TWO q-SUMMATION FORMULAS AND q-ANALOGUES OF SERIES EXPANSIONS FOR CERTAIN CONSTANTS

BING HE AND HONGCUN ZHAI

Abstract. From two q-summation formulas we deduce certain series expansion formulas involving the q-gamma function. With these formulas we can give q-analogues of series expansions for certain constants.

1. Introduction

Throughout this paper we always assume that $|q| < 1$. The q-gamma function $\Gamma_q(x)$, first introduced by Thomae and later by Jackson, is defined as [5, p. 20]

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \tag{1.1}$$

where $(z;q)_\infty$ is given by

$$(z;q)_\infty = \prod_{n=0}^{\infty}(1-zq^n).$$

When $q \to 1$, the q-gamma function reduces to the classical gamma function $\Gamma(x)$ which is defined by [1]: for Re $x > 0$,

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt.$$

From the definition of the q-gamma function we know that

$$\frac{(q^x;q)_n}{(1-q)^n} = \frac{\Gamma_q(x+n)}{\Gamma_q(x)}, \tag{1.2}$$

where $(z;q)_n$ is the q-shifted factorial given by

$$(z;q)_0 = 1,\ (z;q)_n = \prod_{k=0}^{n-1}(1-zq^k) \text{ for } n \geq 1.$$

We now extend the definition of $(q^x;q)_n$ to any complex α.

Definition. For any complex α, we define the general q-shifted factorial by

$$(q^x;q)_\alpha = \frac{\Gamma_q(x+\alpha)}{\Gamma_q(x)}(1-q)^\alpha \tag{1.3}$$

2000 *Mathematics Subject Classification.* 33D05, 33D15, 65B10.
Key words and phrases. q-analogue; series expansions for constants; q-summation formula.
The first author is the corresponding author.

For brevity, we denote $\frac{(q^x;q)_\alpha}{(1-q)^\alpha}$ by $(x|q)_\alpha$, namely, $(x|q)_\alpha = \frac{\Gamma_q(x+\alpha)}{\Gamma_q(x)}$. For any non-negative integer n, we have

$$(x|q)_n = \prod_{k=0}^{n-1}[x+k]_q$$

and

$$(x|q)_{-n} = \frac{\Gamma_q(x-n)}{\Gamma_q(x)} = \frac{(1-q)^n}{(q^{x-n};q)_n},$$

where $[z]_q$ is the q-integer defined by

$$[z]_q = \frac{1-q^z}{1-q}.$$

In particular,

$$(x|q)_0 = 1,\ (x|q)_1 = [x]_q,\ (x|q)_{-1} = \frac{1}{[x-1]_q}.$$

Gosper in [7] introduced q-analogues of $\sin x$ and π :

$$\sin_q(\pi x) := q^{(x-1/2)^2}\frac{(q^{2-2x};q^2)_\infty(q^{2x};q^2)_\infty}{(q;q^2)_\infty^2}$$

and

$$\pi_q := (1-q^2)q^{1/4}\frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2}.$$

They satisfy the following relations:

$$\lim_{q\to1}\sin_q x = \sin x, \quad \lim_{q\to1}\pi_q = \pi$$

and

$$\Gamma_{q^2}(x)\Gamma_{q^2}(1-x) = \frac{\pi_q}{\sin_q(\pi x)}q^{x(x-1)}. \tag{1.4}$$

When $q\to1$, the last identity reduces to the Euler reflection formula [1, (1.2.1)]:

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}.$$

Ramanujan [13] recorded without proof 17 series expansions for $1/\pi$, among which, the proof of the first three was briefly sketched in [12]. The first complete proof of all 17 formulas was found by the Borwein brothers [3]. D.V. Chudnovsky and G.V. Chudnovsky [4] proved some of the Ramanujan's series representations for $1/\pi$ independently and established new series as well. The readers can refer to the paper [2] for the history of the Ramanujan-type series for $1/\pi$. Recently, using certain properties of the general rising shifted factorial and the gamma function, Liu in [10, 11] supplied many series expansion formula for $1/\pi$. q-Analogues of two Ramanujan-type series for $1/\pi$ were established by Guo and Liu [9] using q-WZ pairs and some basic hypergeometric identities.

Our motivation for the present work emanates from [9, 10, 11]. In this paper we shall deduce from two q-summation formulas certain series expansion formulas involving the q-gamma function. These formulas allow us to give q-analogues of series expansions for certain constants. These series expansion formulas are as follows.

Theorem 1.1. *For any complex number* α *and* $\mathrm{Re}(c-a-b)>0$ *we have*

$$\sum_{n=0}^{\infty}\frac{(\alpha|q^2)_{a+n}(1-\alpha|q^2)_{b+n}}{[n]_{q^2}!\Gamma_{q^2}(c+n+1)}q^{2(c-a-b)n}$$
$$=\frac{(\alpha|q^2)_a(1-\alpha|q^2)_b\Gamma_{q^2}(c-a-b)}{(1-\alpha|q^2)_{c-a}(\alpha|q^2)_{c-b}}q^{-\alpha(\alpha-1)}\cdot\frac{\sin_q(\pi\alpha)}{\pi_q},$$

where $[n]_q!$ *is given by*

$$[0]_q!=1,\quad [n]_q!=\prod_{k=1}^{n}[k]_q \text{ for } n\geq 1.$$

Theorem 1.2. *For* $\mathrm{Re}(a+b+c+d+1+\alpha-\beta-\gamma-\delta)>0$ *we have*

$$\sum_{n=0}^{\infty}\frac{(1-q^{4n+2a+2\alpha})(\alpha|q^2)_{a+n}(\beta|q^2)_{n-b}(\gamma|q^2)_{n-c}(\delta|q^2)_{n-d}}{(1-q^2)[n]_{q^2}!(1+\alpha-\beta|q^2)_{a+b+n}(1+\alpha-\gamma|q^2)_{a+c+n}(1+\alpha-\delta|q^2)_{a+d+n}}q^{An}$$
$$=\frac{\Gamma_{q^2}(1+\alpha-\beta)\Gamma_{q^2}(1+\alpha-\gamma)\Gamma_{q^2}(1+\alpha-\delta)\Gamma_{q^2}(2+\alpha-\beta-\gamma-\delta)}{\Gamma_{q^2}(\alpha)\Gamma_{q^2}(1+\alpha-\beta-\gamma)\Gamma_{q^2}(1+\alpha-\beta-\delta)\Gamma_{q^2}(1+\alpha-\gamma-\delta)}$$
$$\times\frac{(\beta|q^2)_{-b}(\gamma|q^2)_{-c}(\delta|q^2)_{-d}(2+\alpha-\beta-\gamma-\delta|q^2)_{a+b+c+d-1}}{(1+\alpha-\beta-\gamma|q^2)_{a+b+c}(1+\alpha-\beta-\delta|q^2)_{a+b+d}(1+\alpha-\gamma-\delta|q^2)_{a+c+d}},$$

where $A=2(a+b+c+d+1+\alpha-\beta-\gamma-\delta)$.

The next section is devoted to our proof of Theorems 1.1 and 1.2. In Section 3 we deduce q-analogues of certain series expansions for $1/\pi$. In the last section several q-analogues of series expansions for π^2 are also obtained.

2. Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1. Recall from [5, (1.5.1)] the q-Gauss summation formula:

$$\sum_{n=0}^{\infty}\frac{(a;q)_n(b;q)_n}{(q;q)_n(c;q)_n}(c/ab)^n=\frac{(c/a;q)_\infty(c/b;q)_\infty}{(c;q)_\infty(c/ab;q)_\infty},\quad |c/ab|<1.$$

Making the substitutions: $q\to q^2,\ a\to q^{2a},\ b\to q^{2b},\ c\to q^{2c}$ in the above identity and using (1.1) and (1.2) we have

$$\sum_{n=0}^{\infty}\frac{\Gamma_{q^2}(a+n)\Gamma_{q^2}(b+n)}{[n]_{q^2}!\Gamma_{q^2}(c+n)}q^{2(c-a-b)n}=\frac{\Gamma_{q^2}(a)\Gamma_{q^2}(b)\Gamma_{q^2}(c-a-b)}{\Gamma_{q^2}(c-a)\Gamma_{q^2}(c-b)}.$$

Replacing $a,\ b,\ c$ by $a+\alpha,\ b+1-\alpha,\ c+1$ respectively in the above formula we get

$$\sum_{n=0}^{\infty}\frac{\Gamma_{q^2}(a+n+\alpha)\Gamma_{q^2}(b+n+1-\alpha)}{[n]_{q^2}!\Gamma_{q^2}(c+n+1)}q^{2(c-a-b)n}$$
$$=\frac{\Gamma_{q^2}(a+\alpha)\Gamma_{q^2}(b+1-\alpha)\Gamma_{q^2}(c-a-b)}{\Gamma_{q^2}(c-a+1-\alpha)\Gamma_{q^2}(c-b+\alpha)}.\tag{2.1}$$

It follows from (1.3) that

$$\Gamma_{q^2}(a+\alpha)=(\alpha|q^2)_a\Gamma_{q^2}(\alpha),$$
$$\Gamma_{q^2}(b+1-\alpha)=(1-\alpha|q^2)_b\Gamma_{q^2}(1-\alpha),$$
$$\Gamma_{q^2}(a+n+\alpha)=(\alpha|q^2)_{a+n}\Gamma_{q^2}(\alpha),$$
$$\Gamma_{q^2}(b+n+1-\alpha)=(1-\alpha|q^2)_{b+n}\Gamma_{q^2}(1-\alpha),$$
$$\Gamma_{q^2}(c-a+1-\alpha)=(1-\alpha|q^2)_{c-a}\Gamma_{q^2}(1-\alpha),$$
$$\Gamma_{q^2}(c-b+\alpha)=(\alpha|q^2)_{c-b}\Gamma_{q^2}(\alpha).$$

Substituting these formulas into (2.1) and simplifying we arrive at

$$\sum_{n=0}^{\infty}\frac{(\alpha|q^2)_{a+n}(1-\alpha|q^2)_{b+n}}{[n]_{q^2}!\Gamma_{q^2}(c+n+1)}q^{2(c-a-b)n}=\frac{(\alpha|q^2)_a(1-\alpha|q^2)_b\Gamma_{q^2}(c-a-b)}{(1-\alpha|q^2)_{c-a}(\alpha|q^2)_{c-b}\Gamma_{q^2}(\alpha)\Gamma_{q^2}(1-\alpha)}.$$

From this identity and (1.4) we can deduce the result readily. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. Recall the following summation formula for the basic hypergeometric series [5, (2.7.1)]:

$$ {}_6\phi_5\left(\begin{matrix}a, qa^{\frac12}, -qa^{\frac12}, b, c, d\\ a^{\frac12}, -a^{\frac12}, aq/b, aq/c, aq/d\end{matrix}; q, \frac{aq}{bcd}\right)=\frac{(aq, aq/bc, aq/bd, aq/cd; q)_\infty}{(aq/b, aq/c, aq/d, aq/bcd; q)_\infty}, \tag{2.2}$$

where $\left|\frac{aq}{bcd}\right|<1$ and ${}_6\phi_5$ is the basic hypergeometric series given by

$${}_6\phi_5\left(\begin{matrix}a_1, a_2, a_3, a_4, a_5, a_6\\ b_1, b_2, b_3, b_4, b_5\end{matrix}; q, z\right)=\sum_{n=0}^{\infty}\frac{(a_1, a_2, a_3, a_4, a_5, a_6; q)_n}{(q, b_1, b_2, b_3, b_4, b_5; q)_n}z^n.$$

Replacing (q, a, b, c, d) by $(q^2, q^{2a}, q^{2b}, q^{2c}, q^{2d})$ in (2.2) and employing (1.2) and (1.3) we have

(2.3)
$$\sum_{n=0}^{\infty}\frac{(1-q^{4n+2a})\Gamma_{q^2}(a+n)\Gamma_{q^2}(b+n)\Gamma_{q^2}(c+n)\Gamma_{q^2}(d+n)}{(1-q^2)[n]_{q^2}!\Gamma_{q^2}(1+a-b+n)\Gamma_{q^2}(1+a-c+n)\Gamma_{q^2}(1+a-d+n)}q^{2n(1+a-b-c-d)}$$
$$=\frac{\Gamma_{q^2}(b)\Gamma_{q^2}(c)\Gamma_{q^2}(d)\Gamma_{q^2}(1+a-b-c-d)}{\Gamma_{q^2}(1+a-b-c)\Gamma_{q^2}(1+a-b-d)\Gamma_{q^2}(1+a-c-d)}.$$

It follows from (1.1) that

$$\Gamma_{q^2}(a+n+\alpha)=(\alpha|q^2)_{a+n}\Gamma_{q^2}(\alpha), \Gamma_{q^2}(n-b+\beta)=(\beta|q^2)_{n-b}\Gamma_{q^2}(\beta),$$
$$\Gamma_{q^2}(n-c+\gamma)=(\gamma|q^2)_{n-c}\Gamma_{q^2}(\gamma), \Gamma_{q^2}(n-d+\delta)=(\delta|q^2)_{n-d}\Gamma_{q^2}(\delta),$$
$$\Gamma_{q^2}(\beta-b)=(\beta|q^2)_{-b}\Gamma_{q^2}(\beta), \Gamma_{q^2}(\gamma-c)=(\gamma|q^2)_{-c}\Gamma_{q^2}(\gamma), \Gamma_{q^2}(\delta-d)=(\delta|q^2)_{-d}\Gamma_{q^2}(\delta),$$
$$\Gamma_{q^2}(a+b+n+1+\alpha-\beta)=(1+\alpha-\beta|q^2)_{a+b+n}\Gamma_{q^2}(1+\alpha-\beta),$$
$$\Gamma_{q^2}(a+c+n+1+\alpha-\gamma)=(1+\alpha-\gamma|q^2)_{a+c+n}\Gamma_{q^2}(1+\alpha-\gamma),$$
$$\Gamma_{q^2}(a+d+n+1+\alpha-\delta)=(1+\alpha-\delta|q^2)_{a+d+n}\Gamma_{q^2}(1+\alpha-\delta),$$
$$\Gamma_{q^2}(a+b+c+1+\alpha-\beta-\gamma)=(1+\alpha-\beta-\gamma|q^2)_{a+b+c}\Gamma_{q^2}(1+\alpha-\beta-\gamma),$$
$$\Gamma_{q^2}(a+b+d+1+\alpha-\beta-\delta)=(1+\alpha-\beta-\delta|q^2)_{a+b+d}\Gamma_{q^2}(1+\alpha-\beta-\delta),$$
$$\Gamma_{q^2}(a+c+d+1+\alpha-\gamma-\delta)=(1+\alpha-\gamma-\delta|q^2)_{a+c+d}\Gamma_{q^2}(1+\alpha-\gamma-\delta)$$

and

$$\begin{aligned}&\Gamma_{q^2}(a+b+c+d+1+\alpha-\beta-\gamma-\delta)\\&=(2+\alpha-\beta-\gamma-\delta|q^2)_{a+b+c+d-1}\Gamma_{q^2}(2+\alpha-\beta-\gamma-\delta).\end{aligned}$$

Making the substitutions: $a \to a+\alpha$, $b \to \beta - b$, $c \to \gamma - c$, $d \to \delta - d$ in (2.3) and then substituting the above identities into the resulting equation we can easily deduce the result. This finishes the proof of Theorem 1.2. □

3. q-Analogues of certain series expansions for $1/\pi$

In this section we employ Theorems 1.1 and 1.2 to deduce q-analogues of certain series expansions for $1/\pi$.

Setting $\alpha = \frac{1}{2}$ in Theorem 1.1 and using the fact $\sin_q \frac{\pi}{2} = 1$ we get

Theorem 3.1. *For* $\mathrm{Re}(c-a-b) > 0$ *we have*

$$\sum_{n=0}^{\infty} \frac{(1/2|q^2)_{a+n}(1/2|q^2)_{b+n}}{[n]_{q^2}!\Gamma_{q^2}(c+n+1)} q^{2(c-a-b)n} = \frac{(1/2|q^2)_a(1/2|q^2)_b\Gamma_{q^2}(c-a-b)}{(1/2|q^2)_{c-a}(1/2|q^2)_{c-b}} \cdot \frac{q^{1/4}}{\pi_q}.$$

We put $a = b = 0$ and $c = l$ in Theorem 3.1 to arrive at

Corollary 3.1. *If* l *is positive integer, then*

$$\sum_{n=0}^{\infty} \frac{(1/2|q^2)_n^2}{[n]_{q^2}!(l|q^2)_{n+1}} q^{2ln} = \frac{q^{1/4}}{\pi_q (1/2|q^2)_l^2}.$$

Example 3.1. $(l = 1)$ We have

$$\sum_{n=0}^{\infty} \frac{(1/2|q^2)_n^2}{[n]_{q^2}![n+1]_{q^2}!} q^{2n} = \frac{(1+q)^2 q^{1/4}}{\pi_q}.$$

This expansion for $1/\pi_q$ is a q-analogue of the series for $1/\pi$ [6, p. 174]:

$$\sum_{n=0}^{\infty} \frac{(1/2)_n^2}{n!(n+1)!} = \frac{4}{\pi},$$

where $(1/2)_n$ is the shifted factorial given by

$$(1/2)_0 = 1,\ (1/2)_n = \prod_{k=0}^{n-1}(1/2+k) \text{ for } n \geq 1.$$

Actually, this expansion for $1/\pi_q$ was also obtained by Guo [8, (1.8)].

Example 3.2. $(l = 2)$ We have

$$\sum_{n=0}^{\infty} \frac{(1/2|q^2)_n^2}{[n]_{q^2}![n+2]_{q^2}!} q^{4n} = \frac{(1+q)^4 q^{1/4}}{\pi_q (1+q+q^2)^2}.$$

This series expansion for $1/\pi_q$ can be considered as a q-analogue of the series for $1/\pi$:

$$\sum_{n=0}^{\infty} \frac{(1/2)_n^2}{n!(n+2)!} = \frac{16}{9\pi}.$$

We set $a = b = -1$ and $c = l$ in Theorem 3.1 to deduce

Corollary 3.2. *If l is a non-negative integer, then*

$$q^2(1+q)^2[l+1]_{q^2}+q^{2l+4}+[l+1]_{q^2}!\sum_{n=1}^{\infty}\frac{(1/2|q^2)_n^2}{[n+1]_{q^2}![l+n+1]_{q^2}!}q^{2(l+2)(n+1)}$$
$$=\frac{(1+q)^2[l+1]_{q^2}!^2}{\pi_q(1/2|q^2)_{l+1}^2}q^{9/4}.$$

Example 3.3. ($l=0$) We have

$$q^2(1+q)^2+q^4+\sum_{n=1}^{\infty}\frac{(1/2|q^2)_n^2}{[n+1]_{q^2}!^2}q^{4n+4}=\frac{(1+q)^4}{\pi_q}q^{9/4}.$$

This series expansion for $1/\pi_q$ can be regarded as a q-analogue of the series for $1/\pi$ [6, p. 174]:

$$5+\sum_{n=1}^{\infty}\frac{(1/2)_n^2}{(n+1)!^2}=\frac{16}{\pi}.$$

Example 3.4. ($l=1$) We have

$$q^2(1+q)^2(1+q^2)+q^6+(1+q^2)\sum_{n=1}^{\infty}\frac{(1/2|q^2)_n^2}{[n+1]_{q^2}![n+2]_{q^2}!}q^{6n+6}=\frac{(1+q)^6(1+q^2)^2}{\pi_q(1+q+q^2)^2}q^{9/4}.$$

This expansion for $1/\pi_q$ is also a q-analogue of the series for $1/\pi$:

$$9+2\sum_{n=1}^{\infty}\frac{(1/2)_n^2}{(n+1)!(n+2)!}=\frac{256}{9\pi}.$$

Remark 3.1. Besides those formulas displayed in Theorem 3.1 and its consequences, we can give some other new series expansions for $1/\pi_q$ with the change of α. We shall not display them out one by one in this paper.

Theorem 3.2. *For* $\mathrm{Re}(a+b+c+d)>0$ *we have*

$$\sum_{n=0}^{\infty}\frac{(1-q^{4n+2a+1})(1/2|q^2)_{a+n}(1/2|q^2)_{n-b}(1/3|q^2)_{n-c}(2/3|q^2)_{n-d}}{(1-q^2)[n]_{q^2}!(1|q^2)_{a+b+n}(7/6|q^2)_{a+c+n}(5/6|q^2)_{a+d+n}}q^{2(a+b+c+d)n}$$
$$=\frac{(1/2|q^2)_{-b}(1/3|q^2)_{-c}(2/3|q^2)_{-d}(1|q^2)_{a+b+c+d-1}}{(1/3|q^2)_{a+b+d}(2/3|q^2)_{a+b+c}(1/2|q^2)_{a+c+d}}\cdot\frac{[1/6]_{q^2}(q^{4/3},q^{2/3};q^2)_\infty q^{1/4}}{(q^{1/3},q^{5/3};q^2)_\infty\pi_q}.$$

Proof. It follows from (1.4) that

$$\Gamma_{q^2}^2(1/2)=\pi_q q^{-1/4}, \tag{3.1}$$
$$\Gamma_{q^2}(1/3)\Gamma_{q^2}(2/3)=\frac{\pi_q}{\sin_q(\pi/3)}q^{-2/9},$$
$$\Gamma_{q^2}(7/6)\Gamma_{q^2}(5/6)=[1/6]_{q^2}\Gamma_{q^2}(1/6)\Gamma_{q^2}(5/6)$$
$$=\frac{\pi_q}{\sin_q(\pi/6)}[1/6]_{q^2}q^{-5/36}.$$

Then, by the definition of $\sin_q$,

$$\frac{\Gamma_{q^2}(7/6)\Gamma_{q^2}(5/6)}{\Gamma_{q^2}(1/3)\Gamma_{q^2}(2/3)}=\frac{\sin_q(\pi/3)}{\sin_q(\pi/6)}[1/6]_{q^2}q^{1/12}=\frac{(q^{4/3},q^{2/3};q^2)_\infty[1/6]_{q^2}}{(q^{1/3},q^{5/3};q^2)_\infty}. \tag{3.2}$$

Therefore, the result follows easily by setting $(\alpha,\beta,\gamma,\delta)=(1/2,1/2,1/3,2/3)$ in Theorem 1.2 and applying the identities $\Gamma_q(1)=1$, (3.1) and (3.2). □

Taking $(a,b,c,d)=(1,0,0,0)$ in Theorem 3.2 we can get

Example 3.5. We have

$$\sum_{n=0}^{\infty}\frac{(1-q^{4n+3})(1-q^{2n+1})(1/2|q^2)_n^2(1/3|q^2)_n(2/3|q^2)_n}{(1-q^2)(1-q^{2n+2})([n]_{q^2}!)^2(7/6|q^2)_{1+n}(5/6|q^2)_{1+n}}q^{2n}$$
$$=\frac{[1/6]_{q^2}(q^{4/3},q^{2/3};q^2)_\infty q^{1/4}}{[1/3]_{q^2}[2/3]_{q^2}[1/2]_{q^2}(q^{1/3},q^{5/3};q^2)_\infty \pi_q}.$$

This series expansion for $1/\pi_q$ can be regarded as a q-analogue of the series for $1/\pi$:

$$\sum_{n=0}^{\infty}\frac{(4n+3)(2n+1)(1/2)_n^2(1/3)_n(2/3)_n}{(n+1)(6n+1)(6n+5)(6n+7)(n!)^2(1/6)_n(5/6)_n}=\frac{\sqrt{3}}{6\pi}.$$

Putting $(a,b,c,d)=(0,0,0,1)$ in Theorem 3.2 we can deduce that

Example 3.6. We have

$$\frac{q^{2/3}}{(1+q)[1/3]_{q^2}[5/6]_{q^2}}-\sum_{n=1}^{\infty}\frac{(1-q^{4n+1})(1/2|q^2)_n^2(1/3|q^2)_n(2/3|q^2)_{n-1}}{(1-q^2)([n]_{q^2}!)^2(7/6|q^2)_n(5/6|q^2)_{1+n}}q^{2n}$$
$$=\frac{[1/6]_{q^2}}{[1/3]_{q^2}^2[1/2]_{q^2}}\cdot\frac{(q^{4/3},q^{2/3};q^2)_\infty q^{11/12}}{(q^{1/3},q^{5/3};q^2)_\infty \pi_q}.$$

This series expansion for $1/\pi_q$ can be considered as a q-analogue of the series for $1/\pi$:

$$1-\frac{5}{18}\sum_{n=1}^{\infty}\frac{(4n+1)(1/2)_n^2(1/3)_n(2/3)_{n-1}}{(n!)^2(7/6)_n(5/6)_{1+n}}=\frac{5}{\sqrt{3}\pi}.$$

4. q-Analogues of series expansions for π^2

In this section we use Theorem 1.2 to give q-analogues of some series expansions for π^2.

Theorem 4.1. *For* $\mathrm{Re}(a+b+c+d-1/2)>0$ *we have*

$$\sum_{n=0}^{\infty}\frac{(1-q^{4n+2a})(1|q^2)_{a+n-1}(1/2|q^2)_{n-b}(1/2|q^2)_{n-c}(1/2|q^2)_{n-d}}{(1-q^2)[n]_{q^2}!(1/2|q^2)_{a+b+n}(1/2|q^2)_{a+c+n}(1/2|q^2)_{a+d+n}}q^{2(a+b+c+d)n-n}$$
$$=\frac{\pi_q^2(1/2|q^2)_{-b}(1/2|q^2)_{-c}(1/2|q^2)_{-d}(1/2|q^2)_{a+b+c+d-1}}{(1|q^2)_{a+b+c-1}(1|q^2)_{a+b+d-1}(1|q^2)_{a+c+d-1}q^{1/2}}$$

Proof. It can be dedeuced from $\Gamma_q(x+1)=[x]_q\Gamma_q(x)$ and Theorem 1.2 that

$$\sum_{n=0}^{\infty}\frac{(1-q^{4n+2a+2\alpha})(\alpha+1|q^2)_{a+n-1}(\beta|q^2)_{n-b}(\gamma|q^2)_{n-c}(\delta|q^2)_{n-d}}{(1-q^2)[n]_{q^2}!(1+\alpha-\beta|q^2)_{a+b+n}(1+\alpha-\gamma|q^2)_{a+c+n}(1+\alpha-\delta|q^2)_{a+d+n}}q^{An}$$
$$=\frac{\Gamma_{q^2}(1+\alpha-\beta)\Gamma_{q^2}(1+\alpha-\gamma)\Gamma_{q^2}(1+\alpha-\delta)\Gamma_{q^2}(2+\alpha-\beta-\gamma-\delta)}{\Gamma_{q^2}(\alpha+1)\Gamma_{q^2}(2+\alpha-\beta-\gamma)\Gamma_{q^2}(2+\alpha-\beta-\delta)\Gamma_{q^2}(2+\alpha-\gamma-\delta)}$$
$$\times\frac{(\beta|q^2)_{-b}(\gamma|q^2)_{-c}(\delta|q^2)_{-d}(2+\alpha-\beta-\gamma-\delta|q^2)_{a+b+c+d-1}}{(2+\alpha-\beta-\gamma|q^2)_{a+b+c-1}(2+\alpha-\beta-\delta|q^2)_{a+b+d-1}(2+\alpha-\gamma-\delta|q^2)_{a+c+d-1}}.$$

Then the result follows readily from by setting $(\alpha,\beta,\gamma,\delta)=(0,1/2,1/2,1/2)$ in the above identity and applying the identities $\Gamma_q(1)=1$ and (3.1). □

Taking $(a,b,c,d)=(1,0,0,0)$ in Theorem 4.1 we can obtain

Example 4.1. We have

$$\sum_{n=0}^{\infty}\frac{(1+q^{2n+1})q^n}{(1-q^{2n+1})^2}=\frac{\pi_q^2}{(1-q^2)^2q^{1/2}}.$$

This series expansion for π_q^2 can be regarded as a q-analogue of the series for π^2 :

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}=\frac{\pi^2}{8}.$$

Actually, this expansion for π_q^2 has been obtained by Sun [14, (1.2)].

Setting $(a,b,c,d)=(1,1,1,0)$ in Theorem 4.1 we can derive

Example 4.2. We have

$$\sum_{n=0}^{\infty}\frac{(1+q^{2n+1})q^{5n}}{(1-q^{2n-1})^2(1-q^{2n+1})^2(1-q^{2n+3})^2}=\frac{\pi_q^2(1+q+q^2)q^{3/2}}{(1+q^2)(1-q^2)^6}.$$

This series expansion for π_q^2 can also be considered as a q-analogue of the series for π^2 :

$$\sum_{n=0}^{\infty}\frac{1}{(2n-1)^2(2n+1)^2(2n+3)^2}=\frac{3\pi^2}{256}.$$

Putting $(a,b,c,d)=(1,1,1,1)$ in Theorem 4.1 we can deduce

Example 4.3. We have

$$\frac{(1+q)q^3}{(1-q)^5(1-q^3)^3}-\sum_{n=1}^{\infty}\frac{(1+q^{2n+1})q^{7n}}{(1-q^{2n-1})^3(1-q^{2n+1})^2(1-q^{2n+3})^3}$$
$$=\frac{\pi_q^2(1+q+q^2)(1+q+q^2+q^3+q^4)q^{5/2}}{(1+q^2)^3(1-q^2)^8}.$$

This series expansion for π_q^2 is also a q-analogue of the series for π^2 :

$$\frac{1}{27}-\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3(2n+1)^2(2n+3)^3}=\frac{15\pi^2}{4096}.$$

Remark 4.1. Besides those formulas displayed in Theorems 3.2 and 4.1 and their consequences, we can give a general series expansion for $1/\pi_q^2$ by taking $(\alpha,\beta,\gamma,\delta)=(1/2,1/2,1/2,1/2)$ in Theorem 1.2, from which many series expansions for $1/\pi_q^2$ can be deduced. We shall not display them out one by one in this work.

Acknowledgements

The first author was partially supported by the National Natural Science Foundation of China (Grant No. 11801451). The second author was supported by the National Natural Science Foundation of China (Grant No. 11371184) and the Natural Science Foundation of Henan Province (Grant No. 162300410086, 2016B259, 172102410069).

References

[1] G.E. Andrews, R. Askey and R. Roy, Special Functions, Encyclopedia of Mathematics and its Applications, Vol. 71. Cambridge University Press, Cambridge, 1999.

[2] N.D. Baruah, B.C. Berndt and H.H. Chan, Ramanujan's series for $1/\pi$: A survey, Amer. Math. Monthly 116 (2009), 567–587.

[3] J.M. Borwein and P.B. Borwein, Pi and the AGM. Wiley, New York, 1987.

[4] D.V. Chudnovsky, G.V. Chudnovsky, Approximation and complex multiplication according to Ramanujan, in: G.E. Andrews, R.A. Askey, B.C. Berndt, K.G. Ramanathan, R.A. Rankin (Eds.), Ramanujan Revisited, Academic Press, Boston, 1988, pp. 375–472.

[5] G. Gasper and M. Rahman, Basic Hypergeometric Series, Cambridge University Press, Cambridge, 1990.

[6] J.W.L. Glaisher, On series for $1/\pi$ and $1/\pi^2$, Quart. J. Pure Appl. Math. 37 (1905), 173–198.

[7] R.W. Gosper, Experiments and discoveries in q-trigonometry, in: F.G. Garvan, M.E.H. Ismail (Eds.), Symbolic Computation, Number Theory, Special Functions, Physics and Combinatorics, Kluwer, Dordrecht, Netherlands, 2001, pp.79–105.

[8] V.J.W. Guo, A q-Analogue of the (I.2) Supercongruence of Van Hamme, International Journal of Number Theory, doi: 10.1142/S1793042118501701.

[9] V.J.W. Guo and J.-C. Liu, q-analogues of two Ramanujan-type formulas for $1/\pi$, arXiv: 1802.01944v2.

[10] Z.-G. Liu, A summation formula and Ramanujan type series, J. Math. Anal. Appl. 389(2) (2012), 1059–1065.

[11] Z.-G. Liu, Gauss summation and Ramanujan-type series for $1/\pi$. Int. J. Number Theory, 8(2) (2012), 289–297.

[12] S. Ramanujan, Collected Papers, Cambridge University Press, Cambridge, 1927, reprinted by Chelsea, New York, 1962, reprinted by the American Mathematical Society, Providence, RI, 2000.

[13] S. Ramanujan, Modular equations and approximations to π, Quart. J. Pure Appl. Math. 45 (1914), 350–372.

[14] Z.-W. Sun, Two q-analogues of Euler's formula $\zeta(2) = \pi^2/6$, arXiv: 1802.01473v3.

School of Mathematics and Statistics, Central South University, Changsha 410083, Hunan, People's Republic of China

E-mail address: yuhe001@foxmail.com; yuhelingyun@foxmail.com

Department of Mathematics, Luoyang Normal University, Luoyang 471934, People's Republic of China

E-mail address: zhai_hc@163.com