MITTAG-LEFFLER FUNCTORS OF MODULES

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Abstract. Finite modules, finitely presented modules and Mittag-Leffler modules are characterized by their behaviour by tensoring with direct products of modules. In this paper, we study and characterize the functors of modules that preserve direct products.

1. Introduction

Let R be an associative ring with unit, we will say that $\mathbb{M}$ is an $\mathcal{R}$-*module* (right $\mathcal{R}$-module) if $\mathbb{M}$ is a covariant additive functor from the category of R-modules (respectively, right R-modules) to the category of abelian groups.

Any right R-module M produces an $\mathcal{R}$-module. Namely, the *quasi-coherent* $\mathcal{R}$-*module* $\mathcal{M}$ associated with a right R-module M is defined by

$$\mathcal{M}(S) = M \otimes_R S,$$

for any R-module S. It is significant to note that the category of right R-modules is equivalent to the category of quasi-coherent $\mathcal{R}$-modules. Therefore, we can study modules through their functorial incarnation.

On the other hand, given an $\mathcal{R}$-module $\mathbb{M}$, $\mathbb{M}^*$ is the right $\mathcal{R}$-module defined as follows:

$$\mathbb{M}^*(N) := \mathrm{Hom}_{\mathcal{R}}(\mathbb{M}, \mathcal{N}),$$

for any right R-module N.

A relevant fact is that quasi-coherent modules are reflexive, that is, the canonical morphism of $\mathcal{R}$-modules $\mathcal{M} \to \mathcal{M}^{**}$ is an isomorphism ([9]).

Quasi-coherent modules preserve direct limits, that is:

$$\mathcal{M}(\lim_{\substack{\to \\ i\in I}} N_i) = M \otimes_R \lim_{\substack{\to \\ i\in I}} N_i = \lim_{\substack{\to \\ i\in I}} (M \otimes_R N_i) = \lim_{\substack{\to \\ i\in I}} \mathcal{M}(N_i)$$

for any direct system of R-modules $\{N_i\}_{i\in I}$, where I is an upward directed set. Watts ([11, Th 1.]) proved that an $\mathcal{R}$-module is quasi-coherent iff it is a right exact functor and preserves direct limits. An $\mathcal{R}$-module $\mathbb{M}$ preserves direct limits iff there exists an exact sequence of morphisms of $\mathcal{R}$-modules

$$\oplus_{i\in I} \mathcal{P}_i^* \to \oplus_{j\in J} \mathcal{Q}_j^* \to \mathbb{M} \to 0,$$

where P_i and Q_j are finitely presented R-modules, for every i, j. Besides, this exact sequence is a projective presentation of $\mathbb{M}$ (Thm 3.6). Let $\langle Qs\text{-}ch\rangle$ be the category of $\mathcal{R}$-modules that preserve direct limits. $\langle Qs\text{-}ch\rangle$ is the smallest full subcategory of the category of $\mathcal{R}$-modules containing quassi-coherent modules stable by kernels, cokernels and direct limits. It can be proved that the category $\langle Qs\text{-}ch\rangle$ is equivalent to the category of functors from the category of finitely presented R-modules to the category of abelian groups. In case that R is a field, an $\mathcal{R}$-module preserves direct limits iff it is quasi-coherent.

The aim of this paper is to extend the notions of finite, finitely presented and Mittag-Leffler modules to $\langle Qs\text{-}ch\rangle$ and give different characterizations of these functors.

Any R-module is a direct limit of finitely presented R-modules and it is well known that an R-module M is finitely presented iff $\mathcal{M}$ preserves direct products.

Definition 1.1. *We will say that $\mathbb{M} \in \langle Qs\text{-}ch\rangle$ is an FP module if it preserves direct products.*

Every $\mathcal{R}$-module $\mathbb{M} \in \langle Qs\text{-}ch\rangle$ is a direct limit of FP modules (4.22). FP modules are characterized as follows.

Theorem 1.2. *$\mathbb{F} \in \langle Qs\text{-}ch\rangle$ is an FP module iff any of the following statements holds*

(1) $\mathrm{Hom}_{\mathcal{R}}(\mathbb{F}, \lim\limits_{\substack{\rightarrow\\ i\in I}} \mathbb{M}_i) = \lim\limits_{\substack{\rightarrow\\ i\in I}} \mathrm{Hom}_{\mathcal{R}}(\mathbb{F}, \mathbb{M}_i)$, *for any direct system of $\mathcal{R}$-modules* $\{\mathbb{M}_i\}_{i\in I}$.
(2) $\mathbb{F}$ *is reflexive and* $\mathbb{F}^* \in \langle Qs\text{-}ch\rangle$.
(3) *There exists an exact sequence of $\mathcal{R}$-modules*
$$\mathcal{P}_1^* \to \mathcal{P}_2^* \to \mathbb{F} \to 0,$$
where P_1 and P_2 are finitely presented R-modules.
(4) *There exists an exact sequence of $\mathcal{R}$-modules*
$$0 \to \mathbb{F} \to \mathcal{Q}_1 \to \mathcal{Q}_2,$$
where Q_1 and Q_2 are finitely presented right R-modules.

The category of FP modules is an abelian category. However, in general, the category of finitely presented R-modules is not abelian. If $\mathbb{F}$ is an FP module, then $\mathbb{F}^*$ is an FP module. However, in general, if P is a finitely presented R-module, then P^* is not a finitely presented R-module.

Definition 1.3. *We will say that an $\mathcal{R}$-module $\mathbb{M} \in \langle Qs\text{-}ch\rangle$ is an ML module if the natural morphism $\mathbb{M}(\prod_i S_i) \to \prod_i \mathbb{M}(S_i)$ is injective for any set $\{S_i\}$ of R-modules.*

A right R-module M is a Mittag-Leffler module iff $\mathcal{M}$ is an ML module. ML modules are chatacterized as follows.

Theorem 1.4. *Let $\mathbb{M} \in \langle Qs\text{-}ch\rangle$. The following statements are equivalent:*

(1) $\mathbb{M}$ *is an ML module.*
(2) $\mathbb{M}$ *is a direct limit of FP submodules.*
(3) *The kernel of every morphism $\mathbb{F} \to \mathbb{M}$ is an FP module, for any FP module* $\mathbb{F}$.

If M is a Mittag-Leffler module it is not true, as a general rule, that M is a direct limit of finitely presented submodules, nor is it true that the image of a morphism of R-modules $P \to M$ is a finitely presented module (where P is a finitely presented R-module).

Definition 1.5. *Let $\mathbb{M}$ be an ML $\mathcal{R}$-module. $\mathbb{M}$ is said to be an SML module if for any FP submodule $\mathbb{F} \subseteq \mathbb{M}$ the dual morphism $\mathbb{M}^* \to \mathbb{F}$ is an epimorphism.*

M is a right strict Mittag-Leffler R-module iff $\mathcal{M}$ is an SML module (6.9). SML modules are chatacterized as follows.

Theorem 1.6. *Let $\mathbb{M} \in \langle Qs\text{-}ch\rangle$. The following statements are equivalent:*

1. $\mathbb{M}$ *is an SML module.*
2. $\mathbb{M}$ *is a direct limit of FP submodules $\mathbb{F}_i$, and the morphism $\mathbb{M}^* \to \mathbb{F}_i^*$ is an epimorphism, for any i.*
3. *There exists a monomorphism $\mathbb{M} \hookrightarrow \prod_{i\in I} \mathcal{P}_i$, where P_i is a finitely presented (right) module, for each $i \in I$.*

In particular, if M is a strict Mittag-Leffler R-module, then it is a pure submodule of a direct product of finitely presented R-modules (this result can be found in [5]).

Finally we prove the following theorem.

Theorem 1.7. *Let M be an R-module. Then,*

1. *M is a Mittag-Leffler module iff the kernel of any morphism $\prod_{\mathbb{N}} \mathcal{R} \to \mathcal{M}$ preserves direct products.*
2. *M is a strict Mittag-Leffler module iff the cokernel of any morphism $\mathcal{M}^* \to \oplus_{\mathbb{N}}\mathcal{R}$ is isomorphic to an $\mathcal{R}$-submodule of a quasi-coherent module.*

This paper is self contained. Functorial characterizations of flat Mittag-Leffler modules and flat strict Mittag-Leffler modules are given in [9].

2. Preliminaries

Remark 2.1. *For the rest of the paper, every definition or statement is given with one module structure (left or right) on each of the modules appearing in that definition or statement; we leave to the reader to do the respective definition or statement by interchanging the left and right structures.*

Notation 2.2. *Let $\mathbb{M}$ be a functor of $\mathcal{R}$-modules. For simplicity, we will sometimes use $m \in \mathbb{M}$ to denote $m \in \mathbb{M}(S)$. Given $m \in \mathbb{M}(S)$ and a morphism of R-modules $S \to S'$, we will often denote by m its image by the morphism $\mathbb{M}(S) \to \mathbb{M}(S')$.*

Remark 2.3. *Direct limits, inverse limits of $\mathcal{R}$-modules and kernels, cokernels, images, etc., of morphisms of $\mathcal{R}$-modules are regarded in the category of $\mathcal{R}$-modules. Besides,*

$$(\lim_{\substack{\rightarrow\\ i\in I}} \mathbb{M}_i)(S) = \lim_{\substack{\rightarrow\\ i\in I}} (\mathbb{M}_i(S)),\ (\lim_{\substack{\leftarrow\\ j\in J}} \mathbb{M}_j)(S) = \lim_{\substack{\leftarrow\\ j\in J}} (\mathbb{M}_j(S)),$$
$$(\operatorname{Ker} f)(S) = \operatorname{Ker} f_S,\ (\operatorname{Coker} f)(S) = \operatorname{Coker} f_S,\ (\operatorname{Im} f)(S) = \operatorname{Im} f_S,$$

(where I is an upward directed set and J a downward directed set).

We will denote by $\operatorname{Hom}_{\mathcal{R}}(\mathbb{M}, \mathbb{M}')$ the family of all morphisms of $\mathcal{R}$-modules from $\mathbb{M}$ to $\mathbb{M}'$.

Proposition 2.4. [9, 2.11] *Let $\mathbb{M}$ be a (left) $\mathcal{R}$-module and let $\mathbb{N}$ be a (right) $\mathcal{R}$-module. Then,*

$$\operatorname{Hom}_{\mathcal{R}}(\mathbb{M}, \mathbb{N}^*) = \operatorname{Hom}_{\mathcal{R}}(\mathbb{N}, \mathbb{M}^*),\ f \mapsto \tilde{f},$$

where $\tilde{f}$ is defined as follows: $\tilde{f}(n)(m) := f(m)(n)$, for any $m \in \mathbb{M}$ and $n \in \mathbb{N}$.

Definition 2.5. *Let $\mathbb{M}$ be an $\mathcal{R}$-module. We will say that $\mathbb{M}^*$ is the dual (right) $\mathcal{R}$-module of $\mathbb{M}$. We will say that an $\mathcal{R}$-module $\mathbb{M}$ is reflexive if the natural morphism*

$$\mathbb{M} \to \mathbb{M}^{**}, m \mapsto \tilde{m} \text{ (for any } m \in \mathbb{M}(S))$$

is an isomorphism, where $\tilde{m}_{S'}(w) := w_S(m)$ (for any $w \in \mathbb{M}^*(S') = \mathrm{Hom}_{\mathcal{R}}(\mathbb{M}, \mathcal{S}')$).

Proposition 2.6. *Let $\mathbb{M}$ and $\mathbb{M}'$ be reflexive functors of $\mathcal{R}$-modules, $f\colon \mathbb{M} \to \mathbb{M}'$ a morphism of $\mathcal{R}$-modules and $f^*\colon \mathbb{N}^* \to \mathbb{M}^*$ the dual morphism. Then, $\mathrm{Ker}\, f = (\mathrm{Coker}\, f^*)^*$.*

2.1. **Quasi-coherent modules.**

Definition 2.7. *Let M (resp. N, V, etc.) be a right R-module. We will denote by $\mathcal{M}$ (resp. $\mathcal{N}$, $\mathcal{V}$, etc.) the $\mathcal{R}$-module defined by $\mathcal{M}(S) := M \otimes_R S$ (resp. $\mathcal{N}(S) := N \otimes_R S$, $\mathcal{V}(S) := V \otimes_R S$, etc.). $\mathcal{M}$ will be called the quasi-coherent $\mathcal{R}$-module associated with M.*

Proposition 2.8. [9, 2.4] *The functors*

$$\begin{array}{c} \text{Category of right } R\text{-modules} \ \to \ \text{Category of quasi-coherent } \mathcal{R}\text{-modules} \\ M \mapsto \mathcal{M} \\ \mathcal{M}(R) \leftarrow\!\shortmid \mathcal{M} \end{array}$$

stablish an equivalence of categories. In particular,

$$\mathrm{Hom}_{\mathcal{R}}(\mathcal{M}, \mathcal{M}') = \mathrm{Hom}_R(M, M').$$

For another, slightly different, version of this proposition see [1, 1.12].

Let $f_R\colon M \to N$ be a morphism of R-modules and $f\colon \mathcal{M} \to \mathcal{N}$ the associated morphism of $\mathcal{R}$-modules. Let $C = \mathrm{Coker}\, f_R$, then $\mathrm{Coker}\, f = \mathcal{C}$, which is a quasi-coherent module.

Let $\mathbb{M}$ be an $\mathcal{R}$-module. Observe that $\mathbb{M}(R)$ is naturally a right R-module: Given $r \in R$, consider the morphism of R-modules $\cdot r\colon R \to R$. Then,

$$m \cdot r := \mathbb{M}(\cdot r)(m), \text{ for any } m \in \mathbb{M}(R).$$

Proposition 2.9. *For every $\mathcal{R}$-module $\mathbb{M}$ and every right R-module M, it is satisfied that*

$$\mathrm{Hom}_{\mathcal{R}}(\mathcal{M}, \mathbb{M}) = \mathrm{Hom}_R(M, \mathbb{M}(R)),\ f \mapsto f_R.$$

Notation 2.10. *Let $\mathbb{M}$ be an $\mathcal{R}$-module. We will denote by $\mathbb{M}_{qc}$ the quasi-coherent module associated with the R-module $\mathbb{M}(R)$, that is,*

$$\mathbb{M}_{qc}(S) := \mathbb{M}(R) \otimes_R S.$$

Given $s \in S$, consider the morphism of R-modules $R \stackrel{\cdot s}{\to} S$, $r \mapsto r \cdot s$. Then, we have the morphism $\mathbb{M}(\cdot s)\colon \mathbb{M}(R) \to \mathbb{M}(S)$, $m \mapsto \mathbb{M}(\cdot s)(m) =: m \cdot s$.

Proposition 2.11. [9, 2.7] *For each $\mathcal{R}$-module $\mathbb{M}$ one has the natural morphism*

$$\mathbb{M}_{qc} \to \mathbb{M},\ m \otimes s \mapsto m \cdot s,$$

for any $m \otimes s \in \mathbb{M}_{qc}(S) = \mathbb{M}(R) \otimes_R S$, and a functorial equality

$$\mathrm{Hom}_{\mathcal{R}}(\mathcal{N}, \mathbb{M}_{qc}) = \mathrm{Hom}_{\mathcal{R}}(\mathcal{N}, \mathbb{M}),$$

for any quasi-coherent $\mathcal{R}$-module $\mathcal{N}$.

Obviously, an $\mathcal{R}$-module $\mathbb{M}$ is a quasi-coherent module iff the natural morphism $\mathbb{M}_{qc} \to \mathbb{M}$ is an isomorphism.

Theorem 2.12. [9, 2.14] *Let M a right R-module and let M' be an R-module. Then,*

$$M \otimes_R M' = \mathrm{Hom}_{\mathcal{R}}(\mathcal{M}^*, \mathcal{M}'),\ m \otimes m' \mapsto m \tilde{\otimes} m',$$

where $m \tilde{\otimes} m'(w) := w(m) \otimes m'$, for any $w \in \mathcal{M}^*$.

Note 2.13. *It is easy to prove that the morphism*

$$f = \sum_{i=1}^{n} m_i \otimes m_i' \in \mathrm{Hom}_{\mathcal{R}}(\mathcal{M}^*, \mathcal{M}') = M \otimes_R M',$$

is equal to the composite morphism $\mathcal{M}^* \overset{g}{\to} \mathcal{L} \overset{h}{\to} \mathcal{M}'$, where L is the free module of basis $\{l_1, \dots, l_n\}$, $g := \sum_i m_i \otimes l_i \in \mathrm{Hom}_{\mathcal{R}}(\mathcal{M}^*, \mathcal{L}) = M \otimes_R L$ and $h(l_i) := m_i'$ for any i. Observe that $\mathrm{Im}\, f \subseteq \mathrm{Im}\, h$.

If we make $\mathcal{M}' = \mathcal{R}$ in the previous theorem, we obtain the following theorem.

Theorem 2.14. [9, 2.16] *Let M be a right R-module. Then, the canonical morphism*

$$\mathcal{M} \to \mathcal{M}^{**},$$

is an isomorphism. That is, quasi-coherent modules are reflexive.

Definition 2.15. *Let M be an R-module. $\mathcal{M}^*$ will be called the $\mathcal{R}$-module scheme associated with M.*

Theorem 2.16. [9, 2.10] *Let $\{\mathbb{M}_i\}$ be a direct system of $\mathcal{R}$-modules. Then,*

$$\mathrm{Hom}_{\mathcal{R}}(\mathcal{N}^*, \lim_{\substack{\to \\ i}} \mathbb{M}_i) = \lim_{\substack{\to \\ i}} \mathrm{Hom}_{\mathcal{R}}(\mathcal{N}^*, \mathbb{M}_i),$$

for any R-module N.

2.2. **Dual module of a direct product of $\mathcal{R}$-modules.**

Proposition 2.17. *Let $\{\mathbb{M}_i\}_{i \in I}$ be a set of $\mathcal{R}$-modules and let $\mathbb{N}$ be an $\mathcal{R}$-module that preserves direct sums. Then, the natural morphism*

$$\oplus_{i \in I} \mathrm{Hom}_{\mathcal{R}}(\mathbb{M}_i, \mathbb{N}) \to \mathrm{Hom}_{\mathcal{R}}(\prod_{i \in I} \mathbb{M}_i, \mathbb{N}), (f_i)_{i \in I} \mapsto \sum_{i \in I} f_i,$$

is an isomorphism, where $(\sum_{i \in I} f_i)(m_i) := \sum_{i \in I} f_i(m_i)$ for any $(m_i) \in \prod_{i \in I} \mathbb{M}_i$.

Proof. The morphism $(f_i)_{i \in I} \mapsto \sum_{i \in I} f_i$ is obviously injective.

For any $i' \in I$, we have the obvious inclusion morphism $\mathbb{M}_{i'} \subseteq \prod_{i \in I} \mathbb{M}_i$.

Given $f \in \mathrm{Hom}_{\mathcal{R}}(\prod_{i \in I} \mathbb{M}_i, \mathbb{N})$, put $J := \{i \in I \colon f_i := f_{|\mathbb{M}_i} \neq 0\}$. For each $j \in J$, there exist an R-module S_j and $m_j \in \mathbb{M}_j(S_j)$ such that $0 \neq f_{j_{S_j}}(m_j) \in N \otimes_R S_j$. $\mathrm{Hom}_{\mathcal{R}}(\mathcal{S}_j^*, \mathbb{M}_j) = \mathbb{M}_j(S_j)$, by the Yoneda Lemma. Hence, we have the morphism $g_j \colon \mathcal{S}_j^* \to \mathbb{M}_j$ defined by m_j. Consider the morphism

$$g \colon \prod_{j \in J} \mathcal{S}_j^* \to \prod_{j \in J} \mathbb{M}_j,\ g((w_j)_{j \in J}) := (g_j(w_j))_{j \in J}.$$

Put $h := f \circ g$. On one hand $h_{|\mathcal{S}_j^*}(Id_j) = f_{j_{S_j}}(m_j) \neq 0$, for any $j \in J$, where $Id_j \in \mathcal{S}_j^*(S_j) = \mathrm{Hom}_R(S_j, S_j)$ is the identity morphism. On the other hand

$$\begin{aligned} \mathrm{Hom}_{\mathcal{R}}(\prod_{j \in J} \mathcal{S}_j^*, \mathbb{N}) &= \mathrm{Hom}_{\mathcal{R}}((\oplus_{j \in J} \mathcal{S}_j)^*, \mathbb{N}) = \mathbb{N}(\oplus_{j \in J} S_j) \\ &= \oplus_{j \in J} \mathbb{N}(S_j) = \oplus_{j \in J} \mathrm{Hom}_{\mathcal{R}}(\mathcal{S}_j^*, \mathbb{N}). \end{aligned}$$

Hence, $h = \sum_{j\in J} h_{|\mathcal{S}_j^*}$, where $h_{|\mathcal{S}_j^*} = 0$, for all $j \in J$ except for a finite number of them. Then, $\#J < \infty$.

Finally, let us prove that $f = \sum_{j\in J} f_j$: Let $m = (m_i) \in \prod_{i\in I} \mathbb{M}_i(S)$, Consider □

Corollary 2.18. *Let $\{\mathbb{M}_i\}_{i\in I}$ be a set of $\mathcal{R}$-modules. Then, $(\prod_{i\in I} \mathbb{M}_i)^* = \oplus_{i\in I}\mathbb{M}_i^*$.*

Corollary 2.19. *Let $\{\mathbb{M}_i\}_{i\in I}$ be a set of reflexive $\mathcal{R}$-modules. Then, $\oplus_i\mathbb{M}_i$ and $\prod_i \mathbb{M}_i$ are reflexive $\mathcal{R}$-modules.*

3. Functors that preserve direct limits

Definition 3.1. *Let $\mathbb{M}$ be an $\mathcal{R}$-module. We will say that $\mathbb{M}$ preserves direct limits if $\mathbb{M}(\lim\limits_{\stackrel{\rightarrow}{i}} S_i) = \lim\limits_{\stackrel{\rightarrow}{i}} \mathbb{M}(S_i)$ for every direct system of R-modules $\{S_i\}_{i\in I}$.*

Example 3.2. *Quasi-coherent modules preserve direct limits.*

Proposition 3.3. *Let P be a finitely presented (right) module and $\{\mathbb{M}_i\}$ a direct system of $\mathcal{R}$-modules. Then,*

$$\mathrm{Hom}_{\mathcal{R}}(\mathcal{P}, \lim_{\stackrel{\rightarrow}{i}} \mathbb{M}_i) = \lim_{\stackrel{\rightarrow}{i}} \mathrm{Hom}_{\mathcal{R}}(\mathcal{P}, \mathbb{M}_i).$$

In particular, $\mathcal{P}^*$ preserves direct limits.

Proof. By 2.9, $\mathrm{Hom}_{\mathcal{R}}(\mathcal{P}, \lim\limits_{\stackrel{\rightarrow}{i}} \mathbb{M}_i) = \mathrm{Hom}_R(P, \lim\limits_{\stackrel{\rightarrow}{i}} \mathbb{M}_i(R)) = \lim\limits_{\stackrel{\rightarrow}{i}} \mathrm{Hom}_R(P, \mathbb{M}_i(R))$ $= \lim\limits_{\stackrel{\rightarrow}{i}} \mathrm{Hom}_{\mathcal{R}}(\mathcal{P}, \mathbb{M}_i)$. □

Proposition 3.4. *An R-module M is finitely presented iff $\mathcal{M}^*$ preserves direct limits.*

Proof. $\Leftarrow$) Any R-module is a direct limit of finitely presented modules. Write $M = \lim\limits_{\stackrel{\rightarrow}{i}} P_i$, where P_i is a finitely presented module, for any i. Observe that

$$Id \in \mathrm{Hom}_{\mathcal{R}}(\mathcal{M}, \mathcal{M}) = \mathcal{M}^*(M) = \lim_{\stackrel{\rightarrow}{i}} \mathcal{M}^*(P_i) = \lim_{\stackrel{\rightarrow}{i}} \mathrm{Hom}_{\mathcal{R}}(\mathcal{M}, \mathcal{P}_i),$$

hence Id factors through a morphism $f_i\colon M \to P_i$, for some i. Then, M is a direct summand of P_i, and it is finitely presented.

$\Rightarrow$) It is well known. □

Let $\mathbb{M}$ and $\mathbb{M}'$ be $\mathcal{R}$-modules. If $\mathbb{M}$ preserves direct limits, then $\mathrm{Hom}_{\mathcal{R}}(\mathbb{M}, \mathbb{M}')$ is a set: Choose a set A of representatives of the isomorphism classes of finitely presented R-modules. Any morphism $f\colon \mathbb{M} \to \mathbb{M}'$ is determined by the set $\{f_S\}_{S\in A}$ since given an R-module T we can write $T = \lim\limits_{\stackrel{\rightarrow}{i\in I}} S_i$, where $S_i \in A$ for any $i \in I$, and the diagram

$$\begin{array}{ccc} \mathbb{M}(\lim\limits_{\stackrel{\rightarrow}{i}} S_i) = \mathbb{M}(S) & \xrightarrow{f_S} & \mathbb{M}'(S) = \mathbb{M}'(\lim\limits_{\stackrel{\rightarrow}{i}} S_i) \\ \Vert & & \uparrow \\ \lim\limits_{\stackrel{\rightarrow}{i}} \mathbb{M}(S_i) & \xrightarrow{[f_{S_i}]} & \lim\limits_{\stackrel{\rightarrow}{i}} \mathbb{M}'(S_i) \end{array}$$

is commutative. Therefore, $\mathrm{Hom}_{\mathcal{R}}(\mathbb{M}, \mathbb{M}') \subset \prod_{S \in A} \mathrm{Hom}_{gr}(\mathbb{M}(S), \mathbb{M}'(S))$.

Proposition 3.5. *If $\mathbb{M}_1$ and $\mathbb{M}_2$ preserve direct limits and $f \colon \mathbb{M}_1 \to \mathbb{M}_2$ is a morphism of $\mathcal{R}$-modules, then $\mathrm{Ker}\, f$, $\mathrm{Im}\, f$ and $\mathrm{Coker}\, f$ preserve direct limits.*

Proposition 3.6. *If $\{\mathbb{M}_i\}_{i \in I}$ is a direct system of $\mathcal{R}$-modules that preserve direct limits, then $\lim\limits_{\stackrel{\rightarrow}{i}} \mathbb{M}_i$ preserves direct limits.*

Choose a set A of representatives of the isomorphism classes of finitely presented R-modules. $\{\mathcal{P}^*\}_{P \in A}$ is a family of generators of the category of $\mathcal{R}$-modules that preserve direct limits: Let $\mathbb{M}$ and $\mathbb{N}$ be $\mathcal{R}$-modules that preserve direct limits and suppose that $\mathbb{N} \underset{\neq}{\subset} \mathbb{M}$. Then, there exist an R-module S and $m \in \mathbb{M}(S)$, such that $m \notin \mathbb{N}(S)$. $S = \lim\limits_{\stackrel{\rightarrow}{i \in I}} P_i$ is a direct limit of finitely presented R-modules $P_i \in A$, and $\mathbb{M}(S) = \lim\limits_{\stackrel{\rightarrow}{i}} \mathbb{M}(P_i)$. Hence, there exist $i \in I$ and $m_i \in \mathbb{M}(P_i)$ such that m_i is mapped to m by the morphism $\mathbb{M}(P_i) \to \mathbb{M}(S)$. Obviously, $m_i \notin \mathbb{N}(P_i)$. Finally, observe that $\mathrm{Hom}_{\mathcal{R}}(\mathcal{P}_i^*, \mathbb{N}) = \mathbb{N}(P_i) \underset{\neq}{\subset} \mathbb{M}(P_i) = \mathrm{Hom}_{\mathcal{R}}(\mathcal{P}_i^*, \mathbb{M})$.

Therefore, $\mathbb{U} := \oplus_{P \in A} \mathcal{P}^*$ is a generator of the category of $\mathcal{R}$-modules that preserve direct limits.

Theorem 3.7. *Let $\mathbb{M}$ be an $\mathcal{R}$-module. $\mathbb{M}$ preserves direct limits iff there exists an exact sequence of morphisms of $\mathcal{R}$-modules*

$$\oplus_{i \in I} \mathcal{Q}_i^* \to \oplus_{j \in J} \mathcal{P}_j^* \to \mathbb{M} \to 0,$$

where P_i, Q_j are finitely presented R-modules, for each $i \in I$ and $j \in J$.

Proof. $\Leftarrow$) $\oplus_i \mathcal{P}_i^*$ and $\oplus_j \mathcal{Q}_j^*$ preserve direct limits by 3.3 and 3.6. $\mathbb{M}$ preserves direct limits by 3.5.

$\Rightarrow$) Put $I := \mathrm{Hom}_{\mathcal{R}}(\mathbb{U}, \mathbb{M})$. Hence, the natural morphism

$$\pi \colon \oplus_I \mathbb{U} \to \mathbb{M}$$

is an epimorphism. By Proposition 3.5, $\mathrm{Ker}\, \pi$ preserve direct limits. Again, there exists an epimorphism $\oplus_J \mathbb{U} \to \mathrm{Ker}\, \pi$, and we conclude.

□

Definition 3.8. *We will denote by $\langle$Qs-ch$\rangle$ the full subcategory of the category of $\mathcal{R}$-modules whose objects are the $\mathcal{R}$-modules that preserve direct limits.*

Theorem 3.9. *$\langle$Qs-ch$\rangle$ is the smallest full subcategory of the category of $\mathcal{R}$-modules stable by kernels, cokernels, direct limits and isomorphisms (if an $\mathcal{R}$-module is isomorphic to an object of the subcategory then it belongs to the subcategory) that contains the $\mathcal{R}$-module $\mathcal{R}$ (or quassi-coherent modules).*

Proof. Quassi-coherent modules preserve direct limits. By Theorem 3.7, Proposition 3.5 and Proposition 3.6 we have only to prove that $\mathcal{P}^* \in \langle Qs\text{-}ch\rangle$, for any finitely presented R-module P. Consider an exact sequence of R-module morphisms $R^n \to R^m \to P \to 0$. Dually, $0 \to \mathcal{P}^* \to \mathcal{R}^m \to \mathcal{R}^n$ is exact and $\mathcal{P}^* \in \langle Qs\text{-}ch\rangle$. □

Corollary 3.10. *Let K be a field and $\mathbb{M}$ an $\mathcal{K}$-module. $\mathbb{M} \in \langle Qs\text{-}ch\rangle$ iff $\mathbb{M}$ is quasi-coherent.*

Corollary 3.11. *If $\mathbb{M} \in \langle Qs\text{-}ch\rangle$ then $\mathbb{M}^*$ preserves direct products.*

Proof. By 3.7, there exists an exact sequence of morphisms of $\mathcal{R}$-modules $\oplus_i \mathcal{Q}_i^* \to \oplus_j \mathcal{P}_j^* \to \mathbb{M} \to 0$, where P_i, Q_j are finitely presented R-modules, for every i, j. Taking dual $\mathcal{R}$-modules, we have the exact sequence of morphisms

$$0 \to \mathbb{M}^* \to \prod_j \mathcal{P}_j \to \prod_i \mathcal{Q}_i$$

$\mathbb{M}^*$ preserves direct products since $\prod_j \mathcal{P}_j$ and $\prod_i \mathcal{Q}_i$ preserve direct products. □

Proposition 3.12. [11, Th 1.] *Let $\mathbb{M}$ be an $\mathcal{R}$-module. $\mathbb{M}$ is a quasi-coherent $\mathcal{R}$-module iff $\mathbb{M} \in \langle Qs\text{-}ch\rangle$ and it is a right-exact functor.*

Proof. $\Leftarrow$) Consider the natural morphism $\mathbb{M}_{qc} \to \mathbb{M}$. Given $\oplus_I R$, observe that

$$\mathbb{M}_{qc}(\oplus_I R) = \oplus_I \mathbb{M}_{qc}(R) = \oplus_I \mathbb{M}(R) = \mathbb{M}(\oplus_I R).$$

Let N be an R-module and let $L_1 \to L_2 \to N \to 0$ be an exact sequence of morphisms of R-modules, where L_1 and L_2 are free R-modules. Then, $\mathbb{M}_{qc}(N) = \mathbb{M}(N)$ since $\mathbb{M}$ and $\mathbb{M}_{qc}$ are right exact. Therefore, the natural morphism $\mathbb{M}_{qc} \to \mathbb{M}$ is an isomorphism. □

$\oplus_{i\in I}\mathcal{P}_i^*$ is a projective $\mathcal{R}$-module, since

$$\mathrm{Hom}_{\mathcal{R}}(\oplus_{i\in I}\mathcal{P}_i^*, \mathbb{M}) = \prod_{i\in I} \mathbb{M}(P_i), \text{ for any } \mathcal{R}\text{-module } \mathbb{M}.$$

Then, $\langle Qs\text{-}ch\rangle$ has enough projective $\mathcal{R}$-modules. $\mathbb{U}$ is a generator of this category, hence, it has enough injective objects, by [4, Theorem 1.10.1].

Let $\mathbb{M}$ be an $\mathcal{R}$-module. Consider the obvious morphism $\pi\colon \oplus_{\mathrm{Hom}_{\mathcal{R}}(\mathbb{U},\mathbb{M})} \mathbb{U} \to \mathbb{M}$. Observe that π_P is surjective for any finitely presented R-module P. Likewise, consider the obvious morphism $\pi'\colon \oplus_{\mathrm{Hom}_{\mathcal{R}}(\mathbb{U},\mathrm{Ker}\,\pi)} \mathbb{U} \to \mathrm{Ker}\,\pi$. Again, π'_P is surjective for any finitely presented R-module P. Let ϕ be the composite morphism

$$\oplus_{\mathrm{Hom}_{\mathcal{R}}(\mathbb{U},\mathrm{Ker}\,\pi)}\mathbb{U} \xrightarrow{\pi'} \mathrm{Ker}\,\pi \subset \oplus_{\mathrm{Hom}_{\mathcal{R}}(\mathbb{U},\mathbb{M})}\mathbb{U}$$

Put $\mathbb{M}_{\langle Qs\text{-}ch\rangle} := \mathrm{Coker}\,\phi$. Observe that there is a natural morphism $\mathbb{M}_{\langle Qs\text{-}ch\rangle} \stackrel{i_{\mathbb{M}}}{\to} \mathbb{M}$ and that $\mathbb{M}_{\langle Qs\text{-}ch\rangle} \in \langle Qs\text{-}ch\rangle$. Besides,

$$\mathbb{M}_{\langle Qs\text{-}ch\rangle}(P) = \mathbb{M}(P),$$

for any finitely presented R-module P. Hence, if $\mathbb{M} \in \langle Qs\text{-}ch\rangle$, then $\mathbb{M}_{\langle Qs\text{-}ch\rangle} = \mathbb{M}$.

Observe that the assignation $\mathbb{M} \rightsquigarrow \mathbb{M}_{\langle Qs\text{-}ch\rangle}$ is functorial, that is, given a morphism a morphism $f\colon \mathbb{N} \to \mathbb{M}$, we can define a natural morphism $f_{\langle Qs\text{-}ch\rangle}\colon \mathbb{N}_{\langle Qs\text{-}ch\rangle} \to \mathbb{M}_{\langle Qs\text{-}ch\rangle}$. Besides, the diagram

$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{\quad f \quad} & \mathbb{M} \\ {\scriptstyle i_{\mathbb{N}}}\uparrow & & \uparrow{\scriptstyle i_{\mathbb{M}}} \\ \mathbb{N}_{\langle Qs\text{-}ch\rangle} & \xrightarrow{\ f_{\langle Qs\text{-}ch\rangle}\ } & \mathbb{M}_{\langle Qs\text{-}ch\rangle} \end{array}$$

is commutative. Obviously, we have the following proposition.

Proposition 3.13. *The functorial morphism*

$$\mathrm{Hom}_{\mathcal{R}}(\mathbb{N}, \mathbb{M}_{\langle Qs\text{-}ch\rangle}) \to \mathrm{Hom}_{\mathcal{R}}(\mathbb{N}, \mathbb{M}),\ \ f \mapsto i_{\mathbb{M}} \circ f,$$

is an isomorphism, for any $\mathcal{R}$-module $\mathbb{N} \in \langle Qs\text{-}ch\rangle$.

If $0 \to \mathbb{M}' \to \mathbb{M} \to \mathbb{M}'' \to 0$ is an exact sequence of morphisms of $\mathcal{R}$-modules, then

$$0 \to \mathbb{M}'_{\langle Qs\text{-}ch\rangle} \to \mathbb{M}_{\langle Qs\text{-}ch\rangle} \to \mathbb{M}''_{\langle Qs\text{-}ch\rangle} \to 0$$

is an exact sequence of morphisms of $\mathcal{R}$-modules.

Every R-module M is functorially a direct limit of finitely presented R-modules: Put $I = M$ and let $\pi\colon \oplus_I R \to M$ be the obvious epimorphism. For each finite subset $J \subseteq I$ let π_J be the obvious composition $\oplus_J R \hookrightarrow \oplus_I R \stackrel{\pi}{\to} M$. Let K_M be the set of pairs (J, N), where J is a finite subset of I and N is a finite submodule of $\mathrm{Ker}\,\pi_J$. K_M is a directed set: $(J, N) \leq (J', N')$ if $J \subseteq J'$ and $N \subseteq N'$. Given (J, N), (J', N'), let $J'' := J \cup J'$ and $N'' := N + N'$, then(J, N),$(J', N') \leq (J'', N'')$. It is easy to check that $M = \varinjlim\limits_{(J,N)\in K_M} (\oplus_J A)/N$. Let us denote $(\oplus_J A)/N = P_{(J,N)}$.

Let **Funct** be the category of covariant and additive functors from the category of finitely presented R-modules to the category of abelian groups. Given $\mathbb{G} \in$ **Funct**, let $i_*\mathbb{G}$ be the $\mathcal{R}$-module defined by $i_*\mathbb{G}(M) = \varinjlim\limits_{k\in K_M} \mathbb{G}(P_k)$. Given an $\mathcal{R}$-module $\mathbb{M}$ let $i^*\mathbb{M} \in$ **Funct** be defined by $i^*\mathbb{M}(P) := \mathbb{M}(P)$. Reader can check that i^* and i_* stablish a categorical equivalence between **Funct** and $\langle Qs\text{-}ch\rangle$.

4. FP modules

Definition 4.1. *An $\mathcal{R}$-module $\mathbb{M} \in \langle Qs\text{-}ch\rangle$ is said to be an F module if the natural morphism $\mathbb{M}(\prod_{i\in I} S_i) \to \prod_{i\in I} \mathbb{M}(S_i)$ is an epimorphism, for any set $\{S_i\}_{i\in I}$ of R-modules.*

Proposition 4.2. $\mathbb{M} \in \langle Qs\text{-}ch\rangle$ *is an F module iff there exists an epimorphism $\mathcal{P}^* \to \mathbb{M}$, where P is a finitely presented R-module.*

Proof. $\Rightarrow$) There exists an epimorphism $\pi\colon \oplus_j \mathcal{Q}_j^* \to \mathbb{M}$, by Theorem 3.7. Put $W := \prod_j Q_j$. Consider the projection $W \to Q_j$, then we have the natural morphism $\mathcal{Q}_j^* \to \mathcal{W}^*$ and the morphism $\oplus_j \mathcal{Q}_j^* \to \mathcal{W}^*$. The composite map

$$\mathrm{Hom}_{\mathcal{R}}(\mathcal{W}^*, \mathbb{M}) = \mathbb{M}(\prod_j Q_j) \to \prod_j \mathbb{M}(Q_j) = \prod_j \mathrm{Hom}_{\mathcal{R}}(\mathcal{Q}_j^*, \mathbb{M}) = \mathrm{Hom}_{\mathcal{R}}(\oplus_j \mathcal{Q}_j^*, \mathbb{M}),$$

is surjective. Then, π factors through an epimorphism $\pi'\colon \mathcal{W}^* \to \mathbb{M}$. Let $\{P_i\}$ be a direct system of finitely presented R-modules such that $W = \lim\limits_{\stackrel{\rightarrow}{i}} P_i$. Then,

$$\mathrm{Hom}_{\mathcal{R}}(\mathcal{W}^*, \mathbb{M}) = \mathbb{M}(W) = \lim_{\stackrel{\rightarrow}{i}} \mathbb{M}(P_i) = \lim_{\stackrel{\rightarrow}{i}} \mathrm{Hom}_{\mathcal{R}}(\mathcal{P}_i^*, \mathbb{M}),$$

and π' factors through an epimorphism $f\colon \mathcal{P}_i^* \to \mathbb{M}$.

□

Example 4.3. *M is a finite module iff $\mathcal{M}$ is an F module.*

Example 4.4. *Module schemes preserve direct products:*

$$\mathcal{N}^*(\prod_{i\in I} S_i) = \mathrm{Hom}_R(N, \prod_{i\in I} S_i) = \prod_{i\in I} \mathrm{Hom}_R(N, S_i) = \prod_{i\in I} \mathcal{N}^*(S_i).$$

Proposition 4.5. *A quasi-coherent module $\mathcal{M}$ preserves direct products iff M is a finitely presented (right) R-module.*

Proposition 4.6. *If $\mathbb{M}_1$ and $\mathbb{M}_2$ preserve direct products and $f\colon \mathbb{M}_1 \to \mathbb{M}_2$ is a morphism of $\mathcal{R}$-modules, then $\mathrm{Ker}\, f$, $\mathrm{Im}\, f$ and $\mathrm{Coker}\, f$ preserve direct products.*

Definition 4.7. *An $\mathcal{R}$-module $\mathbb{F} \in \langle Qs\text{-}ch\rangle$ will be said to be an FP module if it preserves direct products.*

Example 4.8. *Let P be a finitely presented R-module. Then, $\mathcal{P}$ and $\mathcal{P}^*$ are FP modules.*

Proposition 4.9. *Suppose that $\mathbb{F}_1$ and $\mathbb{F}_2$ are FP modules. If $f\colon \mathbb{F}_1 \to \mathbb{F}_2$ is a morphism of $\mathcal{R}$-modules, then $\mathrm{Ker}\, f$, $\mathrm{Im}\, f$ and $\mathrm{Coker}\, f$ are FP modules.*

Proposition 4.10. *Let $\mathbb{F}$ be an $\mathcal{R}$-module. $\mathbb{F}$ is an FP module iff there exists an exact sequence of $\mathcal{R}$-modules $\mathcal{P}^* \to \mathcal{Q}^* \to \mathbb{F} \to 0$, where P and Q are finitely presented R-modules.*

Proof. $\Leftarrow$) It is an immediate consequence of Proposition 4.9.

$\Rightarrow$) There exists an epimorphism $\pi\colon \mathcal{Q}* \to \mathbb{F}$, where Q is a finitely presented R-module, by Proposition 4.2. $\mathrm{Ker}\,\pi$ is an FP module by Proposition 4.9. Again, there exists an epimorphism $\mathcal{P}^* \to \mathrm{Ker}\,\pi$, for some finitely presented R-module P. We are done.

□

Proposition 4.11. *Let $\mathbb{F}$ be an $\mathcal{R}$-module. $\mathbb{F}$ is a projective and FP module iff $\mathbb{F} \simeq \mathcal{P}^*$ for a finitely presented R-module P.*

Proof. $\Rightarrow$) By Proposition 4.10, there exists an epimorphism $\mathcal{Q}^* \to \mathbb{F}$, where Q is a finitely presented R-module. Then, $\mathcal{Q}^* \simeq \mathbb{F} \oplus \mathbb{G}$, for some $\mathcal{R}$-module $\mathbb{G}$, since $\mathbb{F}$ is projective. In particular, $\mathbb{F}$ is reflexive since $\mathcal{Q}^*$ is reflexive. Taking dual modules, $\mathcal{Q} \simeq \mathbb{F}^* \oplus \mathbb{G}^*$. Then,

$$(\mathbb{F}^*)_{qc} \oplus {\mathbb{G}^*}_{qc} \simeq \mathcal{Q}_{qc} = \mathcal{Q} \simeq \mathbb{F}^* \oplus \mathbb{G}^*$$

Therefore, $(\mathbb{F}^*)_{qc} = \mathbb{F}^*$ and $(\mathbb{F}^*)_{qc}(R)$ is a finitely presented R-module since it is a direct summand of Q. Finally, $\mathbb{F} = ((\mathbb{F}^*)_{qc})^*$.

□

Proposition 4.12. *If $\mathbb{F}$ is an FP module then $\mathbb{F}^*$ is an FP (right) module.*

Proof. There exists an exact sequence of morphisms of $\mathcal{R}$-modules $\mathcal{P}^* \to \mathcal{Q}^* \to \mathbb{F} \to 0$, where P and Q are finitely presented R-modules. Taking dual modules, we obtain the exact sequence $0 \to \mathbb{F}^* \to \mathcal{Q} \to \mathcal{P}$. Hence, $\mathbb{F}^*$ is an FP (right) module by Proposition 4.9.

□

Proposition 4.13. [8, 7.14] *If* $0 \to \mathcal{M}_1 \overset{i'}{\to} \mathcal{M}_2 \overset{\pi'}{\to} \mathcal{M}_3 \to 0$ *is an exact sequence of morphisms of* $\mathcal{R}$*-modules and* M_3 *is a finitely presented module, then this exact sequence splits.*

Corollary 4.14. *Let* P *be a finitely presented* R*-module and* M *an* R*-module. Then,*

$$\mathrm{Ext}^1_{\mathcal{R}}(\mathcal{P}, \mathcal{M}) = 0.$$

Proof. If

$$0 \to \mathcal{M} \to \mathbb{M} \to \mathcal{P} \to 0$$

is an exact sequence of morphisms of $\mathcal{R}$-modules, then $\mathbb{M}$ is quasi-coherent since $\mathbb{M} \in \langle Qs\text{-}ch\rangle$ and it is right exact. By Proposition 4.13, the sequence of morphisms splits. Hence, $\mathrm{Ext}^1_{\mathcal{R}}(\mathcal{P}, \mathcal{M}) = 0$.

□

Lemma 4.15. *Let* $f\colon \mathcal{V}_2 \to \mathcal{V}_1$ *be a morphism of* $\mathcal{R}$*-modules between quasi-coherent modules. Then,* f *is an epimorphism iff* $f^*\colon \mathcal{V}_1^* \to \mathcal{V}_2^*$ *is a monomophism.*

Proof. $\Leftarrow$) $\mathrm{Coker}\, f$ is the quasi-coherent module associated with $\mathrm{Coker}\, f_R$, and $(\mathrm{Coker}\, f)^* = \mathrm{Ker}\, f^* = 0$. Then, $\mathrm{Coker}\, f = (\mathrm{Coker}\, f)^{**} = 0$.

□

Corollary 4.16. *Let* P *be a finitely presented* R*-module and* M *an* R*-module. Then,*

$$\mathrm{Ext}^i_{\mathcal{R}}(\mathcal{P}, \mathcal{M}) = 0, \ \textit{for any } i > 0.$$

Proof. Let $R^n \to P$ be an epimorphism and let $\pi\colon \mathcal{R}^n \to \mathcal{P}$ be the induced morphism. Observe that $\mathrm{Ext}^1_{\mathcal{R}}(\mathcal{P}, \mathcal{M}) = 0$, by Corollary 4.14 and $\mathrm{Ext}^{i+1}_{\mathcal{R}}(\mathcal{P}, \mathcal{M}) = \mathrm{Ext}^i_{\mathcal{R}}(\mathrm{Ker}\,\pi, \mathcal{M})$ for any $i \geq 1$.

$\mathrm{Ker}\,\pi$ is an FP (right) module, by 4.9. There exists an epimorphism $g\colon \mathcal{Q}^* \to \mathrm{Ker}\,\pi$, where Q is a finitely presented R-module, by 4.10. Let g' be the composite morphism $\mathcal{Q}^* \to \mathrm{Ker}\,\pi \subseteq \mathcal{R}^n$. $\mathrm{Ker}\, g = \mathrm{Ker}\, g' = (\mathrm{Coker}\, g'^*)^*$, by 2.6. $\mathrm{Coker}\, g'^*$ is equal to the quasi-coherent $\mathcal{R}$-module associated with $\mathrm{Coker}\, g'^*_R =: Q'$, which is a finitely presented R-module. We have the exact sequence of morphisms

$$0 \to \mathcal{Q}'^* \to \mathcal{Q}^* \to \mathrm{Ker}\,\pi \to 0$$

Then, $\mathrm{Ext}^i_{\mathcal{R}}(\mathrm{Ker}\,\pi, \mathcal{M}) = 0$, for $i > 1$. Taking dual $\mathcal{R}$-modules we have the exact sequence of morphisms

$$0 \to (\mathrm{Ker}\,\pi)^* \to \mathcal{Q} \to \mathcal{Q}' \to 0,$$

by 4.15. Hence, $\mathrm{Ext}^1_{\mathcal{R}}(\mathrm{Ker}\,\pi, \mathcal{M}) = 0$.

□

Theorem 4.17. *Let* $\mathbb{F}$ *be an FP module. Then,* $\mathbb{F}$ *is reflexive and*

$$\mathrm{Ext}^i_{\mathcal{R}}(\mathbb{F}, \mathcal{M}) = 0, \ \textit{for any } i > 0 \textit{ and for any (right) } \mathcal{R}\textit{-module } M.$$

Proof. By 4.10, there exists an exact sequence of morphisms $\mathcal{P}_2^* \xrightarrow{f} \mathcal{P}_1^* \xrightarrow{g} \mathbb{F} \to 0$, where P_1 and P_2 are finitely prensented R-modules. Taking dual $\mathcal{R}$-modules, we have the exact sequence of morphisms

$$0 \to \mathbb{F}^* \to \mathcal{P}_1 \xrightarrow{f^*} \mathcal{P}_2$$

Put $P_3 := \operatorname{Coker} f_R^*$, which is a finitely presented R-module. Then, we have the exact sequence of morphisms of $\mathcal{R}$-modules

$$0 \to \mathbb{F}^* \to \mathcal{P}_1 \xrightarrow{f^*} \mathcal{P}_2 \to \mathcal{P}_3 \to 0$$

By Corollary 4.16, it is easy to prove that $\operatorname{Ext}^i_{\mathcal{R}}(\mathbb{F}^*, \mathcal{M}) = 0$, for any $i > 0$ and for any R-module M. Hence, the sequence of morphisms

$$0 \to \mathcal{P}_3^* \to \mathcal{P}_2^* \xrightarrow{f} \mathcal{P}_1^* \to \mathbb{F}^{**} \to 0$$

is exact and $\mathbb{F} = \mathbb{F}^{**}$. Finally, $\mathbb{F}$ is the dual module of $\mathbb{F}^*$, which is an FP (right) module, by 4.12. We have just proved that $\operatorname{Ext}^i_{\mathcal{R}}(\mathbb{F}, \mathcal{M}) = 0$, for any $i > 0$ and for any (right) R-module M.

□

Corollary 4.18. *Let $\mathbb{F}_1$, $\mathbb{F}_2$ and $\mathbb{F}_3$ be FP modules. If $\mathbb{F}_1 \to \mathbb{F}_2 \to \mathbb{F}_3$ is an exact sequence of morphisms of $\mathcal{R}$-modules, then the dual sequence $\mathbb{F}_3^* \to \mathbb{F}_2^* \to \mathbb{F}_1^*$ is exact.*

Corollary 4.19. *Let $\mathbb{F}$ be a right $\mathcal{R}$-module. $\mathbb{F}$ is an FP (right) module iff there exists an exact sequence of morphisms of $\mathcal{R}$-modules*

$$0 \to \mathbb{F} \to \mathcal{P} \to \mathcal{Q},$$

where P and Q are finitely presented R-modules.

Proof. $\Leftarrow$) It is an immediate consequence of 4.9.

$\Rightarrow$) $\mathbb{F}^*$ is an FP module, by 4.12. By 4.10, there exists an exact sequence of morphisms of $\mathcal{R}$-modules $\mathcal{Q}^* \to \mathcal{P}^* \to \mathbb{F}^*$, where P and Q are finitely presented R-modules. Taking dual $\mathcal{R}$-modules, we have the exact sequence

$$0 \to \mathbb{F} \overset{4.17}{=} \mathbb{F}^{**} \to \mathcal{P} \to \mathcal{Q}.$$

□

Corollary 4.20. *Let $\mathbb{M}$ be an $\mathcal{R}$-module. $\mathbb{M}$ is an FP module iff it is reflexive and $\mathbb{M}, \mathbb{M}^* \in \langle Qs\text{-}ch \rangle$.*

Proof. $\Rightarrow$) It is an immediate consequence of Theorem 4.17.

$\Leftarrow$) By Corollary 3.11, $\mathbb{M}^*$ preserves direct products, therefore it is an FP module. By Proposition 4.12, $\mathbb{M} = \mathbb{M}^{**}$ is an FP module.

□

Lemma 4.21. *Let $\mathbb{F}$ be an FP module and $\{\mathbb{M}_i\}$ a direct system of $\mathcal{R}$-modules. Then,* $\operatorname{Hom}_{\mathcal{R}}(\mathbb{F}, \varinjlim_i \mathbb{M}_i) = \varinjlim_i \operatorname{Hom}_{\mathcal{R}}(\mathbb{F}, \mathbb{M}_i)$.

Proof. By 4.10, there exists an exact sequence of morphisms $\mathcal{P}^* \to \mathcal{Q}^* \to \mathbb{F} \to 0$, where P and Q are finitely presented R-modules. By 2.16, $\operatorname{Hom}_{\mathcal{R}}(\mathcal{P}^*, \varinjlim_i \mathbb{M}_i) =$

$\lim\limits_{\substack{\rightarrow\\ i}} \mathrm{Hom}_{\mathcal{R}}(\mathcal{P}^*, \mathbb{M}_i)$, for any finitely presented $\mathcal{R}$-module P. Now it is easy to prove that $\mathrm{Hom}_{\mathcal{R}}(\mathbb{F}, \lim\limits_{\substack{\rightarrow\\ i}} \mathbb{M}_i) = \lim\limits_{\substack{\rightarrow\\ i}} \mathrm{Hom}_{\mathcal{R}}(\mathbb{F}, \mathbb{M}_i)$.

□

Theorem 4.22. *Let $\mathbb{M}$ be an $\mathcal{R}$-module. $\mathbb{M} \in \langle Qs\text{-}ch\rangle$ iff it is a direct limit of FP modules.*

Proof. ⇒) By 3.7, there exists an exact sequence of morphism of $\mathcal{R}$-modules

$$\oplus_{i\in I}\mathcal{P}_i^* \to \oplus_{j\in J}\mathcal{Q}_j^* \to \mathbb{M} \to 0,$$

where P_i, Q_j are finitely presented R-modules, for any i, j. Let F (respectively G) be the set of all finite subsets of I (respectively J). By 4.21, given $I' \in F$ there exists $J' \in G$ such that the composite morphism $\oplus_{i\in I'}\mathcal{P}_i^* \hookrightarrow \oplus_{i\in I}\mathcal{P}_i^* \to \oplus_{j\in J}\mathcal{Q}_j^*$ factors through $\oplus_{j\in J'}\mathcal{Q}_j^*$, since $\oplus_{i\in I'}\mathcal{P}_i^* = (\oplus_{i\in I'}\mathcal{P}_i)^*$ is an FP module. We will say that $J' \geq I'$ and we will denote

$$\mathbb{F}_{J'\geq I'} := \mathrm{Coker}[\oplus_{i\in I'}\mathcal{P}_i^* \to \oplus_{j\in J'}\mathcal{Q}_j^*].$$

Let H be the set of pairs (J', I'), where $J' \in G$, $I' \in F$ and $J' \geq I'$. Now, it is easy to check that $\mathbb{M} = \lim\limits_{\substack{\rightarrow\\ (J',I')\in H}} \mathbb{F}_{J'\geq I'}$.

⇐) It is an immediate consequence of 3.6.

□

Corollary 4.23. $\mathbb{M} \in \langle Qs\text{-}ch\rangle$ *is an FP module iff*

$$\mathrm{Hom}_{\mathcal{R}}(\mathbb{M}, \lim_{\substack{\rightarrow\\ i\in I}} \mathbb{M}_i) = \lim_{\substack{\rightarrow\\ i\in I}} \mathrm{Hom}_{\mathcal{R}}(\mathbb{M}, \mathbb{M}_i)$$

for any direct system of $\mathcal{R}$-modules $\{\mathbb{M}_i\}_{i\in I}$.

Proof. ⇒) It is Lemma 4.21.

⇐) By Theorem 4.22, $\mathbb{M} = \lim\limits_{\substack{\rightarrow\\ i\in I}} \mathbb{F}_i$, where $\mathbb{F}_i$ is a FP modules, for any i. The identity morphism $\mathbb{M} \to \mathbb{M} = \lim\limits_{\substack{\rightarrow\\ i\in I}} \mathbb{F}_i$ factors through a morphism $\mathbb{M} \to \mathbb{F}_i$, for some $i \in I$. Then, $\mathbb{M}$ is a direct summand of $\mathbb{F}_i$. By Proposition 4.9, $\mathbb{M}$ is an FP module.

□

Proposition 4.24. *Let I be an injective R-module. Then,*

$$\mathrm{Hom}_{\mathcal{R}}(\mathbb{M}, \mathcal{I}) = \mathrm{Hom}_R(\mathbb{M}(R), I),$$

for any $\mathcal{R}$-module $\mathbb{M} \in \langle Qs\text{-}ch\rangle$. In particular, $\mathcal{I}$ is an injective object of $\langle Qs\text{-}ch\rangle$.

Proof. Let $\mathbb{F}$ be an FP $\mathcal{R}$-module. By Corollary 4.19, there exists an exact sequence of $\mathcal{R}$-module morphisms

$$0 \to \mathbb{F} \to \mathcal{P} \to \mathcal{Q}$$

where P and Q are finitely presented R-modules. By Corollary 4.18, we have the exact sequence morphisms of groups

$$\mathrm{Hom}_{\mathcal{R}}(\mathcal{Q}, \mathcal{I}) \to \mathrm{Hom}_{\mathcal{R}}(\mathcal{P}, \mathcal{I}) \to \mathrm{Hom}_{\mathcal{R}}(\mathbb{F}, \mathcal{I}) \to 0$$

On the other hand,

$$\mathrm{Hom}_R(Q, I) \to \mathrm{Hom}_R(P, I) \to \mathrm{Hom}_R(\mathbb{F}(R), I) \to 0$$

is exact, because I is an injective R-module. Hence, $\mathrm{Hom}_{\mathcal{R}}(\mathbb{F}, \mathcal{I}) = \mathrm{Hom}_R(\mathbb{F}(R), I)$. If $\mathbb{M}' \in \langle Qs\text{-}ch\rangle$, then $\mathbb{M}' = \lim\limits_{\stackrel{\to}{i}} \mathbb{F}_i$, where $\mathbb{F}_i$ are FP $\mathcal{R}$-modules, by Theorem 4.22. Then,

$$\mathrm{Hom}_{\mathcal{R}}(\mathbb{M}', \mathcal{I}) = \lim_{\stackrel{\leftarrow}{i}} \mathrm{Hom}_{\mathcal{R}}(\mathbb{F}_i, \mathcal{I}) = \lim_{\stackrel{\leftarrow}{i}} \mathrm{Hom}_R(\mathbb{F}_i(R), \mathcal{I}) = \mathrm{Hom}_R(\mathbb{M}'(R), I)$$

□

Definition 4.25. *An R-module M is said to be pure-injective if for any pure morphism $N \hookrightarrow N'$ the induced morphism*

$$\mathrm{Hom}_R(N', M) \to \mathrm{Hom}_R(N, M)$$

is surjective.

Proposition 4.26. $\mathbb{M}$ *is an injective object of $\langle Qs\text{-}ch\rangle$ iff* $\mathbb{M}$ *is the quasi-coherent $\mathcal{R}$-module associated with a pure-injective R-module.*

Proof. $\Rightarrow$) First, let us prove that $\mathbb{M}$ is quassi-coherent. By Proposition 3.12, we have to prove that $\mathbb{M}$ is a right-exact functor. Let $N_1 \stackrel{f}{\to} N_2 \stackrel{g}{\to} N_3 \to 0$ be an exact sequence of R-module morphisms. We have to prove that the sequence of morphisms $\mathbb{M}(N_1) \to \mathbb{M}(N_2) \to \mathbb{M}(N_3) \to 0$ is exact. Put $N_1 := \lim\limits_{\stackrel{\to}{i}} P_i$, where P_i are finitely presented R-modules and $N_{3i} := N_2/f(P_i)$. We have the exact sequences $P_i \to N_2 \to N_{3i} \to 0$ and $\lim\limits_{\stackrel{\to}{i}} P_i = N_1$ and $\lim\limits_{\stackrel{\to}{i}} N_{3i} = N_3$. We have only to prove that that the sequence of morphisms $\mathbb{M}(P_i) \to \mathbb{M}(N_2) \to \mathbb{M}(N_{3i}) \to 0$ is exact. That is, we can suppose that N_1 is a finitely presented R-module. Put $N_2 = \lim\limits_{\stackrel{\to}{j}} P_j$. The morphism $f\colon N_1 \to N_2$ factors through a morphism $f_j\colon N_1 \to P_j$. Let $f_{j'}$ be the composite morphism $N_1 \stackrel{f'}{\to} N_j \to N_{j'}$ for any $j' \geq j$. Put $N_{3j'} := N_2/f_{j'}(N_1)$. We have the exact sequences $N_1 \to P_{j'} \to N_{3j'} \to 0$ and $\lim\limits_{\stackrel{\to}{j'>j}} P_{j'} = N_2$ and $\lim\limits_{\stackrel{\to}{j'>j}} N_{3j'} = N_3$. We have only to prove that that the sequence of morphisms $\mathbb{M}(N_1) \to \mathbb{M}(P_{j'}) \to \mathbb{M}(N_{3j'}) \to 0$ is exact. That is, we can suppose that N_1, N_2 and N_3 are finitely presented R-modules. The sequence of $\mathcal{R}$-module morphisms $0 \to \mathcal{N}_3^* \to \mathcal{N}_2^* \to \mathcal{N}_1^*$ is an exact sequence. Then, taking $\mathrm{Hom}_{\mathcal{R}}(-, \mathbb{M})$ the sequence

$$\mathbb{M}(N_1) \to \mathbb{M}(N_2) \to \mathbb{M}(N_3) \to 0$$

is exact. We are done.

Put $\mathbb{M} := \mathcal{M}$. If $N \hookrightarrow N'$ is a pure morphism, then the induced morphism $\mathcal{N} \hookrightarrow \mathcal{N}'$ is a monomorphism. Then, the morphism

$$\mathrm{Hom}_R(N', M) = \mathrm{Hom}_{\mathcal{R}}(\mathcal{N}', \mathcal{M}) \to \mathrm{Hom}_{\mathcal{R}}(\mathcal{N}, \mathcal{M}) = \mathrm{Hom}_R(N, M)$$

is surjective and M is pure-injective.

$\Leftarrow$) Let M be pure-injective. $\langle Qs\text{-}ch\rangle$ has enough injective modules. Let $i\colon \mathcal{M} \hookrightarrow \mathcal{M}'$ be a monomorphism, where $\mathcal{M}'$ is an injective object of $\langle Qs\text{-}ch\rangle$. The morphism i has a retraction since M is pure-injective. $\mathcal{M}$ is an injective object of $\langle Qs\text{-}ch\rangle$ since it is a direct summand of $\mathcal{M}'$.

□

5. Mittag-Leffler modules

Mittag-Leffler conditions were first introduced by Grothendieck in [3], and deeply studied by some authors, such as Raynaud and Gruson in [7]. Recently, Drinfeld suggested to employ them in infinite dimensional algebraic geometry (see [2] and [6])

Definition 5.1. *We will say that an $\mathcal{R}$-module $\mathbb{M} \in \langle Qs\text{-}ch \rangle$ is an ML module if the natural morphism $\mathbb{M}(\prod_{i\in I} S_i) \to \prod_{i\in I} \mathbb{M}(S_i)$ is injective for any set $\{S_i\}_{i\in I}$ of R-modules.*

Example 5.2. *FP modules are ML modules*

Example 5.3. *M is an Mittag-Leffler module iff $\mathcal{M}$ is a ML module(*[10, Tag 059H]*).*

Proposition 5.4. *Let $\mathbb{M}$ and $\mathbb{M}'$ be ML modules and $f\colon \mathbb{M} \to \mathbb{M}'$ a morphism of $\mathcal{R}$-modules. Then,* $\operatorname{Ker} f$ *and* $\operatorname{Im} f$ *are ML modules.*

Proof. $\operatorname{Ker} f$ and $\operatorname{Im} f$ preserve direct limits, by 3.5. If $\mathbb{F}'$ is an $\mathcal{R}$-submodule of an ML module $\mathbb{F}$, then the morphism $\mathbb{F}'(\prod_i S_i) \to \prod_i \mathbb{F}'(S_i)$ is injective for any set of $\{S_i\}_{i\in I}$ R-modules. Hence, $\operatorname{Ker} f$ and $\operatorname{Im} f$ are ML modules.

□

Lemma 5.5. *Let $\mathbb{M}$ be an ML module, $\mathbb{F}$ an FP module and $f\colon \mathbb{F} \to \mathbb{M}$ a morphism of $\mathcal{R}$-modules. Then,* $\operatorname{Ker} f$ *and* $\operatorname{Im} f$ *are FP modules.*

Proof. $\operatorname{Ker} f$ preserves direct limits by 3.5. Let $\{S_i\}_{i\in I}$ be a set of R-modules. Consider the commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 \longrightarrow & \operatorname{Ker} f(\prod_{i\in I} S_i) & \longrightarrow & \mathbb{F}(\prod_{i\in I} S_i) & \longrightarrow & \mathbb{M}(\prod_{i\in I} S_i) \\
& \downarrow & & \| & & \downarrow \\
0 \longrightarrow & \prod_{i\in I} \operatorname{Ker} f(S_i) & \longrightarrow & \prod_{i\in I} \mathbb{F}(S_i) & \longrightarrow & \prod_{i\in I} \mathbb{M}(S_i)
\end{array}$$

Hence, $\operatorname{Ker} f(\prod_{i\in I} S_i) = \prod_{i\in I} \operatorname{Ker} f(S_i)$ and $\operatorname{Ker} f$ is an FP module. $\operatorname{Im} f$ is isomorphic to the cokernel of the monomorphism $\operatorname{Ker} f \to \mathbb{F}$, which is an FP module by 4.9.

□

Lemma 5.6. *If $\{\mathbb{M}_i, f_{ij}\}$ is a direct system of ML modules and f_{ij} is a monomorphism for any $i \le j$, then $\lim\limits_{\substack{\to\\ i}} \mathbb{M}_i$ is an ML module.*

Proof. $\lim\limits_{\substack{\to\\ i}} \mathbb{M}_i \in \langle Qs\text{-}ch \rangle$. Besides, the composite morphism

$$\begin{aligned}
(\lim_{\substack{\to\\ i}} \mathbb{M})(\prod_j S_j) = \lim_{\substack{\to\\ i}} \mathbb{M}_i(\prod_j S_j) &\hookrightarrow \lim_{\substack{\to\\ i}} \prod_j \mathbb{M}_i(S_j) \\
\hookrightarrow \prod_j \lim_{\substack{\to\\ i}} \mathbb{M}_i(S_j) &= \prod_j (\lim_{\substack{\to\\ i}} \mathbb{M}_i)(S_j)
\end{aligned}$$

is injective, for any set $\{S_j\}_{j\in J}$ of R-modules.

□

Proposition 5.7. *An $\mathcal{R}$-module $\mathbb{M}$ is an ML module iff $\mathbb{M}$ is a direct limit of FP submodules.*

Proof. $\Rightarrow$) By 3.7, there exists an epimorphism $\pi\colon \oplus_{i\in I} \mathcal{P}_i^* \to \mathbb{M}$. Let F be the set of all finite subsets of I. Given $J \in F$, put $\mathbb{F}_J := \pi(\oplus_{i\in J}\mathcal{P}_i^*)$, which is an FP module by 5.5. Obviously, $\mathbb{M} = \lim\limits_{\substack{\to \\ J\in F}} \mathbb{F}_J$.

$\Leftarrow$) It is an immediate consequence of 5.6. □

Proposition 5.8. $\mathbb{M} \in \langle Qs\text{-}ch\rangle$ *is an ML module iff for every FP module $\mathbb{F}$ and every morphism $f\colon \mathbb{F} \to \mathbb{M}$, $\operatorname{Im} f$ is an FP module.*

Proof. $\Rightarrow$) It is Lemma 5.5.

$\Leftarrow$) By 3.7, there exists an epimorphism $\pi\colon \oplus_{i\in I} \mathcal{P}_i^* \to \mathbb{M}$. Let F be the set of all finite subsets of I. Given $J \in F$, put $\mathbb{F}_J := \pi(\oplus_{i\in J}\mathcal{P}_i^*)$, which is an FP module. Obviously, $\mathbb{M} = \lim\limits_{\substack{\to \\ J\in F}} \mathbb{F}_J$. By 5.7, $\mathbb{M}$ is an ML module.

□

We can now generalize a crucial closure property of the category of Mittag-Leffler modules (see [6, Prop 2.2])

Corollary 5.9. *If $\{\mathbb{M}_i, f_{ij}\}_{i,j\in I}$ is a direct system of ML modules and $\lim\limits_{\substack{\to \\ n\in\mathbb{N}}} \mathbb{M}_{i_n}$ is an ML module for any ordered subset $\{i_1 \leq i_2 \leq \cdots \leq i_n \leq \cdots\}$ of I, then $\lim\limits_{\substack{\to \\ i\in I}} \mathbb{M}_i$ is an ML module.*

Proof. Let $\mathbb{F}$ be an FP module and $f\colon \mathbb{F} \to \lim\limits_{\substack{\to \\ i}} \mathbb{M}_i$ a morphism of $\mathcal{R}$-modules. By Lemma 4.21, f factors though a morphism $f_i\colon \mathbb{F} \to \mathbb{M}_i$. Put $f_j := f_{ij} \circ f_i$ for any $j \geq i$ and put $\mathbb{K}_j := \operatorname{Ker} f_j$. Observe that $\mathbb{K}_j \subseteq \mathbb{K}_{j'}$, for any $j' \geq j \geq i$. Let

$$i \leq j_1 < j_2 < \cdots < j_n < \cdots$$

be an ordered subset of I. Observe that $\mathbb{K}' := \cup_{n\in\mathbb{N}}\mathbb{K}_{j_n}$ equals the kernel of the natural morphism $\mathbb{F} \to \lim\limits_{\substack{\to \\ n\in\mathbb{N}}} \mathbb{M}_{j_n}$. Hence, $\mathbb{K}'$ is an FP module. The identity morphism $Id\colon \mathbb{K}' \to \mathbb{K}'$ factors through a morphism $\mathbb{K}' \to \mathbb{K}_{j_n}$, by Lemma 4.21. Then, $\mathbb{K}' = \mathbb{K}_{j_n}$ and $\mathbb{K}_{j_n} = \mathbb{K}_{j_{n+r}}$, for any $r > 0$. Therefore, there exists $j \in I$ such that $\operatorname{Ker} f_j = \operatorname{Ker} f$. Hence, $\operatorname{Im} f = \operatorname{Im} f_j$ which is an FP module. Then, $\lim\limits_{\substack{\to \\ i\in I}} \mathbb{M}_i$ is an ML module.

□

Theorem 5.10. *Let $\mathbb{M} \in \langle Qs\text{-}ch\rangle$. The following statements are equivalent*

1. $\mathbb{M}$ *is an ML module.*
2. *The kernel of every morphism of $\mathcal{R}$-modules $\mathbb{F} \to \mathbb{M}$ is an FP module, for any FP module $\mathbb{F}$.*
3. *The kernel of every morphism $\mathcal{N}^* \to \mathbb{M}$ is isomorphic to a quotient of a module scheme, for any R-module N.*
4. *The kernel of every morphism $\mathcal{N}^* \to \mathbb{M}$ preserves direct products, for any R-module N.*

Proof. (1) $\iff$ (2) $\operatorname{Im} f$ is an FP module iff $\operatorname{Ker} f$ is an FP module, by 4.9, and $\operatorname{Im} f$ is an FP module iff $\mathbb{M}$ is an ML module, by 5.8.

(2) $\Rightarrow$ (3) Let $f\colon \mathcal{N}^* \to \mathbb{M}$ be a morphism of $\mathcal{R}$-modules. There exists a direct system $\{P_i\}$ of finitely presented R-modules such that $N = \lim\limits_{\substack{\to \\ i}} P_i$. Observe, that

$$\operatorname{Hom}_{\mathcal{R}}(\mathcal{N}^*, \mathbb{M}) = \mathbb{M}(N) = \lim_{\substack{\to \\ i}} \mathbb{M}(P_i) = \lim_{\substack{\to \\ i}} \operatorname{Hom}_{\mathcal{R}}(\mathcal{P}_i^*, \mathbb{M}).$$

Therefore, f factors through a morphism $g\colon \mathcal{P}_i^* \to \mathbb{M}$. $\operatorname{Ker} g$ is an FP module since $\mathcal{P}_i^*$ is an FP module. There exists an epimorphism $\mathcal{Q}^* \to \operatorname{Ker} g$ by 4.10 Consider the morphism $\pi_1\colon \mathcal{N}^* \times_{\mathcal{P}_i^*} \mathcal{Q}^* \to \mathcal{N}^*$, $\pi_1(w,v) = w$. It is easy to check that $\operatorname{Im} \pi_1 = \operatorname{Ker} f$. Finally, observe that $\mathcal{N}^* \times_{\mathcal{P}_i^*} \mathcal{Q}^* = (\mathcal{N} \oplus_{\mathcal{P}} \mathcal{Q})^*$ and observe that $\mathcal{N} \oplus_{\mathcal{P}} \mathcal{Q}$ is a quasi-coherent module since it is equal to the cokernel of a morphism $\mathcal{P} \to \mathcal{N} \oplus \mathcal{Q}$.

(3) $\Rightarrow$ (4) $\operatorname{Ker}[\mathcal{N}^* \to \mathbb{M}] \simeq \operatorname{Im}[\mathcal{N}'^* \to \mathcal{N}^*]$, for some R-module N'. Hence, $\operatorname{Ker}[\mathcal{N}^* \to \mathbb{M}]$ preserves direct products, by 4.6.

(4) $\Rightarrow$ (1) Let $\mathbb{F}$ be an FP module and $f\colon \mathbb{F} \to \mathbb{M}$ a morphism of $\mathcal{R}$-modules By 4.10, there exists an epimorphism $\mathcal{P}^* \to \mathbb{F}$. Let g be the composite morphism $\mathcal{P}^* \to \mathbb{F} \to \mathbb{M}$. Then, $\operatorname{Im} f = \operatorname{Im} g$. $\operatorname{Im} g$ preserves direct limits, by 3.5. $\operatorname{Ker} g$ preserves direct products by Hypothesis. $\operatorname{Im} g = \operatorname{Coker}[\operatorname{Ker} g \to \mathcal{P}^*]$ preserves direct products, by 4.6. Therefore, $\operatorname{Im} f = \operatorname{Im} g$ is an FP module and $\mathbb{M}$ is an ML module, by 5.8.

□

Theorem 5.11. *Let M be a right R-module. M is a Mittag-Leffler module iff the image of every morphism $f\colon \mathcal{R}^n \to \mathcal{M}$ is an FP module, for any n.*

Proof. $\Rightarrow$) By 4.9, $\operatorname{Im} f$ is an FP module.

$\Leftarrow$) Obviously $\mathcal{M}$ is a direct limit of FP modules. By Proposition 5.7, $\mathcal{M}$ is an ML module, hence M is a Mittag-Leffler module.

□

Corollary 5.12. *Let M be a right R-module. M is a Mittag-Leffler module iff the image of every morphism of $\mathcal{R}$-modules $f\colon \prod_{\mathbb{N}} \mathcal{R} \to \mathcal{M}$ is an FP module (or preserves direct products).*

Proof. Any morphism $f\colon \prod_{\mathbb{N}} \mathcal{R} \to \mathcal{M}$ is the composite morphism of an epimorphism $\prod_{\mathbb{N}} \mathcal{R} \to \mathcal{R}^n$ and a morphism $g\colon \mathcal{R}^n \to \mathcal{M}$, by 2.12. Observe that $\operatorname{Im} f = \operatorname{Im} g$ and $\operatorname{Im} g$ is an FP module iff it preserves direct products.

□

Corollary 5.13. *Let M be an R-module. M is a Mittag-Leffler module iff the kernel of every morphism of $\mathcal{R}$-modules $f\colon \prod_{\mathbb{N}} \mathcal{R} \to \mathcal{M}$ preserves direct products.*

Proof. It is an immediate consequence of 5.12, since $\operatorname{Ker} f$ preserves direct products iff $\operatorname{Im} g$ preserves directs products.

□

6. Strict Mittag-Leffler modules

Definition 6.1. *We will say that an $\mathcal{R}$-module $\mathbb{M} \in \langle Qs\text{-}ch\rangle$ is an SML module if it is an ML $\mathcal{R}$-module and for every FP submodule $\mathbb{F} \subseteq \mathbb{M}$ the dual morphism $\mathbb{M}^* \to \mathbb{F}^*$ is an epimorphism.*

Theorem 6.2. *Let $\mathbb{M}$ be an $\mathcal{R}$-module. $\mathbb{M}$ is an SML module iff there exists a direct system $\{\mathbb{F}_i\}_{i\in I}$ of FP submodules of $\mathbb{M}$ such that $\mathbb{M} = \lim\limits_{\substack{\rightarrow\\ i\in I}} \mathbb{F}_i$ and the natural morphism $\mathbb{M}^* \to \mathbb{F}_i^*$ is an epimorphism, for each $i \in I$.*

Proof. $\Rightarrow$) It is an immediate consequence of Proposition 5.7.

$\Leftarrow$) By Proposition 5.7, $\mathbb{M}$ is an ML module. Let $\mathbb{F} \subseteq \mathbb{M}$ be an FP submodule. This monomorphism factors through a monomorphism $\mathbb{F} \to \mathbb{F}_i$, by Lemma 4.21. Dually, $\mathbb{M}^* \to \mathbb{F}_i^*$ is an epimorphism by the hypothesis, and $\mathbb{F}_i^* \to \mathbb{F}^*$ is an epimorphism by Corollary 4.18. Therefore, the morphism $\mathbb{M}^* \to \mathbb{F}^*$ is an epimorphism.

□

Proposition 6.3. *A reflexive $\mathcal{R}$-module $\mathbb{M} \in \langle$Qs-ch$\rangle$ is an SML module if for every FP (right) module $\mathbb{F}$ the image of every morphism $f\colon \mathbb{M}^* \to \mathbb{F}$ is an FP (right) module.*

Proof. $\Rightarrow$) Consider the dual morphism $f^*\colon \mathbb{F}^* \to \mathbb{M}$. $\operatorname{Im} f^*$ is an FP module by Proposition 5.8. Dually, $\mathbb{F} \to (\operatorname{Im} f^*)^*$ is an epimorphism by Corollary 4.18, and the morphism $(\operatorname{Im} f^*)^* \to \mathbb{M}$ is a monomorphism. Hence, $\operatorname{Im} f = (\operatorname{Im} f^*)^*$, which is an FP module by Proposition 4.12.

$\Leftarrow$) Let $g\colon \mathbb{F} \to \mathbb{M}$ be an $\mathcal{R}$-module morphism. Again, $\operatorname{Im} g = (\operatorname{Im} g^*)^*$ which is an FP module. Hence, $\mathbb{M}$ is an ML $\mathcal{R}$-module, by Proposition 5.8. If g is a monomorphism, $\operatorname{Im} g^* = (\operatorname{Im} g^*)^{**} = (\operatorname{Im} g)^* = \mathbb{F}^*$, g^* is an epimorphism and $\mathbb{M}$ is an SML module.

□

Corollary 6.4. *Let M be a right R-module. $\mathcal{M}$ is an SML module iff for every finitely generated submodule $N \subseteq M$ the image $\tilde{N}$ of the associated morphism $\mathcal{N} \to \mathcal{M}$ is an FP module and the morphism $\mathcal{M}^* \to \tilde{N}^*$ is an epimorphism.*

Proof. $\Rightarrow$) $\tilde{N}$ is an FP module by 5.11. $\mathcal{M}^* \to \tilde{N}^*$ is an epimorphism, by 6.1.

$\Leftarrow$) It is an immediate consequence of 6.2.

□

Proposition 6.5. *Let $\mathbb{M} \in \langle$Qs-ch$\rangle$ be a reflexive $\mathcal{R}$-module. $\mathbb{M}$ is an SML module iff for any R-module N and any morphism $g\colon \mathcal{N}^* \to \mathbb{M}$, $\operatorname{Coker} g^*$ is isomorphic to an $\mathcal{R}$-submodule of a quasi-coherent (right) module.*

Proof. $\Rightarrow$) Let $\{\mathbb{F}_i\}_{i\in I}$ be a direct system of FP submodules of $\mathbb{M}$ such that $\mathbb{M} = \lim\limits_{\substack{\rightarrow\\ i}} \mathbb{F}_i$. By 2.16, g factors through a morphism $g_i\colon \mathcal{N}^* \to \mathbb{F}_i$, for some $i \in I$. Therefore, g^* is the composite morphism of the epimorphism $\mathbb{M}^* \to \mathbb{F}_i^*$ and $g_i^*\colon \mathbb{F}_i^* \to \mathcal{N}$. Hence, $\operatorname{Im} g^* = \operatorname{Im} g_i^*$ and $\operatorname{Coker} g^* = \operatorname{Coker} g_i^*$. By 4.19, there exist finitely presented R-modules P and Q and an exact sequence of morphisms $0 \to \mathbb{F}_i^* \overset{r}{\to} \mathcal{P} \overset{s}{\to} \mathcal{Q}$. By 4.18, there exists a morphism $t\colon \mathcal{P} \to \mathcal{N}$ such that $g_i^* = t \circ r$. Considering the obvious morphisms

$$\mathcal{N}/\operatorname{Im} g_i^* \hookrightarrow \mathcal{N}/\operatorname{Im} g_i^* \oplus_{\mathcal{P}/\operatorname{Im} r} \mathcal{Q} \simeq \mathcal{N} \oplus_{\mathcal{P}} \mathcal{Q}.$$

we obtain that $\operatorname{Coker} g^*$ is a submodule of a quasi-coherent $\mathcal{R}$-module.

$\Leftarrow$) Let $\mathbb{F}$ be an FP module and $f\colon \mathbb{F} \to \mathbb{M}$ a morphism of $\mathcal{R}$-modules. Let $f^*\colon \mathbb{M}^* \to \mathbb{F}^*$ be the dual morphism. We have to prove that $\operatorname{Im} f^*$ is an FP (right) module. $\operatorname{Im} f^*$ preserves direct products since $\mathbb{M}^*$ and $\mathbb{F}^*$ preserve direct products. There exist an R-module M and a monomorphism $\operatorname{Coker} f^* \hookrightarrow \mathcal{M}$. Let g be the composite morphism $\mathbb{F}^* \to \operatorname{Coker} f^* \hookrightarrow \mathcal{M}$. $\operatorname{Im} f^* = \operatorname{Ker} g$, which preserves direct limits, by 3.5. Therefore, $\operatorname{Im} f^*$ is an FP (right) module.

□

Proposition 6.6. *$\mathcal{M}$ is an SML module iff the cokernel of every morphism $\mathcal{M}^* \to \oplus_{\mathbb{N}}\mathcal{R}$ is isomorphic to an $\mathcal{R}$-submodule of a quasi-coherent (right) module.*

Proof. ⇒) The morphism $\mathcal{M}^* \to \oplus_{\mathbb{N}}\mathcal{R}$ factors through a direct summand $\mathcal{R}^n$ of $\oplus_{\mathbb{N}}\mathcal{R}$. $\operatorname{Coker}[\mathcal{M}^* \to \mathcal{R}^n]$ is isomorphic to an $\mathcal{R}$-submodule of a quasi-coherent (right) module, by 6.5.

⇐) Let $f\colon \mathcal{R}^n \to \mathcal{M}$ be a morphism of $\mathcal{R}$-modules and let $f^*\colon \mathcal{M}^* \to \mathcal{R}^n$ be the dual morphism. $\operatorname{Coker} f^*$ preserves direct products by 4.6. There exist an R-module N and a monomorphism $\operatorname{Coker} f^* \hookrightarrow \mathcal{N}$. $\operatorname{Coker} f^* = \operatorname{Im}[\mathcal{R}^n \to \mathcal{N}]$ preserves direct limits, by 3.5. Therefore, $\operatorname{Coker} f^*$ is an FP (right) module. $\operatorname{Im} f^* = \operatorname{Ker}[\mathcal{R}^n \to \operatorname{Coker} f^*]$ is an FP (right) $\mathcal{R}$-module, by 4.9. We have the epimorphism and the monomorphism $\mathcal{M}^* \twoheadrightarrow \operatorname{Im} f^* \hookrightarrow \mathcal{R}^n$. By 4.18, we have the epimorphism and the monomorphism

$$\mathcal{R}^n \twoheadrightarrow (\operatorname{Im} f^*)^* \hookrightarrow \mathcal{M}$$

Hence, $\operatorname{Im} f = (\operatorname{Im} f^*)^*$ is an FP module. By 6.4, $\mathcal{M}$ is an SML module.

□

Theorem 6.7. *$\mathbb{M} \in \langle Qs\text{-}ch\rangle$ is an SML module iff there exists a monomorphism $\mathbb{M} \to \prod_{j\in J} \mathcal{P}_j$, where P_j is a finitely presented (right) module, for each $j \in J$.*

Proof. ⇒) Choose a set A of representatives of the isomorphism classes of finitely presented (right) R-modules. Let B be the set of pairs $(\mathcal{P}, g)$, where $P \in A$ and $g \in \operatorname{Hom}_{\mathcal{R}}(\mathbb{M}, \mathcal{P})$. The "canonical" morphism

$$G\colon \mathbb{M} \to \prod_{(\mathcal{P},g)\in B} \mathcal{P},\ G(m) := (g(m))_{(\mathcal{P},g)}$$

is a monomorphism: There exists a direct system $\{\mathbb{F}_i\}$ of FP submodules of $\mathbb{M}$, such that $\mathbb{M} = \lim\limits_{\substack{\to\\ i\in I}} \mathbb{F}_i$ and the natural morphism $\mathbb{M}^* \to \mathbb{F}_i^*$ is an epimorphism, for any i. There exist a finitely presented (right) R-module $Q \in A$ and a monomorphism $g_i\colon \mathbb{F}_i \hookrightarrow \mathcal{Q}$, by 4.19. There exists $f_i \in \operatorname{Hom}_{\mathcal{R}}(\mathbb{M}, \mathcal{Q})$ such that $f_{i|\mathbb{F}_i} = g_i$, since the morphism

$$\operatorname{Hom}_{\mathcal{R}}(\mathbb{M}, \mathcal{Q}) = \mathbb{M}^*(Q) \to \mathbb{F}_i^*(Q) = \operatorname{Hom}_{\mathcal{R}}(\mathbb{F}_i, \mathcal{Q})$$

is surjective. Let $\pi_{(\mathcal{Q},f_i)}\colon \prod_{(\mathcal{P},g)\in B} \mathcal{P} \to \mathcal{Q}$ be the projection onto the factor indexed by $(\mathcal{Q}, f_i)$. Therefore, $G_{|\mathbb{F}_i}$ is a monomorphism since $\pi_{(\mathcal{Q},f_i)} \circ G_{|\mathbb{F}_i} = f_{i|\mathbb{F}_i} = g_i$. Hence, G is a monomorphism.

⇐) Let $i\colon \mathbb{M} \hookrightarrow \prod_{j\in J} \mathcal{P}_j$ be a monomorphism, where P_j is a finitely presented (right) R-module for each $j \in J$. $\prod_j \mathcal{P}_j$ preserves direct products. Hence, $\mathbb{M}(\prod_i S_i) \to \prod_i \mathbb{M}(S_i)$ is injective, for any set $\{S_i\}$ of R-modules. Therefore, $\mathbb{M}$ is an ML module.

Let $\mathbb{F}$ be an FP $\mathcal{R}$-module and $\mathbb{F} \hookrightarrow \mathbb{M}$ a monomorphism. By 6.1, we have to prove that the dual morphism $\mathbb{M}^* \to \mathbb{F}^*$ is an epimorphism. We can suppose that $\mathbb{M} = \prod_j \mathcal{P}_j$. We have the commutative diagram (see Appendix)

$$\begin{array}{ccc} \mathbb{F}\circ\mathbb{D} & \hookrightarrow & \mathbb{M}\circ\mathbb{D} \\ \Big\| {\scriptstyle 7.5} & & \Big\| {\scriptstyle 7.5} \\ \mathbb{D}\circ\mathbb{F}^* & \longrightarrow & \mathbb{D}\circ\mathbb{M}^* \end{array}$$

Then, the morphism $\mathbb{D}\circ\mathbb{F}^* \to \mathbb{D}\circ\mathbb{M}^*$ is a monomorphism. Hence, the morphism $\mathbb{M}^* \to \mathbb{F}^*$ is an epimorphism.

□

Definition 6.8. [7, II 2.3.2] *An R-module M is said to be a strict Mittag-Leffler R-module if for every finitely generated submodule $N \overset{i}{\subseteq} M$ there exist a finitely presented R-module P and a commutative diagram of morphisms of R-modules*

$$\begin{array}{ccccc} & & M & & \\ & \overset{i}{\nearrow} & & \overset{f}{\searrow} & \\ N & & & & P \\ & \underset{i}{\searrow} & & \underset{g}{\swarrow} & \\ & & M & & \end{array}$$

Theorem 6.9. *M is a strict Mittag-Leffler right R-module iff $\mathcal{M}$ is an SML module.*

Proof. $\Rightarrow$) For every finitely generated R-submodule $N \overset{i_R}{\subseteq} M$ there exist a finitely presented (right) R-module P and a commutative diagram of morphisms of $\mathcal{R}$-modules

$$\begin{array}{ccccc} & & \mathcal{M} & & \\ & \overset{i}{\nearrow} & & \overset{f}{\searrow} & \\ \mathcal{N} & & & & \mathcal{P} \\ & \underset{i}{\searrow} & & \underset{g}{\swarrow} & \\ & & \mathcal{M} & & \end{array}$$

(where i is the morphism induced by i_R). Hence, $\tilde{N} := \operatorname{Im} i \simeq \operatorname{Im}(f\circ i)$, which is an FP module by 5.11. The morphism $\mathcal{M}^* \to \tilde{N}^*$ is an epimorphism since the composite morphism $\mathcal{P}^* \to \mathcal{M}^* \to \tilde{N}^*$ is an epimorphism, by 4.18. By 6.4, $\mathcal{M}$ is an SML module.

$\Leftarrow$) Let $N \subseteq M$ be a finitely generated R-submodule. The image $\tilde{N}$ of the induced morphism $\mathcal{N} \to \mathcal{M}$ is an FP module, by 6.4. Let $\tilde{i}\colon \tilde{N} \hookrightarrow \mathcal{M}$ be the inclusion morphism. There exist a finitely presented (right) R-module P and a monomorphism $j\colon \tilde{N} \hookrightarrow \mathcal{P}$, by 4.19. There exist a morphism $f\colon \mathcal{M} \to \mathcal{P}$ such that $j = f\circ i$ since the morphism $\mathcal{M}^* \to \tilde{N}^*$ is an epimorphism (by 6.3) and $j \in \tilde{N}^*(P)$. There exists a morphism $g\colon \mathcal{P} \to \mathcal{M}$ such that $i = g\circ j$ since the morphism $\mathcal{P}^* \to \tilde{N}^*$ is an epimorphism by 6.3 and $i \in \tilde{N}^*(M)$. The diagram

$$\begin{array}{ccccccc} & & & & \mathcal{M} & & \\ & & & \overset{\tilde{i}}{\nearrow} & & \overset{f}{\searrow} & \\ \mathcal{N} & \longrightarrow & \tilde{N} & & \overset{j}{\dashrightarrow} & & \mathcal{P} \\ & & & \underset{\tilde{i}}{\searrow} & & \underset{g}{\swarrow} & \\ & & & & \mathcal{M} & & \end{array}$$

is commutative. Hence, M is a strict Mittag-Leffler right module.

□

7. Appendix: Functor $\mathbb{D}$

Let $\mathbb{D}$ be the contravariant additive functor from the category of abelian groups to the category of abelian groups defined by

$$\mathbb{D}(N) = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Q}/\mathbb{Z}),$$

for any abelian group N. $\mathbb{D}$ is an exact functor and $\mathbb{D}(N) = 0$ iff $N = 0$.

We will say that $\mathbb{H}$ is a contravariant $\mathcal{R}$-module if $\mathbb{H}$ is a contravariant additive functor from the category of R-modules to the category abelian groups.

Proposition 7.1. *Let $\mathbb{M}$ be an $\mathcal{R}$-module and $\mathbb{H}$ a contravariant $\mathcal{R}$-module. Then, there exists a natural isomorphism*

$$\mathrm{Hom}_{\mathcal{R}}(\mathbb{M}, \mathbb{D} \circ \mathbb{H}) = \mathrm{Hom}_{\mathcal{R}}(\mathbb{H}, \mathbb{D} \circ \mathbb{M}).$$

In particular, if $\mathbb{M}$ is projective then $\mathbb{D} \circ \mathbb{M}$ is injective, and if $\mathbb{H}$ is projective then $\mathbb{D} \circ \mathbb{H}$ is injective.

Proof. $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{M}(S), \mathrm{Hom}_{\mathbb{Z}}(\mathbb{H}(S), \mathbb{Q}/\mathbb{Z})) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{H}(S), \mathrm{Hom}_{\mathbb{Z}}(\mathbb{M}(S), \mathbb{Q}/\mathbb{Z}))$, for any R-module S.

□

Example 7.2. *Given an R-module N', let $N'_\bullet$ be the contravariant $\mathcal{R}$-module defined by*

$$N'_\bullet(S) := \mathrm{Hom}_R(S, N')$$

Observe that $N'_\bullet$ is a projective $\mathcal{R}$-module since $\mathrm{Hom}_{\mathcal{R}}(N'_\bullet, \mathbb{H}) = \mathbb{H}(N')$. Let N be a right R-module. $\mathbb{D} \circ \mathcal{N} = \mathbb{D}(N)_\bullet$. Hence, $\mathbb{D}^2 \circ \mathcal{N} = \mathbb{D} \circ \mathbb{D}(N)_\bullet$ and it is an injective (right) $\mathcal{R}$-module by Proposition 7.1. By Proposition 3.13, $(\mathbb{D}^2 \circ \mathcal{N})_{\langle \text{Qs-ch} \rangle}$ is an injective object of $\langle$Qs-ch$\rangle$. By Proposition 4.26, $(\mathbb{D}^2 \circ \mathcal{N})_{\langle \text{Qs-ch} \rangle}(R) = \mathbb{D}^2(N)$ is a pure-injective R-module.

Given a module N, let $i_N \colon N \to \mathbb{D}^2(N)$ be the natural morphism defined by $i_N(n)(w) := w(n)$, for any $n \in N$ and $w \in \mathbb{D}(N) = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Q}/\mathbb{Z})$. The morphism i_N is a monomorphism and it is easy to check that the composite morphism

$$\mathbb{D}(N) \xrightarrow{i_{\mathbb{D}(N)}} \mathbb{D}^3(N) \xrightarrow{\mathbb{D}(i_N)} \mathbb{D}(N)$$

is the identity morphism.

Proposition 7.3. *Let $\mathbb{M}$ be a right $\mathcal{R}$-module and $\mathbb{H}$ a contravariant $\mathcal{R}$-module. Then, there exists a natural isomorphism*

$$\mathrm{Hom}_{\mathcal{R}}(\mathbb{M} \circ \mathbb{D}, \mathbb{H}) = \mathrm{Hom}_{\mathcal{R}}(\mathbb{M}, \mathbb{H} \circ \mathbb{D}).$$

In particular, if $\mathbb{M}$ is projective then $\mathbb{M} \circ \mathbb{D}$ is projective, and if $\mathbb{H}$ is injective then $\mathbb{H} \circ \mathbb{D}$ is injective.

Proof. Given a morphism $f \colon \mathbb{M} \circ \mathbb{D} \to \mathbb{H}$ composing with $\mathbb{D}$ we obtain the morphism $\mathbb{M} \circ \mathbb{D}^2 \to \mathbb{H} \circ \mathbb{D}$. Let f' be the composite morphism $\mathbb{M} \to \mathbb{M} \circ \mathbb{D}^2 \to \mathbb{H} \circ \mathbb{D}$, that is, $f'_N = f_{\mathbb{D}(N)} \circ \mathbb{M}(i_N)$, for any R-module N.

Given a morphism $g \colon \mathbb{M} \to \mathbb{H} \circ \mathbb{D}$ composing with $\mathbb{D}$ we obtain the morphism $\mathbb{M} \circ \mathbb{D} \to \mathbb{H} \circ \mathbb{D}^2$. Let $\widehat{g}$ be the composite morphism $\mathbb{M} \circ \mathbb{D} \to \mathbb{H} \circ \mathbb{D}^2 \to \mathbb{H}$, , that is, $\widehat{g}_N = \mathbb{H}(i_N) \circ g_{\mathbb{D}(N)}$, for any R-module N.

We have to prove that $f=\widehat{f'}$ and $g=\widehat{g}\,'$. The diagram

$$\begin{array}{ccccc}
\mathbb{M}(\mathbb{D}(N)) & \xrightarrow{\mathbb{M}(i_{\mathbb{D}(N)})} & \mathbb{M}(\mathbb{D}^3(N)) & \xrightarrow{f_{\mathbb{D}^2(N)}} & \mathbb{H}(\mathbb{D}^2(N)) \\
 & \searrow^{Id} & \downarrow{\scriptstyle \mathbb{M}(\mathbb{D}(i_N))} & & \downarrow{\scriptstyle \mathbb{H}(i_N)} \\
 & & \mathbb{M}(\mathbb{D}(N)) & \xrightarrow{f_N} & \mathbb{H}(N)
\end{array}$$

is commutative. Hence, $f_N=\mathbb{H}(i_N)\circ f_{\mathbb{D}^2(N)}\circ \mathbb{M}(i_{\mathbb{D}(N)})=\mathbb{H}(i_N)\circ f'_{\mathbb{D}(N)}=\widehat{f'}_N$.

Likewise, $g=\widehat{g}\,'$.

□

Let N be a right R-module and let $c\colon N\otimes_R \mathbb{D}(N)\to \mathbb{Q}/\mathbb{Z}$ be defined by $c(n\otimes w):=w(n)$. We have a natural morphism

$$\mathbb{M}^*\circ\mathbb{D}\to\mathbb{D}\circ\mathbb{M},\ f\mapsto\tilde f \text{ for any } f\in\mathbb{M}^*(\mathbb{D}(N))$$

where $\tilde f\in(\mathbb{D}\circ\mathbb{M})(N)=\operatorname{Hom}_{\mathbb{Z}}(\mathbb{M}(N),\mathbb{Q}/\mathbb{Z})$ is defined by $\tilde f(m):=c(f_N(m))$. By Proposition 7.3, we have the natural morphism

$$\mathbb{M}^*\to\mathbb{D}\circ\mathbb{M}\circ\mathbb{D},\ f\mapsto\tilde f,\ \text{ for any } f\in\mathbb{M}^*(N)=\operatorname{Hom}_{\mathcal{R}}(\mathbb{M},\mathcal{N}),$$

where $\tilde f\in\mathbb{D}(\mathbb{M}(\mathbb{D}(N)))=\operatorname{Hom}_{\mathbb{Z}}(\mathbb{M}(\mathbb{D}(N)),\mathbb{Q}/\mathbb{Z})$ is defined as $\tilde f(m)=c(f_{\mathbb{D}(N)}(m))$.

Proposition 7.4. *The natural morphism*

$$\mathbb{M}^*\to\mathbb{D}\circ\mathbb{M}\circ\mathbb{D}$$

is a monomorphism. In particular, $\mathbb{M}^*$ *is a well defined functor, that is,* $\mathbb{M}^*(N)$ *is a set for any* R*-module* N*.*

Proof. The composite morphism

$$\begin{aligned}\mathbb{M}^*(N)=\operatorname{Hom}_{\mathcal{R}}(\mathbb{M},\mathcal{N})&\hookrightarrow\operatorname{Hom}_{\mathcal{R}}(\mathbb{M},\mathbb{D}^2\circ\mathcal{N})=\operatorname{Hom}_{\mathcal{R}}(\mathbb{D}\circ\mathcal{N},\mathbb{D}\circ\mathbb{M})\\ &=\operatorname{Hom}_{\mathcal{R}}(\mathbb{D}(N)_\bullet,\mathbb{D}\circ\mathbb{M})=(\mathbb{D}\circ\mathbb{M})(\mathbb{D}(N))=(\mathbb{D}\circ\mathbb{M}\circ\mathbb{D})(N)\end{aligned}$$

is injective. □

By Proposition 7.3, we have the natural morphism

$$\mathbb{M}\circ\mathbb{D}\to\mathbb{D}\circ\mathbb{M}^*,\ m\mapsto\tilde m \text{ for any } m\in\mathbb{M}(\mathbb{D}(N))$$

where $\tilde m\in(\mathbb{D}\circ\mathbb{M}^*)(N)=\operatorname{Hom}_{\mathbb{Z}}(\operatorname{Hom}_{\mathcal{R}}(\mathbb{M},\mathcal{N}),\mathbb{Q}/\mathbb{Z})$ is defined by $\tilde m(g):=c(g_{\mathbb{D}(N)}(m))$, and the triangle

$$\begin{array}{ccc}
\mathbb{M}\circ\mathbb{D} & \longrightarrow & \mathbb{D}\circ\mathbb{M}^* \\
 & \searrow \quad \nearrow & \\
 & \mathbb{M}^{**}\circ\mathbb{D} &
\end{array}$$

is commutative.

Proposition 7.5. *If* $\mathbb{M}$ *preserves direct limits, then the natural morphism*

$$\mathbb{M}^*\circ\mathbb{D}\to\mathbb{D}\circ\mathbb{M}$$

is an isomorphism.

Proof. The contravariant functors $\mathbb{M} \rightsquigarrow \mathbb{M}^* \circ \mathbb{D}$, $\mathbb{D} \circ \mathbb{M}$ are left-exact and transform direct sums into direct products. By Theorem 3.7, we have only to check that $\mathcal{P} \circ \mathbb{D} = \mathbb{D} \circ \mathcal{P}^*$, for any finitely presented R-module. The functors $M \rightsquigarrow \mathcal{M} \circ \mathbb{D}$, $\mathbb{D} \circ \mathcal{M}^*$ are right exact and transform finite direct sums into finite direct sums and $\mathcal{R} \circ \mathbb{D} = \mathbb{D} \circ \mathcal{R}^*$, and we conclude.

□

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