LOCALLY ANALYTIC VECTORS AND OVERCONVERGENT (φ, τ)-MODULES

HUI GAO AND LÉO POYETON

Abstract. Let p be a prime, let K be a complete discrete valuation field of characteristic 0 with a perfect residue field of characteristic p, and let G_K be the Galois group. Let π be a fixed uniformizer of K, let K_∞ be the extension by adjoining to K a system of compatible p^n-th roots of π for all n, and let L be the Galois closure of K_∞. Using these field extensions, Caruso constructs the (φ, τ)-modules, which classify p-adic Galois representations of G_K. In this paper, we study locally analytic vectors in some period rings with respect to the p-adic Lie group $\mathrm{Gal}(L/K)$, in the spirit of the work by Berger and Colmez. Using these locally analytic vectors, and using the classical overconvergent (φ, Γ)-modules, we can establish the overconvergence property of the (φ, τ)-modules.

Contents

1. Introduction

1.1. **Overview and main theorem.** Let p be a prime, and let K be a complete discrete valuation field of characteristic 0 with a perfect residue field of characteristic p. We fix an algebraic closure $\overline{K}$ of K and set $G_K := \mathrm{Gal}(\overline{K}/K)$. In p-adic Hodge theory, we use various "linear algebra" tools to study p-adic representations of G_K. A key idea in p-adic Hodge theory is to first restrict the Galois representations to some subgroups of G_K. For example, the classical (φ, Γ)-modules are constructed by using the subgroup $G_{p^\infty} := \mathrm{Gal}(\overline{K}/K_{p^\infty})$ where K_{p^∞} is the extension of K by adjoining a compatible system of p^n-th primitive roots of 1 for all n (cf. Notation 1.1.1 below). Later, it becomes clear that it is also important to study other possible theories arising from other subgroups. In this paper, we will study the (φ, τ)-modules, which are constructed by using the subgroup $G_\infty := \mathrm{Gal}(\overline{K}/K_\infty)$ where K_∞ is the extension of K by adjoining a compatible system of p^n-th roots of a fixed uniformizer of K for all n (cf. Notation 1.1.1 below).

The (φ, τ)-modules, firstly constructed by Caruso (cf. [Car13]), originated from works by Breuil and Kisin (cf. e.g., [Bre99, Kis06]); they look quite similar to the (φ, Γ)-modules, but in certain situations (in particular, if we consider the semi-stable representations), give much more useful information than the later. For example, these semi-stable (φ, τ)-modules (called Kisin modules, or Breuil-Kisin modules, or $(\varphi, \hat{G})$-modules in various contexts) can

Date: March 19, 2019.

2010 *Mathematics Subject Classification.* Primary 11F80, 11S20.

Key words and phrases. Locally analytic vectors, Overconvergence, (φ, τ)-modules.

be used to study Galois deformation rings (cf. [Kis08]), to classify semi-stable (integral) Galois representations (cf. [Liu10]), and to study integral models of Shimura varieties (cf. [Kis10]), to name just a few. In contrast, the (φ, Γ)-modules can only achieve very partial results in the aforementioned situations. However, the (φ, Γ)-modules have their own advantages; for example, they can be used to interprete Iwasawa cohomology (cf. [CC99]), to prove p-adic monodromy theorem (cf. [Ber02]), and most fantastically, to construct p-adic Langlands correspondence in the $\mathrm{GL}_2(\mathbb{Q}_p)$-situation (cf. [Col10b]). To explore other possible applications of the (φ, τ)-modules (and also the (φ, Γ)-modules), it is desirable to establish more parallel properties and build more links between these two theories. In this paper, we will study the overconvergence property of the (φ, τ)-modules; the analogous property of the (φ, Γ)-modules, first established by Cherbonnier and Colmez (cf. [CC98]), played a fundamental role in almost all applications of the (φ, Γ)-modules.

Let us be more precise now.

Notation 1.1.1. Let k be the (perfect) residue field of K, let $W(k)$ be the ring of Witt vectors, and let $K_0 := W(k)[1/p]$. Thus K is a totally ramified finite extension of K_0; write $e := [K : K_0]$. Let C_p be the p-adic completion of $\overline{K}$. Let v_p be the valuation on C_p such that $v_p(p) = 1$. For any subfield $Y \subset C_p$, let $\mathcal{O}_Y$ be its ring of integers.

Let $\pi \in K$ be a uniformizer, and let $E(u) \in W(k)[u]$ be the irreducible polynomial of π over K_0. Define a sequence of elements $\pi_n \in \overline{K}$ inductively such that $\pi_0 = \pi$ and $(\pi_{n+1})^p = \pi_n$. Define $\mu_n \in \overline{K}$ inductively such that μ_1 is a primitive p-th root of unity and $(\mu_{n+1})^p = \mu_n$. Let

$$K_\infty := \cup_{n=1}^\infty K(\pi_n), \quad K_{p^\infty} = \cup_{n=1}^\infty K(\mu_n), \quad L := \cup_{n=1}^\infty K(\pi_n, \mu_n).$$

Let

$$G_\infty := \mathrm{Gal}(\overline{K}/K_\infty), \quad G_{p^\infty} := \mathrm{Gal}(\overline{K}/K_{p^\infty}), \quad G_L := \mathrm{Gal}(\overline{K}/L), \quad \hat{G} := \mathrm{Gal}(L/K).$$

Let V be a finite dimensional $\mathbb{Q}_p$-vector space equipped with a continuous $\mathbb{Q}_p$-linear G_K-action. In [Car13], using the theory of field of norms for the field K_∞, Caruso associates to V an étale (φ, τ)-module (if one uses the field K_{p^∞} instead, one would get the usual étale (φ, Γ)-module); this induces an equivalence between the category of p-adic representations of G_K and the category of étale (φ, τ)-modules. An étale (φ, τ)-module is a triple $\hat{D} = (D, \varphi_D, \hat{G})$ (see Def. 6.2.2 for more details). Here, we only mention that D is a finite dimensional vector space over the field $\mathbf{B}_{K_\infty} := \mathbf{A}_{K_\infty}[1/p]$ where

$$\mathbf{A}_{K_\infty} := \{ \sum_{i=-\infty}^{+\infty} a_i u^i : a_i \in W(k), v_p(a_i) \to +\infty, \text{ as } i \to -\infty \},$$

and φ_D is a certain map $D \to D$ (here, we ignore the discussion of the $\hat{G}$-data). We say that $\hat{D}$ is *overconvergent* if we can "descend" the module D to a φ-stable submodule $D^\dagger$ over a subring $\mathbf{B}^\dagger_{K_\infty}$ (called the overconvergent subring) of $\mathbf{B}_{K_\infty}$, where

$$\mathbf{B}^\dagger_{K_\infty} := \{ \sum_{i=-\infty}^{+\infty} a_i u^i \in \mathbf{B}_{K_\infty}, v_p(a_i) + i\alpha \to +\infty \text{ for some } \alpha > 0, \text{ as } i \to -\infty \}.$$

The following is our main theorem.

Theorem 1.1.2. *For any finite dimensional $\mathbb{Q}_p$-representation V of G_K, its associated (φ, τ)-module is overconvergent.*

Remark 1.1.3. (1) Thm. 1.1.2 is originally proposed as a question by Caruso in [Car13, §4], as an analogue of the classical overconvergence theorem for étale (φ, Γ)-modules by Cherbonnier and Colmez ([CC98]).

(2) In a previous joint work by the first named author and T. Liu, Thm 1.1.2 is established when K is a finite extension of $\mathbb{Q}_p$, using a completely different method (see [GL]); a key ingredient in *loc. cit.* is the construction of "loose crystalline lifts" of torsion Galois representations, which requires the finiteness of k (see e.g., [GL, Rem. 1.1.2]).

(3) There does not seem to be any obvious comparison between the proof in this paper and that in [GL]. The main idea in [GL] is to "approximate" a general p-adic Galois representation by torsion crystalline representations; whereas we do not use any torsion representations in the current paper.

Remark 1.1.4. (1) In an upcoming work by the first named author, the overconvergence property will also be established for (φ,τ)-modules attached to an arithmetic family of Galois representations V_S over a rigid analytic space S (we need to assume $K/\mathbb{Q}_p$ finite there). Furthermore, we will use these family of overconvergent (φ,τ)-modules to study sheaves of Fontaine periods (e.g., as in [Bel15]).

(2) Using ideas and methods in this paper, it also seems very plausible to formulate and prove overconvergence results for *geometric* families of (φ,τ)-modules, in analogy with results in [KL].

(3) In contrast, the methods in [GL] can not be generalized to families (either arithmetic or geometric) of Galois representations.

Remark 1.1.5. We refer to [GL, §1.2] for some discussions of the importance and usefulness of overconvergence results in p-adic Hodge theory. In particular, in *loc. cit.*, we mentioned about the *link* between the category of all Galois representations and the category of geometric (i.e., semi-stable, crystalline) representations. Indeed, in *loc. cit.*, we used this link to prove the overconvergence theorem. In the current paper, we do not use any semi-stable representations; instead, some results we obtain in the current paper will be used to study semi-stable representations. One result worth mentioning is Thm. 3.4.4(4) (see also Rem. 3.4.5), where we show certain ring of locally analytic vectors is related with the ring $\mathcal{O}_{[0,1)}$ in [Kis06]. We will report some progress (in particular, on the theory of $(\varphi,\hat{G})$-modules) in a future work by the first named author and T. Liu.

1.2. **Strategy of proof.** The key ingredient for the proof of Thm. 1.1.2 is the calculation of locally analytic vectors in some period rings, in the spirit of the work by Berger and Colmez ([BC16, Ber16]). The philosophy that overconvergence of Galois representations is related with locally analytic vectors is first observed by Colmez, in the framework of p-adic Langlands correspondence (cf. [Col10b, Intro. 13.3]). For example, overconvergent (φ,Γ)-modules (cf. [CC98]) are closely related with locally analytic vectors in the p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ (cf. [LXZ12, Col14]), i.e., via the "*locally analytic p-adic Langlands correspondence*".

To study the p-adic Langlands correspondence for $\mathrm{GL}_2(F)$ where $F/\mathbb{Q}_p$ is a finite extension, Berger recently proves overconvergence of the Lubin-Tate (φ,Γ)-modules (cf. [Ber16]). The key idea in *loc. cit.*, very roughly speaking, is that there should exist "enough" locally analytic vectors in the Lubin-Tate (φ,Γ)-modules. To find these locally analytic vectors, one first "enlarges" the space of Lubin-Tate (φ,Γ)-modules over a bigger period ring; then there are indeed enough locally analytic vectors, by *using the classical overconvergent* (φ,Γ)-*modules as an input* (cf. [Ber16, Thm. 9.1]). One then descends from the bigger space of locally analytic vectors to the level of Lubin-Tate (φ,Γ)-modules, via a monodromy theorem (cf. [Ber16, §6]).

The key idea in our paper is similar to that in [Ber16]. Indeed, (very roughly speaking), we first "enlarge" the space of the (φ,τ)-module over the big period ring $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig},L}$ (which is $\mathrm{Gal}(\overline{K}/L)$-invariant of the well-known ring $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$); there are enough locally analytic vectors on this level, by *using the classical overconvergent* (φ,Γ)-*modules as an input* again (cf. the proof of Thm. 6.2.6). To descend these locally analytic vectors to the level of (φ,τ)-modules, we can use a Tate-Sen descent or a monodromy descent (see Prop. 6.1.6 and Rem. 6.1.7 for more details).

As the strategy suggests, one needs to compute locally analytic vectors in some period rings (e.g., $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig},L}$). In the case of (φ,Γ)-modules, the concerned p-adic Lie group is $\mathrm{Gal}(K_{p^\infty}/K)$ (see Notation 1.1.1), which is one-dimensional. In the case of Lubin-Tate (φ,Γ)-modules, the

p-adic Lie group is $\mathcal{O}_F^\times$, which is of dimension $[F:\mathbb{Q}_p]$. In general, it would be very difficult to calculate locally analytic vectors for p-adic Lie groups of dimension higher than one. In [Ber16], Berger considers firstly the "F-analytic" locally analytic vectors, which behave similar to the one-dimensional case. He then uses these "F-analytic" locally analytic vectors to determine the full space of $\mathcal{O}_F^\times$-locally analytic vectors. In our paper, the concerned p-adic Lie group is $\hat{G} = \mathrm{Gal}(L/K)$, which is of dimension two. The key observation is that we need to firstly consider $\hat{G}$-locally analytic vectors which are *furthermore* $\mathrm{Gal}(L/K_\infty)$-*invariant*; these locally analytic vectors then again behave similar to the one-dimensional case. Indeed, we have:

Theorem 1.2.1. *Let* $(\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},L})^{\tau\text{-pa},\gamma=1}$ *denote the set of* $\mathrm{Gal}(L/K_{p^\infty})$*-(pro)-locally analytic vectors which are furthermore fixed by* $\mathrm{Gal}(L/K_\infty)$*. Then we have*

$$(\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},L})^{\tau\text{-pa},\gamma=1} = \cup_{m\geq 0}\varphi^{-m}(\mathbf{B}^\dagger_{\mathrm{rig},K_\infty}),$$

where $\mathbf{B}^\dagger_{\mathrm{rig},K_\infty}$ *is the "Robba ring with coefficients in* K_0*" (cf. Def. 3.4.6).*

With the above theorem established, we can also completely determine the $\hat{G}$-locally analytic vectors in $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},L}$; since the statement is too technical, we refer the reader to Thm. 5.3.5.

1.3. **Structure of the paper.** In §2, we study the rings $\widetilde{\mathbf{B}}^I$ and $\mathbf{B}^I$ (where I is an interval), as well as their $\mathrm{Gal}(\overline{K}/K_\infty)$-invariants which are denoted as $\widetilde{\mathbf{B}}^I_{K_\infty}$ and $\mathbf{B}^I_{K_\infty}$. In §3, we compute locally analytic vectors in $\widetilde{\mathbf{B}}^I_{K_\infty}$; and in §4, we need to carry out similar calculations when we replace K_∞ with a finite extension. In §5, we compute the $\hat{G}$-locally analytic vectors in $\widetilde{\mathbf{B}}^I_L$. All these calculations will be used in §6 to carry out the descent of locally analytic vectors, giving us the desired overconvergence result.

1.4. **Notations.**

1.4.1. *Convention on ring notations.* In this paper, we will use many rings. Let us mention some of the conventions about how we choose the notations; it also serves as a brief index of ring notations.

(1) In §1.4.2, we define some basic rings. We also compare them with notations commonly used in integral p-adic Hodge theory (see Rem. 1.4.3).
(2) In §2.1, we define the rings $\widetilde{\mathbf{A}}^I$ and $\widetilde{\mathbf{B}}^I$ (where I is an interval), which are exactly the same as $\widetilde{\mathbf{A}}^I$ and $\widetilde{\mathbf{B}}^I$ in [Ber08] (which are $\widetilde{\mathbf{A}}_I$ and $\widetilde{\mathbf{B}}_I$ in [Ber02]). (See also the table in [Ber08, §1.1] for a comparison of notations with those of Colmez and Kedlaya).
(3) When Y is a ring with a G_K-action, $X \subset \overline{K}$ is a subfield, we use Y_X to denote the $\mathrm{Gal}(\overline{K}/X)$-invariants of Y. Some examples include when $Y = \widetilde{\mathbf{A}}^I, \widetilde{\mathbf{B}}^I, \mathbf{A}^I, \mathbf{B}^I$ and $X = L, K_\infty, M$ where M/K_∞ is a finite extension. This "style of notation" imitates that of [Ber08], which uses the subscript $*_K$ to denote G_{p^∞}-invariants.
(4) In §2.2, we define the rings $\mathbf{A}^I$ and $\mathbf{B}^I$ and study their G_∞-invariants: $\mathbf{A}^I_{K_\infty}$ and $\mathbf{B}^I_{K_\infty}$. These rings "correspond" to those rings studied in [Col08, §6.3, §7]. Our $\mathbf{A}^I$ and $\mathbf{B}^I$ are *different* from $\mathbf{A}^I$ and $\mathbf{B}^I$ in [Col08] (cf. Rem. 1.4.3); fortunately, we are mostly interested in $\mathbf{A}^I_{K_\infty}$ and $\mathbf{B}^I_{K_\infty}$, and since we are using K_∞ as subscripts, confusions are avoided.

1.4.2. *Period rings.* Let $\widetilde{\mathbf{E}}^+ := \varprojlim \mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}$ where the transition maps are $x \mapsto x^p$, let $\widetilde{\mathbf{E}} := \mathrm{Fr}\widetilde{\mathbf{E}}^+$. An element of $\widetilde{\mathbf{E}}$ can be uniquely represented by $(x^{(n)})_{n\geq 0}$ where $x^{(n)} \in C_p$ and $(x^{(n+1)})^p = (x^{(n)})$; let $v_{\widetilde{\mathbf{E}}}$ be the usual valuation where $v_{\widetilde{\mathbf{E}}}(x) := v_p(x^{(0)})$. Let

$$\widetilde{\mathbf{A}}^+ := W(\widetilde{\mathbf{E}}^+), \quad \widetilde{\mathbf{A}} := W(\widetilde{\mathbf{E}}), \quad \widetilde{\mathbf{B}}^+ := \widetilde{\mathbf{A}}^+[1/p], \quad \widetilde{\mathbf{B}} := \widetilde{\mathbf{A}}[1/p],$$

where $W(\cdot)$ means the ring of Witt vectors. There is a unique surjective ring homomorphism $\theta : \widetilde{\mathbf{A}}^+ \to \mathcal{O}_{C_p}$, which lifts the projection $\widetilde{\mathbf{E}}^+ \to \mathcal{O}_{\overline{K}}/p$ onto the first factor in the inverse limit. Let $\mathbf{B}_{\mathrm{dR}}^+$ be the $\mathrm{Ker}\theta[1/p]$-adic completion of $\widetilde{\mathbf{B}}^+$ (so the θ-map extends to $\mathbf{B}_{\mathrm{dR}}^+$). Let $\underline{\varepsilon} = \{\mu_n\}_{n\geq 0} \in \widetilde{\mathbf{E}}^+$, let $[\underline{\varepsilon}] \in \widetilde{\mathbf{A}}^+$ be its Teichmüller lift, and let $t := \log([\underline{\varepsilon}]) \in \mathbf{B}_{\mathrm{dR}}^+$ as usual.

Let $\underline{\pi} := \{\pi_n\}_{n\geq 0} \in \widetilde{\mathbf{E}}^+$. Let $\mathbf{E}_{K_\infty}^+ := k[\![\underline{\pi}]\!]$, $\mathbf{E}_{K_\infty} := k((\underline{\pi}))$, and let $\mathbf{E}$ be the separable closure of $\mathbf{E}_{K_\infty}$ in $\widetilde{\mathbf{E}}$. By the theory of field of norms (cf. §4), $\mathrm{Gal}(\mathbf{E}/\mathbf{E}_{K_\infty}) \simeq G_\infty$. Furthermore, the completion of $\mathbf{E}$ with respect to $v_{\widetilde{\mathbf{E}}}$ is $\widetilde{\mathbf{E}}$.

Let $[\underline{\pi}] \in \widetilde{\mathbf{A}}^+$ be the Teichmüller lift of $\underline{\pi}$. Let $\mathbf{A}_{K_\infty}^+ := W[\![u]\!]$ with Frobenius φ extending the arithmetic Frobenius on $W(k)$ and $\varphi(u) = u^p$. There is a $W(k)$-linear Frobenius-equivariant embedding $\mathbf{A}_{K_\infty}^+ \hookrightarrow \widetilde{\mathbf{A}}^+$ via $u \mapsto [\underline{\pi}]$. Let $\mathbf{A}_{K_\infty}$ be the p-adic completion of $\mathbf{A}_{K_\infty}^+[1/u]$. Our fixed embedding $\mathbf{A}_{K_\infty}^+ \hookrightarrow \widetilde{\mathbf{A}}^+$ determined by $\underline{\pi}$ uniquely extends to a φ-equivariant embedding $\mathbf{A}_{K_\infty} \hookrightarrow \widetilde{\mathbf{A}}$, and we identify $\mathbf{A}_{K_\infty}$ with its image in $\widetilde{\mathbf{A}}$. We note that $\mathbf{A}_{K_\infty}$ is a complete discrete valuation ring with uniformizer p and residue field $\mathbf{E}_{K_\infty}$.

Let $\mathbf{B}_{K_\infty} := \mathbf{A}_{K_\infty}[1/p]$. Let $\mathbf{B}$ be the p-adic completion of the maximal unramified extension of $\mathbf{B}_{K_\infty}$ inside $\widetilde{\mathbf{B}}$, and let $\mathbf{A} \subset \mathbf{B}$ be the ring of integers. Let $\mathbf{A}^+ := \widetilde{\mathbf{A}}^+ \cap \mathbf{A}$. Then we have:

$$(\mathbf{A})^{G_\infty} = \mathbf{A}_{K_\infty}, \quad (\mathbf{B})^{G_\infty} = \mathbf{B}_{K_\infty}, \quad (\mathbf{A}^+)^{G_\infty} = \mathbf{A}_{K_\infty}^+.$$

Remark 1.4.3. (1) The following rings (and their "$\mathbf{B}$-variants") that we defined above,

$$\widetilde{\mathbf{E}}^+, \quad \widetilde{\mathbf{E}}, \quad \widetilde{\mathbf{A}}^+, \quad \widetilde{\mathbf{A}}, \quad \mathbf{A}_{K_\infty}^+, \quad \mathbf{A}_{K_\infty}, \quad \mathbf{A}, \quad \mathbf{A}^+$$

are precisely the following rings which are commonly used in integral p-adic Hodge theory (e.g., in [GL]):

$$R, \quad \mathrm{Fr}R, \quad W(R), \quad W(\mathrm{Fr}R), \quad \mathfrak{S}, \quad \mathcal{O}_{\mathcal{E}}, \quad \mathcal{O}_{\widehat{\mathcal{E}}^{\mathrm{ur}}}, \quad \mathfrak{S}^{\mathrm{ur}}.$$

(2) The rings $\mathbf{A}$ and $\mathbf{B}$ (and their variants, e.g., $\mathbf{A}^I, \mathbf{B}^I$, in §2.2) are *different* from the "$\mathbf{A}$" and "$\mathbf{B}$" in [Ber08] or [Col08]. Indeed, they are the same algebraic rings, but with different structures (e.g., Frobenius structure). In the proof of our final main theorem (Thm. 6.2.6), we will use the font $\mathbb{A}$, $\mathbb{B}$ to denote those rings in the (φ, Γ)-module setting.

1.4.4. *Valuations and norms.* A non-Archimedean valuation of a ring A is a map $v : A \to \mathbb{R} \cup \{+\infty\}$ such that $v(x) = +\infty \Leftrightarrow x = 0$ and $v(x+y) \geq \inf\{v(x), v(y)\}$. It is called *sub-multiplicative* (resp. *multiplicative*) if $v(xy) \geq v(x) + v(y)$ (resp. $v(xy) = v(x) + v(y)$), for all x, y. All the valuations in this paper are sub-multiplicative (some are multiplicative). Given a matrix $T = (t_{i,j})_{i,j}$ over A, let $v(T) := \min\{v(t_{i,j})\}$. A non-Archimedean valuation v on A induces a non-Archimedean norm where $\|a\| := p^{-v(a)}$, and vice versa.

1.4.5. *Some other notations.* Throughout this paper, we reserve φ to denote Frobenius operator. We sometimes add subscripts to indicate on which object Frobenius is defined. For example, $\varphi_{\mathfrak{M}}$ is the Frobenius defined on $\mathfrak{M}$. We always drop these subscripts if no confusion arises. We use $\mathrm{M}_d(A)$ (resp. $\mathrm{GL}_d(A)$) to denote the set of $d \times d$-matrices (resp. invertible $d \times d$-matrices) with entries in A.

Acknowledgement. We would like to dedicate this paper to Professor Jean-Marc Fontaine and Professor Jean-Pierre Wintenberger, with great admiration. We thank Laurent Berger and Tong Liu for many useful discussions. The influence of the work of Laurent Berger and Pierre Colmez in this paper will be evident to the reader. We thank the anonymous referees for several useful comments. H.G. is partially supported by a postdoctoral position in University of Helsinki, funded by Academy of Finland through Kari Vilonen. L.P. is currently a PhD student of Laurent Berger at the ENS de Lyon.

2. A study of some rings

In this section, we study some rings which are denoted as $\widetilde{\mathbf{B}}^I$ and $\mathbf{B}^I$ (where I is an interval). In particular, we study their G_∞-invariants (see 1.1.1 for G_∞), which are denoted as $\widetilde{\mathbf{B}}^I_{K_\infty}$ and $\mathbf{B}^I_{K_\infty}$. The results will be used in Section 3 to further determine the link between these rings. All results in this section are analogues of their G_{p^∞}-versions, established in [Ber02, Col08]; the proofs are also similar.

2.1. **The ring $\widetilde{\mathbf{B}}^I$ and its G_∞-invariants.** Let $\overline{\pi} = \underline{\varepsilon} - 1 \in \widetilde{\mathbf{E}}^+$ (this is not $\underline{\pi}$), and let $[\overline{\pi}] \in \widetilde{\mathbf{A}}^+$ be its Teichmüller lift. When A is a p-adic complete ring, we use $A\{X, Y\}$ to denote the p-adic completion of $A[X, Y]$. As in [Ber02, §2], we define the following rings.

Definition 2.1.1. (1) Let

$$\begin{aligned} \widetilde{\mathbf{A}}^{[r,s]} : &= \widetilde{\mathbf{A}}^+\{\frac{p}{[\overline{\pi}]^r}, \frac{[\overline{\pi}]^s}{p}\}, \text{ when } r \leq s \in \mathbb{Z}^{\geq 0}[1/p], s > 0; \\ \widetilde{\mathbf{A}}^{[r,+\infty]} : &= \widetilde{\mathbf{A}}^+\{\frac{p}{[\overline{\pi}]^r}\}, \text{ when } r \in \mathbb{Z}^{\geq 0}[1/p]; \\ \widetilde{\mathbf{A}}^{[+\infty,+\infty]} : &= \widetilde{\mathbf{A}}. \end{aligned}$$

Here, to be rigorous, $\widetilde{\mathbf{A}}^+\{p/[\overline{\pi}]^r, [\overline{\pi}]^s/p\}$ is defined as $\widetilde{\mathbf{A}}^+\{X, Y\}/([\overline{\pi}]^r X - p, pY - [\overline{\pi}]^s, XY - [\overline{\pi}]^{s-r})$, and similarly for $\widetilde{\mathbf{A}}^+\{p/[\overline{\pi}]^r\}$ (and other similar occurrences later); see [Ber02, §2] for more details.

(2) If I is one of the closed intervals above, then let $\widetilde{\mathbf{B}}^I := \widetilde{\mathbf{A}}^I[1/p]$.

Remark 2.1.2. We do not define $\widetilde{\mathbf{A}}^{[0,0]}$. Indeed, we will refrain from using the interval $[0, 0]$ throughout the paper; see Rem. 2.1.9 and Rem. 2.2.6 for more remarks concerning $[0, 0]$.

2.1.3. If I is one of the closed intervals above, then $\widetilde{\mathbf{A}}^I$ is p-adically separated and complete; we use V^I to denote its p-adic valuation (which is sub-multiplicative). When $I \subset J$ are two closed intervals as above, then by [Ber02, Lem. 2.5], there exists a natural (continuous) embedding $\widetilde{\mathbf{A}}^J \hookrightarrow \widetilde{\mathbf{A}}^I$; we identify $\widetilde{\mathbf{A}}^J$ with its image (as algebraic rings) in this case.

Definition 2.1.4. When $r \in \mathbb{Z}^{\geq 0}[1/p]$, let

$$\widetilde{\mathbf{B}}^{[r,+\infty)} := \bigcap_{n \geq 0} \widetilde{\mathbf{B}}^{[r,s_n]}$$

where $s_n \in \mathbb{Z}^{>0}[1/p]$ is any sequence increasing to $+\infty$. We equip $\widetilde{\mathbf{B}}^{[r,+\infty)}$ with its natural Fréchet topology.

Lemma 2.1.5. *(1) Let $I \subset J$ be as in §2.1.3. If $0 \notin J$, then $\widetilde{\mathbf{B}}^J$ is dense in $\widetilde{\mathbf{B}}^I$ with respect to V^I.*

(2) Suppose $r \leq s \in \mathbb{Z}^{\geq 0}[1/p]$ and $s > 0$, then $\widetilde{\mathbf{B}}^{[0,s]}$ is closed in $\widetilde{\mathbf{B}}^{[r,s]}$ with respect to $V^{[r,s]}$.

(3) Suppose $0 \leq s_1 \leq s_2 \leq s \leq +\infty$ and $s_2 > 0$, then the closure of $\widetilde{\mathbf{B}}^{[0,s]}$ in $\widetilde{\mathbf{B}}^{[s_1,s_2]}$ (with respect to $V^{[s_1,s_2]}$) is $\widetilde{\mathbf{B}}^{[0,s_2]}$.

(4) When $r \in \mathbb{Z}^{\geq 0}[1/p]$, $\widetilde{\mathbf{B}}^{[r,+\infty)}$ is complete with respect to its Fréchet topology, and contains $\widetilde{\mathbf{B}}^{[r,+\infty]}$ as a dense subring.

Proof. Item 1 is easy. To prove Item 2, it suffices to show that

$$\widetilde{\mathbf{A}}^{[0,s]} \cap p\widetilde{\mathbf{A}}^{[r,s]} = p\widetilde{\mathbf{A}}^{[0,s]} \tag{2.1.1}$$

This is indeed [Ber16, Lem. 3.2(3)]; however, in *loc. cit.*, the definitions of $\widetilde{\mathbf{A}}^{[0,s]}$ and $\widetilde{\mathbf{A}}^{[r,s]}$ rely on the valuations W^I (denoted as "$V(x, I)$" in *loc. cit.*) which we will recall in Def. 2.1.8. Here we give a "direct" proof using the *explicit* structure of these rings per our Def. 2.1.1. Let $x \in \widetilde{\mathbf{A}}^{[r,s]}$ such that $px \in \widetilde{\mathbf{A}}^{[0,s]}$. We can decompose $x = x^- + x^+$ with $x^- \in \widetilde{\mathbf{A}}^{[r,+\infty]}$

and $x^+ \in \widetilde{\mathbf{A}}^{[0,s]}$ (the decomposition is not unique). It suffices to show that $px^- \in p\widetilde{\mathbf{A}}^{[0,s]}$. But indeed,

$$\begin{aligned} px^- &\in p\widetilde{\mathbf{A}}^{[r,+\infty]} \cap \widetilde{\mathbf{A}}^{[0,s]} \\ &= p\widetilde{\mathbf{A}}^{[r,+\infty]} \cap (\widetilde{\mathbf{A}}^{[r,+\infty]} \cap \widetilde{\mathbf{A}}^{[0,s]}) \\ &\subset p\widetilde{\mathbf{A}}^{[r,+\infty]} \cap (\widetilde{\mathbf{A}}^{[s,+\infty]} \cap \widetilde{\mathbf{A}}^{[0,s]}) \\ &= p\widetilde{\mathbf{A}}^{[r,+\infty]} \cap \widetilde{\mathbf{A}}^{[0,+\infty]}, \text{ by [Ber02, Lem. 2.15]} \\ &\subset p\widetilde{\mathbf{A}} \cap \widetilde{\mathbf{A}}^{[0,+\infty]} \\ &= p\widetilde{\mathbf{A}}^{[0,+\infty]}. \end{aligned}$$

To prove Item 3, simply note that $\widetilde{\mathbf{B}}^{[0,s]}$ is contained in $\widetilde{\mathbf{B}}^{[0,s_2]}$ but its closure contains $\widetilde{\mathbf{B}}^{[0,s_2]}$, and then apply Item 2. (Note that Items 2 and 3 correct the statements above [Ber02, Rem. 2.6], as Berger never explicitly requires $0 \notin J$.) Item 4 is [Ber02, Lem. 2.19] (the proof there works for $r = 0$ as well). □

Remark 2.1.6. (1) For any interval I such that $\widetilde{\mathbf{A}}^I$ and $\widetilde{\mathbf{B}}^I$ are defined, there is a natural bijection (called Frobenius) $\varphi : \widetilde{\mathbf{A}}^I \to \widetilde{\mathbf{A}}^{pI}$ which is valuation-preserving.

(2) For $n \in \mathbb{Z}^{\geq 0}$, let $r_n := (p-1)p^{n-1}$. Let

$$I_c := \{[r_\ell, r_k], [r_\ell, +\infty], [0, r_k], [0, +\infty]\}, \text{ where } \ell \leq k \text{ run through } \mathbb{Z}^{\geq 0}.$$

By item (1), in many situations, it would suffice to study $\widetilde{\mathbf{A}}^I$ (and $\widetilde{\mathbf{B}}^I$) for $I \in I_c$ or $I = [+\infty, +\infty]$. The cases for I a general closed interval can be deduced using Frobenius operation; the cases for $I = [r, +\infty)$ can be deduced by taking Fréchet completion.

Convention 2.1.7. From now on, whenever we define rings with an interval as superscript (such as $\widetilde{\mathbf{A}}^I$, or $\mathbf{A}^I$, $\mathcal{A}^I$ etc. in the following), we always define in the general case with $\inf(I), \sup(I) \in \{Z^{\geq 0}[1/p], +\infty\}$. But we will only compute (the explicit structure of) these rings with $\inf(I), \sup(I) \in \{0, r_\ell, r_k, +\infty\}$ (when applicable); the general case can always be easily deduced using Frobenius operations.

There is another type of valuation W^I on $\widetilde{\mathbf{B}}^{[r,+\infty]}$, which we quickly recall. A particularly useful fact is that $W^{[s,s]}$ are *multiplicative* valuations (not just sub-multiplicative), see Lem. 2.1.10 below.

Definition 2.1.8. Suppose $r \in \mathbb{Z}^{\geq 0}[1/p]$, and let $x = \sum_{i \geq i_0} p^i [x_i] \in \widetilde{\mathbf{B}}^{[r,+\infty]}$ ($\subset \widetilde{\mathbf{B}}^{[+\infty,+\infty]}$). Denote $w_k(x) := \inf_{i \leq k}\{v_{\widetilde{\mathbf{E}}}(x_i)\}$. See [Col08, §5.1] for the properties of w_k; in particular, we have $w_k(x+y) \geq \inf\{w_k(x), w_k(y)\}$ with equality when $w_k(x) \neq w_k(y)$. For $s \geq r$ and $s > 0$, let

$$W^{[s,s]}(x) := \inf_{k \geq k_0}\{k + \frac{p-1}{ps} \cdot v_{\widetilde{\mathbf{E}}}(x_k)\} = \inf_{k \geq k_0}\{k + \frac{p-1}{ps} \cdot w_k(x)\};$$

this is a well-defined valuation (cf. [Col08, Prop. 5.4]). For $I \subset [r, +\infty)$ a non-empty closed interval such that $I \neq [0,0]$, let

$$W^I(x) := \inf_{\alpha \in I, \alpha \neq 0}\{W^{[\alpha,\alpha]}(x)\}.$$

Remark 2.1.9. We do not define "$W^{[0,0]}$". Indeed when $r = 0$, then $\widetilde{\mathbf{B}}^{[0,+\infty]} = \widetilde{\mathbf{B}}^+$. It might seem that we could define " $W^{[0,0]}(x) := \inf_{x_k \neq 0}\{k\}$," which is precisely the p-adic valuation of $\widetilde{\mathbf{B}}^+$. However, this valuation is "*incompatible*" with the valuations $W^{[s,s]}$ for $s > 0$. Indeed, one observes that the valuations $W^{[s,s]}$ behave *continuously* with respect to $s > 0$; but this continuity breaks for "$s = 0$". Indeed, $W^{[s,s]}(x)$ do not converge to the aforementioned "$W^{[0,0]}(x)$" when $s \to 0$; this phenomenon is best explained using the geometric picture of the "degeneration of annuli to a closed disk", cf. Rem 2.2.6. Alternatively, it might seem that we could define " $W^{[0,0]}(x) := +\infty, \forall x$"; however this is not a valuation anymore (cf. §1.4.4).

Lemma 2.1.10. *Suppose $r \leq s \in \mathbb{Z}^{\geq 0}[1/p]$ and $s > 0$, then the following holds.*

(1) When $r > 0$, $\widetilde{\mathbf{A}}^{[r,+\infty]}$ and $\widetilde{\mathbf{A}}^{[r,+\infty]}[1/[\overline{\pi}]]$ are complete with respect to $W^{[r,r]}$.
(2) $W^{[s,s]}(xy) = W^{[s,s]}(x) + W^{[s,s]}(y), \forall x, y \in \widetilde{\mathbf{B}}^{[r,+\infty]}$.
(3) Let $x \in \widetilde{\mathbf{B}}^{[r,+\infty]}$.
  (a) When $r > 0$, $W^{[r,s]}(x) = \inf\{W^{[r,r]}(x), W^{[s,s]}(x)\}$.
  (b) When $r = 0$, $W^{[r,s]}(x)(= W^{[0,s]}(x)) = W^{[s,s]}(x)$.
(4) For $x \in \widetilde{\mathbf{B}}^{[r,+\infty]}$, we have $V^{[r,s]}(x) = \lfloor W^{[r,s]}(x) \rfloor$, where $V^{[r,s]}(x)$ is defined by considering x as an element in $\widetilde{\mathbf{B}}^{[r,s]}$.
(5) The completion of $\widetilde{\mathbf{B}}^{[r,+\infty]}$ with respect to $W^{[r,s]}$ is isomorphic to $\widetilde{\mathbf{B}}^{[r,s]}$ as topological rings. (Thus, we can extend $W^{[r,s]}$ to $\widetilde{\mathbf{B}}^{[r,s]}$).

Proof. All these results are well-known. Item 1 is [Col08, Prop. 5.6]; note that the ring "$\widetilde{\mathbf{A}}^{(0,r]}$" in *loc. cit.* is our $\widetilde{\mathbf{A}}^{[(p-1)/(pr),+\infty]}[1/[\overline{\pi}]]$, and the ring of integers in $\widetilde{\mathbf{A}}^{(0,r]}$ is precisely our $\widetilde{\mathbf{A}}^{[(p-1)/(pr),+\infty]}$. Item 2 is [Ber10, Lem. 21.3]. Item 3(a) (the maximum modulus principle) is [Ber02, Cor. 2.20]; indeed, it follows easily by looking at the definition of $W^{[\alpha,\alpha]}(x)$. Item 3(b) follows from similar observation, by noting that $x \in \widetilde{\mathbf{B}}^{[0,+\infty]}$ implies $v_{\widetilde{\mathbf{E}}}(x_k) \geq 0$ for all $k \geq k_0$ in Def. 2.1.8. Item 4 is [Ber02, Lem. 2.7]; the proof works for $r = 0$ as well (which Berger did not explicitly mention). Item 5 follows from Item 4 and Lem. 2.1.5.

□

Remark 2.1.11. Let $r > 0$.

(1) Suppose $x \in \widetilde{\mathbf{B}}^{[r,+\infty]}$, then $W^{[r,r]}(x) \geq 0$ does not imply that $x \in \widetilde{\mathbf{A}}^{[r,+\infty]}$, it only implies that $x \in \widetilde{\mathbf{A}}^{[r,r]}$. However, if $x \in \widetilde{\mathbf{A}}^{[r,+\infty]}[1/[\overline{\pi}]]$, then $W^{[r,r]}(x) \geq 0$ if and only if $x \in \widetilde{\mathbf{A}}^{[r,+\infty]}$.
(2) In comparison to Lem. 2.1.10(1), $\widetilde{\mathbf{B}}^{[r,+\infty]}$ is not complete with respect to $W^{[r,r]}$; indeed, its completion is $\widetilde{\mathbf{B}}^{[r,r]}$ by Lem. 2.1.10(5).
(3) In comparison to Lem. 2.1.10(5), the completion of $\widetilde{\mathbf{A}}^{[r,+\infty]}$ with respect to $W^{[r,s]}$ is strictly contained in $\widetilde{\mathbf{A}}^{[r,s]}$ (which is already the case when $r = s$ by Lem. 2.1.10(1)). Also note that $\widetilde{\mathbf{A}}^{[r,s]}$ is complete with respect to $W^{[r,s]}$, since it is the ring of integers in $\widetilde{\mathbf{B}}^{[r,s]}$. (Thus, $\widetilde{\mathbf{A}}^{[r,+\infty]}$ is a closed subset of $\widetilde{\mathbf{A}}^{[r,r]}$ with respect to $W^{[r,r]}$).

Let I be an interval. When $\widetilde{\mathbf{B}}^I$ (resp. $\widetilde{\mathbf{A}}^I$) is defined, let $\widetilde{\mathbf{B}}^I_{K_\infty} := (\widetilde{\mathbf{B}}^I)^{G_\infty}$ (resp. $\widetilde{\mathbf{A}}^I_{K_\infty} := (\widetilde{\mathbf{A}}^I)^{G_\infty}$). Recall that as in [Ber02, §2.2], when $r_n \in I$, there exists $\iota_n : \widetilde{\mathbf{B}}^I \hookrightarrow \mathbf{B}^+_{\mathrm{dR}}$. Let $\theta : \mathbf{B}^+_{\mathrm{dR}} \to C_p$ be the usual map.

Lemma 2.1.12. *Let $q := ([\underline{\varepsilon}]^p - 1)/([\underline{\varepsilon}] - 1)$. Suppose $I = [r_\ell, r_k]$ or $[0, r_k]$. We have*

(1) $\mathrm{Ker}(\theta \circ \iota_k : \widetilde{\mathbf{A}}^I \to C_p) = \frac{\varphi^{k-1}(q)}{p}\widetilde{\mathbf{A}}^I = \frac{\varphi^k(E(u))}{p}\widetilde{\mathbf{A}}^I$,
  $\mathrm{Ker}(\theta \circ \iota_k : \widetilde{\mathbf{B}}^I \to C_p) = \varphi^{k-1}(q)\widetilde{\mathbf{B}}^I = \varphi^k(E(u))\widetilde{\mathbf{B}}^I$.
(2) $\mathrm{Ker}(\theta \circ \iota_k : \widetilde{\mathbf{A}}^I_{K_\infty} \to C_p) = \frac{\varphi^k(E(u))}{p}\widetilde{\mathbf{A}}^I_{K_\infty}$,
  $\mathrm{Ker}(\theta \circ \iota_k : \widetilde{\mathbf{B}}^I_{K_\infty} \to C_p) = \varphi^k(E(u))\widetilde{\mathbf{B}}^I_{K_\infty}$.

Proof. Item (1) is easily deduced from [Ber02, Prop. 2.12], because $E(u)$ and $\varphi^{-1}(q)$ generate the same ideal in $\widetilde{\mathbf{A}}^+$ (i.e., the kernel of the θ-map in §1.4.2). Item (2) is an easy consequence of (1). □

In the following, we study more detailed structure of the rings $\widetilde{\mathbf{B}}^I_{K_\infty}$ and $\widetilde{\mathbf{A}}^I_{K_\infty}$. These results (Lem. 2.1.13, Prop. 2.1.14 and Prop. 2.1.16) will not be used in this paper. We still include them here because they are standard and will be useful in the future; also, they serve as prelude to the computation of the rings $\mathbf{B}^I_{K_\infty}$ and $\mathbf{A}^I_{K_\infty}$ in next subsection.

Lemma 2.1.13. *Suppose $\ell \le k$, then we have the following short exact sequence*

$$0 \to \widetilde{\mathbf{B}}^{[0,+\infty]}_{K_\infty} \to \widetilde{\mathbf{B}}^{[r_\ell,+\infty]}_{K_\infty} \oplus \widetilde{\mathbf{B}}^{[0,r_k]}_{K_\infty} \to \widetilde{\mathbf{B}}^{[r_\ell,r_k]}_{K_\infty} \to 0, \tag{2.1.2}$$

where the second arrow is $x \mapsto (x,x)$, and the third arrow is $(a,b) \mapsto a-b$.

Proof. The proof is analogous to [Ber02, Lem. 2.27]. By the proof of [Ber02, Lem. 2.18], we have

$$0 \to \widetilde{\mathbf{B}}^{[0,+\infty]} \to \widetilde{\mathbf{B}}^{[r_\ell,+\infty]} \oplus \widetilde{\mathbf{B}}^{[0,r_k]} \to \widetilde{\mathbf{B}}^{[r_\ell,r_k]} \to 0.$$

Take G_∞-invariants, and consider the long exact sequence, it suffices to show that the map

$$\delta : \widetilde{\mathbf{B}}^{[r_\ell,r_k]}_{K_\infty} \to H^1(G_\infty, \widetilde{\mathbf{B}}^+) \tag{2.1.3}$$

is the zero map. By exactly the same argument as in [Ber02, Lem. 2.27], it suffices to show that $H^1(G_\infty, \mathfrak{m}_{\widetilde{\mathbf{E}}^+}) = 0$ (where $\mathfrak{m}_{\widetilde{\mathbf{E}}^+}$ is the maximal ideal of $\widetilde{\mathbf{E}}^+$); and this is an analogue of [Col98, Prop. IV.1.4(iii)]. Indeed, the ring $\widetilde{\mathbf{E}}^+$ satisfies the conditions (C1), (C2) and (C3) in [Col98, IV.1] with respect to our APF extension K_∞ (note that the K_∞ in *loc. cit.* is our K_{p^∞}); the proof is verbatim as in [Col98, Rem. IV.1.1(iii)], since the theory of fields of norms for our extension K_∞ also works (see e.g. [Bre99, §2] for a detailed development). □

Proposition 2.1.14. *(1) $\widetilde{\mathbf{A}}^{[0,r_k]}_{K_\infty} = \widetilde{\mathbf{A}}^+_{K_\infty}\{\frac{\varphi^k(E(u))}{p}\} = \widetilde{\mathbf{A}}^+_{K_\infty}\{\frac{u^{ep^k}}{p}\}$.*

(2) $\widetilde{\mathbf{A}}^{[r_\ell,+\infty]}_{K_\infty} = \widetilde{\mathbf{A}}^+_{K_\infty}\{\frac{p}{u^{ep^\ell}}\}$.

(3) $\widetilde{\mathbf{B}}^{[r_\ell,r_k]}_{K_\infty} = \widetilde{\mathbf{A}}^+_{K_\infty}\{\frac{p}{u^{ep^\ell}}, \frac{u^{ep^k}}{p}\}[\frac{1}{p}]$.

Proof. Item (1) is an analogue of [Ber02, Lem. 2.29]. By applying φ^{-k} to all rings, it suffices to prove it when $k=0$. By definition of $\widetilde{\mathbf{A}}^{[0,r_k]}$, it is obvious that $\widetilde{\mathbf{A}}^+_{K_\infty}\{\frac{\varphi^k(E(u))}{p}\} \subset \widetilde{\mathbf{A}}^{[0,r_k]}_{K_\infty}$; it suffices to show the inclusion is identity. Since $\widetilde{\mathbf{E}}^+_{K_\infty}/u^e\widetilde{\mathbf{E}}^+_{K_\infty}$ has a basis of u^i for $i \in \mathbb{Z}[1/p] \cap [0,e)$, we can easily deduce that $\theta : \widetilde{\mathbf{A}}^+_{K_\infty} \to \mathcal{O}_{\hat{K_\infty}}$ is surjective. Given $x \in \widetilde{\mathbf{A}}^{[0,r_0]}_{K_\infty}$, we recursively define two sequences $x_i \in \widetilde{\mathbf{A}}^{[0,r_0]}_{K_\infty}$ and $a_i \in \widetilde{\mathbf{A}}^+_{K_\infty}$ as follows:

- let $x_0 = x$;
- choose any $a_i \in \widetilde{\mathbf{A}}^+_{K_\infty}$ such that $\theta(a_i) = \theta(x_i) \in \mathcal{O}_{\hat{K_\infty}}$;
- let $x_{i+1} := (x_i - a_i) \cdot \frac{p}{E(u)}$, then $x_{i+1} \in \widetilde{\mathbf{A}}^{[0,r_0]}_{K_\infty}$ by Lem. 2.1.12.

Then it is easy to check that $x = \sum_{i\ge 0} a_i (E(u)/p)^i$ with $a_i \to 0$.

For Item (2), again it suffices to consider the case $\ell = 0$. Let $x \in \widetilde{\mathbf{A}}^{[r_0,+\infty]}_{K_\infty}$, write it as $x = \sum_{k\ge 0} p^k [x_k]$, then clearly $x_k \in (\widetilde{\mathbf{E}})^{G_\infty}$. Since $(pr_0)/(p-1)\cdot k + v_{\widetilde{\mathbf{E}}}(x_k) \to +\infty$ as $k \to +\infty$, so $k + v_{\widetilde{\mathbf{E}}}(x_k) \to +\infty$, and so $v_{\widetilde{\mathbf{E}}}(x_k \cdot \underline{\pi}^{ek}) \to +\infty$. Then one can easily show that $x \in \widetilde{\mathbf{A}}^+_{K_\infty}\{\frac{p}{u^e}\}$.

Consider Item (3). By Lem. 2.1.13, any element $x \in \widetilde{\mathbf{B}}^{[r_\ell,r_k]}_{K_\infty}$ can be written as a sum $x = a+b$ with $a \in \widetilde{\mathbf{B}}^{[r_\ell,+\infty]}_{K_\infty}$ and $b \in \widetilde{\mathbf{B}}^{[0,r_k]}_{K_\infty}$, so we can apply Items (1) and (2) to conclude. □

Remark 2.1.15. We do not know if we have

$$\widetilde{\mathbf{A}}^{[r_\ell,r_k]}_{K_\infty} = \widetilde{\mathbf{A}}^+_{K_\infty}\{\frac{p}{u^{ep^\ell}}, \frac{u^{ep^k}}{p}\}. \tag{2.1.4}$$

Equivalently, we do not know if the "$\widetilde{\mathbf{A}}$"-version of (2.1.2) (by changing all $\widetilde{\mathbf{B}}$ there to $\widetilde{\mathbf{A}}$) holds. Indeed, to show that the δ-map in (2.1.3) is the zero map following [Ber02, Lem. 2.27], it is critical to use the fact that u is invertible in $\widetilde{\mathbf{B}}^{[r_\ell,r_k]}_{K_\infty}$ (which fails in $\widetilde{\mathbf{A}}^{[r_\ell,r_k]}_{K_\infty}$). We tend to think that (2.1.4) holds. In particular, the "$\mathbf{A}$"-version of (2.1.4) holds, cf. Prop.2.2.10; the proof critically relies on the *unique* decompostion $f = f^- + f^+$ in Lem. 2.2.5, which fails inside $\widetilde{\mathbf{A}}^{[r_\ell,r_k]}_{K_\infty}$. Fortunately, (2.1.4) is perhaps useless anyway; e.g., the "K_{p^∞}-version" was never studied in [Ber02]. In contrast, Prop. 2.2.10 (indeed Cor. 2.2.11) plays a key role in our Thm. 3.4.4.

Proposition 2.1.16. *(1) The ring $\widetilde{\mathbf{B}}^{[r_\ell,+\infty]}_{K_\infty}$ is dense in $\widetilde{\mathbf{B}}^{[r_\ell,+\infty)}_{K_\infty}$ for the Fréchet topology.*

(2) The ring $\widetilde{\mathbf{B}}^{[0,+\infty]}_{K_\infty}$ is dense in $\widetilde{\mathbf{B}}^{[0,+\infty)}_{K_\infty}$ for the Fréchet topology.

Proof. The proof (for both Items) is verbatim as the proof of [Ber02, Prop. 2.30], by changing q there to $E(u)$. □

2.2. The ring $\mathbf{B}^I$ and its G_∞-invariants.

Definition 2.2.1. (1) When $r \in \mathbb{Z}^{\geq 0}[1/p]$, let

$$\mathbf{A}^{[r,+\infty]} := \mathbf{A} \cap \widetilde{\mathbf{A}}^{[r,+\infty]}, \quad \mathbf{B}^{[r,+\infty]} := \mathbf{B} \cap \widetilde{\mathbf{B}}^{[r,+\infty]}.$$

(2) When $r, s \in \mathbb{Z}^{\geq 0}[1/p], s \neq 0$, let $\mathbf{B}^{[r,s]}$ be the closure of $\mathbf{B}^{[r,+\infty]}$ in $\widetilde{\mathbf{B}}^{[r,s]}$ with respect to $W^{[r,s]}$ (By Rem. 2.1.2, there is no $\widetilde{\mathbf{B}}^{[0,0]}$ hence no $\mathbf{B}^{[0,0]}$). Let $\mathbf{A}^{[r,s]} := \mathbf{B}^{[r,s]} \cap \widetilde{\mathbf{A}}^{[r,s]}$, which is the ring of integers in $\mathbf{B}^{[r,s]}$.

(3) When $r \in \mathbb{Z}^{\geq 0}[1/p]$, let

$$\mathbf{B}^{[r,+\infty)} := \bigcap_{n\geq 0} \mathbf{B}^{[r,s_n]}$$

where $s_n \in \mathbb{Z}^{>0}[1/p]$ is any sequence increasing to $+\infty$.

Definition 2.2.2. For $r \in \mathbb{Z}^{\geq 0}[1/p]$, let $\mathcal{A}^{[r,+\infty]}(K_0)$ be the ring consisting of infinite series $f = \sum_{k\in\mathbb{Z}} a_k T^k$ where $a_k \in W(k)$ such that f is a holomorphic function on the annulus defined by

$$v_p(T) \in (0, \quad \frac{p-1}{ep} \cdot \frac{1}{r}].$$

(Note that when $r = 0$, it implies that $a_k = 0, \forall k < 0$). Let $\mathcal{B}^{[r,+\infty]}(K_0) := \mathcal{A}^{[r,+\infty]}(K_0)[1/p]$.

Definition 2.2.3. Suppose $f = \sum_{k\in\mathbb{Z}} a_k T^k \in \mathcal{B}^{[r,+\infty]}(K_0)$.

(1) When $s \geq r$, $s > 0$, let

$$\mathcal{W}^{[s,s]}(f) := \inf_{k\in\mathbb{Z}} \{v_p(a_k) + \frac{p-1}{ps} \cdot \frac{k}{e}\}.$$

(2) For $I \subset [r, +\infty)$ a non-empty closed interval, let

$$\mathcal{W}^I(f) := \inf_{\alpha\in I, \alpha\neq 0} \mathcal{W}^{[\alpha,\alpha]}(f). \tag{2.2.1}$$

It is well-known that $\mathcal{W}^{[s,s]}$ for any $s > 0$ is an multiplicative valuation; thus $\mathcal{W}^I$ is an sub-multiplicative valuation.

Definition 2.2.4. For $r \leq s \in \mathbb{Z}^{\geq 0}[1/p], s \neq 0$, let $\mathcal{B}^{[r,s]}(K_0)$ be the completion of $\mathcal{B}^{[r,+\infty]}(K_0)$ with respect to $\mathcal{W}^{[r,s]}$. Let $\mathcal{A}^{[r,s]}(K_0)$ be the ring of integers in $\mathcal{B}^{[r,s]}(K_0)$ with respect to $\mathcal{W}^{[r,s]}$.

Lemma 2.2.5. *(1) $\mathcal{B}^{[r_\ell,+\infty]}(K_0)$ is complete with respect to $\mathcal{W}^{[r_\ell,r_\ell]}$, and $\mathcal{A}^{[r_\ell,+\infty]}(K_0)$ is the ring of integers with respect to this valuation. Furthermore, we have*

$$\mathcal{A}^{[r_\ell,+\infty]}(K_0) = W(k)[\![T]\!]\{\frac{p}{T^{ep^\ell}}\}. \tag{2.2.2}$$

(2) We have $\mathcal{W}^{[0,r_k]}(x) = \mathcal{W}^{[r_k,r_k]}(x)$. Furthermore, $\mathcal{B}^{[0,r_k]}(K_0)$ is the ring consisting of infinite series $f = \sum_{k\in\mathbb{Z}} a_k T^k$ where $a_k \in K_0$ such that f is a holomorphic function on the closed disk *defined by*

$$v_p(T) \in [\frac{p-1}{ep} \cdot \frac{1}{r_k}, \quad +\infty].$$

Indeed, we have

$$\mathcal{A}^{[0,r_k]}(K_0) = W(k)[\![T]\!]\{\frac{T^{ep^k}}{p}\}, \text{ and } \mathcal{B}^{[r,s]}(K_0) = \mathcal{A}^{[r,s]}(K_0)[1/p]. \tag{2.2.3}$$

(3) *For $I = [r,s] = [r_\ell, r_k]$, we have $\mathcal{W}^I(x) = \inf\{\mathcal{W}^{[r,r]}(x), \mathcal{W}^{[s,s]}(x)\}$. Furthermore, $\mathcal{B}^{[r,s]}(K_0)$ is the ring consisting of infinite series $f = \sum_{k\in\mathbb{Z}} a_k T^k$ where $a_k \in K_0$ such that f is a holomorphic function on the annulus defined by*

$$v_p(T) \in [\frac{p-1}{ep}\cdot\frac{1}{s}, \quad \frac{p-1}{ep}\cdot\frac{1}{r}].$$

Indeed, we have

$$\mathcal{A}^{[r_\ell,r_k]}(K_0) = W(k)[\![T]\!]\{\frac{p}{T^{ep^\ell}}, \frac{T^{ep^k}}{p}\}, \text{ and } \mathcal{B}^{[r,s]}(K_0) = \mathcal{A}^{[r,s]}(K_0)[1/p]. \tag{2.2.4}$$

Proof. Everything is elementary and well-known; we only sketch how to prove (2.2.4). Let $f = \sum_{k\in\mathbb{Z}} a_k T^k \in \mathcal{A}^{[r_\ell,r_k]}(K_0)$, then we can decompose $f = f^- + f^+$ uniquely where $f^- = \sum_{k<0} a_k T^k$ and $f^+ = \sum_{k\geq 0} a_k T^k$. Since the valuations $\mathcal{W}^{[s,s]}$ are defined in a "term-wise" fashion (i.e., $\mathcal{W}^{[s,s]}(f) = \inf_k \mathcal{W}^{[s,s]}(a_k T^k)$), it is easy to see that $f^- \in \mathcal{A}^{[r_\ell,+\infty]}(K_0)$ and $f^+ \in \mathcal{A}^{[0,r_k]}(K_0)$; then we can conclude using (2.2.2) and (2.2.3). □

Remark 2.2.6. When $r = 0$ in Def. 2.2.3, it actually makes perfect sense to define

$$\mathcal{W}^{[0,0]}(f) := v_p(a_0). \tag{2.2.5}$$

Indeed, the valuations $\mathcal{W}^{[s,s]}(f)$ (for $s > 0$) correspond to the Gauss norms on the *circle* of radius $p^{-(p-1)/eps}$, and this "$\mathcal{W}^{[0,0]}(f)$" corresponds precisely to the norm on the zero point. Using (2.2.5), we could even modify (2.2.1) (when $0 \in I$) to be

$$\mathcal{W}^{[0,s]}(f) := \inf_{\alpha\in[0,s]} \mathcal{W}^{[\alpha,\alpha]}(f). \tag{2.2.6}$$

But these two definitions give the same valuation (namely, $\mathcal{W}^{[s,s]}(f)$), because the zero point is not on the boundary (of the relevant closed disk) anyway! Since we do not have a "compatible" $W^{[0,0]}$ by Rem. 2.1.9, it is better for us to completely ignore "$\mathcal{W}^{[0,0]}$".

Lemma 2.2.7. *Let $r \leq s \in \mathbb{Z}^{\geq 0}[1/p]$, $s > 0$.*

(1) *The map $f(T) \mapsto f(u)$ induces ring isomorphisms*

$$\begin{aligned} \mathcal{A}^{[0,+\infty]}(K_0) &\simeq \mathbf{A}^{[0,+\infty]}_{K_\infty}, \text{ when } r = 0; \\ \mathcal{A}^{[r,+\infty]}(K_0) &\simeq \mathbf{A}^{[r,+\infty]}_{K_\infty}[1/u], \text{ when } r > 0. \end{aligned}$$

Furthermore, given $f \in \mathcal{A}^{[r,+\infty]}(K_0)$, we have

$$\mathcal{W}^{[s,s]}(f(T)) = W^{[s,s]}(f(u)).$$

(2) *The map $f(T) \mapsto f(u)$ induces isometric isomorphisms*

$$\begin{aligned} \mathcal{A}^{[0,s]}(K_0) &\simeq \mathbf{A}^{[0,s]}_{K_\infty}, \text{ when } r = 0; \\ \mathcal{A}^{[r,s]}(K_0) &\simeq \mathbf{A}^{[r,s]}_{K_\infty}, \text{ when } r > 0. \end{aligned}$$

Before we prove the lemma, we introduce the section s and use it to build an approximating sequence.

2.2.8. *The section s.* Denote

$$s : \mathbf{A}_{K_\infty}/p \to \mathbf{A}_{K_\infty}$$

the section where for $\overline{x} = \sum_{i \gg -\infty} \overline{a_i} u^i$, let $s(\overline{x}) := \sum_{i \gg -\infty} [\overline{a_i}] u^i$. One can see that $s(\overline{x}) \in \mathbf{A}_{K_\infty}^{[r,+\infty]}[1/u]$ for any $r \geq 0$. Furthermore, for any $k \geq 0$, we have

$$w_k(s(\overline{x})) = \inf_i \{w_k([\overline{a_i}]u^i)\} = \frac{1}{e} \min\{i : \overline{a_i} \neq 0\} = v_{\widetilde{\mathbf{E}}}(\overline{x}), \tag{2.2.7}$$

where the first identity holds because $w_k([\overline{a_i}]u^i)$ are distinct for different i.

2.2.9. *An approximating sequence.* Let $r \geq 0$, given $x \in \mathbf{A}_{K_\infty}^{[r,+\infty]}[1/u]$, define a sequence $\{x_n\}$ in $\mathbf{A}_{K_\infty}^{[r,+\infty]}[1/u]$ where $x_0 = x$ and $x_{n+1} := p^{-1}(x_n - s(\overline{x_n}))$. Then $x = \sum_{n \geq 0} p^n \cdot s(\overline{x_n})$, and we have that

$$\begin{aligned} w_k(x_{n+1}) &= w_{k+1}(px_{n+1}) \\ &\geq \inf\{w_{k+1}(x_n), w_{k+1}(s(\overline{x_n}))\} \\ &= \inf\{w_{k+1}(x_n), w_0(x_n)\}, \text{ by (2.2.7)}, \\ &= w_{k+1}(x_n). \end{aligned}$$

Iterating the above process, we get

$$w_0(x_n) \geq w_n(x_0) = w_n(x). \tag{2.2.8}$$

Proof of Lem. 2.2.7. Lem. 2.2.7 is an analogue of [Col08, Prop. 7.5], and the proof uses similar ideas. It suffices to prove Item (1).

Part 1. Given $f(T) = \sum_{k \in \mathbb{Z}} a_k T^k \in \mathcal{A}^{[r,+\infty]}(K_0)$, then for any $s \in [r, +\infty), s > 0$,

$$W^{[s,s]}(f(u)) \geq \inf_k \{W^{[s,s]}(a_k u^k)\} = \inf_k \{v_p(a_k) + \frac{p-1}{ps} \cdot \frac{k}{e}\} = \mathcal{W}^{[s,s]}(f(T)).$$

When $r > 0$, $v_p(a_k) + \frac{p-1}{pr} \cdot \frac{k}{e} \to +\infty$ for $k \to +\infty$ or $k \to -\infty$. By Lem. 2.1.10, $\mathbf{A}_{K_\infty}^{[r,+\infty]}[1/u]$ is complete with respect to $W^{[r,r]}$; thus $f(u) \in \mathbf{A}_{K_\infty}^{[r,+\infty]}[1/u]$ when $r > 0$. When $r = 0$, then it is clear that $f(u) \in \mathbf{A}_{K_\infty}^{[0,+\infty]}$. Also, it is obvious that the map $f(T) \mapsto f(u)$ is injective.

Part 2. Given $x \in \mathbf{A}_{K_\infty}^{[r,+\infty]}[1/u]$ when $r > 0$ (resp. $x \in \mathbf{A}_{K_\infty}^{[0,+\infty]}$ when $r = 0$), let $\{x_n\}$ be the sequence constructed in §2.2.9 (note that when $x \in \mathbf{A}_{K_\infty}^{[0,+\infty]}$, then $x_n \in \mathbf{A}_{K_\infty}^{[0,+\infty]}, \forall n$). Let $f_n(T)$ be the formal series $\sum_{k \in \mathbb{Z}} f_{n,k} T^k$ such that $f_n(u) = s(\overline{x_n})$, and let $f(T) := \sum_{n \geq 0} p^n f_n(T)$. By (2.2.8),

$$v_{\widetilde{\mathbf{E}}}(\overline{x_n}) = w_0(x_n) \geq w_n(x),$$

so the expression for $s(\overline{x_n})$ would be of the form $\sum_{i \geq e w_n(x)} [\overline{a_i}] u^i$ (recall that $v_{\widetilde{\mathbf{E}}}(u) = 1/e$). Thus $f_n(T) = \sum_{i \geq e w_n(x)} [\overline{a_i}] T^i$, and so

$$\mathcal{W}^{[s,s]}(p^n f_n(T)) \geq \mathcal{W}^{[s,s]}(p^n T^{\lceil e w_n(x) \rceil}) \geq n + \frac{p-1}{ps} \cdot \frac{1}{e} \cdot e w_n(x) \geq W^{[s,s]}(x).$$

When $r > 0$, $n + \frac{p-1}{pr} \cdot w_n(x) \to +\infty$ when $n \to +\infty$, so $f(T)$ converges in $\mathcal{A}^{[r,+\infty]}(K_0)$. (When $r = 0$, $f(T)$ automatically converges in $\mathcal{A}^{[0,+\infty]}(K_0)$). It is clear $f(u) = x$, and $\mathcal{W}^{[s,s]}(f(T)) \geq W^{[s,s]}(x)$. Combined with Part 1, this concludes the proof. ☐

Proposition 2.2.10. *We have*

$$\begin{aligned}
\mathbf{A}_{K_\infty}^{[0,+\infty]} &= \mathbf{A}_{K_\infty}^{+}, \\
\mathbf{A}_{K_\infty}^{[0,r_k]} &= \mathbf{A}_{K_\infty}^{+}\{\frac{u^{ep^k}}{p}\}, \\
\mathbf{A}_{K_\infty}^{[r_\ell,+\infty]} &= \mathbf{A}_{K_\infty}^{+}\{\frac{p}{u^{ep^\ell}}\}, \\
\mathbf{A}_{K_\infty}^{[r_\ell,r_k]} &= \mathbf{A}_{K_\infty}^{+}\{\frac{p}{u^{ep^\ell}}, \frac{u^{ep^k}}{p}\}.
\end{aligned}$$

Proof. This easily follows from Lem. 2.2.7 and Lem. 2.2.5. ☐

Corollary 2.2.11. *Suppose* $[r,s]=[r_\ell,r_k]\subset[r',s]=[r_{\ell'},r_k]$, *then* $\mathbf{A}_{K_\infty}^{[r,s]}\cap\widetilde{\mathbf{A}}^{[r',s]}=\mathbf{A}_{K_\infty}^{[r',s]}$.

Proof. Let $f\in\mathbf{A}_{K_\infty}^{[r,s]}\cap\widetilde{\mathbf{A}}^{[r',s]}$. By Prop. 2.2.10, we can always write $f=f_1+f_2$, where $f_1\in\mathbf{A}_{K_\infty}^{[r,+\infty]}$ and $f_2\in\mathbf{A}_{K_\infty}^{[0,s]}$; it then suffices to show that $f_1\in\mathbf{A}_{K_\infty}^{[r',s]}$. But indeed we can show that $f_1\in\mathbf{A}_{K_\infty}^{[r',+\infty]}$, using similar argument as in [CC98, Lem. II.2.2]. ☐

3. Locally analytic vectors of some rings

The main result in this section is to calculate locally analytic vectors in $(\widetilde{\mathbf{B}}^I)^{G_\infty}=\widetilde{\mathbf{B}}^I_{K_\infty}$. Actually, there is no group action on $(\widetilde{\mathbf{B}}^I)^{G_\infty}$ since G_∞ is not normal in G_K; what we do instead is to calculate locally analytic vectors in $\widetilde{\mathbf{B}}^I_L:=(\widetilde{\mathbf{B}})^{\mathrm{Gal}(\overline{K}/L)}$ (with respect to the $\mathrm{Gal}(L/K)$-action) that are *furthermore* G_∞*-invariant.*

3.1. **Theory of locally analytic vectors.** Let us recall the theory of locally analytic vectors, see [BC16, §2.1] and [Ber16, §2] for more details. Recall that a $\mathbb{Q}_p$-Banach space W is a $\mathbb{Q}_p$-vector space with a complete non-Archimedean norm $\|\cdot\|$ such that $\|aw\|=\|a\|_p\|w\|, \forall a\in\mathbb{Q}_p, w\in W$, where $\|a\|_p$ is the usual p-adic norm on $\mathbb{Q}_p$. Recall the multi-index notations: if $\mathbf{c}=(c_1,\ldots,c_d)$ and $\mathbf{k}=(k_1,\ldots,k_d)\in\mathbb{N}^d$ (here $\mathbb{N}=\mathbb{Z}^{\geq 0}$), then we let $\mathbf{c}^{\mathbf{k}}=c_1^{k_1}\cdot\ldots\cdot c_d^{k_d}$.

3.1.1. Let G be a p-adic Lie group, and let $(W,\|\cdot\|)$ be a $\mathbb{Q}_p$-Banach representation of G. Let H be an open subgroup of G such that there exist coordinates $c_1,\ldots,c_d: H\to\mathbb{Z}_p$ giving rise to an analytic bijection $\mathbf{c}: H\to\mathbb{Z}_p^d$. We say that an element $w\in W$ is an H-analytic vector if there exists a sequence $\{w_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{N}^d}$ with $w_{\mathbf{k}}\to 0$ in W, such that

$$g(w)=\sum_{\mathbf{k}\in\mathbb{N}^d}\mathbf{c}(g)^{\mathbf{k}}w_{\mathbf{k}}, \quad \forall g\in H.$$

Let $W^{H\text{-an}}$ denote the space of H-analytic vectors. $W^{H\text{-an}}$ injects into $\mathcal{C}^{\mathrm{an}}(H,W)$ (the space of analytic functions on H valued in W), and we endow it with the induced norm, which we denote as $\|\cdot\|_H$. We have $\|w\|_H=\sup_{\mathbf{k}\in\mathbb{N}^d}\|w_{\mathbf{k}}\|$, and $W^{H\text{-an}}$ is a Banach space.

We say that a vector $w\in W$ is *locally analytic* if there exists an open subgroup H as above such that $w\in W^{H\text{-an}}$. Let $W^{G\text{-la}}$ (or W^{la} when there is no confusion) denote the space of such vectors. We have $W^{\mathrm{la}}=\cup_H W^{H\text{-an}}$ where H runs through open subgroups of G. We can endow W^{la} with the inductive limit topology, so that W^{la} is an LB space.

Lemma 3.1.2. *Keep the notations as in §3.1.1. If W is furthermore a ring such that $\|xy\|\leq\|x\|\cdot\|y\|$ for $x,y\in W$, then*

(1) $W^{H\text{-an}}$ is a ring, and $\|xy\|_H\leq\|x\|_H\cdot\|y\|_H$ if $x,y\in W^{H\text{-an}}$.
(2) Suppose $w\in W^\times$ and $w\in W^{G\text{-la}}$, then $1/w\in W^{G\text{-la}}$. (In particular, if W is a field, then $W^{G\text{-la}}$ is also a field.)

Proof. Item (1) is [BC16, Lem. 2.5(i)]. Item (2) is stronger than [BC16, Lem. 2.5(ii)], but this stronger statement is proved in *loc. cit.*. ☐

3.1.3. Keep the notations as in §3.1.1. By the paragraph preceding [BC16, Lem. 2.4], there exists some (not unique) open compact subgroup G_1 of G such that there exist local coordinates $\widetilde{\mathbf{c}} : G_1 \to \mathbb{Z}_p^d$, which furthermore satisfy $\widetilde{\mathbf{c}}(G_n) = (p^n\mathbb{Z}_p)^d$ where $G_n := G_1^{p^{n-1}}$. Then we have $W^{\mathrm{la}} = \cup_n W^{G_n\text{-an}}$.

Lemma 3.1.4. *([BC16, Lem. 2.4]) Keep the notations as in §3.1.3. Suppose $w \in W^{G_n\text{-an}}$, then for all $m \geq n$, $w \in W^{G_m\text{-an}}$ and $\|w\|_{G_m} \leq \|w\|_{G_n}$. Furthermore, $\|w\|_{G_m} = \|w\|$ when $m \gg 0$.*

3.1.5. Let W be a Fréchet space, whose topology is defined by a sequence $\{p_i\}_{i\geq 1}$ of semi-norms. Let W_i denote the Hausdorff completion of W for p_i, so that $W = \varprojlim_{i\geq 1} W_i$. If W is a Fréchet representation of G, then a vector $w \in W$ is called *pro-analytic* if its image $\pi_i(w)$ in W_i is a locally analytic vector for all i. We denote by W^{pa} the set of such vectors. We can extend this definition to LF spaces (cf. [Ber16, §2]).

Proposition 3.1.6. *Let G be a p-adic Lie group, let $\hat{B}$ be a Banach (resp. Fréchet) G-ring, and $B \subset \hat{B}$ a subring (but not necessarily G-stable). Let W be a free B-module of finite rank, let $\hat{W} := \hat{B} \otimes_B W$, and suppose there is a $\hat{B}$-semi-linear G-action on $\hat{W}$. Let $B^{\mathrm{la}} := B \cap \hat{B}^{\mathrm{la}}$ and $W^{\mathrm{la}} := W \cap \hat{W}^{\mathrm{la}}$ (resp. $B^{\mathrm{pa}} := B \cap \hat{B}^{\mathrm{pa}}$ and $W^{\mathrm{pa}} := W \cap \hat{W}^{\mathrm{pa}}$).*

If W has a B-basis $w_1, \ldots, w_d$ such that $g \mapsto \mathrm{Mat}(g)$ is a globally analytic *(resp.* pro-analytic*) function $G \to \mathrm{GL}_d(\hat{B}) \subset \mathrm{M}_d(\hat{B})$, then*

$$W^{\mathrm{la}} = \oplus_{j=1}^d B^{\mathrm{la}} \cdot w_j \quad (\text{resp. } W^{\mathrm{pa}} = \oplus_{j=1}^d B^{\mathrm{pa}} \cdot w_j).$$

Proof. By [BC16, Prop. 2.3] (resp. [Ber16, Prop. 2.4]), we have $\hat{W}^{\mathrm{la}} = \oplus_{j=1}^d \hat{B}^{\mathrm{la}} \cdot w_j$ (resp. $\hat{W}^{\mathrm{pa}} = \oplus_{j=1}^d \hat{B}^{\mathrm{pa}} \cdot w_j$), then we can take intersections with W to conclude. □

In the following, we give a useful criterion to determine analytic vectors for the p-adic Lie group $\mathbb{Z}_p$.

Lemma 3.1.7. *Suppose $(W, \|\cdot\|)$ is a $\mathbb{Q}_p$-Banach representation of $\mathbb{Z}_p$. Let τ be a topological generator of $\mathbb{Z}_p$, and let $\log \tau$ denote the (formally written) series $(-1) \cdot \sum_{k\geq 1}(1-\tau)^k/k$. Given $x \in W$, then $x \in W^{\mathbb{Z}_p\text{-an}}$ if and only if the following hold:*

(1) the series $(\log \tau)(x)$ converges in W, and inductively, $(\log \tau)^i(x) := (\log \tau)((\log \tau)^{i-1}(x))$ converges in W for all $i \geq 2$;
(2) $\|(\log \tau)^i(x)/i!\| \to 0$ as $i \to +\infty$;
(3) for all $a \in \mathbb{Z}_p$,

$$\tau^a(x) = \sum_{i=0}^{\infty} a^i \cdot \frac{(\log \tau)^i(x)}{i!}. \tag{3.1.1}$$

If the above holds, then $\log \tau(x) \in W^{\mathbb{Z}_p\text{-an}}$, and for all $a \in \mathbb{Z}_p$, we have $(\log \tau^a)(x) = a \cdot \log \tau(x)$.

Proof. This is standard, cf. [ST02, §3]. □

Lemma 3.1.8. *Suppose $(W, \|\cdot\|)$ is a $\mathbb{Q}_p$-Banach representation of $\mathbb{Z}_p$ such that $\|g(w)\| = \|w\|, \forall g \in \mathbb{Z}_p, w \in W$ (i.e., $\|\cdot\|$ is an* invariant *norm). Let $x \in W$. Let τ be a topological generator of $\mathbb{Z}_p$. If there exists some $r < \inf\{1/e, p^{-1/(p-1)}\}$ (here e is Euler's number $2.718\ldots$), some $R > 0$ and $k_0 \in \mathbb{Z}^{\geq 0}$, such that*

$$\|(1-\tau^a)^k(x)\| \leq R, \quad \text{for all } a \in \mathbb{Z}_p, k < k_0; \tag{3.1.2}$$

$$\|(1-\tau^a)^k(x)\| \leq r^k, \quad \text{for all } a \in \mathbb{Z}_p, k \geq k_0, \tag{3.1.3}$$

then $x \in W^{\mathbb{Z}_p\text{-an}}$.

Proof. **Step 0**: *Partial log.* Let A be a $\mathbb{Q}_p$-algebra. Given $a \in A$, denote

$$\log_m a := \sum_{i=1}^{p^m-1} \frac{(1-a)^i}{i} \in A.$$

If A is furthermore a Banach algebra, and $\|\frac{(1-a)^i}{i}\| \to 0$ when $i \to +\infty$, then we denote $\log a := (-1)\cdot \sum_{i=1}^{+\infty} \frac{(1-a)^i}{i}$ (and say $\log a$ is well-defined). Suppose $a, b \in A$ such that $ab = ba$, then we have the identity:

$$\frac{(1-ab)^i}{i} = \frac{(1-a)^i}{i} + \sum_{j=1}^{i} \binom{i-1}{j-1} \cdot a^j (1-a)^{i-j} \cdot \frac{(1-b)^j}{j}.$$

So we have (cf. [Car13, Eqn. (3.4)]):

$$\log_m(ab) = \log_m a + \sum_{j=1}^{p^m-1} \left(a^j \cdot \sum_{i=j}^{p^m-1} \binom{i-1}{j-1} \cdot (1-a)^{i-j} \right) \cdot \frac{(1-b)^j}{j}.$$

Note that (cf. the equation below [Car13, Eqn. (3.4)])

$$(1-X)^j \cdot \sum_{i=j}^{p^m-1} \binom{i-1}{j-1} X^{i-j} \in 1 + X^{p^m-j}\mathbb{Z}_p[X].$$

Apply the above identity with $X = 1-a$, then we get

$$\log_m(ab) - \log_m a - \log_m b = \sum_{j=1}^{p^m-1} f_j(1-a)\cdot(1-a)^{p^m-j} \cdot \frac{(1-b)^j}{j}, \tag{3.1.4}$$

where $f_j(X) \in \mathbb{Z}_p[X]$ are some polynomials.

Step 1: *Logarithm of x.* Using condition (3.1.2) and (3.1.3), it is clear that for any $a \in \mathbb{Z}_p$, $(\log \tau^a)(x)$ is well-defined. Furthermore, there exists some $r' > 0$, such that

$$\|(\log \tau^a)(x)\| < r', \quad \forall a \in \mathbb{Z}_p. \tag{3.1.5}$$

We claim that

$$(\log \tau^a)(x) = a \cdot (\log \tau)(x), \quad \forall a \in \mathbb{Z}_p. \tag{3.1.6}$$

To prove (3.1.6), we first show that

$$(\log \tau^{\alpha+\beta})(x) = (\log \tau^{\alpha})(x) + (\log \tau^{\beta})(x), \quad \forall \alpha, \beta \in \mathbb{Z}_p. \tag{3.1.7}$$

Using (3.1.4), we have

(3.1.8)

$$(\log_m \tau^{\alpha+\beta})(x) - (\log_m \tau^{\alpha})(x) - (\log_m \tau^{\beta})(x) = \sum_{j=1}^{p^m-1} f_j(1-\tau^\alpha)\cdot(1-\tau^\alpha)^{p^m-j}\cdot \frac{(1-\tau^\beta)^j}{j}(x).$$

Since $\|\cdot\|$ is an invariant norm, it is easy to see that

$$\|(f(\tau))(w)\| \le \|w\|, \quad \forall w \in W, f(X) \in \mathbb{Z}_p[X] \text{ a polynomial.} \tag{3.1.9}$$

When $p^m/2 \ge k_0$ (so $\max\{j, p^m - j\} \ge k_0, \forall j$), the norm of the right hand side of (3.1.8) is bounded by $p^m r^{p^m/2}$ (using (3.1.3) and (3.1.9)). Let $m \to +\infty$, and so (3.1.7) is proved. Now given $a \in \mathbb{Z}_p$, let $a = a_m + p^m b_m$ where $a_m \in \mathbb{Z}, b_m \in \mathbb{Z}_p$. By (3.1.7),

$$(\log \tau^a)(x) = (\log \tau^{a_m})(x) + (\log \tau^{p^m b_m})(x) = a_m \cdot (\log \tau)(x) + p^m \cdot (\log \tau^{b_m})(x).$$

Use (3.1.5), and let $m \to +\infty$, we can conclude (3.1.6).

Step 2: *General term of a summation.* Consider the summation $\sum_{k=0}^{\infty} \frac{(\log \tau^a)^k(x)}{k!}$ where $a \in \mathbb{Z}_p$, then its "general term" is of the form:

$$\frac{1}{k!}\frac{(1-\tau^a)^{i_1+\cdots+i_k}(x)}{i_1\cdot\cdots\cdot i_k}, \text{ where } i_j \ge 1.$$

Suppose $\sum i_j = n$, then $n \geq k$. Let

$$r_k := \sup_{n \geq k} \left\{ \| \frac{1}{k!} \frac{(1-\tau^a)^n(x)}{i_1 \cdots i_k} \|, \text{ where } \sum i_j = n \right\}.$$

Note that we have

$$\| \frac{1}{k!} \frac{(1-\tau^a)^n(x)}{i_1 \cdots i_k} \| \leq r^n \cdot p^{\frac{k}{p-1}} \cdot (\frac{n}{k})^k, \text{ when } n \geq k_0.$$

Fix a k, consider the function $f(X) = r^X \cdot X^k$ with $X \geq k$. Its logarithm is $X \ln r + k \ln X$, which has derivative $\ln r + k/X < 0$ since $r < 1/e$. Thus we conclude that

$$\| \frac{1}{k!} \frac{(1-\tau^a)^n(x)}{i_1 \cdots i_k} \| \leq r^k \cdot p^{\frac{k}{p-1}} \cdot (\frac{k}{k})^k = (rp^{\frac{1}{p-1}})^k, \text{ when } n \geq k_0.$$

This implies that $r_k < +\infty, \forall k$. Furthermore,

$$r_k \leq (rp^{\frac{1}{p-1}})^k, \text{ when } k \geq k_0,$$

and so $\lim_k r_k \to 0$ since $r < p^{-\frac{1}{p-1}}$. This implies that the summation $\sum_{k=0}^{\infty} \frac{(\log \tau^a)^k(x)}{k!}$ converges absolutely.

Step 3: *Conclusion.* Using Step 2 and (3.1.6) in Step 1, it is easy to show that all the itemized conditions in Lem. 3.1.7 are satisfied; in particular, the equality (3.1.1) holds because by Step 2, we can "re-arrange" the order of the summation. Thus $x \in W^{\mathbb{Z}_p\text{-an}}$. □

Notation 3.1.9. If $(W, \|\cdot\|)$ is a p-adically separated and complete normed $\mathbb{Z}_p$-module such that $\|aw\| = \|a\|_p \|w\|$ for all $a \in \mathbb{Z}_p$ and $w \in W$, and such that $W[1/p]$ (with the naturally induced norm) is a $\mathbb{Q}_p$-Banach space, then we say $(W, \|\cdot\|)$ is a $\mathbb{Z}_p$-Banach space for brevity. If furthermore such W carries a continuous action by a p-adic Lie group G, then we denote $W^{G\text{-la}} := (W[1/p])^{G\text{-la}} \cap W$.

3.2. Locally analytic representations of $\hat{G}$.

Let $\hat{G} = \mathrm{Gal}(L/K)$ be as in Notation 1.1.1. In this subsection, we mainly set up some notations with respect to representations of $\hat{G}$.

Notation 3.2.1. (1) Recall that:

- if $K_\infty \cap K_{p^\infty} = K$, then $\mathrm{Gal}(L/K_{p^\infty})$ and $\mathrm{Gal}(L/K_\infty)$ topologically generate $\hat{G}$ (cf. [Liu08, Lem. 5.1.2]);
- if $K_\infty \cap K_{p^\infty} \supsetneq K$, then necessarily $p = 2$, and $\mathrm{Gal}(L/K_{p^\infty})$ and $\mathrm{Gal}(L/K_\infty)$ topologically generate an open subgroup (denoted as $\hat{G}'$) of $\hat{G}$ of index 2 (cf. [Liu10, Prop. 4.1.5]).

(2) Note that:

- $\mathrm{Gal}(L/K_{p^\infty}) \simeq \mathbb{Z}_p$, and let $\tau \in \mathrm{Gal}(L/K_{p^\infty})$ be a topological generator;
- $\mathrm{Gal}(L/K_\infty)$ $(\subset \mathrm{Gal}(K_{p^\infty}/K) \subset \mathbb{Z}_p^\times)$ is not necessarily pro-cyclic when $p = 2$.

If we let $\Delta \subset \mathrm{Gal}(L/K_\infty)$ be the torsion subgroup, then $\mathrm{Gal}(L/K_\infty)/\Delta$ is pro-cyclic; choose $\gamma' \in \mathrm{Gal}(L/K_\infty)$ such that its image in $\mathrm{Gal}(L/K_\infty)/\Delta$ is a topological generator.

(3) Let $\tau_n := \tau^{p^n}$ and $\gamma'_n := (\gamma')^{p^n}$. Let $\hat{G}_n \subset \hat{G}$ be the subgroup topologically generated by τ_n and γ'_n. These $\hat{G}_n$ satisfy the property in §3.1.3.

Notation 3.2.2. (1) Given a $\hat{G}$-representation W, we use

$$W^{\tau=1}, \quad W^{\gamma=1}$$

to mean

$$W^{\mathrm{Gal}(L/K_{p^\infty})=1}, \quad W^{\mathrm{Gal}(L/K_\infty)=1}.$$

And we use

$$W^{\tau\text{-la}}, \quad W^{\tau_n\text{-an}}, \quad W^{\gamma\text{-la}}$$

to mean

$$W^{\mathrm{Gal}(L/K_{p^\infty})\text{-la}}, \quad W^{\mathrm{Gal}(L/(K_{p^\infty}(\pi_n)))\text{-la}}, \quad W^{\mathrm{Gal}(L/K_\infty)\text{-la}}.$$

(2) Let
$$\nabla_\tau := \frac{\log \tau^{p^n}}{p^n} \text{ for } n \gg 0, \quad \nabla_\gamma := \frac{\log g}{\log_p \chi_p(g)} \text{ for } g \in \mathrm{Gal}(L/K_\infty) \text{ close enough to } 1$$
be the two differential operators (acting on $\hat{G}$-locally analytic representations).

Remark 3.2.3. Note that we never define γ to be an element of $\mathrm{Gal}(L/K_\infty)$; although when $p > 2$ (or in general, when $\mathrm{Gal}(L/K_\infty)$ is pro-cyclic), we could have defined it as a topological generator of $\mathrm{Gal}(L/K_\infty)$. In particular, although "$\gamma = 1$" might be slightly ambiguous (but only when $p = 2$), we use the notation for brevity.

Lemma 3.2.4. *Let $W^{\tau\text{-la},\gamma=1} := W^{\tau\text{-la}} \cap W^{\gamma=1}$, then*
$$W^{\tau\text{-la},\gamma=1} \subset W^{\hat{G}\text{-la}}.$$

Proof. This can be deduced from the fact that any element $g \in \hat{G}$ (or $g \in \hat{G}'$ when $K_\infty \cap K_{p^\infty} \neq K$, cf. Notation 3.2.1) can be uniquely written as a product $g_1 g_2$ for some $g_1 \in \mathrm{Gal}(L/K_\infty), g_2 \in \mathrm{Gal}(L/K_{p^\infty})$. □

Remark 3.2.5. (1) Let $W^{\gamma\text{-la},\tau=1} := W^{\gamma\text{-la}} \cap W^{\tau=1}$, then
$$W^{\gamma\text{-la},\tau=1} = \left((W)^{\mathrm{Gal}(L/K_{p^\infty})}\right)^{\mathrm{Gal}(K_{p^\infty}/K)\text{-la}} \subset W^{\hat{G}\text{-la}}$$
because $\mathrm{Gal}(L/K_{p^\infty})$ is normal in $\hat{G}$.

(2) We do not know if the inclusion $W^{\hat{G}\text{-la}} \subset W^{\gamma\text{-la}} \cap W^{\tau\text{-la}}$ is an equality (very probably not, see next item).

(3) We thank Laurent Berger for informing us of the following example. Let $G_1 = G_2 = \mathbb{Z}_p$, and let $G = G_1 \times G_2$. Let W be the space of continuous $\mathbb{Q}_p$-valued functions on G with the action of G by translations. Let $f(x,y) = 0$ if $(x,y) = 0$ and $f(x,y) = (x^2 \cdot y^2)/(x^2 + py^2)$ otherwise. Then $f \in W^{G_1\text{-la}} \cap W^{G_2\text{-la}}$, but $f \notin W^{G\text{-la}}$. (Note that by Hartog's theorem, the analogous phenomenon does not happen over the usual complex numbers).

3.3. **Locally analytic vectors in $\hat{L}$.** Let $\hat{L}$ be the p-adic completion of L (cf. Notation 1.1.1). As in [BC16, §4.4], consider the 2-dimensional $\mathbb{Q}_p$-representation of G_K (associated to our choice of $\{\pi_n\}_{n\geq 0}$) such that $g \mapsto \begin{pmatrix} \chi(g) & c(g) \\ 0 & 1 \end{pmatrix}$ where χ is the p-adic cyclotomic character. Since the co-cycle $c(g)$ becomes trivial over C_p, there exists $\alpha \in C_p$ (indeed, $\alpha \in \hat{L}$) such that $c(g) = g(\alpha)\chi(g) - \alpha$. This implies $g(\alpha) = \alpha/\chi(g) + c(g)/\chi(g)$ and so $\alpha \in \hat{L}^{\hat{G}\text{-la}}$. Now similarly as in the beginning of [BC16, §4.2], let $\alpha_n \in L$ such that $\|\alpha - \alpha_n\|_p \leq p^{-n}$. Then there exists $r(n) \gg 0$ such that if $m \geq r(n)$, then $\|\alpha - \alpha_n\|_{\hat{G}_m} = \|\alpha - \alpha_n\|_p$ and $\alpha - \alpha_n \in \hat{L}^{\hat{G}_m\text{-an}}$ (see Notation 3.2.1 for $\hat{G}_m$). We can furthermore suppose that $\{r(n)\}_n$ is an increasing sequence.

Definition 3.3.1. Let $(H, \|\cdot\|)$ be a $\mathbb{Q}_p$-Banach algebra such that $\|\cdot\|$ is sub-multiplicative, and let $W \subset H$ be a $\mathbb{Q}_p$-subalgebra. Let T be a variable, and let $W\{\{T\}\}_n$ be the vector space consisting of $\sum_{k\geq 0} a_k T^k$ with $a_k \in W$, and $p^{nk} a_k \to 0$ when $k \to +\infty$. For $h \in H$ such that $\|h\| \leq p^{-n}$, denote $W\{\{h\}\}_n$ the image of the evaluation map $W\{\{T\}\}_n \to H$ where $T \mapsto h$.

Proposition 3.3.2. *(1) $\hat{L}^{\hat{G}\text{-la}} = \cup_{n\geq 1} K(\mu_{r(n)}, \pi_{r(n)})\{\{\alpha - \alpha_n\}\}_n$.*
(2) $\hat{L}^{\hat{G}\text{-la},\nabla_\gamma=0} = L$.
(3) $\hat{L}^{\tau\text{-la},\gamma=1} = K_\infty$.

Proof. Item (1) is [BC16, Prop. 4.12]; we quickly recall the proof here. Suppose $x \in \hat{L}^{\hat{G}_n\text{-an}}$. For $i \geq 0$, let
$$y_i = \sum_{k\geq 0} (-1)^k (\alpha - \alpha_n)^k \nabla_\tau^{k+i}(x) \binom{k+i}{k},$$

then there exists $m \geq n$ such that $y_i \in \hat{L}^{\hat{G}_m\text{-an}}$, and $x = \sum_{i\geq 0} y_i(\alpha - \alpha_n)^i$ in $\hat{L}^{\hat{G}_m\text{-an}}$. Then the fact $\nabla_\tau(y_i) = 0$ will imply that $y_i \in K(\mu_m, \pi_m)$, concluding (1).

Consider Item (2). By [BC16, Prop. 6.3], there exists a non-zero element $\beta \in C_p \otimes \mathrm{Lie}\, \hat{G}$ such that $\beta = 0$ on $\hat{L}^{\hat{G}\text{-la}}$. Write $\beta = a\nabla_\tau + b\nabla_\gamma$ with $a, b \in C_p$. We have $a \neq 0$ since $\nabla_\gamma \neq 0$ on K_{p^∞}; similarly $b \neq 0$. Thus, the condition $\nabla_\gamma = 0$ in Item (2) implies $\nabla_\tau = 0$, and so $y_i = 0$ for $i \geq 1$, concluding (2).

Item (3) easily follows from (2). □

3.4. Locally analytic vectors in $\widetilde{\mathbf{B}}^I_{K_\infty}$.

Lemma 3.4.1. *Suppose* $I = [r_\ell, r_k]$ *or* $[0, r_k]$.

1. $\widetilde{\mathbf{A}}^{[0,r_k]} = \widetilde{\mathbf{A}}^+\{\frac{\varphi^k(E(u))}{p}\}$.
2. $p\widetilde{\mathbf{A}}^I \cap \frac{\varphi^k(E(u))}{p}\widetilde{\mathbf{A}}^I = \varphi^k(E(u))\widetilde{\mathbf{A}}^I$.
3. $p\widetilde{\mathbf{A}}^I \cap \widetilde{\mathbf{A}}^{[0,r_k]} = p\widetilde{\mathbf{A}}^{[0,r_k]}$.
4. *If* $y \in \widetilde{\mathbf{A}}^{[0,r_k]} + p\widetilde{\mathbf{A}}^I$ *and* $y_i \in \widetilde{\mathbf{A}}^+$ *such that* $y - \sum_{i=0}^{j-1} y_i(\frac{\varphi^k(E(u))}{p})^i$ *is in* $(\mathrm{Ker}(\theta \circ \iota_k))^j$ *for all* $j \geq 1$. *Then there exists some* $j \geq 1$ *such that* $y - \sum_{i=0}^{j-1} y_i(\frac{\varphi^k(E(u))}{p})^i \in p\widetilde{\mathbf{A}}^I$.

Proof. These are easy analogues of [Ber16, Lem. 3.1, Lem. 3.2, Prop. 3.3]; let us sketch the proofs for the reader's convenience.

Item (1) easily follows from Def. 2.1.1 (or see [Ber16, Lem. 3.1] for a quick development).

For Item (2), suppose px belongs to left hand side, then px and hence x belongs to the kernel of $\theta \circ \iota_k : \widetilde{\mathbf{A}}^I \to C_p$; one then concludes by Lem. 2.1.12(1).

Item (3) is vacuous when $I = [0, r_k]$. When $I = [r_\ell, r_k]$, this is [Ber16, Lem. 3.2(3)] (or our Eqn. (2.1.1)).

Consider Item (4). By Item (1), there exists some $j \geq 1$ and some $a_i \in \widetilde{\mathbf{A}}^+$ such that

$$y - \sum_{i=0}^{j-1} a_i(\frac{\varphi^k(E(u))}{p})^i \in p\widetilde{\mathbf{A}}^I \tag{3.4.1}$$

(note that this is possible for either $I = [r_\ell, r_k]$ or $[0, r_k]$). One then proceeds as in [Ber16, Prop. 3.3], by changing all the Q_k (resp. π, resp. $[r, s]$) in *loc. cit.* to $\varphi^k(E(u))$ (resp. p, resp. I), to show that one can replace the a_i above by y_i without changing the property in (3.4.1).

□

For I a closed interval, note that $(\widetilde{\mathbf{B}}^I_L, W^I)$ is a $\mathbb{Q}_p$-Banach representation of $\hat{G}$ (in particular, note that $W^I(p) = 1$); also note that the valuation W^I is invariant under the Galois action.

Lemma 3.4.2. *Suppose* $I = [r_\ell, r_k]$ *or* $[0, r_k]$.

1. *For each* $n \geq 0$, $\varphi^{-n}(u) \in (\widetilde{\mathbf{B}}^I_L)^{\tau_{n+k}\text{-an}}$. *Thus:*
$$\varphi^{-n}(u) \in (\widetilde{\mathbf{B}}^I_L)^{\tau_{n+k}\text{-an}, \gamma=1} \subset (\widetilde{\mathbf{B}}^I_L)^{\hat{G}\text{-la}}.$$
2. *There exists* $m_0 \geq 0$ *(depending on* k *only) such that*
$$\frac{t}{\varphi^k(E(u))} \in (\widetilde{\mathbf{B}}^I_L)^{\tau_{m_0}\text{-an}}.$$
3. *Suppose* $x \in \widetilde{\mathbf{B}}^I_L$ *such that* $tx \in (\widetilde{\mathbf{B}}^I_L)^{\tau_n\text{-an}}$ *for some* $n \geq 0$, *then* $x \in (\widetilde{\mathbf{B}}^I_L)^{\tau_n\text{-an}}$.
4. *Suppose* $m \geq m_0$. *Then*
$$(\widetilde{\mathbf{B}}^I_L)^{\tau_m\text{-an}, \gamma=1} \cap \varphi^k(E(u))\widetilde{\mathbf{B}}^I_L = \varphi^k(E(u))(\widetilde{\mathbf{B}}^I_L)^{\tau_m\text{-an}, \gamma=1}.$$

Proof. The proof of Item (1) follows similar ideas as in [Ber16, Prop. 4.1]. Let us mention that it is relatively easy to show that $\varphi^{-n}(u)$ is *locally* analytic, e.g., using (3.4.3) below; however it is critical to control the radius of analyticity (which is $p^{-(n+k)}$ in this case) for later application in Thm. 3.4.4. Write v for $[\underline{\varepsilon}]-1 \in \widetilde{\mathbf{A}}^+$. For $a \in \mathbb{Z}_p$, we have

$$\tau^a(\varphi^{-n}(u)) = \varphi^{-n}(u\cdot(1+v)^a) = \varphi^{-n}(u)\cdot\left(\sum_{m=0}^{\infty}\binom{a}{m}\varphi^{-n}(v)^m\right).$$

It suffices to show that the (formally written) summation function (from $\mathbb{Z}_p$ to $\widetilde{\mathbf{B}}_L^I$)

$$T \mapsto \sum_{m\geq 0}\binom{T}{m}\cdot\varphi^{-n}(v)^m \tag{3.4.2}$$

is (well-defined and) analytic on the closed disk (around 0) of radius p^{-h} where $h=n+k$. By [Col10a, Thm. I.4.7]) (due to Amice), the polynomials $\lfloor m/p^h\rfloor!\binom{T}{m}$ for $m\geq 0$ form an orthonormal basis of $\mathrm{LA}_h(\mathbb{Z}_p,\mathbb{Q}_p)$, where $\mathrm{LA}_h(\mathbb{Z}_p,\mathbb{Q}_p)$ is the Banach space of functions on $\mathbb{Z}_p$ that are analytic on all the closed sub-disks of radius p^{-h} (cf. the definition above [Col10a, Rem. I.4.4]). See [Col10a, Def. I.1.3] for the definition of an orthonormal basis; in particular, it implies that the norm of $\lfloor m/p^h\rfloor!\binom{T}{m}$ on the closed disk (around 0) of radius p^{-h} is ≤ 1. Note that since $\varphi^{-n}(v)\in\widetilde{\mathbf{A}}^+$,

$$W^I(\varphi^{-n}(v)) = W^{[r_k,r_k]}(\varphi^{-n}(v)) = \frac{1}{(p-1)p^{n+k-1}}.$$

Thus, the norm of the term $\binom{T}{m}\cdot\varphi^{-n}(v)^m$ on the closed disk of radius p^{-h} is

$$\leq \|\binom{T}{m}\|_{\mathrm{LA}_h(\mathbb{Z}_p,\mathbb{Q}_p)}\cdot p^{W^I(\varphi^{-n}(v)^m)} = p^{v_p(\lfloor m/p^h\rfloor!)}\cdot p^{-\frac{m}{(p-1)p^{n+k-1}}} \leq p^{-\frac{m}{p^h}}.$$

Thus $\binom{T}{m}\cdot\varphi^{-n}(v)^m$ converges to 0 and the analyticity of (3.4.2) is verified.

Consider Item (2). Denote $F := \varphi^k(E(u))$. Since F is a generator of $\mathrm{Ker}(\theta\circ\iota_k : \widetilde{\mathbf{B}}^I \to C_p)$, we have $\frac{t}{F}\in\widetilde{\mathbf{B}}_L^I$. Let $m_0 \gg 0$ such that when $a\in p^{m_0}\mathbb{Z}_p$,

$$(1-\tau^a)(u) = u(1-[\underline{\varepsilon}]^a) = u\cdot p^\theta t\cdot h(p^\theta t), \quad \text{for some } \theta>0, h(X)\in\mathbb{Z}_p[\![X]\!]. \tag{3.4.3}$$

By increasing m_0 if needed, we can further assume that

$$W^I(p^\theta\cdot\frac{t}{F}) = \alpha > 0. \tag{3.4.4}$$

We claim that for all $a\in p^{m_0}\mathbb{Z}_p$, there exists $f_s(X,Y)\in W(k)[\![X,Y]\!]$ (depending on a), such that

$$(1-\tau^a)^s(\frac{t}{F}) = \frac{t(p^\theta t)^s\cdot f_s(u,p^\theta t)}{\prod_{i=0}^s\tau^{ai}(F)}, \quad \forall s\geq 0. \tag{3.4.5}$$

When $s=0$, simply let $f_0=1$. Suppose (3.4.5) is valid for $s-1$, then

$$(1-\tau^a)^s(\frac{t}{F}) = t(p^\theta t)^{s-1}\cdot\frac{\tau^{as}(F)\cdot f_{s-1}-F\cdot\tau^a(f_{s-1})}{\prod_{i=0}^s\tau^{ai}(F)}.$$

Note that

$$\tau^{as}(F)\cdot f_{s-1}-F\cdot\tau^a(f_{s-1}) = (\tau^{as}-1)(F)\cdot f_{s-1}-F\cdot(\tau^a-1)(f_{s-1}).$$

Note that for any $i,j\geq 0$,

$$(\tau^b-1)(u^i(p^\theta t)^j) = p^\theta t\cdot P_{i,j}(u,p^\theta t), \text{ with } P_{i,j}\in W(k)[\![X,Y]\!].$$

Thus it is easy to see that $(\tau^{as}-1)(F) = p^\theta t\cdot G(u,p^\theta t)$ and $(\tau^a-1)(f_{s-1}) = p^\theta t\cdot H(u,p^\theta t)$ with some $G,H\in W(k)[\![X,Y]\!]$, so we can simply let

$$f_s := \frac{\tau^{as}(F)\cdot f_{s-1}-F\cdot\tau^a(f_{s-1})}{p^\theta t},$$

concluding the proof of (3.4.5). By (3.4.5), we have

$$W^I((1-\tau^a)^s(\frac{t}{F})) \geq W^I(p^{-\theta}\cdot(\frac{p^\theta t}{F})^{s+1}) \geq -\theta+(s+1)\alpha. \tag{3.4.6}$$

Thus it is easy to see that for the group generated by $p^{m_0}\tau$ ($\simeq \mathbb{Z}_p$), the conditions (3.1.2) and (3.1.3) in Lem. 3.1.8 are satisfied (if needed, we can increase m_0 to increase α), and we can conclude Item (2).

For Item (3), one can assume that $n=0$ (the general case is similar). Write $I=[r,s]$. Since $W^I = \inf\{W^{[r,r]}, W^{[s,s]}\}$ (or $W^I = W^{[s,s]}$ if $r=0$), and both $W^{[r,r]}$ and $W^{[s,s]}$ are *multiplicative* valuations, it is easy to see that there exists a constant $c(I)>0$ depending on I only, such that

$$W^I(y) \geq W^I(ty) - c(I), \quad \forall y \in \widetilde{\mathbf{B}}_L^I.$$

Using this, and the fact that $(1-\tau^a)(tx) = t\cdot(1-\tau^a)(x)$, it is easy to see that if tx satisfies the itemized conditions in Lem. 3.1.7, then so does x.

For Item (4), suppose $y \in \widetilde{\mathbf{B}}_L^I$ such that $\varphi^k(E(u))\cdot y \in (\widetilde{\mathbf{B}}_L^I)^{\tau_m\text{-an}}$, it suffices to show that $y \in (\widetilde{\mathbf{B}}_L^I)^{\tau_m\text{-an}}$. By Item (2), $\frac{t}{\varphi^k(E(u))}\cdot\varphi^k(E(u))\cdot y = ty$ is an analytic vector, and we can conclude by Item (3). □

Definition 3.4.3. Define

$$\mathbf{A}_{K_\infty,m}^I := \varphi^{-m}(\mathbf{A}_{K_\infty}^{p^m I}), \quad \mathbf{A}_{K_\infty,\infty}^I := \cup_{m\geq 0}\mathbf{A}_{K_\infty,m}^I.$$

Define $\mathbf{B}_{K_\infty,m}^I$ and $\mathbf{B}_{K_\infty,\infty}^I$ similarly.

Theorem 3.4.4. *Suppose $I=[r_\ell,r_k]$ or $[0,r_k]$. Let m_0 be as in Lem. 3.4.2.*

1. *$(\widetilde{\mathbf{A}}_L^I)^{\tau_{m+k}\text{-an},\gamma=1} \subset \mathbf{A}_{K_\infty,m}^I$ for any $m\geq m_0$.*
2. *$(\widetilde{\mathbf{A}}_L^I)^{\tau\text{-la},\gamma=1} = \mathbf{A}_{K_\infty,\infty}^I$.*
3. *$(\widetilde{\mathbf{B}}_L^{[r_\ell,+\infty)})^{\tau\text{-pa},\gamma=1} = \mathbf{B}_{K_\infty,\infty}^{[r_\ell,+\infty)}$.*
4. *$(\widetilde{\mathbf{B}}_L^{[0,+\infty)})^{\tau\text{-pa},\gamma=1} = \mathbf{B}_{K_\infty,\infty}^{[0,+\infty)}$.*

Proof. The proof of Item (1) follows the same strategy as in [Ber16, Thm. 4.4]. (Some error of *loc. cit.* is corrected in the errata, posted on Berger's homepage.) Suppose $x \in (\widetilde{\mathbf{A}}_L^I)^{\tau_{m+k}\text{-an},\gamma=1}$.

- When $I=[0,r_k]$, for each $n\geq 0$, we let $k_n=0$, and let
$$x_n := (\frac{u^{ep^k}}{p})^{k_n}x = x \in \widetilde{\mathbf{A}}^{[0,r_k]} = \widetilde{\mathbf{A}}^{[0,r_k]} + p^n\widetilde{\mathbf{A}}^I.$$
- When $I=[r_\ell,r_k]$, note that $\widetilde{\mathbf{A}}^I = \widetilde{\mathbf{A}}^+\{\frac{p}{u^{ep^\ell}}, \frac{u^{ep^k}}{p}\}$ and note that $k\geq\ell$. Thus for each $n\geq 0$, we can choose $k_n \gg 0$ such that we have
$$x_n := (\frac{u^{ep^k}}{p})^{k_n}x \in \widetilde{\mathbf{A}}^{[0,r_k]} + p^n\widetilde{\mathbf{A}}^I.$$

For either of the above two cases, $x_n \in (\widetilde{\mathbf{A}}_L^I)^{\tau_{m+k}\text{-an},\gamma=1}$ by Lem. 3.4.2(1) (and Lem. 3.1.2). So

$$\theta\circ\iota_k(x_n) \in (\mathcal{O}_{\hat{L}})^{\tau_{m+k}\text{-an},\gamma=1} = \mathcal{O}_{K(\pi_{m+k})},$$

where the last identity follows from similar argument as in [BC16, Thm. 3.2]. Since $\theta\circ\iota_k(\varphi^{-m}(u)) = \pi_{m+k}$, there exists $y_{n,0}\in W(k)[\varphi^{-m}(u)]$ such that

$$\theta\circ\iota_k(x_n) = \theta\circ\iota_k(y_{n,0}).$$

By Lem. 2.1.12,

$$x_n - y_{n,0} = (F/p)\cdot x_{n,1}, \text{ with } x_{n,1}\in\widetilde{\mathbf{A}}^I, \text{ where } F:=\varphi^k(E(u)).$$

By Lem. 3.4.2(1), $y_{n,0} \in (\widetilde{\mathbf{A}}_L^I)^{\tau_{m+k}\text{-an},\gamma=1}$. (As we mentioned in the proof of *loc. cit.*, it is important to know that $y_{n,0}$ is "τ_{m+k}-an" for the argument here to proceed). Thus by Lem. 3.4.2(4), $x_{n,1} \in (\widetilde{\mathbf{A}}_L^I)^{\tau_{m+k}\text{-an},\gamma=1}$. Applying this procedure inductively gives us a sequence $\{y_{n,i}\}_{i\geq 0}$ where $y_{n,i} \in W(k)[\varphi^{-m}(u)]$ such that

$$x_n - \big(y_{n,0} + (F/p)y_{n,1} + \cdots + (F/p)^{i-1}y_{n,i-1}\big) \in (F/p)^i\widetilde{\mathbf{A}}_L^I.$$

By Lem. 3.4.1(4), there exists $j \gg 0$ such that

$$x_n - \big(y_{n,0} + (F/p)y_{n,1} + \cdots + (F/p)^{j-1}y_{n,j-1}\big) \in p\widetilde{\mathbf{A}}_L^I. \tag{3.4.7}$$

Note that the left hand side of (3.4.7) belongs to $\widetilde{\mathbf{A}}_L^{[0,r_k]} + p^n\widetilde{\mathbf{A}}_L^I$ (since $y_{n,i}$ and F/p are in $\widetilde{\mathbf{A}}_L^{[0,r_k]}$), and so it further belongs to

$$(\widetilde{\mathbf{A}}_L^{[0,r_k]} + p^n\widetilde{\mathbf{A}}_L^I) \cap p\widetilde{\mathbf{A}}_L^I = p(\widetilde{\mathbf{A}}_L^{[0,r_k]} + p^{n-1}\widetilde{\mathbf{A}}_L^I), \quad \text{by Lem. 3.4.1(3)}.$$

Let

$$x_n - \big(y_{n,0} + (F/p)y_{n,1} + \cdots + (F/p)^{j-1}y_{n,j-1}\big) = px_n'.$$

Since $y_{n,i} \in (\widetilde{\mathbf{A}}_L^I)^{\tau_{m+k}\text{-an},\gamma=1}$, we have $x_n' \in (\widetilde{\mathbf{A}}_L^I)^{\tau_{m+k}\text{-an},\gamma=1}$. Apply to x_n' the same procedure that we applied to x_n, and proceed inductively. In the end, we will get $\{\tilde{y}_{n,i}\}_{i\leq j_n}$ for some $j_n \gg 0$ where $\tilde{y}_{n,i} \in W(k)[\varphi^{-m}(u)]$, and

$$\tilde{y}_n = \tilde{y}_{n,0} + (F/p)\tilde{y}_{n,1} + \cdots + ((F/p))^{j_n-1}\tilde{y}_{n,j_n-1},$$

such that

$$x_n - \tilde{y}_n \in p^n\widetilde{\mathbf{A}}^I.$$

Let $z_n := (\frac{p}{u^{ep^k}})^{k_n}\tilde{y}_n$, then $z_n \in \varphi^{-m}(\mathbf{A}_{K_\infty}^{p^m[r_k,r_k]})$ (note that here it is critical to use the interval $[r_k,r_k]$ and not $[0,r_k]$ or $[r_\ell,r_k]$, because the element $\frac{p}{u^{ep^k}}$ belongs only to $\mathbf{A}^{[r_k,r_k]}$). We have

$$x - z_n = (\frac{p}{u^{ep^k}})^{k_n}(x_n - \tilde{y}_n) \in p^n\widetilde{\mathbf{A}}^{[r_k,r_k]},$$

and hence z_n converges to x as elements in $\widetilde{\mathbf{A}}^{[r_k,r_k]}$ (with respect to $W^{[r_k,r_k]}$), and so

$$x \in \varphi^{-m}(\mathbf{A}_{K_\infty}^{p^m[r_k,r_k]}).$$

Finally by Cor. 2.2.11, we have

$$x \in \varphi^{-m}(\mathbf{A}_{K_\infty}^{p^m[r_k,r_k]}) \cap \widetilde{\mathbf{A}}^I = \varphi^{-m}(\mathbf{A}_{K_\infty}^{p^mI}) = \mathbf{A}_{K_\infty,m}^I.$$

Consider Item (2). Item (1) already implies that $(\widetilde{\mathbf{A}}_L^I)^{\tau\text{-la},\gamma=1} \subset \mathbf{A}_{K_\infty,\infty}^I$. To show the other direction, it suffices to show that elements in $\mathbf{A}_{K_\infty}^I$ are τ-locally analytic. We claim that for any $f \in \mathbf{A}_{K_\infty}^I$, and for $a \in p^b\mathbb{Z}_p$, we have

$$W^I((1-\tau^a)^s(f) \geq s\alpha \tag{3.4.8}$$

for some α that we can arbitrarily enlarge (after enlarging b); then we can conclude using Lem. 3.1.8. To verify (3.4.8), by linearity and density, it suffices to verify it for the cases $f = u^m(\frac{u^{ep^k}}{p})^n$ for $m \geq 0$ and $n \geq 0$, and (when $I = [r_\ell, r_k]$ the cases $f = u^m(\frac{p}{u^{ep^\ell}})^n$ for $m \geq 0$ and $n \geq 1$. Indeed, we have

$$\begin{aligned} W^I\left((1-\tau^a)^s(u^m(\frac{u^{ep^k}}{p})^n)\right) &= W^I\left(u^m(\frac{u^{ep^k}}{p})^n \cdot (1-[\underline{\varepsilon}]^{aep^kn+am})^s\right) \\ &\geq W^I\left((1-[\underline{\varepsilon}]^{aep^kn+am})^s\right), \text{ since } W^I(u^m(\frac{u^{ep^k}}{p})^n) \geq 0 \\ &\geq s\alpha, \text{ using (3.4.4)}. \end{aligned}$$

The verification for $f = u^m(\frac{p}{u^{ep^\ell}})^n$ is similar.

For Items (3) and (4), one can argue similarly as in [Ber16, Thm. 4.4(3)]. □

Remark 3.4.5. Item (4) of Thm. 3.4.4 (and (1), (2) when $I = [0, r_k]$) will not be used in this paper, but it has potential applications to the study of semi-stable Galois representations; indeed, the ring $\mathbf{B}_{K_\infty}^{[0,+\infty)}$ is precisely the ring $\mathcal{O}_{[0,1)}$ in [Kis06].

Definition 3.4.6. (1) Define the following rings (which are LB spaces):

$$\widetilde{\mathbf{B}}^\dagger := \cup_{r\geq 0}\widetilde{\mathbf{B}}^{[r,+\infty]}, \quad \mathbf{B}^\dagger := \cup_{r\geq 0}\mathbf{B}^{[r,+\infty]}, \quad \mathbf{B}^\dagger_{K_\infty} := \cup_{r\geq 0}\mathbf{B}^{[r,+\infty]}_{K_\infty}.$$

(2) Define the following rings (which are LF spaces):

$$\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} := \cup_{r\geq 0}\widetilde{\mathbf{B}}^{[r,+\infty)}, \quad \mathbf{B}^\dagger_{\mathrm{rig}} := \cup_{r\geq 0}\mathbf{B}^{[r,+\infty)}, \quad \mathbf{B}^\dagger_{\mathrm{rig},K_\infty} := \cup_{r\geq 0}\mathbf{B}^{[r,+\infty)}_{K_\infty}.$$

Corollary 3.4.7. $(\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},L})^{\tau\text{-pa},\gamma=1} = \cup_{m\geq 0}\varphi^{-m}(\mathbf{B}^\dagger_{\mathrm{rig},K_\infty})$.

Remark 3.4.8. In comparison, by [Ber16, Thm. 4.4], we have

$$(\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},L})^{\tau=1,\gamma\text{-pa}} = \cup_{m\geq 0}\varphi^{-m}(\mathbb{B}^\dagger_{\mathrm{rig},K_{p^\infty}}),$$

where $\mathbb{B}^\dagger_{\mathrm{rig},K_{p^\infty}}$ is the ring "$\mathbf{B}^\dagger_{\mathrm{rig},K}$" in [Ber08]. (As we mentioned in Rem. 1.4.3, we use the font "$\mathbb{B}$" to denote the "$\mathbf{B}$"-rings in the (φ, Γ)-module setting).

4. Field of norms, and locally analytic vectors

In this section, when $K_\infty \subset M \subset L$ where M/K_∞ is a finite extension, we calculate $\hat{G}$-locally analytic vectors in $\widetilde{\mathbf{B}}^I_L$ which are furthermore invariant under $\mathrm{Gal}(L/M)$; the results are parallel with the case for $M = K_\infty$.

4.1. **Field of norms.** In this subsection, we briefly recall the theory of field of norms developed by Fontaine and Wintenberger (cf. [FW79, Win83]). To save space, we refer the readers to [Win83] for more details.

In this subsection, let E_1 be a complete discrete valuation field with a perfect residue field of characteristic p. Let $\overline{E_1}$ be a fixed algebraic closure, and let E_1^{ur} be the maximal unramified extension of E_1 contained in $\overline{E_1}$.

If E_2/E_1 is an algebraic extension, let $\mathcal{E}(E_2/E_1)$ be the poset consisting of fields E such that $E_1 \subset E \subset E_2$ and $[E : E_1] < +\infty$. Let

$$X_{E_1}(E_2) := \varprojlim_{E\in\mathcal{E}(E_2/E_1)} E$$

where the transition maps from E' to E (for $E \subset E'$) are the norm maps $N_{E'/E}$. For $\alpha \in X_{E_1}(E_2)$, we denote it as $\alpha = \{\alpha_E\}_{E_1\subset E\subset E_2}$ where $\alpha_E \in E$ and $N_{E'/E}(\alpha_{E'}) = \alpha_E$ when $E \subset E'$. For any $\alpha \in X_{E_1}(E_2)$, the number $v_E(\alpha_E)$ for $E_1^{\mathrm{ur}} \cap E_2 \subset E \subset E_2$ is independent of E (here, v_E is the valuation such that $v_E(E) = \mathbb{Z} \cup \{\infty\}$); denote the number as $v(\alpha)$.

A priori, $X_{E_1}(E_2)$ is only a multiplicative monoid; however, by [Win83, Thm. 2.1.3(1)], we can indeed equip it with a natural additive structure, making $X_{E_1}(E_2)$ into a ring. Furthermore, we have the following.

Theorem 4.1.1. [Win83, Thm. 2.1.3(2)] *Suppose E_2/E_1 is an infinite APF extension (cf. [Win83, §1.2] for the definition of APF (and strict APF) extensions), then there exists an element $u_{E_2/E_1} \in X_{E_1}(E_2)$ such that $v(u_{E_2/E_1}) = 1$, and there exists a (valuation-preserving)* field *isomorphism*

$$X_{E_1}(E_2) \simeq k_{E_2}((u_{E_2/E_1})),$$

where k_{E_2} is the residue field of E_2 (which is a finite extension of k_{E_1}), and $k_{E_2}((u_{E_2/E_1}))$ is equipped with the u_{E_2/E_1}-adic valuation.

Example 4.1.2. Let $K, K_{p^\infty}, K_\infty$ be as in Notation 1.1.1.

(1) When $K = K_0$, the element $\widetilde{\mu} := \{\mu_n\}_{n\geq 1}$ defines an element in $X_K(K_{p^\infty})$, and $\widetilde{\mu} - 1$ is a uniformizer of $X_K(K_{p^\infty})$.

(2) The element $\widetilde{\pi} := \{\pi_n\}_{n\geq 1}$ defines an element in $X_K(K_\infty)$, which is a uniformizer.

Let $E_1 \subset E_2 \subset E_3$ where E_2/E_1 is an infinite APF extension, and E_3/E_2 is finite extension (so E_3/E_1 is also an APF extension). Then by [Win83, §3.1.1], we can naturally define an embedding $X_{E_1}(E_2) \hookrightarrow X_{E_1}(E_3)$ (and we identify $X_{E_1}(E_2)$ with its image).

Theorem 4.1.3. [Win83, Thm. 3.1.2] *If E_3/E_2 is furthermore Galois, then $X_{E_1}(E_3)$ is Galois over $X_{E_1}(E_2)$, and there exists a natural isomorphism*

$$\mathrm{Gal}(X_{E_1}(E_3)/X_{E_1}(E_2)) \simeq \mathrm{Gal}(E_3/E_2).$$

Remark 4.1.4. We can also construct a natural separable closure of $X_{E_1}(E_2)$, see [Win83, Cor. 3.2.3].

For any complete valued filed (A, v_A) with a perfect residue field of characteristic p, let

$$R(A) := \{(x_n)_{n=0}^{\infty} : x_n \in A, x_{n+1}^p = x_n\}.$$

For $x \in R(A)$, let $v_R(x) := v_A(x_0)$. Then $R(A)$ is a perfect field of characteristic p, complete with respect to v_R.

Theorem 4.1.5. [Win83, Thm. 4.2.1] *Suppose E_2/E_1 is an infinite* strict *APF extension. Let $\hat{E}_2$ be the completion of E_2. There exists a natural k_{E_2}-algebra embedding*

$$\Lambda_{E_2/E_1} : X_{E_1}(E_2) \hookrightarrow R(\hat{E}_2) \hookrightarrow R(\hat{\overline{E_1}}).$$

Example 4.1.6. Note that $R(C_p)$ is precisely $\widetilde{\mathbf{E}}$. Using notations in Example 4.1.2, we have

(1) when $K = K_0$, for the embedding $X_K(K_{p^\infty}) \to \widetilde{\mathbf{E}}$, we have $\widetilde{\mu} - 1 \mapsto \underline{\varepsilon} - 1$;
(2) for the embedding $X_K(K_\infty) \to \widetilde{\mathbf{E}}$, we have $\widetilde{\pi} \mapsto \underline{\pi}$.

4.2. **Finite extensions of K_∞ and locally analytic vectors.** Let $K_\infty \subset M \subset L$ where M/K_∞ is a finite extension (which is always Galois). In the following, given a ring A (possibly with superscripts), let A_M denote $\mathrm{Gal}(\overline{K}/M)$-invariants of A.

4.2.1. *Ramification subgroups.* Let G_K^s (where $s \geq -1$) denote the usual (upper numbering) ramification subgroups of G_K. For any $s \geq -1$, let $\overline{K}^{(s)} := \cap_{t>s} \overline{K}^{G_K^t}$. For any $K \subset E \subset \overline{K}$, let $E^{(s)} := E \cap \overline{K}^{(s)}$. Let $c(E) := \inf\{s : E^{(s)} = E\}$ (called the conductor of E). See [Col08, Lem. 4.1] for some properties of $c(E)$. When $n \geq 1$, let $K_n := K(\pi_n)$. By standard computation (e.g., using the formula above [LB10, Prop. 1.1]), we have

$$c(K_n) = (n + \frac{1}{p-1})e. \tag{4.2.1}$$

(Unfortunately, the computation of $c(K_n)$ in [LB10, Prop. 1.4] is incorrect.)

4.2.2. *Finite extensions of K_∞.* Choose an $\alpha \in M$ such that $M = K_\infty[\alpha]$, and let $\widetilde{M} := K[\alpha]$. Define $\widetilde{M}_n := \widetilde{M}(\pi_n)$ (note that $\pi_0 = \pi$ is not necessarily a uniformizer of $\widetilde{M}$). By using exactly the same argument as in [Col08, Lem. 4.2, Cor. 4.3, Rem. 4.4], the following hold:

(1) When $n \geq c(\widetilde{M})$ (where $c(\widetilde{M})$ is the conductor), then $c(\widetilde{M}_n) = \sup\{c(\widetilde{M}), c(K_n)\} = c(K_n)$ by (4.2.1), and hence $\widetilde{M}_{n+1}/\widetilde{M}_n$ is totally ramified of degree p.
(2) When $n \geq c(\widetilde{M})$, $e(\widetilde{M}_{n+1}/K_{n+1}) = e(\widetilde{M}_n/K_n)$ (resp. $f(\widetilde{M}_{n+1}/K_{n+1}) = f(\widetilde{M}_n/K_n)$), where $e(A/B)$ (resp. $f(A/B)$) is the ramification index (resp. inertial degree) of a finite extension. Denote the common numbers as e' (resp. f'), then $e'f' = [M : K_\infty]$.
(3) Let $K' := K^{\mathrm{ur}} \cap M$ where K^{ur} is the maximal unramified extension of K contained in $\overline{K}$, then $[K' : K] = f'$.

4.2.3. *Construction of u_M.* Let k' be the residue field of K', and let $M_0 := \cup_{n\geq 1}K'(\pi_n)$. Then by §4.2.2 and Examples 4.1.2 and 4.1.6, we have $X_K(M_0) \simeq k'((\underline{\pi})) = k'((u))$ (recall $u = [\underline{\pi}]$ as in §1.4.2). Choose any $\overline{u}_M \in X_K(M)$ such that $X_K(M) = k'((\overline{u}_M))$. By Thm. 4.1.3, $X_K(M)$ is a totally ramified extension of $X_K(M_0)$ of degree e', and so $v_{\widetilde{\mathbf{E}}}(\overline{u}_M) = 1/ee'$ if we regard $\overline{u}_M \in \widetilde{\mathbf{E}}$ via Thm. 4.1.5. Let $\overline{P}(X) = X^{e'} + \overline{a}_{e'-1}X^{e'-1} + \cdots + \overline{a}_0$ be the minimal polynomial of $\overline{u}_M$ over $X_K(M_0)$. Since $\overline{u}_M$ is integral over $X_K(M_0)$, $\overline{a}_i \in k'[\![u]\!]$. Let $a_i \in W(k')[\![u]\!]$ be any lift of $\overline{a}_i$, and let $P(X) = X^{e'} + a_{e'-1}X^{e'-1} + \cdots + a_0$. By Hensel's Lemma, $P(X)$ has a unique root (which we denote as u_M) in $\mathbf{A}_M$ which reduces to $\overline{u}_M$ modulo p. (Note that u_M depends on the choices of $\overline{u}_M$ and a_i.)

We have $\mathrm{Gal}(X_K(M)/X_K(K_\infty)) \simeq \mathrm{Gal}(\mathbf{B}_M/\mathbf{B}_{K_\infty}) \simeq \mathrm{Gal}(\widetilde{\mathbf{B}}_M/\widetilde{\mathbf{B}}_{K_\infty})$ (cf. [CC98, §I.3]). Let $v_1, \cdots, v_{f'}$ be a basis of $W(k')$ over $W(k)$, and let $x_{a+f'b} := v_a \cdot u_M^b$ with $1 \leq a \leq f', 0 \leq b \leq e'-1$, then we have

$$\mathbf{A}_M = \oplus_{i=1}^{e'f'}\mathbf{A}_{K_\infty} \cdot x_i,$$

and so (cf. [Ber10, Lem. 24.5]),

$$\widetilde{\mathbf{A}}_M = \oplus_{i=1}^{e'f'}\widetilde{\mathbf{A}}_{K_\infty} \cdot x_i.$$

Lemma 4.2.4. *Let $r > 0$ and let $x = \sum_{k\geq 0} p^k[a_k] \in \widetilde{\mathbf{A}}^{[r,+\infty]}[1/u]$, the following are equivalent:*

(1) $x \in (\widetilde{\mathbf{A}}^{[r,+\infty]})^\times$;
(2) $v_{\widetilde{\mathbf{E}}}(a_0) = 0$, and $k + \frac{p-1}{pr} \cdot v_{\widetilde{\mathbf{E}}}(a_k) > 0, \forall k > 0$;
(3) $v_{\widetilde{\mathbf{E}}}(a_0) = 0$, and $k + \frac{p-1}{pr} \cdot w_k(x) > 0, \forall k > 0$.

Proof. The equivalence between (1) and (2) is proved in [Col08, Lem. 5.9]; see the proof of Lem. 2.1.10 for comparison of notations. The equivalence between (2) and (3) is trivial. □

Lemma 4.2.5. *(1) There exists some constant $r_M > 0$ which depends only on M (and not on the construction of u_M as in §4.2.3), such that:*
(a) $u_M \in \mathbf{A}_M^{[r_M,+\infty]}$, and
(b) $u_M/[\overline{u}_M]$ is a unit in $\widetilde{\mathbf{A}}_M^{[r_M,+\infty]}$.
(c) $P'(u_M)/[P'(\overline{u}_M)]$ is a unit in $\widetilde{\mathbf{A}}_M^{[r_M,+\infty]}$, where $P'(X)$ is the derivative of $P(X)$.
(2) If $I = [r_\ell, r_k]$ or $[r_\ell, +\infty]$ such that $r_\ell \geq r_M$, then

$$\mathbf{B}_M^I = \oplus_{i=1}^{e'f'}\mathbf{B}_{K_\infty}^I \cdot x_i, \qquad \widetilde{\mathbf{B}}_M^I = \oplus_{i=1}^{e'f'}\widetilde{\mathbf{B}}_{K_\infty}^I \cdot x_i.$$

Proof. Item (1) follows from exactly the same argument as [Col08, Lem. 6.4, Lem. 6.5] (where Item (1b) uses Lem. 4.2.4). Item (2) follows from exactly the same argument as [Col08, Lem. 6.11] (i.e., an argument using the trace operator). □

Lemma 4.2.6. *Suppose $r_\ell \geq r_M$, then $x_i \in (\widetilde{\mathbf{A}}_L^{[r_\ell,r_k]})^{\tau\text{-la}}$.*

Remark 4.2.7. The proof of Lem. 4.2.6 is inspired by the argument in the proof of [Ber16, Thm. 4.4(2)]; indeed, we use ideas inspired by the inverse function theorem on [Ser06, Page 73]. However, since the ring $\widetilde{\mathbf{A}}_L^{[r_\ell,r_k]}$ (or $\widetilde{\mathbf{B}}_L^{[r_\ell,r_k]}$) is not a *field* and the norm on it is not *multiplicative*, we cannot directly apply *loc. cit.*. (we thank an anonymous referee for pointing this out). Indeed, the argument in [Ber16, Thm. 4.4(2)] is incomplete. Let us mention that the argument in our proof can be easily adapted to give a corrected proof of *loc. cit.*.

We first start with an easy lemma.

Lemma 4.2.8. *Let $(W, \|\cdot\|)$ be a normed $\mathbb{Z}_p$-algebra. Let $\mathrm{val}(\cdot)$ be the associated valuation on W, and suppose it is* multiplicative. *Let $f(X) = X^n + a_{n-1}X^{n-1} + \ldots + a_0$ where $a_i \in W$ such that $\mathrm{val}(a_i) \geq 0$. Suppose $\rho \in W$ such that $f(\rho) = 0$ and $f'(\rho) \neq 0$ (where $f'(X)$ is the derivative). Suppose $\rho' \in W$ such that $f(\rho') = 0$ and $\mathrm{val}(\rho - \rho') > \mathrm{val}(a_i)$ for all i such that $a_i \neq 0$. Then $\rho = \rho'$ (i.e., within a small neighborhood of ρ, $f(X)$ has no other roots.)*

Proof. Firstly, it is easy to see that $\mathrm{val}(\rho) \geq 0$; then we can easily reduce the lemma to the case $\rho = 0$. That is, we can assume $f(X) = X^n + a_{n-1}X^{n-1} + \ldots + a_1 X$ and $a_1 \neq 0$. Now if $\rho' \neq 0$ and $\mathrm{val}(\rho') > \mathrm{val}(a_i)$ for all i such that $a_i \neq 0$, then $\mathrm{val}(f(\rho')) = \mathrm{val}(a_1\rho') < +\infty$, and hence $f(\rho') \neq 0$. □

Proof of Lem. 4.2.6. The lemma is trivial if $e' = 1$; suppose now $e' \geq 2$. Firstly, by Lem. 3.1.2, it suffices to show that $u_M \in (\widetilde{\mathbf{A}}^I_L)^{\tau\text{-la}}$ (here $I := [r_\ell, r_k]$). Recall we denote $P(X) = X^n + a_{n-1}X^{n-1} \ldots + a_0$ in §4.2.3 (here we write $n := e' \geq 2$ for brevity), where $a_i \in W(k')[\![u]\!]$. Thus for all $\theta \in \mathbb{Z}_p$, $\tau^\theta P(X) := X^n + \tau^\theta(a_{n-1})X^{n-1} \ldots + \tau^\theta(a_0)$ has $\tau^\theta(u_M)$ as a root in $\widetilde{\mathbf{A}}^I$.

For $m \gg 0$ and for each $\beta \in \mathbb{Z}_p$, we will *construct* another root of $\tau^{p^m\beta}P(X)$ of the form

$$y = y(m,\beta) = w_0 + \sum_{k\geq 1}(p^m\beta)^k w'_k = w_0 + \sum_{k\geq 1}\beta^k w_k, \tag{4.2.2}$$

where $w_0 = u_M$ (independent of m), and for each $k \geq 1$ $w_k := w_k(m) := p^{mk}w'_k$ (here w'_k depends only on k and β but not on m) such that

$$w_k \in \widetilde{\mathbf{A}}^I_L, \text{ and hence } \lim_{k\to+\infty} w_k = 0 \text{ by enlarging } m. \tag{4.2.3}$$

Now fix any $s \in I$. By enlarging m if necessary, we can easily make

$$W^{[s,s]}(y - u_M) > W^{[s,s]}(a_i), \forall i \text{ such that } a_i \neq 0, \tag{4.2.4}$$

and

$$W^{[s,s]}(\tau^{p^m\beta}(u_M) - u_M) > W^{[s,s]}(a_i), \forall i \text{ such that } a_i \neq 0. \tag{4.2.5}$$

Here, (4.2.5) is possible because the Galois action on $\widetilde{\mathbf{A}}^{[s,s]}$ is continuous. By (4.2.4) and (4.2.5), we have

$$W^{[s,s]}(y - \tau^{p^m\beta}(u_M)) > W^{[s,s]}(a_i), \forall i \text{ such that } a_i \neq 0.$$

By Lem. 4.2.8 (recall $W^{[s,s]}$ is an multiplicative valuation by Lem. 2.1.10), we can conclude $\tau^{p^m\beta}(u_M) = y$ as elements in $\widetilde{\mathbf{A}}^{[s,s]}$. Since $\widetilde{\mathbf{A}}^I \hookrightarrow \widetilde{\mathbf{A}}^{[s,s]}$ (cf. §2.1.3), we have $\tau^{p^m\beta}(u_M) = y$ as elements in $\widetilde{\mathbf{A}}^I$. Thus $u_M \in (\widetilde{\mathbf{A}}^I_L)^{\tau\text{-la}}$ by definition.

Now we construct y in (4.2.2). Before we do so, we pick some $\delta \gg 0$ such that

$$p^\delta/P'(u_M) \in \widetilde{\mathbf{A}}^I_L, \tag{4.2.6}$$

which is possible because of Lem. 4.2.5(1)(c). Now note that all a_i are locally analytic vectors, so we can write for each i,

$$\tau^{p^m\beta}(a_i) = a_{i,0} + \sum_{j\geq 1}(p^m\beta)^j a'_{i,j} = a_i + \sum_{j\geq 1}\beta^j a_{i,j}, \tag{4.2.7}$$

where again $a_{i,0} = a_i$. By enlarging m if necessary, we can suppose

$$a_{i,j} \in p^{2\delta}\widetilde{\mathbf{A}}^I_L, \quad \forall 0 \leq i \leq n-1, \forall j \geq 1. \tag{4.2.8}$$

Plug (4.2.7) and (4.2.2) into $\tau^{p^m\beta}P(X)$. We get
(4.2.9)

$$(w_0 + \sum_{k\geq 1}\beta^k w_k)^n + (a_{n-1,0} + \sum_{j\geq 1}\beta^j a_{n-1,j})(w_0 + \sum_{k\geq 1}\beta^k w_k)^{n-1} + \cdots + (a_{0,0} + \sum_{j\geq 1}\beta^j a_{0,j}) = 0.$$

We will let the coefficient of β^k to be zero for each $k \geq 0$, and use these equations to solve w_k inductively. Firstly, note that we automatically have

$$\mathrm{Coeff}(\beta^0) = w_0^n + \sum_{i=0}^{n-1} a_{i,0}\cdot w_0^i = P(w_0) = P(u_M) = 0. \tag{4.2.10}$$

For each $k \geq 1$, one can easily compute that

$$\mathrm{Coeff}(\beta^k) = P'(w_0)\cdot w_k + Q_k\left((a_{i,j})_{1\leq i\leq n-1, 0\leq j\leq k-1}, w_0, \cdots, w_{k-1}\right) \tag{4.2.11}$$

where Q_k is a polynomial of the variables $(a_{i,j})_{1\le i\le n-1, 0\le j\le k-1}, w_0, \cdots, w_{k-1}$ with *integer* coefficients. By letting $\mathrm{Coeff}(\beta^k) = 0$, we will show by induction that

$$w_k \in p^\delta \widetilde{\mathbf{A}}_L^I, \quad \forall k \ge 1. \tag{4.2.12}$$

It suffices to show that each monomial in Q_k is divisible by $p^{2\delta}$, since by (4.2.6)

$$p^{2\delta}\widetilde{\mathbf{A}}_L^I \subset P'(w_0)\cdot p^\delta \widetilde{\mathbf{A}}_L^I.$$

When $k=1$, each monomial in Q_1 contains some $a_{i,1}$ as a factor, and hence one can conclude (4.2.12) for $k=1$ using (4.2.8). Suppose (4.2.12) is true for $k-1$, and consider $\mathrm{Coeff}(\beta^k)$ (where now $k\ge 2$). For a monomial in Q_k, if it does not contain any $a_{i,j}$ with $j\ge 1$ as a factor, then it is a product of elements in $\{a_{0,0},\dots,a_{n-1,0}, w_0, w_1, \dots, w_{k-1}\}$; however, one easily observes that such product contains at least two (possibly equal) elements from $\{w_1,\dots,w_{k-1}\}$ (using $k\ge 2$), and hence by induction hypothesis the monomial is divisible by $p^{2\delta}$. Thus, (4.2.12) is verified for k, and this finishes the construction of (4.2.2). □

Theorem 4.2.9. *Suppose $[r,s]=[r_\ell, r_k]$, then*

(1) $(\widetilde{\mathbf{B}}_L^{[r,s]})^{\tau\text{-la},\mathrm{Gal}(L/M)=1} = \cup_{m\ge 0}\varphi^{-m}(\mathbf{B}_M^{p^m[r,s]})$.
(2) $(\widetilde{\mathbf{B}}_L^{[r,+\infty)})^{\tau\text{-pa},\mathrm{Gal}(L/M)=1} = \cup_{m\ge 0}\varphi^{-m}(\mathbf{B}_M^{p^m[r,+\infty)})$.

Proof. It suffices to prove Item (1). Denote $I := [r,s]$. Since φ induces a bijection between $(\widetilde{\mathbf{B}}_L^I)^{\tau\text{-la},\mathrm{Gal}(L/M)=1}$ and $(\widetilde{\mathbf{B}}_L^{pI})^{\tau\text{-la},\mathrm{Gal}(L/M)=1}$, it suffices to consider the case when $r > r_M$. By Lem. 4.2.5(2) and Lem. 4.2.6, it is clear that $\cup_{m\ge 0}\varphi^{-m}(\mathbf{B}_M^{p^m[r,s]}) \subset (\widetilde{\mathbf{B}}_L^{[r,s]})^{\tau\text{-la},\mathrm{Gal}(L/M)=1}$. But we also have

$$\begin{aligned}
(\widetilde{\mathbf{B}}_L^I)^{\tau\text{-la},\mathrm{Gal}(L/M)=1} &= (\widetilde{\mathbf{B}}_M^I)^{\tau\text{-la}} \\
&= (\oplus_{i=1}^{e'f'}\widetilde{\mathbf{B}}_{K_\infty}^I\cdot x_i)^{\tau\text{-la}}, \text{ by Lem. 4.2.5(2)} \\
&= \oplus_{i=1}^{e'f'}(\widetilde{\mathbf{B}}_{K_\infty}^I)^{\tau\text{-la}}\cdot x_i, \text{ by Prop.3.1.6 and Lem.4.2.6} \\
&= \oplus_{i=1}^{e'f'}(\mathbf{B}_{K_\infty,\infty}^I)\cdot x_i, \text{ by Thm.3.4.4} \\
&\subset \cup_{m\ge 0}\varphi^{-m}(\mathbf{B}_M^{p^m[r,s]}), \text{ by Lem.4.2.5(2).}
\end{aligned}$$

□

4.3. Structure of A_M^I.

In this subsection, we study the concrete structure of A_M^I; these results will be used in §6.

Definition 4.3.1. (1) For $0<r<+\infty$, let $\mathcal{A}_M^{[r,+\infty]}(K_0')$ be the ring consisting of infinite series $f=\sum_{k\in\mathbb{Z}} a_k T^k$ where $a_k\in W(k')$ such that f is a holomorphic function on the annulus defined by $0 < v_p(T) \le (p-1)/(e'epr)$. Let $\mathcal{B}_M^{[r,+\infty]}(K_0') := \mathcal{A}_M^{[r,+\infty]}(K_0')[1/p]$.

(2) For $f=\sum_{k\in\mathbb{Z}} a_k T^k \in \mathcal{B}_M^{[r,+\infty]}(K_0')$, and $s\in[r,+\infty)$, let

$$\mathcal{W}_M^{[s,s]}(f) := \inf_{k\in\mathbb{Z}}\{v_p(a_k) + \frac{p-1}{ps}\cdot\frac{k}{e'e}\}.$$

For $I=[a,b]\subset[r,+\infty)$ a non-empty closed interval, let

$$\mathcal{W}_M^{[a,b]}(f) := \inf_{\alpha\in I}\{\mathcal{W}_M^{[\alpha,\alpha]}(f)\}.$$

(3) Let $\mathcal{B}_M^{[r,s]}(K_0')$ be the completion of $\mathcal{B}_M^{[r,+\infty]}(K_0')$ with respect to $\mathcal{W}_M^{[r,s]}$. Let $\mathcal{A}_M^{[r,s]}(K_0')$ be the ring of integers with respect to $\mathcal{W}_M^{[r,s]}$.

Lemma 4.3.2. *For $I=[r,s]\subset(0,+\infty)$, we have $\mathcal{W}_M^I(x) = \inf\{\mathcal{W}_M^{[r,r]}(x), \mathcal{W}_M^{[s,s]}(x)\}$. Furthermore, $\mathcal{B}_M^{[r,s]}(K_0')$ is the ring consisting of infinite series $f=\sum_{k\in\mathbb{Z}} a_k T^k$ where $a_k\in K_0'$ such that f is a holomorphic function on the annulus defined by*

$$v_p(T)\in[\frac{p-1}{e'ep}\cdot\frac{1}{s}, \quad \frac{p-1}{e'ep}\cdot\frac{1}{r}].$$

Proof. This is easy. ☐

Lemma 4.3.3. *Suppose $r > r_M$.*

(1) The map $f(T) \mapsto f(u_M)$ induces a ring isomorphism

$$\mathcal{A}_M^{[r,+\infty]}(K_0') \simeq \mathbf{A}_M^{[r,+\infty]}[1/u_M]$$

such that for $f \in \mathcal{A}_M^{[r,+\infty]}(K_0')$, and all s such that $r \leq s < +\infty$, we have

$$\mathcal{W}_M^{[s,s]}(f(T)) = W^{[s,s]}(f(u_M)).$$

(2) For any $s \geq r$, the map $f(T) \mapsto f(u_M)$ is an isometric isomorphism

$$\mathcal{A}_M^{[r,s]}(K_0') \simeq \mathbf{A}_M^{[r,s]}$$

The proof uses similar strategy as in Lem. 2.2.7. We first study the section s.

4.3.4. *The section s.* Denote

$$s : X_K(M) = \mathbf{A}_M/p \to \mathbf{A}_M$$

the section where for $\overline{x} = \overline{u}_M^b(\sum_{i\geq 0} \bar{a}_i \overline{u}_M^i)$ with $\bar{a}_0 \neq 0$, $s(\overline{x}) := u_M^b \sum_{i\geq 0}[\bar{a}_i]u_M^i$. (When $M = K_\infty$, this is precisely the s in §2.2.8.) Using the expression, one can check that:

(1) $s(\overline{x}) \in \mathbf{A}_M^{[r_M,+\infty]}[1/u_M]$;
(2) $W^{[r_M,r_M]}(s(\overline{x})) = W^{[r_M,r_M]}(u_M^b) = W^{[r_M,r_M]}([\overline{u}_M]^b) = (p-1)(pr_M)^{-1}\cdot v_{\widetilde{\mathbf{E}}}(\overline{x})$, where the first equality is because $\sum_{i\geq 0}[\bar{a}_i]u_M^i$ is a unit in $\mathbf{A}_M^{[r_M,+\infty]}$, and the second equality uses Lem. 4.2.5(1b);
(3) $w_0(s(\overline{x})) = v_{\widetilde{\mathbf{E}}}(\overline{x})$;
(4) since $s(\overline{x})/[\overline{u}_M]^b$ is a unit in $\mathbf{A}_M^{[r_M,+\infty]}$, Lem. 4.2.4(3) implies that when $k \geq 1$,

$$w_k(s(\overline{x})) > v_{\widetilde{\mathbf{E}}}(\overline{x}) - k\cdot pr_M(p-1)^{-1} = w_0(s(\overline{x})) - k\cdot pr_M(p-1)^{-1}. \quad (4.3.1)$$

4.3.5. *An approximating sequence.* Given $x \in \mathbf{A}_M^{[r_M,+\infty]}[1/u_M]$, define a sequence $\{x_n\}$ in $\mathbf{A}_M^{[r_M,+\infty]}[1/u_M]$ where $x_0 = x$ and $x_{n+1} := p^{-1}(x_n - s(\overline{x_n}))$. Note that $x = \sum_{n\geq 0} p^n s(\overline{x_n})$. Similarly as in [Col08, Lem. 7.3], we have

$$\begin{aligned} w_k(x_{n+1}) &\geq \inf\{w_{k+1}(x_n), w_{k+1}(s(\overline{x_n}))\} \\ &\geq \inf\{w_{k+1}(x_n), w_0(s(\overline{x_n})) - (k+1)\cdot pr_M(p-1)^{-1}\}, \text{ by (4.3.1)} \\ &= \inf\{w_{k+1}(x_n), w_0(x_n) - (k+1)\cdot pr_M(p-1)^{-1}\}. \end{aligned}$$

Similarly as in [Col08, Lem. 7.4], by repeatedly using the above, we have

$$v_{\widetilde{\mathbf{E}}}(\overline{x_n}) = w_0(x_n) \geq \inf_{0\leq i\leq n}\{w_i(x) - (n-i)\cdot pr_M(p-1)^{-1}\}. \quad (4.3.2)$$

Proof of Lem. 4.3.3. It suffices to prove Item (1). Given $f(T) \in \mathcal{A}_M^{[r,+\infty]}(K_0')$, then similarly as in (Part 1) of the proof of Lem. 2.2.7, $f(u_M) \in \mathbf{A}_M^{[r,+\infty]}[1/u_M]$, and $W^{[s,s]}(f(u_M)) \geq \mathcal{W}_M^{[s,s]}(f(T))$.

For the other direction, suppose $x \in \mathbf{A}_M^{[r,+\infty]}[1/u_M]$, let $\{x_n\}$ be the sequence constructed in §4.3.5. Let $f_n(T)$ be a formal series such that $f_n(u_M) = s(\overline{x_n})$. Note that $f_n(T)$ is $T^{v_{\widetilde{\mathbf{E}}}(\overline{x_n})/v_{\widetilde{\mathbf{E}}}(\overline{u}_M)}$ times a unit in $\mathbf{A}_M^{[r_M,+\infty]}$ (note that $T^{v_{\widetilde{\mathbf{E}}}(\overline{x_n})/v_{\widetilde{\mathbf{E}}}(\overline{u}_M)}$ makes sense since $\overline{x_n}$

belongs to $X_K(M) = k'((\overline{u}_M))$, and so for any $s \geq r$,

$$\begin{aligned} \mathcal{W}_M^{[s,s]}(p^n f_n(T)) &\geq \mathcal{W}_M^{[s,s]}(p^n T^{v_{\widetilde{\mathbf{E}}}(\overline{x_n})/v_{\widetilde{\mathbf{E}}}(\overline{u}_M)}) \\ &\geq n + \frac{p-1}{ps} \cdot \inf_{0 \leq i \leq n} \{w_i(x) - \frac{(n-i)pr_M}{p-1}\}, \text{ by (4.3.2)} \\ &= \inf_{0 \leq i \leq n} \{\frac{p-1}{ps} \cdot w_i(x) + i + (n-i)(1 - \frac{r_M}{s})\} \\ &\geq \inf_{0 \leq i \leq n} \{\frac{p-1}{ps} \cdot w_i(x) + i\}, \text{ since } s > r_M \\ &\geq W^{[s,s]}(x). \end{aligned}$$

Note that $\inf_{0 \leq i \leq n}\{\frac{p-1}{ps} \cdot w_i(x) + i + (n-i)(1 - \frac{r_M}{s})\}$ converges to $+\infty$ when $n \to +\infty$, so $f(T) = \sum_{n \geq 0} p^n f_n(T)$ converges in $\mathcal{A}_M^{[r,+\infty]}(K_0')$. Clearly $f(u_M) = x$, and $\mathcal{W}_M^{[s,s]}(f(T)) \geq W^{[s,s]}(x)$. □

Proposition 4.3.6. *Suppose $r_\ell > r_M$, then*

$$\mathbf{A}_M^{[r_\ell,+\infty]} = W(k')[\![u_M]\!]\{\frac{p}{u_M^{e'ep^\ell}}\}, \quad \mathbf{A}_M^{[r_\ell,r_k]} = W(k')[\![u_M]\!]\{\frac{p}{u_M^{e'ep^\ell}}, \frac{u_M^{e'ep^k}}{p}\}$$

Proof. It follows from Lem. 4.3.2 and Lem. 4.3.3. □

Corollary 4.3.7. *Suppose $[r,s] \subset [r',s] \subset (r_M,+\infty]$, then $\mathbf{A}_M^{[r,s]} \cap \widetilde{\mathbf{A}}^{[r',s]} = \mathbf{A}_M^{[r',s]}$.*

Proof. This is similar to Cor. 2.2.11, by using Prop. 4.3.6. □

Lemma 4.3.8. *Suppose $r > r_M$. If $x \in \mathbf{A}_M^{[r,+\infty]}[1/u_M]$ and $x \in (\widetilde{\mathbf{A}}^{[r,+\infty]})^\times$, then $x \in (\mathbf{A}_M^{[r,+\infty]})^\times$.*

Proof. Let $\{x_n\}$ be the sequence constructed in §4.3.5, and so $x = \sum_{n \geq 0} p^n s(\overline{x_n})$. By Lem. 4.2.4, $v_{\widetilde{\mathbf{E}}}(\overline{x_0}) = 0$, and so $s(\overline{x_0}) \in (\mathbf{A}_M^{[r,+\infty]})^\times$. It then suffices to show that $1+y \in (\mathbf{A}_M^{[r,+\infty]})^\times$, where $y = \sum_{n \geq 1} p^n s(\overline{x_n})/s(\overline{x_0})$. As we calculated in the proof of Lem. 4.3.3,

$$W^{[r,r]}(p^n s(\overline{x_n})) \geq \inf_{0 \leq i \leq n}\{\frac{p-1}{pr} \cdot w_i(x) + i + (n-i)(1 - \frac{r_M}{r})\} > 0,$$

where the final inequality uses $n \geq 1$ and Lem. 4.2.4. And since $W^{[r,r]}(p^n s(\overline{x_n})) \to +\infty$ when $n \to +\infty$, so $W^{[r,r]}(y) > 0$, and $(1+y)^{-1} \in \mathbf{A}_M^{[r,r]}$. Thus by Cor. 4.3.7, we can conclude that $(1+y)^{-1} \in \mathbf{A}_M^{[r,r]} \cap \widetilde{\mathbf{A}}_M^{[r,+\infty]} = \mathbf{A}_M^{[r,+\infty]}$. □

5. Computation of $\hat{G}$-locally analytic vectors

In this section, we compute the $\hat{G}$-locally analytic vectors in $\widetilde{\mathbf{B}}_L^I$. The strategy is very similar to [Ber16, Thm. 5.4]: we need to find a "formal variable" (denoted as b in the following) which plays the role of $\mathbf{y}$ in [Ber16, Thm. 5.4] (or of α in Prop. 3.3.2(1)). Indeed, the discovery of b is the key observation for our calculations. In the following, we define b, and then use Tate's normalized traces to build an approximating sequence b_n, and use them to determine the set of $\hat{G}$-locally analytic vectors in $\widetilde{\mathbf{B}}_L^I$.

5.1. **The element b.** Let $\lambda := \prod_{n \geq 0} \varphi^n(\frac{E(u)}{E(0)}) \in \mathbf{B}_{K_\infty}^{[0,+\infty)}$. Let $b := \frac{t}{p\lambda}$, then b is precisely the $\mathfrak{t}$ in [Liu08, Example 3.2.3], and $b \in \widetilde{\mathbf{A}}_L^+$. Since $\widetilde{\mathbf{B}}_L^\dagger$ is a field ([Col08, Prop. 5.12]), there exists some $r(b) > 0$ such that $1/b \in \widetilde{\mathbf{B}}_L^{[r(b),+\infty]}$.

Lemma 5.1.1. *If $r_\ell \geq r(b)$, then $b, 1/b \in (\widetilde{\mathbf{B}}_L^{[r_\ell,r_k]})^{\hat{G}\text{-la}}$.*

Proof. Since γ acts on b (resp. $1/b$) via cyclotomic character (resp. inverse of cyclotomic character), it suffices to show that b (resp. $1/b$) is τ-locally analytic (cf. the argument in Lem. 3.2.4). The result for $1/b$ follows from Lem. 3.4.2(3). Then Lem. 3.1.2(2) implies that b is also locally analytic. □

Remark 5.1.2. (1) It seems likely that $b \in (\widetilde{\mathbf{B}}_L^{[r,s]})^{\hat{G}\text{-la}}$ for any $[r,s] \in [0,+\infty)$, just as the element $t/(\varphi^k(E(u)))$ in Lem. 3.4.2(2); but we do not know how to prove it.

(2) The result that $b \in (\widetilde{\mathbf{B}}_L^{[r,s]})^{\hat{G}\text{-la}}$ for $r \geq r(b)$ implies easily that $t/(\varphi^k(E(u))) \in (\widetilde{\mathbf{B}}_L^{[r,s]})^{\hat{G}\text{-la}}$ for $r \geq r(b)$, because the element $\lambda/(\varphi^k(E(u)))$ is locally analytic; this (partial) proof of Lem. 3.4.2(2) avoids use of Lem. 3.1.8. However, we need the full result of Lem. 3.4.2(2) for the calculation in Thm. 3.4.4.

5.2. **Tate's normalized traces.** Recall (see e.g., [Col08, §5.1]) that the weak topology on $\widetilde{\mathbf{A}}$ is the one defined by the semi-valuations w_k, for $k \in \mathbb{N}$, meaning that $x_n \to x$ for the weak topology in $\widetilde{\mathbf{A}}$ if and only if for all $k \in \mathbb{N}$, $w_k(x_n - x) \to +\infty$. In particular, the set $\{p^n\widetilde{\mathbf{A}} + u^k\widetilde{\mathbf{A}}^+\}_{n,k\geq 0}$ forms a basis of neighbourhoods of 0 in $\widetilde{\mathbf{A}}$ for the weak topology. The following lemma is very useful.

Lemma 5.2.1. *Let $r' > 0$ and $x_n \in \widetilde{\mathbf{A}}^{[r',+\infty]}, \forall n \geq 1$. Suppose $x_n \to 0$ in $\widetilde{\mathbf{A}}$ with respect to the weak topology. Then for any $r' < s < +\infty$ (note that it is critical $s \neq r'$), $x_n \to 0$ in $\widetilde{\mathbf{A}}^{[s,+\infty]}$ with respect to the $W^{[s,s]}$-topology.*

Proof. This is implied by [Col08, Prop. 5.8]. Indeed, we can let the "C" in *loc. cit.* to be 0 (see the proof of our Lem. 2.1.10 for comparison of notations). □

In this subsection, we let $K_\infty \subset M \subset L$ where M/K_∞ is a finite extension. For $n \geq 1$ and I an interval, let

$$\mathbf{A}_{M,n} := \varphi^{-n}(\mathbf{A}_M), \quad \mathbf{A}_{M,n}^I := \varphi^{-n}(\mathbf{A}_M^{p^nI}).$$

Denote $J := \mathbb{Z}[1/p] \cap [0,1)$ and for $n \in \mathbb{N}$, let $J_n := \{i \in J : v_p(i) \geq -n\}$.

Lemma 5.2.2.

(1) Every element $x \in \mathbf{E}_{M,n} := \varphi^{-n}(\mathbf{E}_M)$ admits a unique expression $x = \sum_{i\in J_n} u^i a_i(x)$ where $a_i(x) \in \mathbf{E}_M$.

(2) Every element $x \in \widetilde{\mathbf{E}}_M$ admits a unique expression $x = \sum_{i\in J} u^i a_i(x)$ where $a_i(x) \in \mathbf{E}_M$ and $a_i(x) \to 0$ (here convergence is with respect to the usual co-finite filter; i.e., with respect to any ordering of J).

(3) Every element $x \in \mathbf{A}_{M,n}$ admits a unique expression $x = \sum_{i\in J_n} u^i a_i(x)$ where $a_i(x) \in \mathbf{A}_M$.

(4) Every element $x \in \widetilde{\mathbf{A}}_M$ admits a unique expression $x = \sum_{i\in J} u^i a_i(x)$ where $a_i(x) \in \mathbf{A}_M$ and $a_i(x) \to 0$ for the weak topology.

Proof. These are easy analogues of [Col08, Prop. 8.3, Prop. 8.5]. □

We now define, for $n \in \mathbb{Z}^{\geq 0}$, $R_{M,n} : \widetilde{\mathbf{A}}_M \to \widetilde{\mathbf{A}}_M$ by

$$R_{M,n}(x) = \sum_{i\in J_n} u^i a_i(x).$$

Proposition 5.2.3. *(1) For $x \in \widetilde{\mathbf{A}}_M$, we have $R_{M,n}(x) \in \mathbf{A}_{M,n}$ and $R_{M,n}(x) \to x$ for the weak topology.*

(2) Let $r' > 0$ and suppose $x \in \widetilde{\mathbf{A}}_M^{[r',+\infty]}$. Suppose $n \gg 0$ such that $p^n r' > r_M$ (where r_M is as in Lem. 4.2.5), then $R_{M,n}(x) \in \mathbf{A}_{M,n}^{[r',+\infty]}$, and $R_{M,n}(x) \to x$ for both the weak topology and the $W^{[r,s]}$-topology for any $r' < r \leq s < +\infty$. In particular, $\mathbf{A}_{M,\infty}^{[r',+\infty]} := \cup_{m\geq 0}\mathbf{A}_{M,m}^{[r',+\infty]}$ is dense in $\widetilde{\mathbf{A}}_M^{[r',+\infty]}$ for both the weak topology and the $W^{[r,s]}$-topology.

Proof. Item (1) follows from Lem. 5.2.2. For Item (2), the result that $R_{M,n}(x) \in \mathbf{A}_{M,n}^{[r',+\infty]}$ for $n \gg 0$ is analogue of [Col08, Cor. 8.11]. The convergence $R_{M,n}(x) \to x$ with respect to the weak topology follows from Item (1); the convergence for the $W^{[r,s]}$-topology then follows from Lem. 5.2.1 (note that $W^{[r,s]} = \inf\{W^{[r,r]}, W^{[s,s]}\}$). □

5.3. **Approximation of** b. We now build a sequence $\{b_n\}_{n\geq 1}$ to approximate b, which furthermore satisfies $\nabla_\gamma(b_n) = 0$ for all n. In the following, we use $K_\infty \subset_{\text{fin}} M \subset L$ to mean that M is a intermediate extension which is finite over K_∞.

Lemma 5.3.1. *Let W be a $\mathbb{Q}_p$-Banach representation of $\hat{G}$. Then*

$$(W^{\hat{G}\text{-la}})^{\nabla_\gamma=0} = \bigcup_{K_\infty \subset_{\text{fin}} M \subset L} W^{\tau\text{-la}, \mathrm{Gal}(L/M)=1}.$$

Proof. If $x \in W^{\hat{G}\text{-la}}$ such that $\nabla_\gamma(x) = 0$, then there exists $m \geq 0$ such that $x \in W^{\hat{G}_m\text{-an}}$ and $\exp(p^m\nabla_\gamma)(x)$ converges in $W^{\hat{G}_m\text{-an}}$. Thus $x \in W^{\tau\text{-la}, \mathrm{Gal}(L/M)=1}$ for some large M. □

Lemma 5.3.2. *Let $[r,s] \subset (0,+\infty)$ and let $n \geq 1$. Let $x \in \widetilde{\mathbf{A}}_L^+$. Then there exists $w \in (\widetilde{\mathbf{B}}_L^{[r,s]})^{\hat{G}\text{-la}, \nabla_\gamma=0}$, such that $x - w \in p^n\widetilde{\mathbf{A}}_L^{[r,s]}$.*

Proof. Fix some $k \gg 0$ such that $u^k \in p^n\widetilde{\mathbf{A}}_L^{[r,s]}$.

Let $\overline{x} \in \widetilde{\mathbf{E}}_L^+$ be the modulo p reduction of x. By [Win83, Cor. 4.3.4], the set

$$\bigcup_{m\in\mathbb{N}} \varphi^{-m}\left(\bigcup_{K_\infty \subset_{\text{fin}} M \subset L} \mathbf{E}_M^+\right)$$

is dense in $\widetilde{\mathbf{E}}_L^+$ for the $\underline{\pi}$-adic topology, where $\mathbf{E}_M^+$ is the ring of integers of $X_K(M)$. Thus, there exists some $\overline{y}_1 \in \varphi^{-m_1}(\mathbf{E}_{M_1}^+)$ for some m_1 and M_1, such that $\overline{x} - \overline{y}_1 = u^k\overline{z}_1$ where $\overline{z}_1 \in \widetilde{\mathbf{E}}_L^+$. Thus we can write

$$x - [\overline{y}_1] - u^k[\overline{z}_1] = px_1 \text{ for some } x_1 \in \widetilde{\mathbf{A}}_L^+.$$

Now we can repeat the process for x_1 (in the process, we can choose M_2 to contain M_1), so we can write $x_1 - [\overline{y}_2] - u^k[\overline{z}_2] = px_2$. Iterate the process, and let $y = [\overline{y}_1] + p[\overline{y}_2] + \cdots + p^{n-1}[\overline{y}_n]$, then $y \in \widetilde{\mathbf{A}}_{M_n}^+$ and

$$x - y \in p^n\widetilde{\mathbf{A}}_L^+ + u^k\widetilde{\mathbf{A}}_L^+.$$

Pick any r' such that $0 < r' < r$. By Prop. 5.2.3(2), we can choose some $N \gg 0$ (in particular, we require $p^N r' > r_{M_n}$), such that if we let $w := R_{M_n,N}(y)$, then we have

- $w \in \mathbf{A}_{M_n,N}^{[r',+\infty]} \subset \widetilde{\mathbf{A}}_L^{[r',+\infty]} \subset \widetilde{\mathbf{A}}_L^{[r,+\infty]}$, and
- $y - w = p^n a + u^k b$ for some $a \in \widetilde{\mathbf{A}}, b \in \widetilde{\mathbf{A}}^+$ (note that we do not know if $a \in \widetilde{\mathbf{A}}_L$ or $b \in \widetilde{\mathbf{A}}_L^+$), and
- $W^{[r,s]}(y - w) \geq n$.

We claim that $a \in \widetilde{\mathbf{A}}^{[r,s]}$. Since $p^n a = y - w - u^k b \in \widetilde{\mathbf{A}}^{[r,s]}$, it suffices to show that $W^{[r,s]}(a) \geq 0$. But we have

$$W^{[r,s]}(a) = W^{[r,s]}(y - w - u^k b) - n \geq \inf\{W^{[r,s]}(y - w), W^{[r,s]}(u^k b)\} - n \geq 0$$

where we use the assumption $u^k \in p^n\widetilde{\mathbf{A}}_L^{[r,s]}$ (so $W^{[r,s]}(u^k) \geq n$).

Now, we have

$$x - w \in p^n\widetilde{\mathbf{A}}^{[r,s]} + u^k\widetilde{\mathbf{A}}^+ \subset p^n\widetilde{\mathbf{A}}^{[r,s]},$$

and necessarily $x - w \in p^n\widetilde{\mathbf{A}}_L^{[r,s]}$ because $x - w$ is G_L-invariant. Finally, $w \in (\widetilde{\mathbf{B}}_L^{[r,s]})^{\hat{G}\text{-la}, \nabla_\gamma=0}$ by Lem. 5.3.1 (and Thm. 4.2.9). □

5.3.3. *An approximating sequence for b.* Let $I=[r,s]\subset(0,+\infty)$ such that $r\geq r(b)$. For any $n\geq 1$, let $b_n\in(\widetilde{\mathbf{B}}_L^I)^{\hat{G}\text{-la},\nabla_\gamma=0}$ be as in Lem. 5.3.2 such that $b-b_n\in p^n\widetilde{\mathbf{A}}_L^I$. For any fixed n, since both b and b_n are locally analytic, we can choose $m=m(n)\gg 0$ (which depends on n) such that $b-b_n\in(\widetilde{\mathbf{B}}_L^I)^{\hat{G}_m\text{-an}}$ and $\|b-b_n\|_{\hat{G}_m}\leq p^{-n}$.

5.3.4. *A differential operator.* Let $I=[r,s]\subset(0,+\infty)$ such that $r\geq r(b)$. Since $\gamma(b)=\chi(\gamma)\cdot b$, we have $\nabla_\gamma(b)=b$. Since $1/b$ is in $(\widetilde{\mathbf{B}}_L^I)^{\hat{G}\text{-la}}$ by Lem 5.1.1, we can define $\partial_\gamma:(\widetilde{\mathbf{B}}_L^I)^{\hat{G}\text{-la}}\to(\widetilde{\mathbf{B}}_L^I)^{\hat{G}\text{-la}}$ via

$$\partial_\gamma:=\frac{1}{b}\nabla_\gamma.$$

So in particular, we have

$$\partial_\gamma(b-b_n)^k=k(b-b_n)^{k-1},\forall k\geq 1.$$

Theorem 5.3.5. *Let $I=[r,s]\subset(0,+\infty)$ such that $r\geq r(b)$. Suppose $x\in(\widetilde{\mathbf{B}}_L^I)^{\hat{G}\text{-la}}$, then there exists $n,m\geq 1$ and a sequence $\{x_i\}_{i\geq 0}$ in $(\widetilde{\mathbf{B}}_L^I)^{\hat{G}_m\text{-an},\nabla_\gamma=0}$ such that $\|p^{ni}x_i\|_{\hat{G}_m}\to 0$ and $x=\sum_{i\geq 0}x_i(b-b_n)^i$ (which converges in the norm $\|\cdot\|_{\hat{G}_m}$).*

Proof. The proof is similar as [Ber16, Thm. 5.4]. Suppose $m\geq 1$ such that $x\in(\widetilde{\mathbf{B}}_L^I)^{\hat{G}_m\text{-an}}$. Apply [BC16, Lem. 2.6] to the map $\partial_\gamma:(\widetilde{\mathbf{B}}_L^I)^{\hat{G}_m\text{-an}}\to(\widetilde{\mathbf{B}}_L^I)^{\hat{G}_m\text{-an}}$, so there exists $n\geq 1$ such that for all $k\in\mathbb{Z}^{\geq 0}$, we have $\|\partial_\gamma^k(x)\|_{\hat{G}_m}\leq p^{(n-1)k}\|x\|_{\hat{G}_m}$. Increase m if necessary so that $m\geq m(n)$ as in §5.3.3. Let

$$x_i:=\frac{1}{i!}\sum_{k\geq 0}(-1)^k\frac{(b-b_n)^k}{k!}\partial_\gamma^{k+i}(x),$$

then similarly as [Ber16, Thm. 5.4], they satisfy the desired property. □

6. Overconvergence of (φ,τ)-modules

In this section, for a p-adic Galois representation V of G_K of dimension d, we show that its associated (φ,τ)-module is overconvergent. We will construct $\widetilde{D}_L^I(V):=(\widetilde{\mathbf{B}}^I\otimes_{\mathbb{Q}_p}V)^{G_L}$ (see §6.2), which is a finite free module over $\widetilde{\mathbf{B}}_L^I$ of rank d equipped with a $\hat{G}$-action. The key point is to show that $(\widetilde{D}_L^I(V))^{\tau\text{-la},\gamma=1}$ is also finite free over $(\widetilde{\mathbf{B}}_L^I)^{\tau\text{-la},\gamma=1}$ of rank d, i.e., $\widetilde{D}_L^I(V)$ has "enough" (τ-la, $\gamma=1$)-vectors; these vectors will further descend to "overconvergent vectors" in the (φ,τ)-module, via Kedlaya's slope filtration theorem. Using the classical overconvergent (φ,Γ)-module, we already know that $(\widetilde{D}_L^I(V))^{\hat{G}\text{-la}}$ is finite free over $(\widetilde{\mathbf{B}}_L^I)^{\hat{G}\text{-la}}$ of rank d. So we need to take $(\gamma=1)$-invariants in $(\widetilde{D}_L^I(V))^{\hat{G}\text{-la}}$, and show it keeps the correct rank; this is achieved by a Tate-Sen descent *or* a monodromy descent (followed by an étale descent).

In §6.1, we will carry out the descent of locally analytic vectors: the Tate-Sen descent and étale descent uses an axiomatic approach taken from [BC08]; the monodromy descent (in Rem. 6.1.7) follows some similar argument as in [Ber16]. In §6.2, we prove the overconvergence result.

In this section, whenever we write $I=[r,s]\subset(0,+\infty)$, we mean $[r,s]=[r_\ell,r_k]$, cf. Convention 2.1.7.

6.1. **Descent of locally analytic vectors.** Since we will use results from [BC08], it will be convenient to use valuation notations.

Notation 6.1.1. Let W be a $\mathbb{Q}_p$- (or $\mathbb{Z}_p$-) Banach representation (cf. Notation 3.1.9) of a p-adic Lie group G. Suppose there is an analytic bijection $\mathbf{c}:G\to\mathbb{Z}_p^d$ (as in §3.1.1), and suppose $W^{G\text{-an}}=W$. Let val_G denote the valuation on W associated to the norm $\|\cdot\|_G$ (cf. §1.4.4).

Proposition 6.1.2. *Let $(\widetilde{\Lambda}, \|\cdot\|)$ be a $\mathbb{Z}_p$-Banach algebra (cf. Notation 3.1.9), and let* val_Λ *be the valuation associated to $\|\cdot\|$. (Here the notation* val_Λ *follows that of* [BC08, §3.1], *although "*$\mathrm{val}_{\widetilde{\Lambda}}$*" might be a more suggestive one).*

Let H_0 be a profinite group which acts on $\widetilde{\Lambda}$ such that $\mathrm{val}_\Lambda(gx) = \mathrm{val}_\Lambda(x), \forall g \in H_0, x \in \widetilde{\Lambda}$. Let $g \mapsto U_g$ be a continuous cocycle of H_0 in $\mathrm{GL}_d(\widetilde{\Lambda})$.

Suppose $H \subset H_0$ is an open subgroup, and suppose there exists some $a > c_1 > 0$ such that the following conditions are satisfied:

- *(TS1): there exists $\alpha \in \widetilde{\Lambda}^H$ such that $\mathrm{val}_\Lambda(\alpha) > -c_1$ and $\sum_{\sigma \in H_0/H} \sigma(\alpha) = 1$.*
- $\mathrm{val}_\Lambda(U_g - 1) \geq a, \forall g \in H$.

Then there exists $M \in \mathrm{GL}_d(\widetilde{\Lambda})$ such that $\mathrm{val}_\Lambda(M-1) \geq a - c_1$ and the cocycle $g \mapsto M^{-1}U_g g(M)$ is trivial when restricted to H.

Proof. This is a slight variant of [BC08, Cor. 3.2.2]. Indeed, in *loc. cit.*, it requires the condition (TS1) to be satisfied for any pair of open subgroups $H_1 \subset H_2$ in H_0 (cf. [BC08, Def. 3.1.3]); however, in the proof of [BC08, Lem. 3.1.2, Cor. 3.2.2], this condition is used only for one pair. □

Lemma 6.1.3. *Let $c_1 > 0$, let $I = [r, s] \subset (0, +\infty)$, and let $K_\infty \subset M \subset L$ where $[M : K_\infty] < +\infty$. Then there exists $n \gg 0$, and*

$$\alpha \in (\widetilde{\mathbf{B}}_L^I)^{\tau_n\text{-an}, \mathrm{Gal}(L/M)=1},$$

such that the following holds:

- $\mathrm{val}_{\tau_n}(\alpha) = W^I(\alpha) > -c_1$, *here* $\mathrm{val}_{\tau_n} = \mathrm{val}_{<\tau_n>}$ *(cf. Notation 6.1.1);*
- $\sum_{\sigma \in \mathrm{Gal}(M/K_\infty)} \sigma(\alpha) = 1$.

Proof. Denote $\mathrm{Tr} := \sum_{\sigma \in \mathrm{Gal}(M/K_\infty)} \sigma$ the trace operator. By Thm. 4.1.3, $X_K(M)$ is a finite Galois extension of $X_K(K_\infty)$, and so there exists $\beta \in X_K(M)$ such that $\mathrm{Tr}(\beta) = 1$. Note that we necessarily have $v_{\widetilde{\mathbf{E}}}(\beta) \leq 0$.

Suppose $m \gg 0$ (m depends on M and I) such that $p^{-m} r_M < r$ (where $r_M > 0$ as in Lem. 4.2.5), and such that

$$\frac{p-1}{pr}\frac{1}{p^m} v_{\widetilde{\mathbf{E}}}(\beta) > -c_1, \quad \text{and such that} \tag{6.1.1}$$

$$(1 - \frac{r_M}{p^m r}) + \frac{p-1}{p^m pr} v_{\widetilde{\mathbf{E}}}(\beta) > 0. \tag{6.1.2}$$

Let $\gamma = \varphi^{-m}(s(\beta))$ (where s is the map in §4.3.4), then

- since $p^{-m} r_M < r$, $\gamma \in \varphi^{-m}(\mathbf{A}_M^{[r_M, +\infty]}[1/u_M]) \subset \widetilde{\mathbf{A}}^{[r, +\infty]}[1/u]$;
- for any $a \in [r, s]$, by using similar argument as in §4.3.4(2) and apply (6.1.1), we have
$$W^{[a,a]}(\gamma) = W^{[p^m a, p^m a]}(s(\beta)) = \frac{p-1}{p \cdot p^m a} v_{\widetilde{\mathbf{E}}}(\beta) > -c_1,$$
and so $W^I(\gamma) > -c_1$.

Since $\mathrm{Tr}(\varphi^{-m}(\beta)) = 1$, we have $\mathrm{Tr}(\gamma) = 1 + \sum_{k \geq 1} p^k [a_k]$. Furthermore, for any $k \geq 1$,

$$w_k(\mathrm{Tr}(\gamma)) \geq \inf_{\sigma \in \mathrm{Gal}(M/K_\infty)} \{w_k(\sigma(\gamma))\} = w_k(\gamma) = p^{-m} w_k(s(\beta)) > p^{-m} \cdot (v_{\widetilde{\mathbf{E}}}(\beta) - kpr_M(p-1)^{-1}),$$

where the final inequality uses (4.3.1). So when $k \geq 1$,

$$\begin{aligned} k+\frac{p-1}{pr}\cdot w_k(\mathrm{Tr}(\gamma)) &> k+\frac{p-1}{pr}\cdot p^{-m}\cdot (v_{\widetilde{\mathbf{E}}}(\beta)-kpr_M(p-1)^{-1}) \\ &= k(1-\frac{r_M}{p^m r})+\frac{p-1}{pr}\cdot\frac{1}{p^m}v_{\widetilde{\mathbf{E}}}(\beta) \\ &\geq (1-\frac{r_M}{p^m r})+\frac{p-1}{pr}\cdot\frac{1}{p^m}v_{\widetilde{\mathbf{E}}}(\beta), \quad \text{since } 1-\frac{r_M}{p^m r}>0 \\ &> 0, \quad \text{by (6.1.2)}. \end{aligned}$$

By Lem. 4.2.4, $\mathrm{Tr}(\gamma) \in (\widetilde{\mathbf{A}}^{[r,+\infty]})^\times$, and so $\varphi^m(\mathrm{Tr}(\gamma)) \in (\widetilde{\mathbf{A}}^{[p^m r,+\infty]})^\times$. Since $\varphi^m(\gamma) \in \mathbf{A}_M^{[r_M,+\infty]}[1/u_M]$, we obtain

$$\varphi^m(\mathrm{Tr}(\gamma)) \in \mathbf{A}_{K_\infty}^{[r_M,+\infty]} \subset \mathbf{A}_{K_\infty}^{[p^m r,+\infty]}, \text{ since } p^{-m}r_M < r.$$

By Lem. 4.3.8 (note that $p^m r > r_M$), $\varphi^m(\mathrm{Tr}(\gamma)) \in (\mathbf{A}_{K_\infty}^{[p^m r,+\infty]})^\times$, and so $\mathrm{Tr}(\gamma) \in (\varphi^{-m}(\mathbf{A}_{K_\infty}^{[p^m r,+\infty]}))^\times$, and so by Thm. 3.4.4,

$$(\mathrm{Tr}(\gamma))^{-1} \in (\widetilde{\mathbf{B}}_L^I)^{\tau\text{-la},\mathrm{Gal}(L/K_\infty)=1}.$$

Let $\alpha := \gamma\cdot(\mathrm{Tr}(\gamma))^{-1}$. Note that

$$\gamma \in \varphi^{-m}(\mathbf{A}_M^{[r_M,+\infty]}[1/u_M]) \subset \varphi^{-m}(\mathbf{B}_M^{p^m I}) \subset (\widetilde{\mathbf{B}}_L^I)^{\tau\text{-la},\mathrm{Gal}(L/M)=1}, \text{ by Thm.4.2.9.}$$

Thus, we have $\alpha \in (\widetilde{\mathbf{B}}_L^I)^{\tau\text{-la},\mathrm{Gal}(L/M)=1}$. We also note that $W^I(\alpha) = W^I(\gamma) > -c_1$. Finally, the existence of $n \gg 0$ such that $\alpha \in (\widetilde{\mathbf{B}}_L^I)^{\tau_n\text{-an},\mathrm{Gal}(L/M)=1}$ is by definition; the existence of $n \gg 0$ such that $\mathrm{val}_{\tau_n}(\alpha) = W^I(\alpha)$ is by Lem. 3.1.4. □

6.1.4. Let B be a $\mathbb{Q}_p$-Banach algebra, equipped with an action by a finite group G. Let $B^\natural$ denote the ring B with trivial G-action. Suppose that

(1) B is a finite free B^G-module;
(2) there exists a G-equivariant decomposition $B^\natural \otimes_{B^G} B \simeq \oplus_{g\in G} B^\natural \cdot e_g$ such that $e_g^2 = e_g$, $e_g e_h = 0$ for $g \neq h$, and $g(e_h) = e_{gh}$.

Proposition 6.1.5. *Let B and G be as in §6.1.4. Suppose N is a finite free B-module with semi-linear G-action, then*

(1) N^G is a finite free B^G-module;
(2) the map $B \otimes_{B^G} N^G \to N$ is a G-equivariant isomorphism.

Proof. This is [BC08, Prop. 2.2.1]. □

Proposition 6.1.6. *Let $I = [r,s] \subset (0,+\infty)$. Let $\mathcal{M}$ be a finite free $(\widetilde{\mathbf{B}}_L^I)^{\hat{G}\text{-la}}$-module of rank d, with a semi-linear and locally analytic $\hat{G}$-action. Then $(\mathcal{M})^{\mathrm{Gal}(L/K_\infty)}$ is finite free over $(\widetilde{\mathbf{B}}_L^I)^{\tau\text{-la},\gamma=1}$ of rank d, and*

$$(\widetilde{\mathbf{B}}_L^I)^{\hat{G}\text{-la}} \otimes_{(\widetilde{\mathbf{B}}_L^I)^{\tau\text{-la},\gamma=1}} (\mathcal{M})^{\mathrm{Gal}(L/K_\infty)} \simeq \mathcal{M}.$$

Proof. The following proof is via Tate-Sen descent; see Rem. 6.1.7 for another proof via monodromy descent.

Since $\mathrm{Gal}(L/K_\infty)$ is topologically generated by finitely many elements (in most cases, by one element; cf. Notation 3.2.1), there exists a basis $e_1,\cdots,e_d$ of $\mathcal{M}$ such that the co-cycle c associated to the $\mathrm{Gal}(L/K_\infty)$-action on $\mathcal{M}$ (with respect to this basis) is of the form $g \mapsto U_g$ where $U_g \in \mathrm{GL}_d((\widetilde{\mathbf{B}}_L^I)^{\hat{G}_n\text{-an}})$ for some $n \gg 0$.

Let $a > c_1 > 0$. Choose some M such that $K_\infty \subset_{\mathrm{fin}} M \subset L$ and such that

$$\mathrm{val}_{\hat{G}_n}(U_g - 1) \geq a, \text{ when } g \in \mathrm{Gal}(L/M),$$

where $\mathrm{val}_{\hat{G}_n}$ is as in Notation 6.1.1. By Lem. 6.1.3, there exists some $n' \gg 0$ and $\alpha \in (\widetilde{\mathbf{B}}_L^I)^{\tau_{n+n'}\text{-an},\mathrm{Gal}(L/M)=1}$ such that $\mathrm{val}_{\hat{G}_{n+n'}}(\alpha) > -c_1$, and $\sum_{\sigma\in\mathrm{Gal}(M/K_\infty)}\sigma(\alpha) = 1$. Apply Prop. 6.1.2 to the pair

$$(\widetilde{\Lambda}, \mathrm{val}_\Lambda) = ((\widetilde{\mathbf{B}}_L^I)^{\hat{G}_{n+n'}\text{-an}}, \mathrm{val}_{\hat{G}_{n+n'}}),$$

(where $\mathrm{val}_{\hat{G}_{n+n'}}$ is sub-multiplicative by Lem. 3.1.2), the restricted co-cycle $c|_{\mathrm{Gal}(L/M)}$, when considered as evaluated in $\mathrm{GL}_d((\widetilde{\mathbf{B}}_L^I)^{\hat{G}_{n+n'}\text{-an}})$, is trivial after base change. So:

(*) : $(\mathcal{M})^{\mathrm{Gal}(L/M)}$ is finite free over $(\widetilde{\mathbf{B}}_L^I)^{\tau\text{-la},\mathrm{Gal}(L/M)=1}$ of rank d.

Let $G := \mathrm{Gal}(M/K_\infty)$. Fix a basis $e_1', \cdots, e_d'$ of $(\mathcal{M})^{\mathrm{Gal}(L/M)}$, and suppose the co-cycle associated to the G-action on $(\mathcal{M})^{\mathrm{Gal}(L/M)}$ with respect to this basis has value in $\mathrm{GL}_d(\varphi^{-m}(\mathbf{B}_M^{p^mI}))$ for some $m \gg 0$ (using Thm. 4.2.9). Let N_m be the $\varphi^{-m}(\mathbf{B}_M^{p^mI})$-span of $e_1', \cdots, e_d'$.

Via the same argument as in [BC08, Lem. 4.2.5], there exists some $s(M) > 0$ such that if $a > s(M)$, then the pair $(\mathbf{B}_M^{[a,+\infty]}, G)$ satisfies the two conditions in §6.1.4. So when $m \gg 0$ such that $p^m r > s(M)$, then the pair $(\mathbf{B}_M^{p^mI}, G)$, and thus also the pair $(\varphi^{-m}(\mathbf{B}_M^{p^mI}), G)$ satisfy the two conditions in §6.1.4. By Prop. 6.1.5, $(N_m)^G$ is finite free over $\varphi^{-m}(\mathbf{B}_{K_\infty}^{p^mI})$ of rank d; this implies the desired result. □

Remark 6.1.7. Keep the notations in Prop. 6.1.6 above. Suppose *furthermore* that $r \geq r(b)$ (see §5 for $r(b)$), then we can give another proof of Prop. 6.1.6 via monodromy descent. The proof follows similar ideas as in [Ber16, §6].

In this second proof, we only reprove the statement (*) above, namely, we show that there exists some $K_\infty \subset M \subset L$ such that $(\mathcal{M})^{\mathrm{Gal}(L/M)}$ is finite free over $(\widetilde{\mathbf{B}}_L^I)^{\tau\text{-la},\mathrm{Gal}(L/M)=1}$ of rank d. By Lem. 5.3.1, it suffices to show that $(\mathcal{M})^{\nabla_\gamma=0}$ is finite free over $(\widetilde{\mathbf{B}}_L^I)^{\hat{G}\text{-la},\nabla_\gamma=0}$ of rank d, and

$$(\widetilde{\mathbf{B}}_L^I)^{\hat{G}\text{-la}} \otimes_{(\widetilde{\mathbf{B}}_L^I)^{\hat{G}\text{-la},\nabla_\gamma=0}} (\mathcal{M})^{\nabla_\gamma=0} \simeq \mathcal{M}.$$

Let $D_\gamma = \mathrm{Mat}(\partial_\gamma)$ (∂_γ is well-defined because $r \geq r(b)$), then it suffices to show that there exists $H \in \mathrm{GL}_d((\widetilde{\mathbf{B}}_L^I)^{\mathrm{la}})$ such that $\partial_\gamma(H) + D_\gamma H = 0$. For $k \in \mathbb{N}$, let $D_k = \mathrm{Mat}(\partial_\gamma^k)$. For n large enough, the series given by

$$H = \sum_{k\geq 0}(-1)^k D_k \frac{(b-b_n)^k}{k!}$$

converges in $M_d((\widetilde{\mathbf{B}}_L^I)^{\mathrm{la}})$ to a solution of the equation $\partial_\gamma(H) + D_\gamma H = 0$. Moreover, for n big enough, we have $W^I(D_k \cdot (b-b_n)^k/k!) > 0$ for $k \geq 1$, so that $H \in \mathrm{GL}_d((\widetilde{\mathbf{B}}_L^I)^{\mathrm{la}})$.

Remark 6.1.8. The condition $r \geq r(b)$ in the proof of Rem. 6.1.7 is actually harmless for application in our main theorem Thm. 6.2.6 (i.e., in the proof of Thm. 6.2.6, we could equally apply Rem. 6.1.7 instead of Prop. 6.1.6). Indeed, at the very beginning of the proof of Thm. 6.2.6, we could assume the "$\tilde{r}_0$" there to be bigger than $r(b)$.

6.2. **Overconvergence of (φ, τ)-modules.**

Definition 6.2.1.

(1) Let $\mathrm{Mod}^{\varphi}_{\mathbf{A}_{K_\infty}}$ denote the category of finite free $\mathbf{A}_{K_\infty}$-modules M equipped with a $\varphi_{\mathbf{A}_{K_\infty}}$-semi-linear endomorphism $\varphi_M : M \to M$ such that $1 \otimes \varphi : \varphi^* M \to M$ is an isomorphism. Morphisms in this category are just $\mathbf{A}_{K_\infty}$-linear maps compatible with φ's.

(2) Let $\mathrm{Mod}^{\varphi}_{\mathbf{B}_{K_\infty}}$ denote the category of finite free $\mathbf{B}_{K_\infty}$-modules D equipped with a $\varphi_{\mathbf{B}_{K_\infty}}$-semi-linear endomorphism $\varphi_D : D \to D$ such that there exists a finite free $\mathbf{A}_{K_\infty}$-lattice M such that $M[1/p] = D$, $\varphi_D(M) \subset M$, and $(M, \varphi_D|_M) \in \mathrm{Mod}^{\varphi}_{\mathbf{A}_{K_\infty}}$.

We call objects in $\mathrm{Mod}^{\varphi}_{\mathbf{A}_{K_\infty}}$ and $\mathrm{Mod}^{\varphi}_{\mathbf{B}_{K_\infty}}$ finite free *étale φ-modules.*

Definition 6.2.2.

(1) Let $\mathrm{Mod}^{\varphi,\hat{G}}_{\mathbf{A}_{K_\infty},\widetilde{\mathbf{A}}_L}$ denote the category consisting of triples $(M,\varphi_M,\hat{G})$ where
   - $(M,\varphi_M)\in \mathrm{Mod}^{\varphi}_{\mathbf{A}_{K_\infty}}$;
   - $\hat{G}$ is a continuous $\widetilde{\mathbf{A}}_L$-semi-linear $\hat{G}$-action on $\hat{M}:=\widetilde{\mathbf{A}}_L\otimes_{\mathbf{A}_{K_\infty}} M$, and $\hat{G}$ commutes with $\varphi_{\hat{M}}$ on $\hat{M}$;
   - regarding M as an $\mathbf{A}_{K_\infty}$-submodule in $\hat{M}$, then $M\subset \hat{M}^{\mathrm{Gal}(L/K_\infty)}$.

(2) Let $\mathrm{Mod}^{\varphi,\hat{G}}_{\mathbf{B}_{K_\infty},\widetilde{\mathbf{B}}_L}$ denote the category consisting of triples $(D,\varphi_D,\hat{G})$ which contains a lattice (in the obvious fashion) $(M,\varphi_M,\hat{G})\in \mathrm{Mod}^{\varphi,\hat{G}}_{\mathbf{A}_{K_\infty},\widetilde{\mathbf{A}}_L}$.

The category $\mathrm{Mod}^{\varphi,\hat{G}}_{\mathbf{A}_{K_\infty},\widetilde{\mathbf{A}}_L}$ (and $\mathrm{Mod}^{\varphi,\hat{G}}_{\mathbf{B}_{K_\infty},\widetilde{\mathbf{B}}_L}$) are precisely the étale (φ,τ)-modules as in [GL, Def. 2.1.5].

6.2.3. Let $\mathrm{Rep}_{\mathbb{Q}_p}(G_\infty)$ (resp. $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$) denote the category of finite dimensional $\mathbb{Q}_p$-vector spaces V with continuous $\mathbb{Q}_p$-linear G_∞ (resp. G_K)-actions.

- For $D\in \mathrm{Mod}^{\varphi}_{\mathbf{B}_{K_\infty}}$, let
$$V(D):=(\widetilde{\mathbf{B}}\otimes_{\mathbf{B}_{K_\infty}} D)^{\varphi=1},$$
then $V(D)\in \mathrm{Rep}_{\mathbb{Q}_p}(G_\infty)$. If furthermore $(D,\varphi_D,\hat{G})\in \mathrm{Mod}^{\varphi,\hat{G}}_{\mathbf{B}_{K_\infty},\widetilde{\mathbf{B}}_L}$, then $V(D)\in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$.
- For $V\in \mathrm{Rep}_{\mathbb{Q}_p}(G_\infty)$, let
$$D_{K_\infty}(V):=(\mathbf{B}\otimes_{\mathbb{Q}_p} V)^{G_\infty},$$
then $D_{K_\infty}(V)\in \mathrm{Mod}^{\varphi}_{\mathbf{B}_{K_\infty}}$. If furthermore $V\in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$, let
$$\widetilde{D}_L(V):=(\widetilde{\mathbf{B}}\otimes_{\mathbb{Q}_p} V)^{G_L},$$
then $\widetilde{D}_L(V)=\widetilde{\mathbf{B}}_L\otimes_{\mathbf{B}_{K_\infty}} D_{K_\infty}(V)$ has a $\hat{G}$-action, making $(D_{K_\infty}(V),\varphi,\hat{G})$ an étale (φ,τ)-module.

Theorem 6.2.4.

(1) *The functors V and D_{K_∞} induce an exact tensor equivalence between the categories* $\mathrm{Mod}^{\varphi}_{\mathbf{B}_{K_\infty}}$ *and* $\mathrm{Rep}_{\mathbb{Q}_p}(G_\infty)$.

(2) *The functors V and $(D_{K_\infty},\widetilde{D}_L)$ induce an exact tensor equivalence between the categories* $\mathrm{Mod}^{\varphi,\hat{G}}_{\mathbf{B}_{K_\infty},\widetilde{\mathbf{B}}_L}$ *and* $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$.

Proof. (1) is [Fon90, Prop. A 1.2.6] (and using [GL, Lem. 2.1.4]). (2) is due to [Car13] (cf. also [GL, Prop. 2.1.7]). □

Let $V\in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$. Given $I\subset[0,+\infty]$ any interval, let
$$D^I_{K_\infty}(V) := (\mathbf{B}^I\otimes_{\mathbb{Q}_p} V)^{G_\infty},$$
$$\widetilde{D}^I_L(V) := (\widetilde{\mathbf{B}}^I\otimes_{\mathbb{Q}_p} V)^{G_L}.$$

Definition 6.2.5. Let $V\in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$, and let $\hat{D}=(D_{K_\infty}(V),\varphi,\hat{G})$ be the étale (φ,τ)-module associated to it. Say that $\hat{D}$ is *overconvergent* if there exists $r>0$, such that for $I'=[r,+\infty]$,

(1) $D^{I'}_{K_\infty}(V)$ is finite free over $\mathbf{B}^{I'}_{K_\infty}$, and $\mathbf{B}_{K_\infty}\otimes_{\mathbf{B}^{I'}_{K_\infty}} D^{I'}_{K_\infty}(V)\simeq D_{K_\infty}(V)$;

(2) $\widetilde{D}^{I'}_L(V)$ is finite free over $\widetilde{\mathbf{B}}^{I'}_L$ and
$$\widetilde{\mathbf{B}}_L\otimes_{\widetilde{\mathbf{B}}^{I'}_L}\widetilde{D}^{I'}_L(V)\simeq \widetilde{D}_L(V).$$

Theorem 6.2.6. *For any $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$, its associated étale (φ, τ)-module is overconvergent.*

Proof. **Step 1:** *locally analytic vectors in $\widetilde{D}_L^I(V)$.* For $I = [r, s] \subset (0, +\infty)$, let

$$D^I_{K_{p^\infty}}(V) := (\mathbb{B}^I \otimes_{\mathbb{Q}_p} V)^{G_{p^\infty}},$$

where (as we mentioned in Rem. 1.4.3) $\mathbb{B}$ and $\mathbb{B}^I$ are the rings denoted as "$\mathbf{B}$" and "$\mathbf{B}^I$" in [Ber08]. We still have $\mathbb{B} \subset \widetilde{\mathbf{B}}$ and $\mathbb{B}^I \subset \widetilde{\mathbf{B}}^I$. By the main result of [CC98], there exists some $\tilde{r}_0 > 0$, such that when $r \geq \tilde{r}_0$, then $D^I_{K_{p^\infty}}(V)$ is finite free over $\mathbb{B}^I_{K_{p^\infty}}$ of rank d (here $\mathbb{B}^I_{K_{p^\infty}}$ is precisely "$\mathbf{B}^I_K$" in [Ber08]). Furthermore, there exists G_K-equivariant and φ-equivariant isomorphism

$$\widetilde{\mathbf{B}}^I \otimes_{\mathbb{Q}_p} V \simeq \widetilde{\mathbf{B}}^I \otimes_{\mathbb{B}^I_{K_{p^\infty}}} D^I_{K_{p^\infty}}(V). \tag{6.2.1}$$

Also, by [Ber02, §5.1],

$$D^I_{K_{p^\infty}}(V) \subset (\widetilde{D}^I_L(V))^{\tau=1, \gamma\text{-la}} \subset (\widetilde{D}^I_L(V))^{\hat{G}\text{-la}}. \tag{6.2.2}$$

By Prop. 3.1.6, (6.2.2) implies

$$\widetilde{D}^I_L(V)^{\hat{G}\text{-la}} = (\widetilde{\mathbf{B}}^I_L)^{\hat{G}\text{-la}} \otimes_{\mathbb{B}^I_{K_{p^\infty}}} D^I_{K_{p^\infty}}(V). \tag{6.2.3}$$

So in particular $\widetilde{D}^I_L(V)^{\hat{G}\text{-la}}$ is finite free over $(\widetilde{\mathbf{B}}^I_L)^{\hat{G}\text{-la}}$. By Prop. 6.1.6, $\widetilde{D}^I_L(V)^{\tau\text{-la}, \gamma=1}$ is finite free over $(\widetilde{\mathbf{B}}^I_L)^{\tau\text{-la}, \gamma=1}$. By (6.2.1) and (6.2.3), we also have

$$\widetilde{\mathbf{B}}^I \otimes_{(\widetilde{\mathbf{B}}^I_L)^{\tau\text{-la}, \gamma=1}} \widetilde{D}^I_L(V)^{\tau\text{-la}, \gamma=1} \simeq \widetilde{\mathbf{B}}^I \otimes_{\mathbb{Q}_p} V \tag{6.2.4}$$

Step 2: *glueing $\widetilde{D}^I_L(V)^{\tau\text{-la}, \gamma=1}$ as a vector bundle.* For each $X \subset [\tilde{r}_0, +\infty)$ a closed interval, denote $M^X := \widetilde{D}^X_L(V)^{\tau\text{-la}, \gamma=1}$, and $R^X := (\widetilde{\mathbf{B}}^X_L)^{\tau\text{-la}, \gamma=1}$, and so Step 1 says that M^X is finite free over R^X. Let $I = [r, s] \subset [\tilde{r}_0, +\infty)$ such that $I \cap pI$ is non-empty. For each $k \geq 1$, φ^k induces a bijection between $\widetilde{D}^I_L(V)$ and $\widetilde{D}^{p^k I}_L(V)$, and thus also a bijection between M^I and $M^{p^k I}$. Let $m_1, \cdots, m_d$ be a basis of M^I, and so $\varphi(m_1), \cdots, \varphi(m_d)$ is a basis of M^{pI}. Let $J := I \cap pI$, then by using Prop. 3.1.6, we have

$$M^J = R^J \otimes_{R^I} M^I, \quad M^J = R^J \otimes_{R^{pI}} M^{pI}.$$

So if we write $(\varphi(m_1), \cdots, \varphi(m_d)) = (m_1, \cdots, m_d)P$, then $P \in \mathrm{GL}_d(R^J)$, and so $P \in \mathrm{GL}_d(\mathbf{B}^J_{K_\infty, m})$ for some $m \gg 0$.

Let $I_k := p^k I$, $J_k := I_k \cap I_{k+1} = p^k J$. For each $k \geq 1$, let E_k be the $\mathbf{B}^{I_k}_{K_\infty, m}$-span of $\varphi^k(m_i)$. Since $\varphi^k(P) \in \mathrm{GL}_d(\mathbf{B}^{J_k}_{K_\infty, m})$, we have

$$\mathbf{B}^{J_k}_{K_\infty, m} \otimes_{\mathbf{B}^{I_k}_{K_\infty, m}} E_k \simeq \mathbf{B}^{J_k}_{K_\infty, m} \otimes_{\mathbf{B}^{I_{k+1}}_{K_\infty, m}} E_{k+1}.$$

This says that the collection $\{\varphi^m(E_k)\}_{k \geq 1}$ forms a vector bundle over $\mathbf{B}^{[p^m r, +\infty)}_{K_\infty}$ (cf. [Ked05, Def. 2.8.1]), and so by [Ked05, Thm. 2.8.4], there exists $n_1, \cdots, n_d \in \cap_{k \geq 1} \varphi^m(E_k)$, such that if we let

$$D^{[p^m r, +\infty)}_{K_\infty} := \oplus_{i=1}^d \mathbf{B}^{[p^m r, +\infty)}_{K_\infty} \cdot n_i,$$

then

$$\mathbf{B}^{p^m I_k}_{K_\infty} \otimes_{\mathbf{B}^{[p^m r, +\infty)}_{K_\infty}} D^{[p^m r, +\infty)}_{K_\infty} \simeq \varphi^m(E_k).$$

Now, define

$$D^\dagger_{\mathrm{rig}, K_\infty} := \mathbf{B}^\dagger_{\mathrm{rig}, K_\infty} \otimes_{\mathbf{B}^{[p^m r, +\infty)}_{K_\infty}} D^{[p^m r, +\infty)}_{K_\infty}$$

Then by (6.2.4), we have

$$\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_{\mathbf{B}^\dagger_{\mathrm{rig}, K_\infty}} D^\dagger_{\mathrm{rig}, K_\infty} = \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_{\mathbb{Q}_p} V. \tag{6.2.5}$$

Eqn. (6.2.5) implies that $D^\dagger_{\mathrm{rig},K_\infty}$ is pure of slope 0 (cf. [Ked05]). By [Ked05, Thm. 6.3.3], there exists an étale φ-module $D^\dagger_{K_\infty}$ over $\mathbf{B}^\dagger_{K_\infty}$ such that

$$\mathbf{B}^\dagger_{\mathrm{rig},K_\infty} \otimes_{\mathbf{B}^\dagger_{K_\infty}} D^\dagger_{K_\infty} = D^\dagger_{\mathrm{rig},K_\infty}.$$

Step 3: *overconvergence.* We claim that

$$\mathbf{B}_{K_\infty} \otimes_{\mathbf{B}^\dagger_{K_\infty}} D^\dagger_{K_\infty} \simeq D_{K_\infty}(V). \tag{6.2.6}$$

Let $D' := \mathbf{B}_{K_\infty} \otimes_{\mathbf{B}^\dagger_{K_\infty}} D^\dagger_{K_\infty}$. By Thm. 6.2.4(1), it suffices to show that

$$V' := (\widetilde{\mathbf{B}} \otimes_{\mathbf{B}_{K_\infty}} D')^{\varphi=1} \simeq V|_{G_\infty}. \tag{6.2.7}$$

Note that V' is always a G_∞-representation over $\mathbb{Q}_p$ of dimension d. We have

$$\begin{aligned} V' &= (\widetilde{\mathbf{B}} \otimes_{\mathbf{B}^\dagger_{K_\infty}} D^\dagger_{K_\infty})^{\varphi=1} \\ &= (\widetilde{\mathbf{B}}^\dagger \otimes_{\mathbf{B}^\dagger_{K_\infty}} D^\dagger_{K_\infty})^{\varphi=1}, \quad \text{by [KL15, Thm. 8.5.3(d)(e)]}, \\ &\subset (\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},K_\infty}} D^\dagger_{\mathrm{rig},K_\infty})^{\varphi=1} \\ &= (\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_{\mathbb{Q}_p} V)^{\varphi=1}, \quad \text{by (6.2.5)}, \\ &= V. \end{aligned}$$

So (6.2.7) holds for dimension reasons, and so (6.2.6) holds, concluding the overconvergence of φ-action (i.e., Def. 6.2.5(1) is verified).

Finally, note that $\widetilde{\mathbf{B}}^\dagger \otimes_{\mathbf{B}^\dagger_{K_\infty}} D^\dagger_{K_\infty} \simeq \widetilde{\mathbf{B}}^\dagger \otimes_{\mathbb{Q}_p} V$, so if we let

$$\widetilde{D}^\dagger_L(V) := (\widetilde{\mathbf{B}}^\dagger \otimes_{\mathbb{Q}_p} V)^{G_L},$$

then $\widetilde{D}^\dagger_L(V) \simeq \widetilde{\mathbf{B}}^\dagger_L \otimes_{\mathbf{B}^\dagger_{K_\infty}} D^\dagger_{K_\infty}$. This implies the overconvergence of the τ-action (i.e., Def. 6.2.5(2) is verified). □

References

[BC08] Laurent Berger and Pierre Colmez. Familles de représentations de de Rham et monodromie p-adique. *Astérisque*, (319):303–337, 2008. Représentations p-adiques de groupes p-adiques. I. Représentations galoisiennes et (φ,Γ)-modules.

[BC16] Laurent Berger and Pierre Colmez. Théorie de Sen et vecteurs localement analytiques. *Ann. Sci. Éc. Norm. Supér. (4)*, 49(4):947–970, 2016.

[Bel15] Rebecca Bellovin. p-adic Hodge theory in rigid analytic families. *Algebra Number Theory*, 9(2):371–433, 2015.

[Ber02] Laurent Berger. Représentations p-adiques et équations différentielles. *Invent. Math.*, 148(2):219–284, 2002.

[Ber08] Laurent Berger. Construction de (φ,Γ)-modules: représentations p-adiques et B-paires. *Algebra Number Theory*, 2(1):91–120, 2008.

[Ber10] Laurent Berger. Galois representations and (φ,Γ)-modules. *course note in IHP*, 2010.

[Ber16] Laurent Berger. Multivariable (φ,Γ)-modules and locally analytic vectors. *Duke Math. J.*, 165(18):3567–3595, 2016.

[Bre99] Christophe Breuil. Une application de corps des normes. *Compositio Math.*, 117(2):189–203, 1999.

[Car13] Xavier Caruso. Représentations galoisiennes p-adiques et (φ,τ)-modules. *Duke Math. J.*, 162(13):2525–2607, 2013.

[CC98] F. Cherbonnier and P. Colmez. Représentations p-adiques surconvergentes. *Invent. Math.*, 133(3):581–611, 1998.

[CC99] Frédéric Cherbonnier and Pierre Colmez. Théorie d'Iwasawa des représentations p-adiques d'un corps local. *J. Amer. Math. Soc.*, 12(1):241–268, 1999.

[Col98] Pierre Colmez. Théorie d'Iwasawa des représentations de de Rham d'un corps local. *Ann. of Math. (2)*, 148(2):485–571, 1998.

[Col08] Pierre Colmez. Espaces vectoriels de dimension finie et représentations de de Rham. *Astérisque*, (319):117–186, 2008. Représentations p-adiques de groupes p-adiques. I. Représentations galoisiennes et (φ,Γ)-modules.

[Col10a] Pierre Colmez. Fonctions d'une variable p-adique. *Astérisque*, (330):13–59, 2010.

[Col10b] Pierre Colmez. Représentations de $\mathrm{GL}_2(\mathbf{Q}_p)$ et (φ,Γ)-modules. *Astérisque*, (330):281–509, 2010.

[Col14] Pierre Colmez. La série principale unitaire de $\mathrm{GL}_2(\mathbf{Q}_p)$: vecteurs localement analytiques. In *Automorphic forms and Galois representations. Vol. 1*, volume 414 of *London Math. Soc. Lecture Note Ser.*, pages 286–358. Cambridge Univ. Press, Cambridge, 2014.

[Fon90] Jean-Marc Fontaine. Représentations p-adiques des corps locaux. I. In *The Grothendieck Festschrift, Vol. II*, volume 87 of *Progr. Math.*, pages 249–309. Birkhäuser Boston, Boston, MA, 1990.

[FW79] Jean-Marc Fontaine and Jean-Pierre Wintenberger. Le "corps des normes" de certaines extensions algébriques de corps locaux. *C. R. Acad. Sci. Paris Sér. A-B*, 288(6):A367–A370, 1979.

[GL] Hui Gao and Tong Liu. Loose crystalline lifts and overconvergence of étale (φ,τ)-modules. *to appear, Amer. J. Math.*

[Ked05] Kiran S. Kedlaya. Slope filtrations revisited. *Doc. Math.*, 10:447–525, 2005.

[Kis06] Mark Kisin. Crystalline representations and F-crystals. In *Algebraic geometry and number theory*, volume 253 of *Progr. Math.*, pages 459–496. Birkhäuser Boston, Boston, MA, 2006.

[Kis08] Mark Kisin. Potentially semi-stable deformation rings. *J. Amer. Math. Soc.*, 21(2):513–546, 2008.

[Kis10] Mark Kisin. Integral models for Shimura varieties of abelian type. *J. Amer. Math. Soc.*, 23(4):967–1012, 2010.

[KL] Kiran Kedlaya and Ruochuan Liu. Relative p-adic Hodge theory, II: imperfect period rings . *preprint.*

[KL15] Kiran S. Kedlaya and Ruochuan Liu. Relative p-adic Hodge theory: foundations. *Astérisque*, (371):239, 2015.

[LB10] Jérémy Le Borgne. Optimisation du théorème d'Ax-Sen-Tate et application à un calcul de cohomologie galoisienne p-adique. *Ann. Inst. Fourier (Grenoble)*, 60(3):1105–1123, 2010.

[Liu08] Tong Liu. On lattices in semi-stable representations: a proof of a conjecture of Breuil. *Compos. Math.*, 144(1):61–88, 2008.

[Liu10] Tong Liu. A note on lattices in semi-stable representations. *Math. Ann.*, 346(1):117–138, 2010.

[LXZ12] Ruochuan Liu, Bingyong Xie, and Yuancao Zhang. Locally analytic vectors of unitary principal series of $\mathrm{GL}_2(\mathbb{Q}_p)$. *Ann. Sci. Éc. Norm. Supér. (4)*, 45(1):167–190, 2012.

[Ser06] Jean-Pierre Serre. *Lie algebras and Lie groups*, volume 1500 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2006. 1964 lectures given at Harvard University, Corrected fifth printing of the second (1992) edition.

[ST02] Peter Schneider and Jeremy Teitelbaum. Locally analytic distributions and p-adic representation theory, with applications to GL_2. *J. Amer. Math. Soc.*, 15(2):443–468, 2002.

[Win83] Jean-Pierre Wintenberger. Le corps des normes de certaines extensions infinies de corps locaux; applications. *Ann. Sci. École Norm. Sup. (4)*, 16(1):59–89, 1983.

Department of Mathematics and Statistics, University of Helsinki, FI-00014, Finland
E-mail address: hui.gao@helsinki.fi

UMPA, École Normale Supérieure de Lyon, 46 allée d'Italie, 69007 Lyon, France
E-mail address: leo.poyeton@ens-lyon.fr