

ARITHMETICAL PROPERTIES AT THE LEVEL OF IDEMPOTENCE

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ABSTRACT. In this paper we give an attempt to extend some arithmetic properties such as multiplicativity, convolution products to the setting of operators theory. We provide a significant examples which are of interest in number theory. We also give a representation of the Euler differential operator by means of the Euler totient arithmetic function and idempotent elements of some associative unital algebra.

1. INTRODUCTION

In number theory, an arithmetical, or number-theoretic function is a function $\alpha : \mathbb{N} \rightarrow \mathbb{C}$. Their various properties were investigated by several authors and they represent an important research topic up to now. An important property shared by many number-theoretic functions is multiplicativity: an arithmetical function α is said to be multiplicative, if for all relatively prime positive integers n, m , we have

$$\alpha(nm) = \alpha(n)\alpha(m).$$

Examples of important arithmetical functions include: the Euler totient function, denoted φ , and defined as the number of positive integers less than and relatively prime to n . The Möbius function given by

$$\mu(n) = \begin{cases} (-1)^{\omega(n)} & \text{if } n \text{ is a square-free integer} \\ 0 & \text{otherwise,} \end{cases}$$

where $\omega(n)$ is the number of prime factors of n is also a multiplicative functions. Another important multiplicative function is the Ramanujan sum's, which is defined by [8]

$$c_n(j) := \sum_{\substack{\gcd(k,n)=1 \\ 1 \leq k \leq n}} \varepsilon_n^{jk},$$

where ε_n denotes a primitive n -th root of unity and $\gcd(k, n)$ to denote the greatest common divisor of the positive integers k and n . These sums fit naturally with other number-theoretic functions. For instance, one has

$$(1.1) \quad c_n(1) = \mu(n), \quad c_n(n) = \varphi(n),$$

For a more elaborate account on the multiplicative functions, we refer the reader to the texts [1, 2] and the survey articles [6].

In this work we suggest to extend the rang of the number-theoretic function and consider functions f such that:

- their domain are the positive integers and whose range is a subset of an unital associative algebra A over \mathbb{C} ;

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- they satisfy the property

$$(1.2) \quad f(nm) = f(n)f(m), \text{ when } \gcd(n, m) = 1.$$

Our first objective is to give a variety of significant examples, which are of interest in number theory. Notice that if \mathcal{A} is an unital algebra then every number-theoretic function $\alpha : \mathbb{N} \rightarrow \mathbb{C}$ can be identified with the function $n \rightarrow \alpha(n) e$ (e is the unit of \mathcal{A}).

Recall that an element $P \in \mathcal{A}$ such that $P^2 = P$ is called idempotent. A set of idempotent elements P_1, \dots, P_n is called orthogonal if,

$$P_i P_j = \delta_{ij} P_i \quad \text{and} \quad \sum_{i=1}^n P_i = e.$$

In this paper, we consider a set of orthogonal idempotent $P_j(n)$, indexed by two integers $n \geq 1$ and $j \geq 0$, satisfying:

- for every fixed integer n , the sequence $j \rightarrow P_j(n)$ is periodic with period n ;
- for every arithmetic progression $j + n, \dots, j + rn$, we have

$$P_j(n) = \sum_{k=1}^r P_{j+kn}(nr).$$

Under these conditions and for every fixed $j \geq 0$, the function $n \rightarrow P_j(n)$ is multiplicative. Another example considered in this note, consists to replace the root of unity in the Ramanujan sum's by an element s of \mathcal{A} satisfying $s^n = e$. Then, the following sum

$$(1.3) \quad \sum_{\substack{\gcd(k,n)=1 \\ 1 \leq k \leq n}} s^k$$

is a multiplicative function.

2. ARITHMETIC PROPERTIES OF IDEMPOTENTS

2.1. Multiplicative functions. Throughout, \mathcal{A} will be an associative unital algebra over \mathbb{C} . We recall the following definitions and basic facts

Definition 2.1. A function $f : \mathbb{N} \rightarrow \mathcal{A}$ is called multiplicative if

$$f(nm) = f(n)f(m), \text{ when } \gcd(n, m) = 1.$$

Lemma 2.1. *If $f : \mathbb{N} \rightarrow \mathcal{A}$ is multiplicative, then $f(1)$ is idempotent.*

Proof. Taking $n = m = 1$, we have $f(1) = f(1.1) = f(1)f(1)$. So, $f(1)$ is idempotent. \square

The following proposition is a characterization of multiplicative functions.

Proposition 2.2. *For an arithmetical function $f : \mathbb{N} \rightarrow \mathcal{A}$, the following are equivalent:*

1. f is multiplicative,
2. for all $n = p_1^{\alpha_1} \dots p_k^{\alpha_k}$ (the standard form factorization of n), we have

$$f(n) = f(p_1^{\alpha_1}) \dots f(p_k^{\alpha_k}).$$

We naturally extend the convolution product such as Dirichlet product, lcm product and unitary product (see, [1, 14, 12]) as follows:

$$(2.1) \quad (f * g)(n) = \sum_{kl=n} f(k)g(l) \quad (\text{Dirichlet product}).$$

$$(2.2) \quad (f \square g)(n) = \sum_{\text{lcm}(k,l)=n} f(k)g(l) \quad (\text{lcm-product}).$$

$$(2.3) \quad (f \sqcup g)(n) = \sum_{kl=n, \text{gcd}(k,l)=1} f(k)g(l) \quad (\text{unitary product}).$$

where $f, g : \mathbb{N} \rightarrow \mathcal{A}$.

The convolution products defined in (2.1), (2.2) and (2.3) are associative but not commutative in general. If, in addition, if for all integers n, m , and every prime numbers p and q , $f(p^n)$ commutes with $g(q^m)$, then we have

$$f * g = g * f, \quad f \square g = g \square f, \quad f \sqcup g = g \sqcup f.$$

The following propositions are easy to prove.

Proposition 2.3. 1) If f is an arithmetic function on \mathcal{A} , then

$$f * I = I * f = f, \quad f \square I = I \square f = f, \quad f \sqcup I = I \sqcup f = f,$$

where

$$I(n) = \begin{cases} e & \text{if } n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

2) If f is an arithmetical function such that $f(1)$ is invertible element in \mathcal{A} , then f has a unique inverse with respect to the Dirichlet product.

3) If f is a multiplicative function on \mathcal{A} and $b \in \mathcal{A}$ invertible element, then the function $n \mapsto bf(n)b^{-1}$ is multiplicative.

4) If f and g are multiplicative functions, then $f * g$ is also a multiplicative function.

Proposition 2.4. 1) If \mathcal{A} is equipped with a norm $\|\cdot\|$ and f a multiplicative function on \mathcal{A} , then $n \rightarrow \|f(n)\|$ is a multiplicative function.

2) If \mathcal{A} is the algebra of $N \times N$ -matrix and f is multiplicative on \mathbb{C}^N , then function $n \mapsto \det(f(n))$ is multiplicative.

2.2. Idempotents.

Definition 2.2. Let \mathcal{A} be an algebra with unity e . We say that a collection $\{P_j(n)\}_{j=0, n=1}^{\infty}$ is an arithmetic system of idempotent if the following conditions are satisfied:

I) for all $i, j = 1, 2, \dots$, we have $P_i(n)P_j(n) = \delta_{ij}P_i(n)$,

II) for all $n, j = 1, 2, \dots$, we have $P_{j+n}(n) = P_j(n)$,

III) for every arithmetic progression $\{j + n, \dots, j + nr\}$ we have

$$P_j(n) = \sum_{k=1}^r P_{j+kn}(nr).$$

The integer n in $P_j(n)$ is called level of the idempotence.

Theorem 2.5. *Let $\{P_j(n)\}_{j=0, n=1}^{\infty}$ be an arithmetic system of idempotents of \mathcal{A} . Then for arbitrary positive integers n, m, k and l , we have*

$$(2.4) \quad P_k(n)P_l(m) = \begin{cases} P_j(\text{lcm}(n, m)) & \text{if } \gcd(n, m) \mid l - k \\ 0 & \text{otherwise.} \end{cases}$$

where j is the unique solution $\pmod{\text{lcm}(n, m)}$ of the system of congruences

$$(2.5) \quad \begin{cases} j \equiv k \pmod{n} \\ j \equiv l \pmod{m}. \end{cases}$$

Here $\text{lcm}(n, m)$ is the least common multiple of the integer n, m .

Lemma 2.6. *(The Chinese Remainder Theorem)*

The system of congruences

$$(2.6) \quad \begin{cases} x \equiv a \pmod{n} \\ x \equiv b \pmod{m}. \end{cases}$$

is solvable if, and only if $\gcd(n, m) \mid a - b$, any two solutions of the system are incongruent $\pmod{\text{lcm}(n, m)}$.

Proof. From condition III) we can write

$$P_k(n)P_l(m) = \sum_{s=0}^{m-1} \sum_{r=0}^{n-1} P_{k+sn}(nm)P_{l+rm}(nm).$$

If $\gcd(n, m) \nmid l - k$, then $k + sn \neq l + rm$ and $P_k(n)P_l(m) = 0$.

If $\gcd(n, m) \mid l - k$, then by Chinese remainder theorem there exist unique integer $j \pmod{\text{lcm}(n, m)}$ such that

$$(2.7) \quad \begin{cases} j \equiv k \pmod{n} \\ j \equiv l \pmod{m}. \end{cases}$$

Then from II) we get

$$P_k(n)P_l(m) = \sum_{s=0}^{m-1} \sum_{r=0}^{n-1} P_{j+sn}(nm)P_{j+rm}(nm).$$

On the other hand, from I) we see that for $0 \leq r \leq n - 1$ and $0 \leq s \leq m - 1$ the term $P_{j+rn}(nm)P_{j+sm}(nm)$ is different from zero, if and only if $rn = sm$, which is equivalent to $r = r' \frac{n}{\gcd(n, m)}$ and $s = r' \frac{m}{\gcd(n, m)}$, with $0 \leq r' \leq \gcd(n, m) - 1$.

Then

$$P_k(n)P_l(m) = \sum_{r'=0}^{\gcd(n, m)-1} P_{j+r' \text{lcm}(n, m)}(nm) = P_j(\text{lcm}(n, m)).$$

Therefore

$$P_k(n)P_l(m) = \begin{cases} P_j(\text{lcm}(n, m)) & \text{if } \gcd(n, m) \mid l - k \\ 0 & \text{otherwise} \end{cases}$$

where j is the unique integer $\pmod{\text{lcm}(n, m)}$ such that

$$(2.8) \quad \begin{cases} j \equiv l \pmod{n} \\ j \equiv k \pmod{m}. \end{cases}$$

□

Corollary 2.7. *The following hold*

1. *If n and m are relatively prime, we have*

$$P_j(n)P_j(m) = P_j(nm).$$

2. *If $n \mid m$, we have*

$$P_j(n)P_k(m) = \begin{cases} P_k(m) & \text{if } k \equiv j \pmod{n} \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.8. *Let $\alpha, \beta : \mathbb{N} \rightarrow \mathbb{C}$ two arithmetic functions and $j = 0, 1, \dots$, we have*

$$(2.9) \quad \alpha P_j \square \beta P_j = (\alpha \square \beta) P_j,$$

$$(2.10) \quad \alpha P_j \sqcup \beta P_j = (\alpha \sqcup \beta) P_j,$$

$$(2.11) \quad \nu_0 * (\alpha \square \beta) P_j = (\nu_0 * \alpha P_j) (\nu_0 * \beta P_j).$$

In particular,

$$(2.12) \quad P_j \square P_j(n) = M_2(n)P_j(n), \quad P_j \sqcup P_j(n) = 2^{\omega(n)}P_j(n)$$

where

$$(2.13) \quad M_s(n) = \begin{cases} 1 & \text{if } n = 1, \\ \prod_{k=1}^r ((a_s + 1)^s - a_k^s) & \text{if } n = \prod_{k=1}^r p_k^{a_k}, \end{cases}$$

and $\nu_0(n) = 1$.

Proof. Let n be a positive integer, we have

$$\begin{aligned} \alpha P_j \square \beta P_j(n) &= \sum_{\text{lcm}(k,l)=n} \alpha(k)\beta(l)P_j(k)P_j(l) \\ &= \sum_{\text{lcm}(k,l)=n} \alpha(k)\beta(l)P_j(\text{lcm}(k,l)) \\ &= (\alpha \square \beta)(n)P_j(n). \end{aligned}$$

To prove (2.10)

$$\begin{aligned} \alpha P_j \sqcup \beta P_j(n) &= \sum_{kl=n, \text{gcd}(k,l)=1} \alpha(k)\beta(l)P_j(k)P_j(l) \\ &= \sum_{kl=n, \text{gcd}(k,l)=1} \alpha(k)\beta(l)P_j(n) \\ &= (\alpha \sqcup \beta)(n)P_j(n). \end{aligned}$$

The equation (2.11) follows from the following identity [9]

$$(2.14) \quad (\nu_0 * \alpha)(\nu_0 * \alpha) = \nu_0 * (\alpha \square \beta).$$

□

3. RAMANUJAN SUM'S

The function $\alpha : \mathbb{N} \rightarrow \mathbb{C}$ is called even function (mod d) if $\alpha(n) = \alpha(\text{gcd}(n, d))$ for all n , that is if the value $\alpha(n)$ depends only on the $\text{gcd}(n, d)$. Hence if α is even(mod d), then it is sufficient to know the values $f(r)$, where $r \mid d$.

The Ramanujan sum's

$$(3.1) \quad c_n(j) := \sum_{\substack{\text{gcd}(k,n)=1 \\ 1 \leq k \leq n}} \varepsilon_n^{jk}.$$

If α is even (mod d), then it has a Ramanujan-Fourier expansion of the form

$$\alpha(n) = \sum_{r|d} (\mathcal{R}_\alpha)(r) c_r(n) \quad (n \in \mathbb{N}),$$

where the (Ramanujan-)Fourier coefficients $(\mathcal{R}_\alpha)(r)$ are uniquely determined and given by

$$(\mathcal{R}_\alpha)(r) = \sum_{\delta|d} \alpha\left(\frac{d}{\delta}\right) c_\delta\left(\frac{d}{r}\right).$$

Definition 3.1. We say that a set $\{P_j(n) : n \geq 1, j \geq 0\}$ is an arithmetic system of orthogonal idempotent, if

1. $\{P_j(n)\}_{j,n=1}^\infty$ is an arithmetic system of idempotent,
2. for every n

$$(3.2) \quad \sum_{j=0}^{n-1} P_j(n) = e.$$

Consider the sum

$$(3.3) \quad C_j(n) = \sum_{\substack{\gcd(k,n)=1 \\ 1 \leq k \leq n}} \varepsilon_n^{-jk} S^k(n), \quad j = 0, 1, \dots,$$

where $S(n) = \sum_{j=1}^n \varepsilon_n^j P_j(n)$.

For every divisor r of n , we denote by

$$(3.4) \quad T_{r,j}(n) = \sum_{\substack{\gcd(k,n)=\frac{n}{r} \\ 1 \leq k \leq n}} P_{k+j}(n).$$

Note that $T_{n,j}(n)$ is denoted simply by $T_j(n)$.

Proposition 3.1. *The following properties hold*

$$C_j(n) = \mu * \nu_1 P_j(n), \quad \& \quad T_j(n) = \nu_0 * \mu P_j(n),$$

where $\nu_k(n) = n^k$.

Proof. We have

$$\begin{aligned} \mu * \nu_1 P_j(n) &= \sum_{d|n} \mu(d) \frac{n}{d} P_j\left(\frac{n}{d}\right) \\ &= \sum_{d|n} \mu(d) \sum_{k=1}^{\frac{n}{d}} \varepsilon_n^{-jkd} S_n^{kd} \\ &= \sum_{k=1}^n \varepsilon_n^{-jk} S_n^k \sum_{\substack{d|n \\ d|k}} \mu(d) \\ &= \sum_{k=1}^n \varepsilon_n^{-jk} S_n^k \sum_{d|\gcd(k,n)} \mu(d). \end{aligned}$$

From Theorem 2.1 in [1] we have

$$(3.5) \quad \sum_{d|\gcd(k,n)} \mu(d) = \begin{cases} 1 & \text{if } \gcd(k, n) = 1, \\ 0 & \text{if } \gcd(k, n) > 1. \end{cases}$$

Hence

$$(3.6) \quad C_j(n) = \mu * \nu_1 P_j(n).$$

To prove 2)

$$\begin{aligned} T_j(n) &= \sum_{\substack{\gcd(k,n)=1 \\ 1 \leq k \leq n}} P_{j+k}(n) \\ &= \sum_{k=1}^n \sum_{\delta | \gcd(k,n)} \mu(\delta) P_{j+k}(n) \\ &= \sum_{\delta | n} \mu(\delta) \sum_{k=1}^{\frac{n}{\delta}} P_{j+k\delta}(n) \\ &= \sum_{\delta | n} \mu(\delta) P_j(\delta) \end{aligned}$$

□

Corollary 3.2.

$$\begin{aligned} C_j(n) &= n \prod_{l=1}^N (P_j(p^{\alpha_l}) - \frac{1}{p} P_j(p^{\alpha_l-1})), \\ T_j(n) &= \prod_{\substack{p|n \\ p \text{ prime}}} (e - P_j(p)). \end{aligned}$$

Proof. The functions $C_j(n)$ and T_j are multiplicative. So it suffice to compute $C_j(p^k)$ and $T_j(p^k)$ where p is prime number. Form proposition 4.1, we have

$$\begin{aligned} C_j(p^k) &= \sum_{l=0}^k \mu(p^l) p^{k-l} P_j(p^{k-l}) \\ &= p^k P_j(p^k) - p^{k-1} P_j(p^{k-1}) \\ &= p^k (P_j(p^k) - p^{-1} P_j(p^{k-1})), \end{aligned}$$

and

$$T_j(p^k) = \begin{cases} e - P_j(p) & \text{if } k = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Since C_j is multiplicative we have

$$C_j(n) = \prod_{l=1}^N C_j(p^{\alpha_l}) = n \prod_{l=1}^N (P_j(p^{\alpha_l}) - \frac{1}{p} P_j(p^{\alpha_l-1})).$$

Similarly,

$$T_j(n) = \prod_{\substack{p|n \\ p \text{ prime}}} (e - P_j(p))$$

□

Theorem 3.3. For fixed integers n and j , $\{T_{r,j}(n) : r \mid n\}$ is a set of $\tau(n)$ orthogonal idempotent:

$$(3.7) \quad \sum_{r \mid n} T_{r,j}(n) = e \quad \text{and} \quad T_{r,j}(n)T_{r',j}(n) = \delta_{r,r'}T_{r,j}(n)$$

where $\tau(n)$ is the number of divisor of n .

Lemma 3.4. The operators $T_j(n)$ and $C_j(n)$ are related by

$$C_j(n) = \sum_{r \mid n} c_n(n/r)T_{r,j}(n), \quad T_{n,j}(n) = \frac{1}{n} \sum_{r \mid n} c_n(n/r) C_j(r).$$

Proof.

$$(3.8) \quad \sum_{r \mid n} c_n(n/r)T_{r,j}(n) = \sum_{r \mid n} c_n(n/r) \sum_{\gcd(k,n)=\frac{n}{r}} P_{k+j}(n)$$

$$(3.9) \quad = \sum_{l=1}^n \sum_{r \mid n} c_n(n/r) \sum_{\gcd(k,n)=\frac{n}{r}} \varepsilon_n^{-(k+j)l} S_n^l$$

$$(3.10) \quad = \frac{1}{n} \sum_{l=1}^n \varepsilon_n^{-lj} \sum_{r \mid n} c_n(n/r) c_r(l) S_n^l$$

From formula Exercise 2.23 in [3], we have

$$(3.11) \quad \sum_{r \mid n} c_n(n/r) c_r(l) = \begin{cases} n & \text{if } \gcd(l, n) = 1, \\ 0 & \text{if } \gcd(l, n) > 1. \end{cases}$$

Hence

$$(3.12) \quad \sum_{r \mid n} c_n(n/r)T_{r,j}(r) = \sum_{\gcd(l,n)=1} \varepsilon_n^{-lj} S_n^l = C_j(n).$$

Similarly,

$$(3.13) \quad \sum_{r \mid n} c_n(n/r)C_j(r) = \sum_{r \mid n} c_n(n/r) \sum_{\gcd(k,n)=\frac{n}{r}} \varepsilon_n^{-jk} S_n^k$$

$$(3.14) \quad = \sum_{l=1}^n \sum_{r \mid n} c_n(n/r) c_r(l-j) P_l(n)$$

$$(3.15) \quad = \sum_{l=1}^n \sum_{r \mid n} c_n(n/r) c_r(l) P_{l+j}(n)$$

The result follows from formula (see, Exercise 2.23 [3])

$$(3.16) \quad \sum_{r \mid n} c_n\left(\frac{n}{r}\right) c_r(k) = \begin{cases} n & \text{if } \gcd(k, n) = 1, \\ 0 & \text{if } \gcd(k, n) > 1. \end{cases}$$

Hence

$$(3.17) \quad \sum_{r \mid n} c_n(n/r)C_j(r) = n \sum_{\gcd(k,n)=1} P_{k+j}(n) = nT_{n,j}(n).$$

□

Theorem 3.5. *Let $\alpha : \mathbb{N} \rightarrow \mathbb{C}$ be even function (mod n), then for all $j = 0, 1, \dots$, we have*

$$(3.18) \quad \sum_{r|n} \alpha(n/r) C_j(r) = \sum_{r|n} \mathcal{R}(\alpha)(r) T_{r,j}(n).$$

Proof.

$$\begin{aligned} \sum_{r|n} \alpha(r) C_j(r) &= \sum_{r|n} \sum_{\delta|r} c_r(r/\delta) T_{\delta,j}(r) \alpha(n/r) \\ &= \sum_{r|n} \sum_{\delta|r} c_r(r/\delta) \sum_{\substack{\gcd(k,r)=\frac{r}{\delta} \\ 1 \leq k \leq r}} P_{k+j}(r) \alpha(n/r) \\ &= \sum_{r|n} \alpha(n/r) \sum_{k=1}^r c_r(k) P_{k+j}(r) \end{aligned}$$

The Ramanujan sum c_r is periodic function with period equal to r , then from condition III) of the Definition 2.2, we can write

$$\begin{aligned} \sum_{k=1}^r c_r(k) P_{k+j}(r) &= \sum_{k=1}^r \sum_{l=0}^{n/r-1} c_r(k+lr) P_{k+lr+j}(r) \\ &= \sum_{k=1}^n c_r(k) P_{k+j}(n) \\ &= \sum_{\delta|n} c_r(n/\delta) \sum_{\substack{\gcd(k,n)=\frac{n}{\delta} \\ 1 \leq k \leq n}} P_{k+j}(n) \\ &= \sum_{\delta|n} c_r(n/\delta) T_{\delta,j}(n). \end{aligned}$$

Hence

$$\begin{aligned} \sum_{r|n} \alpha(r) C_j(r) &= \sum_{\delta|n} \sum_{r|n} c_r(n/\delta) \alpha(n/r) T_{\delta,j}(n) \\ &= \sum_{\delta|n} \mathcal{R}(\alpha)(\delta) T_{\delta,j}(n). \end{aligned}$$

□

4. EXAMPLE

We denote by H_R the vector space of all analytic functions on the open ball $B(0, R)$ in complex plane \mathbb{C} . H_R endowed with the topology of compact convergence, it is a complete locally convex topological vector spaces.

For $n = 2, 3, \dots$, we denote by $S(n)$ the diagonal operator acting on monomials $e_k(z) = z^k$, ($k = 0, 1, \dots$) as follows

$$S(n)e_k = \varepsilon_n^k e_k, \quad \varepsilon_n = e^{\frac{2i\pi}{n}}.$$

It's clearly that $S(n)$ is a continuous map from H_R into its self satisfying $S^n(n) = i_{H_R}$. The primitive idempotents $P_1(n), \dots, P_n(n)$ related to $S(n)$ are given by

$$(4.1) \quad P_j(n) = \frac{1}{n} \sum_{l=0}^{n-1} \varepsilon_n^{-lj} S^l(n).$$

These obey at the following relations

$$(4.2) \quad i_{H_R} = \sum_{j=1}^n P_j(n), \quad P_i(n)P_j(n) = \delta_{ij}P_j(n).$$

The set $\{P_j(n)\}_{j,n=1}^{\infty}$ is an arithmetic system of orthogonal idempotent on the algebra $L(H_R)$ of all continuous linear maps of the space H_R . To see this it suffice to prove for all positive integers n and m such that n divide m the following identity

$$P_j(n) = \sum_{k=1}^{m/n} P_{j+kn}(m).$$

Indeed, from (4.1) and the following formula

$$(4.3) \quad \sum_{k=1}^r \varepsilon_r^{-lk} = \begin{cases} r, & \text{if } r|l \\ 0, & \text{otherwise,} \end{cases}$$

we can write

$$\begin{aligned} \sum_{k=1}^{m/n} P_{j+kr}(m) &= \frac{1}{m} \sum_{k=1}^{m/n} \sum_{l=1}^m \varepsilon_m^{-l(j+kr)} S^l(m) \\ &= \frac{1}{m} \sum_{l=1}^d \varepsilon_m^{-lj} S^l(m) \sum_{k=1}^{m/n} \varepsilon_m^{-lkr} \\ &= \frac{1}{n} \sum_{l=1}^r \varepsilon_n^{-lj} S^l(n) \\ &= P_j(n). \end{aligned}$$

The operators $C_0(n)$ and $T_0(n)$ acting on the basis $\{e_k\}$ as follows

$$(4.4) \quad C_0(n)e_m = c_n(m)e_m,$$

$$(4.5) \quad T_0(n)e_m = \begin{cases} e_m & \text{if } \gcd(n, m) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, from Proposition 3.1, we have

$$\begin{aligned} T_0(n)e_m &= \sum_{d|n} \mu(d)P_0(d)e_m \\ &= \sum_{d|\gcd(n,m)} \mu(d) e_m \\ &= \begin{cases} e_m & \text{if } \gcd(n, m) = 1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Now, if we compare the trace and the determinant of the matrix $[T_0(n)]_N$ (resp. $[C_0(n)]_N$) of the restriction of $C_0(n)$ (resp. $T_0(n)$) to the subspace generated by $\{e_1, \dots, e_N\}$, we get

$$(4.6) \quad \prod_{k=1}^N c_n(k) = \begin{cases} \prod_{p|n, \text{ prime}} (1-p)^{\lfloor \frac{N}{p} \rfloor} & \text{if } n \text{ is squarefree,} \\ 0 & \text{if otherwise.} \end{cases}$$

$$(4.7) \quad \sum_{\substack{\gcd(k,n)=1 \\ 1 \leq k \leq n}} \lfloor \frac{N-k}{n} \rfloor = N\omega(n) - \sum_{p|n, \text{ prime}} \lfloor \frac{N}{p} \rfloor = \sum_{r|n} \mu(r) \lfloor \frac{N}{r} \rfloor,$$

and

$$(4.8) \quad \sum_{k=1}^N c_n(k) = \sum_{l=1}^k (p_l^{\alpha_l} \lfloor \frac{N}{p_l^{\alpha_l}} \rfloor - p_l^{\alpha_l-1} \lfloor \frac{N}{p_l^{\alpha_l-1}} \rfloor) = \sum_{d|n} d\mu(n/d) \lfloor \frac{N}{d} \rfloor.$$

Let H_0 the subspace of all function $f \in H_R$ such that $f(0) = 0$ and let $\alpha : \mathbb{N} \rightarrow \mathbb{C}$. For $f \in H_0$, we put

$$(4.9) \quad \mathcal{P}(\alpha)f = \sum_{n=1}^{\infty} \alpha(n) P_0(n)f.$$

Proposition 4.1. *Let α be an arithmetic function. Then $\mathcal{P}(\alpha)$ defines a continuous diagonal map form H_0 into itself, if and only if $\limsup_{m \rightarrow \infty} |(\nu_0 * \alpha)(m)|^{1/m} \leq 1$. Furthermore,*

$$(4.10) \quad \mathcal{P}(\epsilon) = i_{H_0}, \quad \mathcal{P}(\alpha \square \beta) = \mathcal{P}(\alpha)\mathcal{P}(\beta).$$

where

$$(4.11) \quad \epsilon(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The operator $\mathcal{P}(\alpha)$ acts on monomials z^m as follows

$$(4.12) \quad \mathcal{P}(\alpha)z^m = \begin{cases} \sum_{n=1}^{\infty} \alpha(n) & \text{if } m = 0, \\ (\nu_0 * \alpha)(m)z^m & \text{otherwise.} \end{cases}$$

Suppose $\mathcal{P}(\alpha)$ defines a continuous linear map on H_0 . It is well-known that

$$f(z) = \sum_{m=1}^{\infty} a_m z^m \in H_R \quad \text{iff} \quad \limsup_{m \rightarrow \infty} |a_m|^{1/m} \leq R.$$

Hence,

$$\begin{aligned} \sum_{m=1}^{\infty} (\nu_0 * \alpha)(m) a_m z^m \in H_R &\Leftrightarrow \limsup_{m \rightarrow \infty} |a_m (\nu_0 * \alpha)(m)|^{1/m} \leq R \\ &\Leftrightarrow \limsup_{m \rightarrow \infty} |(\nu_0 * \alpha)(m)|^{1/m} \leq 1. \end{aligned}$$

This show that

$$\limsup_{m \rightarrow \infty} |(\nu_0 * \alpha)(m)|^{1/m} \leq 1.$$

The converse follows from the Closed Graph Theorem (see, [10] Theorem 2.15, p. 51]). To prove (4.10), we use equation (2.9)

$$\begin{aligned} \mathcal{P}(\alpha \square \beta) &= \sum_{n=1}^{\infty} (\alpha \square \beta)(n) P_0(n) \\ &= \sum_{n=1}^{\infty} (\alpha P_0 \square \beta P_0)(n) \\ &= \sum_{n=1}^{\infty} \alpha(n) P_0(n) \sum_{n=1}^{\infty} \beta(n) P_0(n) \\ &= \mathcal{P}(\alpha) \mathcal{P}(\beta) \end{aligned}$$

□

Many examples of operators acting on the space H_0 such as the Euler operator, the integration operator, the shift operator and the backward shift operator given by

$$(\theta f)(z) = z f'(z), \quad (If)(z) = \int_0^z f(t) dt, \quad (Uf) = z f(z), \quad (U^* f)(z) = \frac{f(z) - f(0)}{z},$$

can be represented by means of the map $\mathcal{P}(\alpha)$ for suitable arithmetic function α . For $\alpha = \varphi$ (where φ is the Euler's totient function), we have

$$(4.13) \quad \mathcal{P}(\varphi) e_m = \nu_0 * (\mu * \nu_1)(m) e_m = \nu_1(m) e_m = m e_m.$$

On the other hand, we have

$$\limsup_{m \rightarrow \infty} |\nu_0 * \varphi(m)|^{1/m} = \limsup_{m \rightarrow \infty} m^{1/m} = 1.$$

Then, $\mathcal{P}(\varphi)$ is a continuous map on H_0 . From (4.13), the mapping $\mathcal{P}(\varphi)$ coincides with the Euler operator $\theta = z \frac{d}{dz}$ on the monomials, hence

$$(4.14) \quad \mathcal{P}(\varphi) = \theta.$$

Moreover, from Proposition 4.1 and the following identity [3]

$$J_r = \underbrace{\varphi \square \dots \square \varphi}_{r\text{-times}},$$

we get

$$\mathcal{P}(J_r) = \mathcal{P}(\varphi) \dots \mathcal{P}(\varphi) = \theta^r.$$

(J_r called Jordan function)

A similarly arguments show that

$$\mathcal{P}(\mu) = e_1 \otimes e_1, \quad \mathcal{P}(\mu * \nu_1) = IU^*.$$

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