r)^\beta, \quad 0 < r < d_\Omega(x) = |x| - 1.$$

Then for $x \notin K_u^0$ we have

$$\int_0^r (s\rho_x(s))^{\frac{1}{1-p}} ds = \int_0^r [s(|x| + s)^\beta]^{\frac{1}{1-p}} ds = |x|^{\frac{2+\beta-p}{1-p}} \int_0^{\frac{r}{|x|}} [t(1+t)^\beta]^{\frac{1}{1-p}} dt.$$

Now from Proposition 1 we get the following explicit estimate at every $x \notin K_u^0$

$$u(x) \geq \left(\frac{\alpha(1-p-q)}{1-p} \right)^{\frac{1-p}{1-p-q}} |x|^{\frac{2+\beta-p}{1-p-q}} \left(\int_0^{\frac{|x|-1}{|x|}} [t(1-t)^\beta]^{\frac{1}{1-p}} dt \right)^{\frac{1-p}{1-p-q}}, \quad x \notin K_u^0.$$

In particular, for any $\gamma > 1$ we get, for $x \notin K_u^0$

$$u(x) \geq C_{p,N,\gamma} |x|^{\frac{2+\beta-p}{1-p-q}}, \quad x \in \mathbb{R}^N \setminus B_\gamma,$$

where

$$C_{p,N,\gamma} = \left(\frac{\alpha(1-p-q)}{1-p} \right)^{\frac{1-p}{1-p-q}} \left(\int_0^{\frac{\gamma-1}{\gamma}} [t(1-t)^\beta]^{\frac{1}{1-p}} dt \right)^{\frac{1-p}{1-p-q}}.$$

Thus similar as in part (i) of Theorem 2, we see that u is eventually constant if $\frac{2+\beta-p}{1-p} > 0$ or equivalently $\beta > p - 2$. Also, similar to the proof of part (iii) of Theorem 2, one can treat the case $\frac{q}{1-p} = 1$ using the above estimate on $\rho(x)$ to show u is eventually constant if $\beta > p - 2$.

(ii) Assume $\frac{q}{1-p} > 1$. If $\beta < 0$ then using the above computations on $\rho_x(r)$ and Proposition 2 we get, for $x \notin K_u^0$ and $\gamma > 1$

$$\inf_{y \in B_r(x)} u(y) \leq C_{p,N,\gamma} r^{\frac{p-2-\beta}{p+q-1}}, \quad x \in \mathbb{R}^N \setminus B_\gamma.$$

Then similar as the proof of part (ii) of Theorem 2, where we also used Lemma 1, we see that when $N = 2$ and $p - 2 < \beta$ then u must be eventually constant. Also when $N \geq 3$, if there exists $x \notin K_u^0$ with $|x|$ sufficiently large, then using the estimate $u(x) \geq C|x|^{2-N}$ for superharmonic functions in exterior domains by Lemma 1, we must have

$$N - 2 \geq \frac{2 + \beta - p}{p + q - 1},$$

or equivalently

$$(N-2)q + p(N-1) \leq N + \beta.$$

Thus u is eventually constant if we have

$$(N-2)q + p(N-1) < N + \beta.$$

Now consider the case ($\beta > 0$). Here we have $\rho_x(r) = (|x| - r)^\beta > (\frac{3}{2})^\beta |x|^\beta$ for $0 < r < \frac{|x|}{2}$ and similar as above we will see that if $(N-2)q + p(N-1) < N + \beta$ then u is eventually constant .

□

3.2 Deadcore supersolutions in bounded domains

Consider equation (1) with $\rho \equiv 1$ and

$$f(u) = u^q + u^r \quad \text{or} \quad f(u) = \max\{u^q, u^r\}$$

with $0 < q < 1 - p < r$. Then we see that in both cases we have $f^{\frac{1}{1-p}} \in L^1(0, a)$ for every $a > 0$ and also $F(\infty) = \int_0^\infty \frac{ds}{f(s)^{\frac{1}{1-p}}} < \infty$.

For example consider (1) with $f(u) = \max\{u^q, u^r\}$, i.e.,

$$-\Delta u = \max\{u^q, u^r\} |\nabla u|^p, \quad x \in \Omega. \quad (31)$$

As a corollary of Proposition 1 we have

Corollary 1. *Let u be a positive supersolution of (31) in an arbitrary domain Ω (bounded or not) and $0 < q < 1 - p < r$. Then u is a dead core solution if*

$$\sup_{x \in \Omega} d_\Omega(x) > \left(\frac{1-p}{1-p-q} + \frac{1-p}{p+r-1} \right)^{\frac{1-p-q}{1-p}} =: \beta.$$

In particular, if $\Omega = B_R$ with $R > \beta$ then $B_{R-\beta}$ is a dead core set of any solution u . Also when $\Omega = \mathbb{R}^N \setminus B_1$ then $\Omega = \mathbb{R}^N \setminus B_{1+\beta}$ is a dead core set of any solution u .

Proof. By the notation of Proposition 1 we have

$$F(\infty) = \int_0^\infty \frac{ds}{f(s)^{\frac{1}{1-p}}} = \int_0^1 \frac{ds}{s^{\frac{q}{1-p}}} + \int_1^\infty \frac{ds}{s^{\frac{r}{1-p}}} = \frac{1-p}{1-p-q} + \frac{1-p}{p+r-1}.$$

Then by Proposition 1 we see that any positive supersolution u must be constant on the set

$$\mathcal{S} := \{x \in \Omega; d_\Omega(x)^{\frac{2-p}{1-p}} \geq F(\infty)\} = \{x \in \Omega; d_\Omega(x) \geq \beta\},$$

where

$$\beta := \left(\frac{1-p}{1-p-q} + \frac{1-p}{p+r-1} \right)^{\frac{1-p-q}{1-p}}.$$

This in particular shows that if a domain Ω satisfies

$$\sup_{x \in \Omega} d_\Omega(x) > \beta,$$

then every supersolution u of ((31)) is a dead core supersolution. Now when $\Omega = B_R$ with $R > \beta$ then we have

$$B_{R-\beta} \subset \{x \in \Omega; d_\Omega(x) \geq \beta\},$$

hence for any supersolution u we have

$$u \equiv C \text{ on } B_{R-\beta}.$$

Similarly when $\Omega = \mathbb{R}^N \setminus B_1$, then for every supersolution u we must have

$$u \equiv C \text{ on } |x| \geq 1 + \beta.$$

□

Now consider the equation

$$-\Delta u = \frac{|\nabla u|^p}{(1-u)^q}, \quad x \in \Omega \quad (q > 1 > p > 0) \quad (32)$$

which is of the form (1) with **singular** nonlinearity

$$f(u) = \frac{1}{(1-u)^q}, \quad q > 1.$$

We have

Corollary 2. *Let $0 \leq u < 1$ be a positive supersolution of (32) in an arbitrary domain Ω (bounded or not), where $0 < p < 1 < q$. Then u is a dead core supersolution if*

$$\sup_{x \in \Omega} d_\Omega(x) > \left(\frac{1-p}{\alpha(1+q-p)} \right)^{\frac{1-p}{2-p}}$$

Proof. Here we have

$$F(t) = \int_0^t \frac{ds}{f(s)^{\frac{1}{1-p}}} = \int_0^t (1-s)^{\frac{q}{1-p}} = \frac{1-p}{1+q-p} [1 - (1-t)^{\frac{1+q-p}{1-p}}].$$

Hence, if $0 < u < 1$ is a supersolution of (32) then by the above results we get

$$(1-u(x))^{\frac{1+q-p}{1-p}} \leq 1 - \frac{\alpha(1+q-p)}{1-p} d_\Omega(x)^{\frac{2-p}{1-p}}, \quad x \notin K_u^0.$$

This in particular shows that if

$$d_\Omega(x) \geq \beta := \left(\frac{1-p}{\alpha(1+q-p)} \right)^{\frac{1-p}{2-p}},$$

then $x \in K_u^0$. Moreover, if

$$\sup_{x \in \Omega} d_\Omega(x) > \beta,$$

then every supersolution $0 < u < 1$ of (32) is a dead core supersolution.

□

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