

BIFURCATIONS FROM FAMILIES OF PERIODIC SOLUTIONS IN PIECEWISE DIFFERENTIAL SYSTEMS

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Abstract. Consider a differential system of the form

$$x' = F_0(t, x) + \sum_{i=1}^k \varepsilon^i F_i(t, x) + \varepsilon^{k+1} R(t, x, \varepsilon),$$

where $F_i : \mathbb{S}^1 \times D \rightarrow \mathbb{R}^m$ and $R : \mathbb{S}^1 \times D \times (-\varepsilon_0, \varepsilon_0) \rightarrow \mathbb{R}^m$ are piecewise C^{k+1} functions and T -periodic in the variable t . Assuming that the unperturbed system $x' = F_0(t, x)$ has a d -dimensional submanifold of periodic solutions with $d < m$, we use the Lyapunov-Schmidt reduction and the averaging theory to study the existence of isolated T -periodic solutions of the above differential system.

Keywords Lyapunov-Schmidt reduction · periodic solution · averaging method · non-smooth differential system · piecewise smooth differential system

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1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

1.1. Introduction. The study of invariant sets, in special isolated periodic solutions, is very important for understanding the dynamics of a differential system. In the present study, we are concerned about isolated T -periodic solutions of non-autonomous

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differential systems written in the form

$$(1) \quad x' = F(t, x; \varepsilon) = F_0(t, x) + \sum_{i=1}^k \varepsilon^i F_i(t, x) + \varepsilon^{k+1} R(t, x, \varepsilon), \quad (t, x) \in \mathbb{R} \times D.$$

Here, the prime denotes the derivative with respect to the independent variable t , all the functions are assumed to be T -periodic in t , D is an open subset of \mathbb{R}^m , and ε is a small parameter. In this regard, the averaging theory serves as an important tool to detect periodic solutions of (1). A classical introduction to the averaging theory can be found in [23, 28].

There are many studies concerning the periodic solutions of system (1). As a fundamental hypothesis, it is usually assumed that the unperturbed system $x' = F_0(t, x)$ has a submanifold of initial conditions $\mathcal{Z} \subset D$ whose orbits are T -periodic. These studies differ among them depending on the regularity of system (1) and on the dimension of \mathcal{Z} . In what follows, we shall quote some of them.

For the case $\dim(\mathcal{Z}) = m$, the classical averaging theory [23, 28] provides sufficient conditions for the existence of periodic solutions of (1) assuming $F_0 = 0$ and some smoothness and boundedness conditions. In [5], the authors extended the former results up to $k = 2$ assuming weaker conditions on the regularity of system (1). In [10], the authors dropped the condition $F_0 = 0$ and developed the averaging theory at any order ($k \geq 1$ being an arbitrary integer) assuming the analyticity of the system (1). The analyticity condition was relaxed in [18] by means of topological methods. The averaging theory was also extended to non-smooth differential systems [13, 14, 16, 17, 20]. The study of non-smooth differential systems is important in many fields of applied sciences since many problems of physics, engineering, economics, and biology are modeled using differential equations with discontinuous right-hand side, see for instance [3, 8, 27]. Thus, there is natural interest in studying the periodic solutions of system (1) when it is not smooth.

For the case $\dim(\mathcal{Z}) < m$, the averaging theory by itself is not enough to analyze the periodic solutions of system (1) and other techniques need to be employed with it, such as the *Lyapunov-Schmidt reduction method*. In the case that F_i 's are smooth functions, we may quote the studies [4, 6, 11, 15]. If the functions F_i are not smooth or even continuous, we have studies [16, 21], where the authors analyzed some classes of these systems.

In the sequel, we describe how the averaging theory and Lyapunov-Schmidt reduction are used for computing isolated periodic solutions of piecewise smooth differential systems.

1.2. Lyapunov-Schmidt reduction. Consider the function

$$(2) \quad g(z, \varepsilon) = \sum_{i=0}^k \varepsilon^i g_i(z) + \mathcal{O}(\varepsilon^{k+1}),$$

where $g_i : D \rightarrow \mathbb{R}^m$ is a C^{k+1} function, $k \geq 1$, for $i = 0, 1, \dots, k$, and D is an open bounded subset of \mathbb{R}^m . For $d < m$, let V be an open bounded subset of \mathbb{R}^d and $\beta : \bar{V} \rightarrow \mathbb{R}^{m-d}$ a C^{k+1} function such that

$$(3) \quad \mathcal{Z} = \{z_\alpha = (\alpha, \beta(\alpha)) : \alpha \in \bar{V}\} \subset D.$$

We assume that

(H_a) function g_0 vanishes on the d -dimensional submanifold \mathcal{Z} of D .

In [6], the authors used the Lyapunov-Schmidt reduction method to develop the *bifurcation function of order i* , for $i = 0, 1, \dots, k$, which for $|\varepsilon| \neq 0$, sufficiently small, control the existence of branches of zeros $z(\varepsilon)$ of system (2) that bifurcate from $z(0) \in \mathcal{Z}$. In this subsection we present the results developed in that study and those that we shall need later on. To do this we need to introduce some notations.

Consider the projections onto the first d coordinates and onto the last $m - d$ coordinates denoted by $\pi : \mathbb{R}^d \times \mathbb{R}^{m-d} \rightarrow \mathbb{R}^d$ and $\pi^\perp : \mathbb{R}^d \times \mathbb{R}^{m-d} \rightarrow \mathbb{R}^{m-d}$, respectively. In addition, for a point $z \in \mathcal{Z}$ we write $z = (a, b) \in \mathbb{R}^d \times \mathbb{R}^{m-d}$.

Let L be a positive integer, let $x = (x_1, x_2, \dots, x_m) \in D$, $t \in \mathbb{R}$ and $y_j = (y_{j1}, \dots, y_{jm}) \in \mathbb{R}^m$ for $j = 1, \dots, L$. Given $G : \mathbb{R} \times D \rightarrow \mathbb{R}^m$ is a sufficiently smooth function, for each $(t, x) \in \mathbb{R} \times D$ we denote by $\partial^L G(t, x)$ a symmetric L -multilinear map which is applied to a ‘‘product’’ of L vectors of \mathbb{R}^m , which we denote as $\bigodot_{j=1}^L y_j \in \mathbb{R}^{mL}$. The definition of this L -multilinear map is

$$\partial^L G(t, x) \bigodot_{j=1}^L y_j = \sum_{i_1, \dots, i_L=1}^n \frac{\partial^L G(t, x)}{\partial x_{i_1} \dots \partial x_{i_L}} y_{1i_1} \dots y_{Li_L}.$$

We define ∂^0 as the identity.

The bifurcation functions $f_i : \bar{V} \rightarrow \mathbb{R}^d$ of order i are defined for $i = 0, 1, \dots, k$ as

$$(4) \quad f_i(\alpha) = \pi g_i(z_\alpha) + \sum_{l=1}^i \sum_{S_l} \frac{1}{c_1! c_2! 2!^{c_2} \dots c_l! l!^{c_l}} \partial_b^L \pi g_{i-l}(z_\alpha) \bigodot_{j=1}^l \gamma_j(\alpha)^{c_j},$$

where the $\gamma_i : V \rightarrow \mathbb{R}^{m-d}$, for $i = 1, 2, \dots, k$, are defined recursively as

$$(5) \quad \begin{aligned} \gamma_1(\alpha) &= -\Delta_\alpha^{-1} \pi^\perp g_1(z_\alpha) \quad \text{and} \\ \gamma_i(\alpha) &= -i! \Delta_\alpha^{-1} \left(\sum_{S'_i} \frac{1}{c_1! c_2! 2!^{c_2} \dots c_{i-1}! (i-1)!^{c_{i-1}}} \partial_b^{L'} \pi^\perp g_0(z_\alpha) \bigodot_{j=1}^{i-1} \gamma_j(\alpha)^{c_j} \right. \\ &\quad \left. + \sum_{l=1}^{i-1} \sum_{S_l} \frac{1}{c_1! c_2! 2!^{c_2} \dots c_l! l!^{c_l}} \partial_b^L \pi^\perp g_{i-l}(z_\alpha) \bigodot_{j=1}^l \gamma_j(\alpha)^{c_j} \right). \end{aligned}$$

We denote by S_l the set of all l -tuples of non-negative integers (c_1, c_2, \dots, c_l) such that $c_1 + 2c_2 + \dots + lc_l = l$, $L = c_1 + c_2 + \dots + c_l$, and by S'_i the set of all $(i-1)$ -tuples

of non-negative integers $(c_1, c_2, \dots, c_{i-1})$ such that $c_1 + 2c_2 + \dots + (i-1)c_{i-1} = i$, $I' = c_1 + c_2 + \dots + c_{i-1}$ and $\Delta_\alpha = \frac{\partial \pi^1 g_0}{\partial b}(z_\alpha)$.

Concerning the zeros of function (2), the following result was proven in [6]:

Theorem 1 ([6], Corollary 1). *Let Δ_α denote the lower right corner $(m-d) \times (m-d)$ matrix of the Jacobian matrix $Dg_0(z_\alpha)$. In addition to hypothesis (H_a) , we assume that*

- (i) *for each $\alpha \in \bar{V}$, $\det \Delta_\alpha \neq 0$; and*
- (ii) *$f_1 = f_2 = \dots = f_{k-1} = 0$ and f_k is not identically zero.*

If there exists $\alpha^ \in V$ such that $f_k(\alpha^*) = 0$ and $\det(Df_k(\alpha^*)) \neq 0$, then there exists a branch of zeros $z(\varepsilon)$ with $g(z(\varepsilon), \varepsilon) = 0$ and $|z(\varepsilon) - z_{\alpha^*}| = \mathcal{O}(\varepsilon)$.*

1.3. The averaged functions. Let $k \geq 1$ be a positive integer. In [6], the authors have used Theorem 1 to provide sufficient conditions for the existence of isolated periodic solutions of the following T -periodic non-autonomous differential system

$$(6) \quad x' = F(t, x, \varepsilon) = F_0(t, x) + \sum_{i=1}^k \varepsilon^i F_i(t, x) + \mathcal{O}(\varepsilon^{k+1}), \quad (t, z) \in \mathbb{R} \times D,$$

Here, F_i 's are C^{k+1} functions T -periodic in the variable t . We assumed that

- (H_b) *all the solutions of the unperturbed system $x' = F_0(t, x)$ starting at points of \mathcal{Z} , defined in (3), are T -periodic.*

Consider the variational equation

$$(7) \quad y' = \frac{\partial F_0}{\partial x}(t, x(t, z, 0))y,$$

where $x(t, z, 0)$ denotes the solution of system (6) when $\varepsilon = 0$. Denote a fundamental matrix of system (7) by $Y(t, z)$. The *average function of order i* of system (6) is defined as

$$(8) \quad g_i(z) = Y^{-1}(T, z) \frac{y_i(T, z)}{i!},$$

where

$$(9) \quad \begin{aligned} y_1(t, z) &= Y(t, z) \int_0^t Y(s, z)^{-1} F_1(s, x(s, z, 0)) ds, \\ y_i(t, z) &= i! Y(t, z) \int_0^t Y(s, z)^{-1} \left(F_i(s, x(s, z, 0)) \right. \\ &\quad + \sum_{S'_i} \frac{1}{b_1! b_2! 2! b_2 \dots b_{i-1}! (i-1)!^{b_{i-1}}} \partial^{I'} F_0(s, x(s, z, 0)) \bigodot_{j=1}^{i-1} y_j(s, z)^{b_j} \\ &\quad \left. + \sum_{l=1}^{i-1} \sum_{S_l} \frac{1}{b_1! b_2! 2! b_2 \dots b_l! l!^{b_l}} \partial^L F_{i-l}(s, x(s, z, 0)) \bigodot_{j=1}^l y_j(s, z)^{b_j} \right) ds. \end{aligned}$$

Now, in the bifurcation functions formulae (4), let the functions g_i 's be given by (8) instead of (2). Concerning the isolated periodic solutions of system (6), the following result was proven in [6]:

Theorem 2 ([6], Corollary 2). *Let Δ_α denote the lower right corner $(m-d) \times (m-d)$ matrix of the Jacobian matrix $Y(0, z_\alpha)^{-1} - Y(T, z_\alpha)^{-1}$. In addition to hypothesis (H_b) , we assume that*

- (i) *for each $\alpha \in \bar{V}$, $\det \Delta_\alpha \neq 0$; and*
- (ii) *$f_1 = f_2 = \dots = f_{k-1} = 0$ and f_k is not identically zero.*

If there exists $\alpha^ \in V$ such that $f_k(\alpha^*) = 0$ and $\det(Df_k(\alpha^*)) \neq 0$, then there exists a T -periodic solution $\varphi(t, \varepsilon)$ of (6) such that $|\varphi(0, \varepsilon) - z_{\alpha^*}| = \mathcal{O}(\varepsilon)$.*

Remark 1. *The functions $y_i(t, z)$ are obtained recurrently as solutions of the following integral equations*

$$(10) \quad \begin{aligned} y_1(t, z) &= \int_0^t \left(F_1(s, x(s, z, 0)) + \partial F_0(s, x(s, z, 0)) y_1(s, z) \right) ds, \\ y_i(t, z) &= i! \int_0^t \left(F_i(s, x(s, z, 0)) + \sum_{l=1}^i \sum_{S_l} \frac{1}{b_1! b_2! 2!^{b_2} \dots b_l! l!^{b_l}} \right. \\ &\quad \left. \cdot \partial^L F_{i-l}(s, x(s, z, 0)) \left(\bigodot_{j=1}^l y_j(s, z)^{b_j} \right) \right) ds, \text{ for } i = 2, \dots, k. \end{aligned}$$

See, for instance, [13, 18, 19].

For more details on the results of this section see [6].

1.4. Standard form and main result. This section is devoted to introduce the class of non-smooth differential systems of our interest as well as our main result.

First of all, we introduce the Filippov's convention for a class of non-smooth differential systems. A *piecewise smooth vector field* defined in an open bounded set $U \subset \mathbb{R}^m$ is a function $X : U \rightarrow \mathbb{R}^n$ which is smooth except on a set Σ of zero measure, called the *discontinuity set*. We assume that $U \setminus \Sigma$ is a finite union of disjoint open sets U_i , $i = 1, 2, \dots, n$, where the restriction $X_i = X|_{U_i}$ can be extended continuously to \bar{U}_i . The local orbit of X at a point $p \in U_i$ is defined as usual for a differential system. The local orbit of X at a point $p \in \Sigma$ follows the Filippov's convention [9]. In what follows, we shall state this convention for the so-called crossing points.

Assume that Σ is locally a codimension one embedded submanifold of \mathbb{R}^n around p . In this case, there exists a small neighborhood U_p and a smooth function $h : U_p \rightarrow \mathbb{R}$, for which 0 is a regular value, such that $\Sigma \cap U_p = h^{-1}(0)$. Denote $U_p^+ = \{z \in U_p : h(z) \geq 0\}$ and $U_p^- = \{z \in U_p : h(z) \leq 0\}$, and write $X^\pm = X|_{U_p^\pm}$. We say that p is a *crossing point* provided that $X^+h(p)X^-h(p) > 0$, where $X^\pm h(p) = \langle \nabla h(p), X^\pm(p) \rangle$. Denote by $\Sigma^c \subset \Sigma$ the set of crossing points, which is open in Σ . For a point $p \in \Sigma^c$, the local

orbit of X at p is given as the concatenation of the local trajectories of X^\pm at p . In this case, we say that the orbit *crosses* the discontinuity set. In this paper, we shall only deal with crossing points. See [9, 12] for the Filippov's convention of local trajectories at points in $\Sigma \setminus \Sigma^c$.

The above convention was stated for vector fields, that is, autonomous differential systems. Nevertheless, it can also be stated for non-autonomous differential systems $x' = F(t, x; \varepsilon)$ just by considering the extended autonomous differential system $(\dot{t}, \dot{x}) = \tilde{F}(t, x; \varepsilon) = (1, F(t, x; \varepsilon))$, $(t, x) \in \mathbb{R} \times \mathbb{R}^n$.

In what follows, we define the class of nonsmooth differential systems of our interest. Let $n > 1$ be a positive integer, $0 < t_1 < t_2 < \dots < t_{n-1} < T$ an ordered sequence of real numbers, and D an open subset of \mathbb{R}^m . For $i = 0, 1, \dots, k$ and $j = 1, 2, \dots, n$, let $F_i^j : \mathbb{R} \times D \rightarrow \mathbb{R}^m$ and $R^j : \mathbb{R} \times D \times (-\varepsilon_0, \varepsilon_0) \rightarrow \mathbb{R}^m$ be C^{k+1} functions T -periodic in the variable t . Define the following piecewise smooth functions

$$F_i(t, x) = \sum_{j=1}^n \chi_{[t_{j-1}, t_j]}(t) F_i^j(t, x), \quad i = 0, 1, \dots, k, \quad \text{and}$$

$$R(t, x, \varepsilon) = \sum_{j=1}^n \chi_{[t_{j-1}, t_j]}(t) R^j(t, x, \varepsilon),$$

where $\chi_A(t)$ is the characteristic function of A defined as

$$\chi_A(t) = \begin{cases} 1 & \text{if } t \in A, \\ 0 & \text{if } t \notin A. \end{cases}$$

Consider the T -periodic non-autonomous differential system

$$(11) \quad x' = F(t, x, \varepsilon) = \sum_{i=0}^k \varepsilon^i F_i(t, x) + \varepsilon^{k+1} R(t, x, \varepsilon).$$

The discontinuity set Σ is given by

$$\Sigma = (\{t = 0 \equiv T\} \cup \{t = t_1\} \cup \dots \cup \{t = t_{n-1}\}) \cap \mathbb{R} \times D,$$

Following the convention above, one can easily see that $\Sigma = \Sigma^c$. Indeed, for each $j = 1, 2, \dots, n$, and $t \in [t_{j-1}, t_j]$, system (11) writes

$$(12) \quad x' = F_j(x, t; \varepsilon) = \sum_{i=0}^k \varepsilon^i F_i^j(t, x) + \varepsilon^{k+1} R^j(t, x, \varepsilon).$$

For $(t_j, x) \in \Sigma$ we have that the connected component of Σ containing (t_j, x) is given by $\Sigma_i = h_j^{-1}(0)$, where $h_j(t, x) = t - t_j$. Considering the extended system $\tilde{F}_j(t, x; \varepsilon) = (1, F_j(t, x; \varepsilon))$, we see that $\tilde{F}_j h_j(t_j, x; \varepsilon) \tilde{F}_{j+1} h_j(t_j, x; \varepsilon) = 1 > 0$.

For each $z \in D$ and $|\varepsilon| \neq 0$ sufficiently small, we denote by $x(\cdot, z, \varepsilon) : [0, t_{(z, \varepsilon)}) \rightarrow \mathbb{R}^m$ the solution of system (11) such that $x(0, z, \varepsilon) = z$, where $[0, t_{(z, \varepsilon)})$ is the interval of

definition of $t \mapsto x(t, z, \varepsilon)$. We shall assume hypothesis (H_b) for system (11), that is, the solutions of the unperturbed system

$$(13) \quad x' = F_0(t, x),$$

starting at points of \mathcal{Z} , defined in (3), are T -periodic.

In [17], the averaging theory was developed for system (11) assuming $\dim(\mathcal{Z}) = m$. Here, we are interested in the case $\dim(\mathcal{Z}) < m$. Accordingly, we shall prove that the bifurcation functions (4) also provides sufficient conditions for the existence of periodic solutions of (11) bifurcating from \mathcal{Z} for $|\varepsilon| \neq 0$ sufficiently small.

In what follows, we shall state our main result which basically says that the simple zeros of the bifurcation functions (4) also control the bifurcation of isolated periodic solutions of the non-smooth system (11). However, in order to obtain the averaged functions (8) we need to compute the fundamental matrix $Y(t, z)$ of the variational equation

$$(14) \quad y' = \frac{\partial}{\partial x} F_0(t, x(t, z, 0))y.$$

Notice that, for each $j = 1, 2, \dots, n$, if $x_j(t, z, \varepsilon)$ denotes the solution of (12) for $t_{j-1} \leq t \leq t_j$, then function $t \mapsto (\partial x_j / \partial z)(t, z, 0)$ is a solution of (14) for $t_{j-1} \leq t \leq t_j$. Recall that the right product of a solution of the variational equation (14) by constant matrix is still a solution. Therefore, a fundamental matrix $Y(t, z)$ of (14) can be built as follows:

$$Y(t, z) = \begin{cases} Y_1(t, z) & \text{if } 0 = t_0 \leq t \leq t_1, \\ Y_2(t, z) & \text{if } t_1 \leq t \leq t_2, \\ \vdots & \\ Y_n(t, z) & \text{if } t_{n-1} \leq t \leq t_n = T, \end{cases}$$

with

$$(15) \quad \begin{aligned} Y_1(t, z) &= \frac{\partial x_1}{\partial z}(t, z, 0), \quad \text{and} \\ Y_j(t, z) &= \frac{\partial x_j}{\partial z}(t, z, 0) \left(\frac{\partial x_j}{\partial z}(t_{j-1}, z, 0) \right)^{-1} Y_{j-1}(t_{j-1}, z), \quad \text{for } j = 2, 3, \dots, n. \end{aligned}$$

Notice that $Y(0, z) = Id$.

Now, we are ready to state our main result.

Theorem A. *Let Δ_α denote the lower right corner $(m-d) \times (m-d)$ matrix of the matrix $Id - Y^{-1}(T, z_\alpha)$. In addition to hypothesis (H_b) , we assume that*

- (i) for each $\alpha \in \bar{V}$, $\det \Delta_\alpha \neq 0$,
- (ii) $f_1 = f_2 = \dots = f_{k-1} = 0$ and f_k is not identically zero.

If there exists $\alpha^ \in V$ such that $f_k(\alpha^*) = 0$ and $\det(Df_k(\alpha^*)) \neq 0$, then there exists a T -periodic solution $\varphi(t, \varepsilon)$ of (11) such that $|\varphi(0, \varepsilon) - z_{\alpha^*}| = \mathcal{O}(\varepsilon)$.*

Remark 2. *The derivatives $\partial^j F_i(s, z)$, which appears in (4), are computed as follows:*

$$\frac{\partial^j F_i}{\partial z}(s, z) = \sum_{j=1}^n \chi_{[t_{j-1}, t_j]}(s) \frac{\partial^j F_i^j}{\partial z^j}(s, z).$$

This paper is organized as follows. In Section 2, we present a wide class of non-smooth differential systems for which Theorem A can be applied. In Section 3, we present the explicit formulae (8) for the average functions of the non-smooth differential system (11). Finally, in Section 4 we prove Theorem A and in Section 5 we provide two applications of Theorem A.

2. ANDRONOV-HOPF EQUILIBRIA

In this section, we provide a class of non-smooth vector fields that can be studied in light of the previous result, namely, non-smooth perturbations of Andronov-Hopf equilibria.

An Andronov-Hopf equilibrium is characterized by a singularity admitting one pair of purely imaginary conjugate eigenvalues such that any other eigenvalue is real and nonvanishing. The Andronov-Hopf bifurcation appears in the study of many differential models describing real phenomena, some of them are non-smooth (see, for instance, [1, 7, 21, 24, 25, 26] and the references quoted therein).

In what follows, we describe this class of non-smooth differential system. Let $D = D_1 \times D_2$ be a subset of \mathbb{R}^{2+m} , where $D_1 \subset \mathbb{R}^2$ and $D_2 \subset \mathbb{R}^m$. Take $n > 1$ a positive integer, $\alpha_0 = 0$, $\alpha_n = 2\pi$ and $\alpha = (\alpha_1, \dots, \alpha_{n-1}) \in \mathbb{T}^{n-1}$ a $(n-1)$ -tuple of angles such that $0 = \alpha_0 < \alpha_1 < \alpha_2 < \dots < \alpha_{n-1} < \alpha_n = 2\pi$. Consider $\mathcal{X}(u, v, w; \varepsilon) = (X_1, X_2, \dots, X_n)$ a n -tuple of smooth vector fields defined on an open bounded neighborhood $D \subset \mathbb{R}^{2+m}$ of the origin and depending on a small parameter ε . For each $j = 1, 2, \dots, n$, let C_j be a subset of D such that the intersection of C_j with the plane $w = 0$ is a sector in the plane uv that is delimited by rays starting at the origin and passing through point $(\cos \alpha_j, \sin \alpha_j)$.

Now let $Z_{\mathcal{X}, \alpha} : D \rightarrow \mathbb{R}^{2+m}$ be a piecewise smooth vector field defined as $Z_{\mathcal{X}, \alpha}(u, v, w; \varepsilon) = X_j(u, v, w; \varepsilon)$ when $(u, v, w) \in C_j$, and consider the following discontinuous piecewise smooth differential system

$$(16) \quad (\dot{u}, \dot{v}, \dot{w})^T = Z_{\mathcal{X}, \alpha}(u, v, w; \varepsilon).$$

The above notation means that at each sector C_j we are considering a smooth differential system of the form

$$(17) \quad (\dot{u}, \dot{v}, \dot{w})^T = X_j(u, v, w; \varepsilon) = X_0(u, v, w) + \sum_{i=1}^k \varepsilon^i X_i^j(u, v, w) + \varepsilon^{k+1} H^j(u, v, w, \varepsilon),$$

where $k > 1$ is an integer.

As our main hypothesis regarding system (16), we assume that the origin is an equilibrium of $Z_{\mathcal{X},\alpha}(u, v, w; \varepsilon)$ and

$$Z_{\mathcal{X},\alpha}(u, v, w; 0) = X_0(u, v, w) = \left(-v + P(u, v, z), u + Q(u, v, z), \lambda w + R(u, v, w) \right),$$

where P, Q, R vanish at the origin and $\lambda = (\tilde{\lambda}_1, \dots, \tilde{\lambda}_m)^T$ is such that $\tilde{\lambda}_s$ is negative, for $s = 1, 2, \dots, m$. Notice that we are assuming that the *unperturbed system* $Z_{\mathcal{X},\alpha}(u, v, w; 0)$ is smooth and admits an Andronov-Hopf equilibrium at the origin.

Now, changing $u \rightarrow \varepsilon U, v \rightarrow \varepsilon V$ and $w \rightarrow \varepsilon W$ we get, for each $j = 1, \dots, n$, a differential system of the form

$$(18) \quad \begin{cases} \dot{U} = -V + \sum_{i=1}^k \varepsilon^i \mathcal{G}_{1,i}(U, V, W) + \varepsilon^{k+1} \mathcal{R}_1(U, V, W, \varepsilon), \\ \dot{V} = U + \sum_{i=1}^k \varepsilon^i \mathcal{G}_{2,i}(U, V, W) + \varepsilon^{k+1} \mathcal{R}_2(U, V, W, \varepsilon), \\ \dot{W} = \lambda W + \sum_{i=1}^k \varepsilon^i \mathcal{G}_{3,i}(U, V, W) + \varepsilon^{k+1} \mathcal{R}_3(U, V, W, \varepsilon), \end{cases}$$

In order to use Theorem A for studying system (16), it has to be written in its standard form. Doing the cylindrical change of variables $U = r \cos \theta, V = r \sin \theta, W = w$ and assuming that each X_j has the form (18), system (17) becomes

$$\begin{aligned} r'(\theta) &= \frac{\dot{r}(t)}{\dot{\theta}(t)} = \sum_{i=0}^k \varepsilon^i F_{1,i}^j(\theta, r, w) + \varepsilon^{k+1} R_1^j(\theta, r, w, \varepsilon), \\ w'(\theta) &= \frac{\dot{w}(t)}{\dot{\theta}(t)} = \sum_{i=0}^k \varepsilon^i F_{2,i}^j(\theta, r, w) + \varepsilon^{k+1} R_2^j(\theta, r, w, \varepsilon). \end{aligned}$$

Now for $\theta \in [\alpha_{j-1}, \alpha_j]$, $F_{i,1}^j, F_{i,2}^j : \mathbb{S}^1 \times D \rightarrow \mathbb{R}^{m+1}$ and $R_1^j, R_2^j : \mathbb{S}^1 \times D \times (-\varepsilon_0, \varepsilon_0) \rightarrow \mathbb{R}^{m+1}$ are C^{k+1} functions, depending on the vector fields X_i^j , and 2π -periodic in the first variable, with D being an open bounded set and $\mathbb{S}^1 \equiv \mathbb{R}/(2\pi\mathbb{Z})$. Denoting

$$\begin{aligned} F_{1,i}(\theta, r, w) &= \sum_{j=1}^n \chi_{[\alpha_{j-1}, \alpha_j]}(\theta) F_{1,i}^j(\theta, r, w), \quad i = 0, 1, \dots, k, \\ F_{2,i}(\theta, r, w) &= \sum_{j=1}^n \chi_{[\alpha_{j-1}, \alpha_j]}(\theta) F_{2,i}^j(\theta, r, w), \quad i = 0, 1, \dots, k, \\ R_1(\theta, r, w, \varepsilon) &= \sum_{j=1}^n \chi_{[\alpha_{j-1}, \alpha_j]}(\theta) R_1^j(\theta, r, w, \varepsilon), \quad \text{and} \\ R_2(\theta, r, w, \varepsilon) &= \sum_{j=1}^n \chi_{[\alpha_{j-1}, \alpha_j]}(\theta) R_2^j(\theta, r, w, \varepsilon), \end{aligned}$$

system (16) writes

$$\begin{aligned} r'(\theta) &= \sum_{i=0}^k \varepsilon^i F_{1,i}(\theta, r, w) + \varepsilon^{k+1} R_1(\theta, r, w, \varepsilon), \\ w'(\theta) &= \sum_{i=0}^k \varepsilon^i F_{2,i}(\theta, r, w) + \varepsilon^{k+1} R_2(\theta, r, w, \varepsilon). \end{aligned}$$

Notice that the unperturbed system is $(r'(\theta), w'(\theta)) = (0, \lambda w)$, so its solution for an initial condition $(r_0, w_0^1, \dots, w_0^m)$ is provided by

$$(r_0, w_0^1 e^{\lambda t}, \dots, w_0^m e^{\lambda t}).$$

Considering the set $\mathcal{Z} = \{(r, 0, \dots, 0); r > 0\} \subset \mathbb{R}^{1+m}$, the solution of the unperturbed system is 2π -periodic for every initial condition $(r_0, w_0^1, \dots, w_0^m) \in \mathcal{Z}$. Moreover, fundamental matrix Y is such that $Id - Y^{-1}$ satisfy that $\det(\Delta_r) \neq 0$ for points in \mathcal{Z} . Hence, system (16) satisfies all hypotheses of Theorem A and, then, we can use our main result to study isolated periodic solutions for systems like (16).

3. AN ALGORITHM FOR THE BIFURCATION FUNCTIONS

In this section, we shall provide an algorithm for computing the bifurcation functions (4) for the non-smooth case. Their expressions are defined recurrently and uses *Bell polynomials*, instead of the set S_l , which are already implemented in computer algebra systems such as Mathematica and Maple. In this way, the computation of the higher order bifurcation functions is significantly simplified. In [22], it was proven that the average functions defined for smooth cases can be computed using *Bell polynomials*. In [17], the authors did the same for the non-smooth case.

For each pair of nonnegative integers (p, q) , the partial Bell polynomial is defined as

$$B_{p,q}(x_1, x_2, \dots, x_{p-q+1}) = \sum_{\tilde{S}_{p,q}} \frac{p!}{b_1! b_2! \dots b_{p-q+1}!} \prod_{j=1}^{p-q+1} \left(\frac{x_j}{j!} \right)^{b_j},$$

where $\tilde{S}_{p,q}$ is the set of all $(p - q + 1)$ -tuple of nonnegative integers $(b_1, b_2, \dots, b_{p-q+1})$ satisfying $b_1 + 2b_2 + \dots + (p - q + 1)b_{p-q+1} = p$, and $b_1 + b_2 + \dots + b_{p-q+1} = q$. Moreover, if g and h are sufficiently smooth functions, Faà di Bruno's formula for the L -th derivative of a composite function gives

$$\frac{d^l}{dx^l} g(h(x)) = \sum_{m=1}^l g^{(m)}(h(x)) B_{l,m}(h'(x), h''(x), \dots, h^{(l-m+1)}(x)).$$

3.1. Average Functions. In this subsection, we develop a recurrence to compute the average function (8) in the particular case of the non-smooth differential equation (11).

So, consider the functions $w_i^j : (t_{j-1}, t_j] \times D \rightarrow \mathbb{R}^m$ defined recurrently for $i = 1, 2, \dots, k$ and $j = 1, 2, \dots, n$, as

(19)

$$\begin{aligned} w_1^1(t, z) &= \int_0^t \left(F_1^1(s, x(s, z, 0)) + \partial F_0^1(s, x(s, z, 0)) w_1^1(s, z) \right) ds, \\ w_i^1(t, z) &= i! \int_0^t \left(F_i^1(s, x(s, z, 0)) + \right. \\ &\quad \left. \sum_{l=1}^i \sum_{S_l} \frac{1}{b_1! b_2! 2!^{b_2} \dots b_l! l!^{b_l}} \cdot \partial^L F_{i-l}^1(s, x(s, z, 0)) \bigodot_{m=1}^l w_m^1(s, z)^{b_m} \right) ds, \\ w_i^j(t, z) &= w_i^{j-1}(t_{j-1}, z) + i! \int_{t_{j-1}}^t \left(F_i^j(s, x(s, z, 0)) + \right. \\ &\quad \left. \sum_{l=1}^i \sum_{S_l} \frac{1}{b_1! b_2! 2!^{b_2} \dots b_l! l!^{b_l}} \cdot \partial^L F_{i-l}^j(s, x(s, z, 0)) \bigodot_{m=1}^l w_m^j(s, z)^{b_m} \right) ds. \end{aligned}$$

Since $F_0 \neq 0$ the recurrence defined in (19) is an integral equation and the next lemma solves it using Bell polynomials.

Lemma 3. *For $i = 1, 2, \dots, k$ and $j = 1, 2, \dots, n$ the recurrence (19) can be written as follows*

$$\begin{aligned} w_1^1(t, z) &= Y_1(t, z) \int_0^t Y_1^{-1}(s, z) F_1^1(s, x(s, z, 0)) ds, \\ w_1^j(t, z) &= Y_j(t, z) \left(Y_j^{-1}(t_{j-1}, z) w_1^{j-1}(t_{j-1}, z) + \int_{t_{j-1}}^t Y_j^{-1}(s, z) F_1^j(s, x(s, z, 0)) ds \right), \\ w_i^1(t, z) &= Y_1(t, z) \int_0^t Y_1^{-1}(s, z) \left(i! F_i^1(s, x(s, z, 0)) \right. \\ &\quad \left. + \sum_{m=2}^i \partial^m F_0^1(s, x(s, z, 0)) \cdot B_{i,m}(w_1^1, \dots, w_{i-m+1}^1), \right. \\ &\quad \left. + \sum_{l=1}^{i-1} \sum_{m=1}^l \frac{i!}{l!} \partial^m F_{i-l}^1(s, x(s, z, 0)) \cdot B_{l,m}(w_1^1, \dots, w_{l-m+1}^1) \right) ds, \end{aligned}$$

$$\begin{aligned}
w_i^j(t, z) = & Y_j(t, z) \left[Y_j^{-1}(t_{j-1}, z) w_i^{j-1}(t_{j-1}, z) + \int_{t_{j-1}}^t Y_j^{-1}(s, z) \left(i! F_i^j(s, x(s, z, 0)) \right. \right. \\
& + \sum_{m=2}^i \partial^m F_0^j(s, x(s, z, 0)) \cdot B_{i,m}(w_1^j, \dots, w_{i-m+1}^j), \\
& \left. \left. + \sum_{l=1}^{i-1} \sum_{m=1}^l \frac{i!}{l!} \partial^m F_{i-l}^j(s, x(s, z, 0)) \cdot B_{l,m}(w_1^j, \dots, w_{l-m+1}^j) \right) ds \right]
\end{aligned}$$

Proof. The idea of the proof is to relate the integral equations (19) to the Cauchy problem and then solve it. For example, if $i = j = 1$ the integral equation is equivalent to the following Cauchy problem

$$\frac{\partial w_1^1}{\partial t}(t, z) = F_1^1(t, x(t, z, 0)) + \partial F_0^1(t, x(t, z, 0)) w_1^1 \quad \text{with } w_1^1(0, z) = 0,$$

and solving this linear differential equation we get the expression of $w_1^1(t, z)$ described in the statement of the lemma. For more details see [17]. \square

The next provides a formula for the average functions (8) for the class of non-smooth differential systems studied in this paper.

Proposition 4. *For $i = 1, 2, \dots, k$, the average function (8) of order i is*

$$g_i(z) = Y_n^{-1}(T, z) \frac{w_i^n(T, z)}{i!}.$$

Proof. For each $i = 1, 2, \dots, k$ we define

$$w_i(t, z) = \sum_{j=1}^n \chi_{[t_{j-1}, t_j]}(t) w_i^j(t, z).$$

Given $t \in [0, T]$ there exists a positive integer \bar{k} such that $t \in (t_{\bar{k}-1}, t_{\bar{k}}]$ and, therefore, $w_i(t, z) = w_i^{\bar{k}}(t, z)$. By applying the proof of Proposition 2 of [17], we obtain (20)

$$\begin{aligned}
w_1(t, z) = & \int_0^t \left(F_1(s, x(s, z, 0)) + \partial F_0(s, x(s, z, 0)) w_1(s, z) \right) ds, \\
w_i(t, z) = & i! \int_0^\theta \left(F_i(s, x(s, z, 0)) + \right. \\
& \left. \sum_{l=1}^i \sum_{S_l} \frac{1}{b_1! b_2! 2!^{b_2} \dots b_l! l!^{b_l}} \cdot \partial^L F_{i-l}(s, x(s, z, 0)) \bigodot_{m=1}^l w_m(s, z)^{b_m} \right) ds.
\end{aligned}$$

Since according to Remark 1 we can consider functions (9) as implicitly provided, we compute the derivatives in the variable t of functions (20) and (10) for $i = 1$, and we see that the functions $w_1(t, z)$ and $y_1(t, z)$ satisfy the same differential equation.

Moreover, for each $i = 2, \dots, k$, the integral equations (10) and (20), which provide respectively y_i and w_i , are defined by the same recurrence. Then, functions y_i and w_i satisfy the same differential equations for $i = 1, 2, \dots, k$, and their initial conditions coincide. Indeed, let $i \in \{1, 2, \dots, k\}$, since $y_i(0, z) = 0$ and, by (20), $w_i(0, z) = 0$, it follows that the initial conditions are the same. Applying the Existence and Uniqueness Theorem on the solutions of the differential system we get $y_i(t, z) = w_i(t, z)$, for all $i \in \{1, 2, \dots, k\}$. \square

3.2. Bifurcation Functions. In this subsection, we shall write the bifurcation functions (4) and the functions $\gamma_i(\alpha)$ provided by (5) in terms of Bell polynomials.

Claim 1. *The bifurcation function (4) is provided by*

$$f_i(\alpha) = \pi g_i(z_\alpha) + \sum_{l=1}^i \sum_{m=1}^l \frac{1}{l!} \partial_b^m \pi g_{i-l}(z_\alpha) B_{l,m}(\gamma_1(\alpha), \dots, \gamma_{l-m+1}(\alpha)),$$

where

$$\begin{aligned} \gamma_1(\alpha) &= -\Delta_\alpha^{-1} \pi^\perp g_1(z_\alpha) \quad \text{and} \\ \gamma_i(\alpha) &= -\Delta_\alpha^{-1} \left(\sum_{l=0}^{i-1} \frac{i!}{l!} \sum_{m=1}^l \partial_b^m \pi^\perp g_{i-l}(z_\alpha) B_{l,m}(\gamma_1(\alpha), \dots, \gamma_{l-m+1}(\alpha)) \right. \\ &\quad \left. + \sum_{m=2}^i \partial_b^m \pi^\perp g_0(z_\alpha) B_{i,m}(\gamma_1(\alpha), \dots, \gamma_{i-m+1}(\alpha)) \right). \end{aligned}$$

Proof. Expression (4) was obtained in [6] by using the Faá di Bruno's formula for the L -th derivative of a composite function. This claim follows by applying the version of Faá di Bruno's formula in terms of Bell polynomials (see [6, 22]). \square

4. PROOF OF THEOREM A

For $j = 1, 2, \dots, n$ let $\xi_j(t, t_{j-1}, z_0, \varepsilon)$ be the solution of the differential system (12) such that $\xi_j(t_{j-1}, t_{j-1}, z_0, \varepsilon) = z_0$. Then, we define the recurrence

$$\begin{aligned} x_1(t, z, \varepsilon) &= \xi_1(t, 0, z, \varepsilon) \\ x_j(t, z, \varepsilon) &= \xi_j(t, t_{j-1}, x_{j-1}(t_{j-1}, z, \varepsilon), \varepsilon), \quad j = 2, \dots, n. \end{aligned}$$

Since we are working in the cross region, it is easy to see that, for $|\varepsilon| \neq 0$ sufficiently small, each $x_j(t, z, \varepsilon)$ is defined for every $t \in [t_{j-1}, t_j]$. Therefore $x(\cdot, z, \varepsilon) : [0, T] \rightarrow \mathbb{R}^m$

is defined as

$$x(t, z, \varepsilon) = \begin{cases} x_1(t, z, \varepsilon) & \text{if } 0 = t_0 \leq t \leq t_1, \\ x_2(t, z, \varepsilon) & \text{if } t_1 \leq t \leq t_2, \\ \vdots & \\ x_j(t, z, \varepsilon) & \text{if } t_{j-1} \leq t \leq t_j, \\ \vdots & \\ x_n(t, z, \varepsilon) & \text{if } t_{n-1} \leq t \leq t_n = T. \end{cases}$$

Notice that $x(t, z, \varepsilon)$ is the solution of the differential equation (13) such that $x(0, z, \varepsilon) = z$. Moreover, the following equality hold

$$x_j(t_{j-1}, z, \varepsilon) = x_{j-1}(t_{j-1}, z, \varepsilon),$$

for $j = 1, 2, \dots, n$.

The next lemma expands the solution $x_j(\cdot, z, \varepsilon)$ around $\varepsilon = 0$.

Lemma 5. *For $j \in \{1, 2, \dots, n\}$ and $t_z^j > t_j$, let $x_j(\cdot, z, \varepsilon) : [t_{j-1}, t_j]$ be the solution of (12). Then*

$$x_j(t, z, \varepsilon) = x_j(t, z, 0) + \sum_{i=1}^k \frac{\varepsilon^i}{i!} w_i^j(t, z) + \mathcal{O}(\varepsilon^{k+1}).$$

Proof. First, having fixed $j \in \{1, 2, \dots, n\}$, we use the continuity of solution $x_j(t, z, \varepsilon)$ and the compactness of set $[t_{j-1}, t_j] \times \overline{D} \times [-\varepsilon_0, \varepsilon_0]$ to arrive at

$$\int_{t_{j-1}}^t R^j(s, x_j(s, z, \varepsilon), \varepsilon) ds = \mathcal{O}(\varepsilon), \quad t \in [t_{j-1}, t_j].$$

Thus, integrating the differential equation (12) from t_{j-1} to t , we get

$$(21) \quad \begin{aligned} x_j(t, z, \varepsilon) &= x_j(t_{j-1}, z, \varepsilon) + \sum_{i=0}^k \varepsilon^i \int_{t_{j-1}}^t F_i^j(s, x_j(s, z, \varepsilon)) ds + \mathcal{O}(\varepsilon^{k+1}), \quad \text{and} \\ x_j(t, z, 0) &= x_j(t_{j-1}, z, 0) + \int_{t_{j-1}}^t F_0^j(s, x_j(s, z, 0)) ds. \end{aligned}$$

Because of the differentiable dependence of the solutions of a differential system on its parameters, the function $\varepsilon \mapsto x_j(t, z, \varepsilon)$ is a C^{k+1} map. The next step is to compute the Taylor expansion of $F_i^j(t, x_j(t, z, \varepsilon))$ around $\varepsilon = 0$, and for this we use the Faá di Bruno's Formula about the l -th derivative of a composite function, which guarantees that, if g and h are sufficiently smooth functions, then

$$\frac{d^l}{d\alpha^l} g(h(\alpha)) = \sum_{S_l} \frac{l!}{b_1! b_2! 2!^{b_2} \dots b_l! l!^{b_l}} g^{(L)}(h(\alpha)) \bigcirc_{j=1}^l \left(h^{(j)}(\alpha) \right)^{b_j},$$

where S_l is the set of all l -tuples of non-negative integers (b_1, b_2, \dots, b_l) satisfying $b_1 + 2b_2 + \dots + lb_l = l$, and $L = b_1 + b_2 + \dots + b_l$.

For each $i = 0, 1, \dots, k-1$, expanding $F_i^j(s, x_j(s, z, \varepsilon))$ around $\varepsilon = 0$ we get
(22)

$$F_i^j(s, x_j(s, z, \varepsilon)) = F_i^j(s, x_j(s, z, 0)) + \sum_{l=1}^{k-i} \sum_{S_l} \frac{\varepsilon^l}{b_1! b_2! 2!^{b_2} \dots b_l! l!^{b_l}} \partial^L F_i^j(s, x_j(s, z, 0)) \bigodot_{m=1}^l r_m^j(s, z)^{b_m},$$

where

$$r_m^j(s, z) = \left. \frac{\partial^m}{\partial \varepsilon^m} x_j(s, z, \varepsilon) \right|_{\varepsilon=0},$$

and for $i = k$

$$(23) \quad F_k^j(s, x_j(s, z, \varepsilon)) = F_k^j(s, x_j(s, z, 0)) + \mathcal{O}(\varepsilon).$$

Substituting (22) and (23) in (21) we get

$$\begin{aligned} x_j(t, z, \varepsilon) = & x_j(t_{j-1}, z, \varepsilon) + \int_{t_{j-1}}^t \left(\sum_{i=0}^k \varepsilon^i F_i^j(s, x_j(s, z, 0)) ds \right. \\ & + \sum_{i=0}^{k-1} \sum_{l=1}^{k-i} \varepsilon^{l+i} \sum_{S_l} \frac{1}{b_1! b_2! 2!^{b_2} \dots b_l! l!^{b_l}} \\ & \left. \cdot \partial^L F_i^j(s, x_j(s, z, 0)) \bigodot_{m=1}^l r_m^j(s, z)^{b_m} \right) ds + \mathcal{O}(\varepsilon^{k+1}). \end{aligned}$$

Then, the proof of the lemma ends using the next two claims.

Claim 2. For $j = 1, 2, \dots, n$ we have

$$x_j(t, z, \varepsilon) = x_j(t, z, 0) + \sum_{i=1}^k \frac{\varepsilon^i}{i!} r_i^j(t, z) + \mathcal{O}(\varepsilon^{k+1}).$$

Claim 3. The equality $r_i^j = w_i^j$ holds for $i = 1, 2, \dots, k$ and $j = 1, 2, \dots, n$.

The proof of Claims 2 and 3 is achievable by following the steps described in the proof of Claims 1 and 2 of [17], respectively. \square

Proof of Theorem A. Consider the displacement function

$$(24) \quad h(z, \varepsilon) = x(T, z, \varepsilon) - z = x_n(T, z, \varepsilon) - z$$

It is easy to see that $x(\cdot, \bar{z}, \bar{\varepsilon})$ is a T -periodic solution if and only if $h(\bar{z}, \bar{\varepsilon}) = 0$. Moreover, to study the zeros of (24) is equivalent to study the zeros of

$$(25) \quad g(z, \varepsilon) = Y_n^{-1}(T, z)h(z, \varepsilon).$$

Based on Lemma 5, we have

$$(26) \quad x_n(T, z, \varepsilon) = x_n(T, z, 0) + \sum_{i=1}^k \frac{\varepsilon^i}{i!} w_i^n(T, z) + \mathcal{O}(\varepsilon^{k+1}),$$

for all $(t, z) \in \mathbb{S}^1 \times D$. By replacing (26) in (25), it follows that

$$\begin{aligned}
(27) \quad g(z, \varepsilon) &= Y_n^{-1}(T, z) \left(x_n(T, z, 0) - z + \sum_{i=1}^k \frac{\varepsilon^i}{i!} w_i^n(T, z) + \mathcal{O}(\varepsilon^{k+1}) \right) \\
&= Y_n^{-1}(T, z)(x_n(T, z, 0) - z) + \sum_{i=1}^k g_i(z) + \mathcal{O}(\varepsilon^{k+1}) \\
&= \sum_{i=0}^k g_i(z) + \mathcal{O}(\varepsilon^{k+1}),
\end{aligned}$$

where $g_0(z) = Y_n^{-1}(T, z)(x_n(T, z, 0) - z)$.

From hypothesis (H_a) the function $g_0(z)$ vanishes on the submanifold \mathcal{Z} , therefore hypothesis (H_a) holds for function (27). In order to take the derivative of $g_0(z)$ with respect to the variable z , we have the next claim.

Claim 4. For every $j \in \{1, 2, \dots, n\}$

$$Y_j(t_j, z) = \frac{\partial x_j}{\partial z}(t_j, z, 0).$$

The proof will be done by induction on j . For $j = 1$ the claim is precisely the definition. Assume that the claim is valid for $j = j_0 - 1$ and we shall prove it for $j = j_0$. Since $x_j(t_{j-1}, z, \varepsilon) = x_{j-1}(t_{j-1}, z, \varepsilon)$ for all $j = 1, 2, \dots, n$ we have

$$\begin{aligned}
Y_{j_0}(t_{j_0}, z) &= \frac{\partial x_{j_0}}{\partial z}(t_{j_0}, z, 0) \left(\frac{\partial x_{j_0}}{\partial z}(t_{j_0-1}, z, 0) \right)^{-1} Y_{j_0-1}(t_{j_0-1}, z) \\
&= \frac{\partial x_{j_0}}{\partial z}(t_{j_0}, z, 0) \left(\frac{\partial x_{j_0-1}}{\partial z}(t_{j_0-1}, z, 0) \right)^{-1} \frac{\partial x_{j_0-1}}{\partial z}(t_{j_0-1}, z, 0) \\
&= \frac{\partial x_{j_0}}{\partial z}(t_{j_0}, z, 0).
\end{aligned}$$

Hence, if $z \in \mathcal{Z}$ then

$$\begin{aligned}
\frac{\partial g_0}{\partial z}(z) &= Y^{-1}(T, z) \left(\frac{\partial x}{\partial z}(T, z, 0) - Id \right) \\
&= Y^{-1}(T, z)(Y(T, z) - Id) \\
&= Id - Y^{-1}(T, z),
\end{aligned}$$

which by assumption has as its lower right corner $(m-d) \times (m-d)$ matrix Δ_α non-singular. From here, the result follows from Proposition 4 and Theorem 1. \square

5. EXAMPLES

This section is devoted to presenting some applications of Theorem A. The first one is a 3D piecewise smooth system for which the plane $y = 0$ is the switching manifold and admits a surface $z = f(x, y)$ foliated by periodic solutions. The second one is a 3D

piecewise smooth system for which the algebraic variety $xy = 0$ is the discontinuity set and the plane $z = 0$ has a piecewise constant center. For these systems, we compute some of the bifurcation functions in order to study the persistence of periodic solutions.

5.1. non-smooth perturbation of a 3D system. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ and $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ be differential functions such that $g(x, y) = f(x, y) + x\partial_y f(x, y) - y\partial_x f(x, y)$. Consider the non-smooth vector field

$$(28) \quad X_\varepsilon(x, y, z) = \begin{cases} X_\varepsilon^+(x, y, z), & y > 0 \\ X_\varepsilon^-(x, y, z), & y < 0 \end{cases}$$

where

$$\begin{aligned} X_\varepsilon^+(x, y, z) &= (-y + \varepsilon(a_0 + a_1z) + \varepsilon^2(a_2 + a_3z), \quad x, \quad -z + g(x, y)), \quad \text{and} \\ X_\varepsilon^-(x, y, z) &= (-y, \quad x + \varepsilon b_1z + \varepsilon^2(b_2 + b_3)z, \quad -z + g(x, y)), \end{aligned}$$

with $a_0, a_1, a_2, b_1, b_2, b_3 \in \mathbb{R}$. Denote the discontinuity set by $\Sigma = \{(x, y, z) \in \mathbb{R}^3 : y = 0\}$.

Notice that surface $z = g(x, y)$ is an invariant set of the unperturbed vector field X_0 . Indeed, considering function $\hat{f}(x, y, z) = z - f(x, y)$, we get

$$\langle \nabla \hat{f}(x, y, z), X_0(x, y, z) \rangle \Big|_{z=f(x, y)} = 0.$$

Moreover, since $X_0(x, y, f(x, y)) = (-y, x, x\partial_y f(x, y) - y\partial_x f(x, y))$, we conclude that the invariant set $z = f(x, y)$ is foliate by periodic solutions.

Next result gives sufficient conditions in order to guarantee the persistence of a periodic solution. Consider the function

$$(29) \quad f_1(r) = a_1 \int_0^\pi f(r \cos \phi, r \sin \phi) \cos \phi d\phi + b_1 \int_\pi^{2\pi} f(r \cos \phi, r \sin \phi) \sin \phi d\phi.$$

Theorem 6. *Consider the piecewise vector field (28). Then, for each $r^* > 0$, such that $f_1(r^*) = 0$ and $f_1'(r^*) \neq 0$, there exists a crossing limit cycle $\varphi(t, \varepsilon)$ of X of period $T_\varepsilon = 2\pi + \mathcal{O}(\varepsilon)$ such that $\varphi(t, \varepsilon) = (x^*, y^*, f(x^*, y^*)) + \mathcal{O}(\varepsilon)$ with $|(x^*, z^*)| = r^*$.*

In order to apply Theorem A for proving Theorem 6, we need to write system (28) in its standard form. Considering cylindrical coordinates $x = r \cos \theta$, $y = r \sin \theta$, $z = z$, the set of discontinuity becomes $\Sigma = \{\theta = 0\} \cup \{\theta = t_1\}$ with $t_0 = 0, t_1 = \pi$ and $t_2 = 2\pi$. The differential system $(\dot{x}, \dot{y}, \dot{z}) = X_\varepsilon^+(x, y, z)$ in cylindrical coordinates writes

$$\begin{aligned} r'(t) &= \varepsilon(a_0 + a_1z) \cos \theta + \varepsilon^2(a_2 + a_3z) \cos \theta, \\ z'(t) &= g(r \cos \theta, r \sin \theta) - z, \\ \theta'(t) &= 1 - \varepsilon \frac{(a_0 + a_1z) \sin \theta}{r} - \varepsilon^2 \frac{(a_2 + a_3z) \sin \theta}{r}, \end{aligned}$$

and the differential system $(\dot{x}, \dot{y}, \dot{z}) = X_\varepsilon^-(x, y, z)$ becomes

$$(30) \quad \begin{aligned} r'(t) &= \varepsilon b_1 z \sin \theta + \varepsilon^2 (b_2 + b_3 z) \sin \theta, \\ z'(t) &= g(r \cos \theta, r \sin \theta) - z, \\ \theta'(t) &= 1 + \varepsilon \frac{b_1 z \cos \theta}{r} + \varepsilon^2 \frac{(a_2 + a_3 z) \cos \theta}{r}. \end{aligned}$$

Notice that, for each $j = 1, 2$ and $t_{j-1} \leq \theta \leq t_j$, we have $\dot{\theta}(t) \neq 0$ for $|\varepsilon| \neq 0$ sufficiently small. Thus, in a sufficiently small neighborhood of the origin we can take θ as the new independent time variable. Accordingly, system (30) becomes

$$\begin{aligned} \dot{r}(\theta) &= \frac{r'(t)}{\theta'(t)} = F_{01}(\theta, r, z) + \varepsilon F_{11}(\theta, r, z) + \varepsilon^2 F_{21}(\theta, r, z) + \mathcal{O}_1(\varepsilon^3), \\ \dot{z}(\theta) &= \frac{z'(t)}{\theta'(t)} = F_{02}(\theta, r, z) + \varepsilon F_{12}(\theta, r, z) + \varepsilon^2 F_{22}(\theta, r, z) + \mathcal{O}_2(\varepsilon^3). \end{aligned}$$

Considering the notation of Theorem A we have $F_i(\theta, r, z) = (F_{i1}(\theta, r, z), F_{i2}(\theta, r, z))$ for each $i \in \{1, 2\}$. Moreover, for each $i \in \{1, 2\}$ the function $F_i(\theta, r, z)$ is written in the form $F_i(\theta, r, z) = \sum_{j=1}^2 \chi_{[t_{j-1}, t_j]}(\theta) F_i^j(\theta, r, z)$.

Defining $\tilde{f}(\theta, r) = f(r \cos \theta, r \sin \theta)$ and $\tilde{g}(\theta, r) = g(r \cos \theta, r \sin \theta)$ we write explicitly the expressions of F_0, F_1^j and F_2^j for $j \in \{1, 2\}$,

$$\begin{aligned} F_0(\theta, r, z) &= (0, \tilde{g}(\theta, r) - z), \\ F_1^1(\theta, r, z) &= \left((a_0 + a_1 z) \cos \theta, \frac{(a_0 + a_1 z) \sin \theta}{r} (\tilde{g}(\theta, r) - z) \right), \\ F_1^2(\theta, r, z) &= \left(b_1 z \sin \theta, -\frac{b_1 z \cos \theta}{r} (\tilde{g}(\theta, r) - z) \right), \\ F_2^1(\theta, r, z) &= \left((a_2 + a_3 z) \cos \theta + \frac{(a_0 + a_1 z)^2 \sin \theta \cos \theta}{r}, \frac{\sin \theta}{r^2} ((a_0 + a_1 z)^2 \sin \theta \right. \\ &\quad \left. + (a_2 + a_3 z)r) (\tilde{g}(\theta, r) - z) \right), \\ F_2^2(\theta, r, z) &= \left((b_2 + b_3 z) \sin \theta - \frac{b_1^2 z^2 \sin \theta \cos \theta}{r}, \frac{\cos \theta}{r^2} (b_1^2 z \cos \theta - (b_2 + b_3 z)r) (\tilde{g}(\theta, r) - z) \right). \end{aligned}$$

The unperturbed system is smooth and its solution $(r(\theta, r_0, z_0), z(\theta, r_0, z_0))$ with initial condition (r_0, z_0) is provided by

$$(31) \quad r(\theta) = \bar{r}(\theta, r_0, z_0) = r_0, \quad z(\theta) = \bar{z}(\theta, r_0, z_0) = e^{-\theta} \left(z_0 + \int_0^\theta e^s \tilde{g}(s, r_0) ds \right).$$

Consequently, a fundamental matrix solution of (14) is provided by

$$Y(\theta, r_0, z_0) = \frac{\partial(\bar{r}, \bar{z})}{\partial(r_0, z_0)}(\theta, r_0, z_0) = \begin{pmatrix} 1 & 0 \\ G(\theta, r_0) & e^{-\theta} \end{pmatrix},$$

where $G(\theta, r_0)$ is the derivative of $\bar{z}(\theta, r_0, z_0)$ with respect to the variable r_0 . Notice that, from (31), $G(\theta, r_0)$ does not depend on z_0 .

Let $\varepsilon_0 > 0$ be a real positive number and consider the set $\mathcal{Z} \subset \mathbb{R}^2$ such that $\mathcal{Z} = \{(r, \tilde{f}(0, r)) : r > \varepsilon_0\}$. Notice that for $(r_0, z_0) = (r_0, \tilde{f}(0, r_0)) \in \mathcal{Z}$ we have $z(\theta, r_0, z_0) = \tilde{f}(\theta, r_0) = f(r_0 \cos \theta, r_0 \sin \theta)$. Indeed, let $w(\theta) = f(r_0 \cos \theta, r_0 \sin \theta)$. Thus

$$\begin{aligned} w'(\theta) &= \partial_x f(r_0 \cos \theta, r_0 \sin \theta)(-r_0 \sin \theta) + \partial_y f(r_0 \cos \theta, r_0 \sin \theta)(r_0 \cos \theta) \\ &= g(r_0 \cos \theta, r_0 \sin \theta) - f(r_0 \cos \theta, r_0 \sin \theta) \\ &= g(r_0 \cos \theta, r_0 \sin \theta) - w(\theta) \\ &= \tilde{g}(\theta, r_0) - w(\theta). \end{aligned}$$

The second equality holds because $g(x, y) = f(x, y) + x\partial_y f(x, y) - y\partial_x f(x, y)$. Hence, for $(r_0, z_0) \in \mathcal{Z}$ the solution $z(\theta, r_0, z_0)$ is 2π -periodic. Moreover,

$$Id - Y^{-1}(2\pi, r, z) = \begin{pmatrix} 0 & 0 \\ \star & 1 - e^{2\pi} \end{pmatrix}.$$

Consequently, $\Delta_\alpha = 1 - e^{2\pi} \neq 0$. Accordingly, all the hypotheses of Theorem A are satisfied.

Proof of Theorem 6. Denote by (r, z_r) a point in \mathcal{Z} , that is $z_r = \tilde{f}(0, r)$. Notice that the bifurcation function of first order is $f_1(r) = \pi g_1(r, z_r)$, where g_1 is defined in (8). Indeed, from definition $f_1(r) = \pi g_1(r, z_r) + \frac{\partial \pi g_0}{\partial b}(r, z_r) \gamma_1(r)$. However,

$$g_0(r, z) = Y^{-1}(2\pi, r, z)((r, z(2\pi, r, z))) - (r, z(0, r, z)) = (0, \star),$$

and then $\pi g_0 \equiv 0$. Moreover,

$$\begin{aligned} w_1^1(\theta, r, z) &= \left(a_0 \sin \theta + a_1 \int_0^\theta z(\phi) \cos \phi d\phi, G(\theta, r) \left(a_0 \sin \theta + a_1 \int_0^\theta z(\phi) \cos \phi d\phi \right) - \right. \\ &\quad \left. e^{-\theta} \int_0^\theta \left(e^\phi G(\phi, r)(a_0 + a_1 z(\phi)) \cos \phi + \sin \phi \frac{e^\phi (\tilde{g}(\phi, r) - z(\phi))(a_0 + a_1 z(\phi))}{r} \right) d\phi \right), \end{aligned}$$

$$\begin{aligned}
w_1^2(\theta, r, z) &= Y(\theta, r, z) \left[Y^{-1}(\pi, r, z) w_1^1(\pi, r, z) + \int_{\pi}^{\theta} Y^{-1}(\phi, r, z) F_1^2(\phi, r(\phi), z(\phi)) d\phi \right] \\
&= Y(\theta, r, z) \left(a_1 \int_0^{\pi} z(\phi) \cos \phi d\phi + b_1 \int_{\pi}^{\theta} z(\phi) \sin \phi d\phi, \right. \\
&\quad \left. \int_0^{\pi} \frac{e^{\phi}((a_0 + a_1 z(\phi))(\sin \phi (g(r \cos \phi, r \sin \phi) - z(\phi)) - r \cos \phi G(\phi, r))}{r} d\phi \right. \\
&\quad \left. + \int_{\pi}^{\theta} -\frac{b_1 e^{\phi} z(\phi) (\cos \phi (g(r \cos \phi, r \sin \phi) - z(\phi)) + r \sin \phi G(\phi, r))}{r} d\phi \right).
\end{aligned}$$

Since $g_1(r, z) = Y^{-1}(2\pi, r, z) w_1^2(2\pi, r, z)$ and $f_1(r) = \pi g_1(r, z_r)$ it follows that

$$(32) \quad f_1(r) = a_1 \int_0^{\pi} f(r \cos \phi, r \sin \phi) \cos \phi d\phi + b_1 \int_{\pi}^{2\pi} f(r \cos \phi, r \sin \phi) \sin \phi d\phi.$$

So, based on Theorem A, each positive simple zero of (29) provides an isolated periodic solution of system (28). This concludes this proof. \square

The next result is an application of Theorem 6. We shall use in its statement the concept of Bessel functions, which are defined as the canonical solutions $y(x)$ of Bessel's differential equation

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - \alpha^2) y = 0, \quad \alpha \in \mathbb{C}.$$

This equation has two linearly independent solutions. Using Frobenius' method we obtain one of these solutions, which is called a *Bessel function of the first kind*, and is denoted by $J_{\alpha}(x)$. More details about this function can be found in [29].

Corollary 7. *Consider the piecewise vector field (28).*

- (a) *If $f(x, y) = \cos x$, then the piecewise smooth vector field X admits a sequence of limit cycles $\varphi_i(t, \varepsilon)$ of X of period T_{ε} such that $T_{\varepsilon} = 2\pi + \mathcal{O}(\varepsilon)$, $\varphi_n(t, \varepsilon) = (x_n^*, y_n^*, \cos(x_n^*)) + \mathcal{O}(\varepsilon)$, and $|(x_n^*, z_n^*)| = n\pi/2$.*
- (b) *If $f(x, y) = \sin x$, then the piecewise smooth vector field X admits a sequence of limit cycles $\varphi_i(t, \varepsilon)$ of X of period T_{ε} such that $T_{\varepsilon} = 2\pi + \mathcal{O}(\varepsilon)$, $\varphi_i(t, \varepsilon) = (x_n^*, y_n^*, \sin(x_n^*)) + \mathcal{O}(\varepsilon)$, and $|(x_n^*, z_n^*)| = r_n^*$, where each r_n is a zero of the Bessel Function of First Kind, $J_1(r)$.*

Proof. For $f(x, y) = \cos x$, the bifurcation function (32) reads $f_1(r) = -(2b_1 \sin r)/r$, and for $f(x, y) = \sin(x)$, the bifurcation function (32) reads $f_1(r) = a_1 \pi J_1(r)$. Therefore, the result follows directly from Theorem 6. \square

Notice that Theorem 6 cannot be applied when f_1 is identically zero, which is the case when $f(x, y) = 2x^2 - y^2$, for instance. For such cases we define the function

(33)

$$\begin{aligned}
 f_2(r) = & \int_0^\pi \left(a_1 \cos s \left(G(s, r) \int_0^s \cos \phi (a_0 + a_1 \tilde{f}(\phi, r)) d\phi \right. \right. \\
 & - e^{-s} \int_0^s e^\phi (a_0 + a_1 \tilde{f}(\phi, r)) (r \cos \phi G(\phi, r) + (\tilde{f}(\phi, r) - \tilde{g}(\phi, r)) d\phi \\
 & \left. \left. + a_2 + a_3 \tilde{f}(\phi, r) + \frac{\sin s}{r} (a_0 + a_1 \tilde{f}(s, r))^2 \right) \right) ds \\
 & + \frac{e^{-2\pi}(1 + e^\pi)}{2(1 - e^{2\pi})} (a_1 e^\pi - b_1) \left[\int_0^\pi e^\phi G(\phi, r) \cos \phi (a_0 + a_1 \tilde{f}(\phi, r)) d\phi \right. \\
 & + \int_0^\pi \frac{e^\phi \sin \phi}{r} (a_0 + a_1 \tilde{f}(\phi, r)) (\tilde{g}(\phi, r) - \tilde{f}(\phi, r)) d\phi \\
 & \left. + b_1 \int_\pi^{2\pi} e^\phi G(\phi, r) \sin \phi \tilde{f}(\phi, r) d\phi + \frac{b_1}{r} \int_\pi^{2\pi} e^\phi \cos \phi (\tilde{g}(\phi, r) - \tilde{f}(\phi, r)) d\phi \right] \\
 & + \int_\pi^{2\pi} \left(\frac{2}{r} (-b_1^2 \cos s (\tilde{f}(s, r))^2 + \sin s (b_2 + b_3 \tilde{f}(s, r))) \right. \\
 & \left. + 2b_1 \sin s \left(G(s, r) \int_0^\pi \cos \phi (a_0 + a_1 \tilde{f}(\phi, r)) + b_1 G(s, r) \int_\pi^s \sin \phi \tilde{f}(\phi, r) d\phi \right. \right. \\
 & \left. \left. + e^{-s} \left(\int_0^\pi -e^\phi \cos \phi G(\phi, r) (a_0 + a_1 \tilde{f}(\phi, r)) + \frac{e^\phi \sin \phi}{r} (\tilde{g}(\phi, r) - \tilde{f}(\phi, r)) d\phi \right. \right. \right. \\
 & \left. \left. \left. + b_1 \int_\pi^s e^\phi \left(\frac{\cos \phi}{r} (\tilde{f}(\phi, r) - \tilde{g}(\phi, r)) - G(\phi, r) \sin \phi \right) d\phi \right) \right) \right) ds.
 \end{aligned}$$

Theorem 8. Consider the piecewise vector field (28). Assume that $f_1 \equiv 0$. Then, for each $r^* > 0$, such that $f_2(r^*) = 0$ and $f_2'(r^*) \neq 0$, there exists a crossing limit cycle $\varphi(t, \varepsilon)$ of X of period T_ε such that $T_\varepsilon = 2\pi + \mathcal{O}(\varepsilon)$, $\varphi(t, \varepsilon) = (x^*, y^*, f(x^*, y^*)) + \mathcal{O}(\varepsilon)$, and $|(x^*, z^*)| = r^*$.

Proof. As we saw before $\pi g_0 \equiv 0$. So, from (4), we compute the bifurcation function of order 2 as

$$(34) \quad f_2(r) = \frac{\partial \pi g_1}{\partial b}(r, z_r) \gamma_1(r) + \pi g_2(r, z_r),$$

where $\gamma_1(r) = -\frac{1}{1 - e^{2\pi}} \pi^\perp g_1(r, z_r)$ and

$$\begin{aligned}
 \pi^\perp g_1(r, z_r) = & \int_0^\pi \frac{e^\phi ((a_0 + a_1 \tilde{f}(\phi, r)) (\sin \phi (g(r \cos \phi, r \sin \phi) - \tilde{f}(\phi, r)) - r \cos \phi G(\phi, r, z))}{r} d\phi \\
 & - b_1 \int_\pi^{2\pi} \frac{e^\phi \tilde{f}(\phi, r) (\cos \phi (g(r \cos \phi, r \sin \phi) - \tilde{f}(\phi, r)) + r \sin \phi G(\phi, r, z))}{r} d\phi.
 \end{aligned}$$

From Proposition 4, we have $g_2(r, z_r) = Y^{-1}(2\pi, r, z)w_2^2(2\pi, r, z)/2$, where $w_i^j(2\pi, r, z)$ is provided in Lemma 3. All these functions may be computed to get (34) as (33). Again, in accordance with Theorem A, each positive simple zero of (33) provides an isolated periodic solution of system (28). This concludes this proof. \square

The next result is an application of Theorem 8.

Corollary 9. *Consider the piecewise vector field (28) and let $f(x, y) = 2x^2 - y^2$. Assuming $a_1^2 + b_1^2 \neq 0$ define*

$$(35) \quad \begin{aligned} A_0 &= \frac{-80b_2(1 - e^\pi)}{(1 + e^\pi)4(15a_1b_1 - b_1^2 - 14a_1^2) - 5\pi(1 - e^\pi)(b_1^1 + 10a_1^2)}, \\ A_1 &= \frac{40a_0((1 + e^\pi)(b_1 - a_1) - a_1\pi(1 - e^\pi))}{(1 + e^\pi)4(15a_1b_1 - b_1^2 - 14a_1^2) - 5\pi(1 - e^\pi)(b_1^1 + 10a_1^2)}, \end{aligned}$$

and $D = -4A_1^3 - 27A_0^2$.

- (i) *If $D > 0$, then the piecewise smooth vector field admits at least one limit cycle. Moreover, if $A_1 < 0$ and $A_0 > 0$, then the piecewise smooth vector field admits at least two limit cycles;*
- (ii) *If $D \leq 0$ and $A_0 < 0$, then the piecewise smooth vector field admits at least one limit cycle.*

Moreover, in both cases we have a limit cycle $\varphi(t, \varepsilon)$ of X of period T_ε such that $T_\varepsilon = 2\pi + \mathcal{O}(\varepsilon)$, $\varphi(t, \varepsilon) = (x_n^*, y_n^*, 2(x_n^*)^2 - (y_n^*)^2) + \mathcal{O}(\varepsilon)$, and $|(x_n^*, z_n^*)| = r_n^*$.

Proof. For $f(x, y) = 2x^2 - y^2$ the bifurcation function (33) becomes

$$(36) \quad \begin{aligned} f_2(r) &= -2b_2 + \frac{a_0((e^\pi(1 - \pi) + 1 + \pi)a_1 - (1 + e^\pi)b_1)}{e^\pi - 1}r \\ &\quad + \frac{(-(e^\pi(56 - 50\pi) + 56 + 50\pi)a_1^2 + 60(1 + e^\pi)a_1b_1 - (e^\pi(4 - 5\pi) + 4 + 5\pi)b_1^2)}{40(e^\pi - 1)}r^3. \end{aligned}$$

Dividing f_2 by $a_1^2 + b_1^2 \neq 0$, we see that the equation $f_2(r) = 0$ is equivalent to $\tilde{f}_2(r) \doteq A_0 + A_1r + r^3 = 0$, where A_0 and A_1 are provided in (35).

Notice that $\tilde{f}_2(r)$ is a polynomial function of degree 3, so it has at least one real root and can be written as $\tilde{f}_2(r) = r^3 - (r_1 + r_2 + r_3)r^2 + (r_1r_2 + r_1r_3 + r_2r_3)r - r_1r_2r_3$, where $r_i, i = 1, 2, 3$ are the zeros of the polynomial. Moreover, the sign of its discriminant $D = -4A_1^3 - 27A_0^2$ carries information about its number of real roots.

If $D > 0$ the polynomial $\tilde{f}_2(r)$ has three simple real roots r_1, r_2 and r_3 . Since the polynomial has no quadratic term, it follows that $r_1 + r_2 + r_3 = 0$ and then at least one of these roots must be positive. Moreover, if $A_1 < 0$ and $A_0 > 0$, then there are two changes of sign between the terms of the polynomial, and then by *Descartes Sign Theorem* we get the two positive roots.

If $D \leq 0$, then there is a pair of complex roots or a double real root. In both cases, the condition $A_0 < 0$ implies that at least one root is positive.

Now, according to Theorem A, each positive simple zero of (36) provides an isolated periodic solution of system (28). This concludes this proof. \square

5.2. non-smooth perturbation of a non-smooth center. In this example, we consider a non-smooth differential system in \mathbb{R}^3 defined in 4 zones ($n = 4$). Consider the non-smooth vector field

$$(37) \quad X(u, v, w) = \begin{cases} X_1(u, v, w) & \text{if } u > 0 \text{ and } v > 0, \\ X_2(u, v, w) & \text{if } u < 0 \text{ and } v > 0, \\ X_3(u, v, w) & \text{if } u < 0 \text{ and } v < 0, \\ X_4(u, v, w) & \text{if } u > 0 \text{ and } v < 0, \end{cases}$$

where

$$\begin{aligned} X_1(u, v, w) &= (-1 + \varepsilon(a_1x + b_1), 1, -w + \varepsilon(c_1x + d_1)), \\ X_2(u, v, w) &= (-1 + \varepsilon(a_2x + b_2), -1, -w + \varepsilon(c_2x + d_2)), \\ X_3(u, v, w) &= (1 + \varepsilon(a_3x + b_3), -1, -w + \varepsilon(c_3x + d_3)), \\ X_4(u, v, w) &= (1 + \varepsilon(a_4x + b_4), 1, -w + \varepsilon(c_4x + d_4)), \end{aligned}$$

with $a_j, b_j, c_j, d_j \in \mathbb{R}$ for all j .

Writing in cylindrical coordinates $u = r \cos \theta$, $v = r \sin \theta$, $w = w$, the discontinuity set is $\Sigma = \{\theta = 0\} \cup \{\theta = t_1\} \cup \{\theta = t_2\} \cup \{\theta = t_3\}$ with $t_0 = 0, t_1 = \pi/2, t_2 = \pi, t_3 = 3\pi/2$ and $t_4 = 2\pi$. For each $j = 1, 2, 3, 4$ the differential system $(\dot{u}, \dot{v}, \dot{w}) = X_j(u, v, w)$ in cylindrical coordinates writes

$$\begin{aligned} r'(t) &= g_j(\theta) + \sum_{i=1}^k \varepsilon^i (a_{ij} r \cos^2 \theta + b_{ij} \cos \theta), \\ w'(t) &= -w + \sum_{i=1}^k \varepsilon^i (c_{ij} r \cos \theta + d_{ij} \cos \theta), \\ \theta'(t) &= \frac{1}{r} \left(\widehat{g}_j(\theta) - \sum_{i=1}^k \varepsilon^i (a_{ij} r \cos \theta \sin \theta + b_{ij} \sin \theta) \right), \end{aligned}$$

where

$$\begin{aligned} g_1(\theta) &= \sin \theta - \cos \theta, & \widehat{g}_1(\theta) &= \sin \theta + \cos \theta, \\ g_2(\theta) &= -(\sin \theta + \cos \theta), & \widehat{g}_2(\theta) &= \sin \theta - \cos \theta, \\ g_3(\theta) &= -\sin \theta + \cos \theta, & \widehat{g}_3(\theta) &= -(\sin \theta + \cos \theta), \\ g_4(\theta) &= \sin \theta + \cos \theta, & \widehat{g}_4(\theta) &= -\sin \theta + \cos \theta. \end{aligned}$$

Notice that, for each $j = 1, 2, 3, 4$ and $t_{j-1} \leq \theta \leq t_j$, $\dot{\theta}(t) \neq 0$ for $|\varepsilon|$ sufficiently small. Thus, in a sufficiently small neighborhood of the origin we can take θ as the new independent time variable by doing $r'(\theta) = \dot{r}(t)/\dot{\theta}(t)$ and $w'(\theta) = \dot{w}(t)/\dot{\theta}(t)$. Taking θ as the new independent time variable, we have

$$(38) \quad \begin{aligned} r'(\theta) &= F_{01}^j(\theta, z) + \varepsilon F_{11}^j(\theta, z) + \mathcal{O}_1(\varepsilon^2), \\ w'(\theta) &= F_{02}^j(\theta, z) + \varepsilon F_{12}^j(\theta, z) + \mathcal{O}_2(\varepsilon^2). \end{aligned}$$

Here, $z = (r, w)$ and the prime denotes the derivative with respect to θ . The expressions of F_{01}^j and F_{02}^j for $j = 1, 2, 3, 4$ are provided by

$$\begin{aligned} F_{01}^1 &= \frac{r(\sin \theta - \cos \theta)}{\sin \theta + \cos \theta}, \quad F_{02}^1 = \frac{-rw}{\sin \theta + \cos \theta}, \quad F_{01}^2 = \frac{r(\sin \theta + \cos \theta)}{\cos \theta - \sin \theta}, \quad F_{02}^2 = \frac{rw}{\cos \theta - \sin \theta}, \\ F_{01}^3 &= \frac{r(\sin \theta - \cos \theta)}{\sin \theta + \cos \theta}, \quad F_{02}^3 = \frac{rw}{\sin \theta + \cos \theta}, \quad F_{01}^4 = \frac{r(\sin \theta + \cos \theta)}{\cos \theta - \sin \theta}, \quad F_{02}^4 = \frac{-rw}{\cos \theta - \sin \theta}. \end{aligned}$$

The expressions of F_{11}^j and F_{12}^j for $j = 1, 2, 3, 4$ are also easily computed. Nevertheless, we shall omit these expressions due to their size.

For each $j \in \{1, 2, 3, 4\}$, the differential system (38) is 2π -periodic in the variable θ and is written in standard form with

$$F_i^j(\theta, z) = \left(F_{i1}^j(\theta, z), F_{i2}^j(\theta, z) \right),$$

for $i = 0, 1$. Now, for each $j \in \{1, 2, 3, 4\}$ we compute the solution $x_j(\theta, z, 0)$ of the unperturbed system

$$\dot{r}(\theta) = F_{01}^j(\theta, z), \quad \dot{w}(\theta) = F_{02}^j(\theta, z).$$

and this solution is

$$\begin{aligned} x_1(\theta, z, 0) &= \left(\frac{r}{\sin \theta + \cos \theta}, we^{-\frac{r \sin \theta}{\sin \theta + \cos \theta}} \right), \\ x_2(\theta, z, 0) &= \left(\frac{-r}{\cos \theta - \sin \theta}, we^{-\frac{r \sin \theta}{\cos \theta - \sin \theta} - 2r} \right), \\ x_3(\theta, z, 0) &= \left(\frac{-r}{\sin \theta + \cos \theta}, we^{-\frac{r \sin \theta}{\sin \theta + \cos \theta} - 2r} \right), \\ x_4(\theta, z, 0) &= \left(\frac{r}{\cos \theta - \sin \theta}, we^{-\frac{r \sin \theta}{\cos \theta - \sin \theta} - 4r} \right). \end{aligned}$$

We note that in each quadrant the denominators of these four solutions never vanish.

Let $0 < r_0 < r_1$ be positive real numbers and consider the set $\mathcal{Z} \subset \mathbb{R}^2$ such that $\mathcal{Z} = \{(\alpha, 0) : r_0 < \alpha < r_1\}$. The solution $x(\theta, z, 0)$ of the unperturbed system $x'(\theta) = F_0(\theta, z)$ satisfies $x(\theta, z, 0) = x_j(\theta, z, 0)$, for $\theta \in [t_{j-1}, t_j]$, and $x(2\pi, z, 0) - x(0, z, 0) = (0, z(1 - e^{-4r}))$. Consequently, for each $z_\alpha \in \mathcal{Z}$, the solution $x(\theta, z, 0)$ is 2π -periodic and system (37) satisfies hypothesis (H_a) . Moreover, the fundamental matrix $Y(\theta, z)$

is provided by

$$Y(\theta, z) = \begin{cases} Y_1(\theta, z) & \text{if } 0 = t_0 \leq \theta \leq \pi/2, \\ Y_2(\theta, z) & \text{if } \pi/2 \leq \theta \leq \pi, \\ Y_3(\theta, z) & \text{if } \pi \leq \theta \leq 3\pi/2, \\ Y_4(\theta, z) & \text{if } 3\pi/2 \leq \theta \leq 2\pi, \end{cases}$$

where $Y_j(t, z)$ are defined by (15). Thus

$$Y_1(\theta, z) = \begin{pmatrix} \frac{1}{g_4(\theta)} & 0 \\ -e^{-\frac{r \sin \theta}{g_4(\theta)} w \sin \theta} & e^{-\frac{r \sin \theta}{g_4(\theta)}} \end{pmatrix},$$

$$Y_4(\theta, z) = \begin{pmatrix} \frac{1}{g_3(\theta)} & 0 \\ -e^{-\frac{r \sin \theta}{g_3(\theta)} - 4r} \frac{w(\sin \theta + 4g_3(\theta))}{g_3(\theta)} & e^{-\frac{r \sin \theta}{g_3(\theta)} - 4r} \end{pmatrix}.$$

Hence,

$$Y_1(0, z)^{-1} - Y_4(2\pi, z)^{-1} = \begin{pmatrix} 0 & 0 \\ -4w & 1 - e^{4r} \end{pmatrix},$$

and then $\det(\Delta_\alpha) = 1 - e^{4r} \neq 0$ if $z_\alpha = (\alpha, 0) \in \mathcal{Z}$. Thus, we can compute the bifurcation functions (4) for system (37). For such we first obtain the functions (19) corresponding to this system,

$$g_0(\theta, z) = (0, w(1 - e^{4r})),$$

$$w_1^4(2\pi, z) = \left(\frac{1}{2}r(r(a_1 + a_2 + a_3 + a_4) + 2(b_1 - b_2 - b_3 + b_4)), \right.$$

$$\frac{1}{3}e^{-4r}(-r^2w(6a_1 + 3a_2 + 2a_3) - 3r(w(4b_1 - 2b_2 - b_3)$$

$$+ e^{2r}(-e^{2r}c_4 + c_2 + c_3) + c_1) + 3(e^r - 1)(e^r(c_2 + d_2)$$

$$+ e^{2r}(c_3 - d_3) + e^{3r}(d_4 - c_4) + c_1 + d_1) \left. \right),$$

and

$$(39) \quad g_1(z) = Y_4(2\pi, z)^{-1} w_1^4(2\pi, z).$$

This way, the bifurcation function (4) corresponding to the function (39) becomes

$$f_1(\alpha) = \frac{1}{2}\alpha(\alpha(a_1 + a_2 + a_3 + a_4) + 2(b_1 - b_2 - b_3 + b_4)),$$

which has a simple zero α^* . So, from Theorem A, we get the existence of an isolated periodic solution of system (38) for ε sufficiently small.

6. CONCLUSION

Establishing the existence of invariant sets, in particular equilibria and periodic solutions, is very important for understanding the dynamics of a differential system. The detection of periodic solutions is far more complicated than equilibria and relies on solving differential equations, which cannot be done in general. For smooth perturbative differential systems, the Melnikov method and the averaging theory are two classical tools that reduce the problem of finding periodic solutions into a simpler problem of finding zeros of a related function. These tools, particularly the averaging theory, have been recently developed for non-smooth perturbative systems.

This manuscript is devoted to develop the averaging theory for a class of non-smooth differential systems. Results relating isolated periodic solutions with zeros of a sequence of bifurcation functions are shown. In Section 2, we show that our results can be used in a wide family of differential systems with applied interests, namely, non-smooth perturbations of vector fields admitting Andronov-Hopf equilibria.

A natural next step for further investigations consists in considering classes of non-smooth differential systems with more general set of discontinuities. For these cases, it has been noticed in [14] that the higher order averaged functions does not control in general the bifurcation of periodic solutions. A complete second order analysis was performed in [2]. It was shown that the second order bifurcation (Melnikov) function is provided as the second-order averaged function added to an increment, which depends on the geometry of the discontinuity set. Currently, the theory lacks a higher order analysis for these cases.

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