RADIAL SYMMETRY OF POSITIVE ENTIRE SOLUTIONS OF A FOURTH ORDER ELLIPTIC EQUATION WITH A SINGULAR NONLINEARITY

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Abstract. The necessary and sufficient conditions for a regular positive entire solution u of the biharmonic equation:

$$-\Delta^2 u = u^{-p} \text{ in } \mathbb{R}^N \ (N \geq 3), \ \ p > 1 \tag{0.1}$$

to be a radially symmetric solution are obtained via the moving plane method (MPM) of a system of equations. It is well-known that for any $a > 0$, (0.1) admits a unique minimal positive entire radial solution $\underline{u}_a(r)$ and a family of non-minimal positive entire radial solutions $u_a(r)$ such that $u_a(0) = \underline{u}_a(0) = a$ and $u_a(r) \geq \underline{u}_a(r)$ for $r \in (0, \infty)$. Moreover, the asymptotic behaviors of $\underline{u}_a(r)$ and $u_a(r)$ at $r = \infty$ are also known. We will see in this paper that the asymptotic behaviors similar to those of $\underline{u}_a(r)$ and $u_a(r)$ at $r = \infty$ can determine the radial symmetry of a general regular positive entire solution u of (0.1). The precisely asymptotic behaviors of $u(x)$ and $-\Delta u(x)$ at $|x| = \infty$ need to be established such that the moving-plane procedure can be started. We provide the necessary and sufficient conditions not only for a regular positive entire solution u of (0.1) to be the minimal entire radial solution, but also for u to be a non-minimal entire radial solution.

1. Introduction

We consider radial symmetry of positive entire solutions of the equation

$$-\Delta^2 u = u^{-p} \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $N = 3$, $1 < p < 3$ and $N \geq 4$, $p > 1$. The necessary and sufficient conditions for a positive entire solution of (1.1) to be a positive entire radially symmetric solution are established.

Equation (1.1) has been extensively studied in recent years, see, for example, [1, 2, 3, 5, 9, 13, 14, 17, 19, 20, 21] and the references therein. It arises in the study of the

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deflection of charged plates in electrostatic actuators in the modeling of electrostatic micro-electromechanical systems (MEMS) (see [18, 22] and the references therein). It is known from [5] that for $N = 3$ and $1 < p < 3$; $N \geq 4$ and $p > 1$ (1.1) admits a singular entire radial solution:

$$U_s(r) = Lr^{\alpha}, \quad r = |x|,$$

where and in the following,

$$\alpha = \frac{4}{p+1}, \quad L = \big[\alpha(2-\alpha)(N-2+\alpha)(N-4+\alpha)\big]^{-\frac{1}{p+1}}. \tag{1.2}$$

Moreover, for any $a > 0$, there is a unique $\tilde{b} := b(a) > 0$ such that the problem

$$\begin{cases} -\Delta^2 u = u^{-p} \ \text{ in } \mathbb{R}^N, \\ u(0) = a, \ u'(0) = 0, \ \Delta u(0) = b, \ u'''(0) = 0 \end{cases} \tag{1.3}$$

has a unique positive radial solution $u_{a,\tilde{b}}(r)$ satisfying

$$\lim_{r \to \infty} r^{-\alpha} u_{a,\tilde{b}}(r) = L. \tag{1.4}$$

It is also known from [5] that for any $b < \tilde{b}$, (1.3) does not admit an entire radial solution; for any $b > \tilde{b}$, (1.3) admits a unique entire radial solution $u_{a,b}(r)$ which has the growth rate $O(r^2)$ at $r = \infty$. Therefore we see that the behaviors of the minimal and non-minimal entire solutions at ∞ are different. A comparison principle (Lemma 3.2 in [20]) ensures that $u_{a,b} > u_{a,\tilde{b}}$ in $(0, \infty)$ for $b > \tilde{b}$. These imply that for any $a > 0$, $u_{a,\tilde{b}}$ is the (unique) minimal positive entire radial solution of (1.1) and $\{u_{a,b}\}_{b>\tilde{b}}$ are a family of entire non-minimal radial solutions of (1.1). Meanwhile, the comparison principle also implies that for any $b_1 > b_2 > \tilde{b}$, $u_{a,b_1} > u_{a,b_2}$ in $(0, \infty)$. The stability of positive entire solutions of (1.1) has also been studied in [14] and the references therein.

In this paper, we are interested in the relationship between the radial symmetry and the asymptotic behavior at ∞ of a positive entire solution of (1.1). We will see that if a positive regular entire solution u of (1.1) admits the asymptotic behavior as that of the minimal entire radial solution of (1.1), it is actually the minimal entire radial solution of (1.1) with respect to some $x_* \in \mathbb{R}^N$. Meanwhile, if a positive regular entire solution u of (1.1) admits the asymptotic behavior as that of a non-minimal entire radial solution of (1.1), it is actually a non-minimal entire radial solution of (1.1) with respect to some $x_* \in \mathbb{R}^N$.

Our main results are the following theorems.

Theorem 1.1. *Let $u \in C^4(\mathbb{R}^N)$ be a positive entire solution of* (1.1) *and*

$$p \in \begin{cases} (1, \frac{N+2}{6-N}], & \text{for } N = 3 \text{ or } 5, \\ (1,3] \cup (7,\infty), & \text{for } N = 4, \\ (1,\infty), & \text{for } N \geq 6. \end{cases} \tag{1.5}$$

Then u is the minimal radial entire solution of (1.1) *with the initial value $u(x_*)$ at some $x_* \in \mathbb{R}^N$ (i.e. $u(x) = u(r)$ with $r = |x - x_*|$) if and only if*

$$\lim_{|x|\to\infty} \Big[|x|^{-\alpha}u(x) - L\Big] = 0. \tag{1.6}$$

Our results for $p = 7$ and $N = 4$; $p \in (\frac{N+2}{6-N}, p^*)$ and $N = 3$, 4 or 5 are a little different, where we denote

$$p^* := \frac{N+3}{5-N} = \begin{cases} 3, & \text{for } N = 3, \\ 7, & \text{for } N = 4, \\ \infty, & \text{for } N = 5. \end{cases} \tag{1.7}$$

Theorem 1.2. *Let $p = 7$ and $N = 4$; $u \in C^4(\mathbb{R}^4)$ be a positive entire solution of* (1.1)*. Then u is the minimal radial entire solution of* (1.1) *with the initial value $u(x_*)$ at some $x_* \in \mathbb{R}^4$ if and only if there exists $0 < \epsilon_0 < \frac{1}{10}$ such that*

$$|x|^{-\alpha}u(x) - L = o\Big(|x|^{-\epsilon_0}\Big) \quad \text{as } |x| \to \infty. \tag{1.8}$$

Theorem 1.3. *Let $p \in (\frac{N+2}{6-N}, p^*)$ and $N = 3$, 4 or 5; $u \in C^4(\mathbb{R}^N)$ be a positive entire solution of* (1.1)*. Then u is the minimal radial entire solution of* (1.1) *with the initial value $u(x_*)$ at some $x_* \in \mathbb{R}^N$ if and only if*

$$|x|^{-\alpha}u(x) - L = o\Big(|x|^{5-N-2\alpha}\Big) \quad \text{as } |x| \to \infty. \tag{1.9}$$

Note that $5 - N - 2\alpha \in (-1, 0)$ when $N = 3$, 4 or 5 and $p \in (\frac{N+2}{6-N}, p^*)$.

The following theorem provides the necessary and sufficient conditions for a positive entire solution of (1.1) to be a non-minimal positive radial entire solution of (1.1).

Theorem 1.4. *Let $u \in C^4(\mathbb{R}^N)$ be a positive entire solution of* (1.1) *with $N = 3$ and $1 < p < 3$; $N \geq 4$ and $p > 1$. Then u is an entire radial solution about some $x_* \in \mathbb{R}^N$, but is not the minimal positive entire radial solution about x_* of* (1.1)*, if and only if there exists $D > 0$ such that*

$$\lim_{|x|\to\infty} \Big[|x|^{-2}u(x) - D\Big] = 0. \tag{1.10}$$

The constant D then determines a particular non-minimal positive entire radial solution.

Theorems 1.1-1.4 show that the asymptotic behavior given in (1.6), (1.8), (1.9) or (1.10) near ∞ of a positive entire solution u of (1.1) determines its radial symmetry with respect to some $x_* \in \mathbb{R}^N$, which seem to be the first such kinds of results for problem (1.1).

Let us comment on some related results. The semilinear equations

$$(P) \qquad -\Delta u = u^p \ \text{ in } \mathbb{R}^N \ (N \geq 3), \ \ p > \frac{N+2}{N-2}$$

and

$$(Q) \qquad \Delta u = u^{-p} \ \text{ in } \mathbb{R}^N \ (N \geq 2), \ \ p > 0$$

have been studied in the past few decades. Some sufficient conditions for a regular positive entire solution of (P) and (Q) to be an entire radial solution are given in [25] for (P) provided $p \in (\frac{N+2}{N-2}, \frac{N+1}{N-3})$ and in [15] for (Q) provided $p > 0$ respectively. The results in [25] were generalized to $p \geq \frac{N}{N-4}$ for $N \geq 5$ in [10]. Recently, the necessary and sufficient conditions for an entire solution u of the equation

$$(P_1) \qquad \Delta^2 u = 8(N-2)(N-4)e^u \ \text{ in } \mathbb{R}^N \ (N \geq 5)$$

to be the entire radial solution of (P_1) with the initial value at some $x_* \in \mathbb{R}^N$ are provided in [11]. Note that (1.1) can be written to the following system of equations:

$$(1.11) \qquad \begin{cases} -\Delta u = w & \text{in } \mathbb{R}^N, \\ -\Delta w = -u^{-p} & \text{in } \mathbb{R}^N. \end{cases}$$

As in [11], we use the moving plane method for a system of equations to obtain our results, but we need to do more delicate estimates for the solution u and Δu near ∞, since (Q) has a more complicated structure of solutions than (P1). We discuss not only the minimal solution but also the non-minimal solutions in this paper. Such estimates we need to do are more complicated since they rely on two parameters p and N. Moreover, for the non-minimal entire radial solution case, the asymptotic behavior (1.10) is not enough to make the moving-plane procedure works, we need to obtain more detailed information of the asymptotic behavior of u based on (1.10). To know more information of the positive entire solutions with asymptotic behavior (1.6) near ∞, we use a Kelvin type transformation:

$$(1.12) \qquad v(y) = |x|^{-\alpha} u(x) - L, \quad y = \frac{x}{|x|^2}$$

and make a fundamental estimate for

$$(1.13) \qquad W(s) := \Big(\int_{S^{N-1}} w^2(s, \theta) d\theta \Big)^{\frac{1}{2}},$$

where $s = |y| = \frac{1}{r}$, $r = |x|$, $w(s,\theta) := v(s,\theta) - \overline{v}(s)$ and

$$\overline{v}(s) = \frac{1}{\omega_{N-1}} \int_{S^{N-1}} v(s,\theta) d\theta, \quad \omega_{N-1} = |S^{N-1}|.$$

The key point is to show that $W(s)$ is Lipschitz continuous, or Hölder continuous in some case, in a neighborhood of $s = 0$.

In Sections 2–5, we deal with positive entire solutions u of (1.1) with the asymptotic behavior (1.6). In the last section, we deal with positive entire solutions u of (1.1) with (1.10). In Section 2, we first introduce some preliminary results about the eigenvalues and eigenfunctions of $\Delta^2_{S^{N-1}}$. Then, using the Kelvin-type transformation given in (1.12) we obtain the information of $v(y)$ near $y = 0$. In Section 3, we derive an important estimate for $W(s)$ (given in (1.13)) near $s = 0$. In Section 4, some estimates for $\overline{v}(s)$ and $v(s, \theta)$ near $s = 0$ are obtained. We present the proofs of Theorems 1.1, 1.2 and 1.3 in Section 5. Finally, we prove Theorem 1.4 in Section 6. In this paper, we use C to denote a positive constant which may change line by line.

2. Preliminaries

In this section, we present some results which will be useful in the following proofs. We use the spherical coordinates $x = (r, \theta)$ as usual. First, let us to show the following lemma (see Lemma 2.1 in [11]).

Lemma 2.1. *Let $(\lambda, Q(\theta))$ be a pair of eigenvalue and eigenfunction of the equation*

$$-\Delta_{S^{N-1}} Q = \lambda Q. \tag{2.1}$$

Then $(\lambda^2, Q(\theta))$ is a pair of eigenvalue and eigenfunction of the equation

$$\Delta^2_{S^{N-1}} Q = \sigma Q. \tag{2.2}$$

Conversely, if $(\sigma, Q(\theta))$ is a pair of eigenvalue and eigenfunction of (2.2) with $\sigma \neq 0$, then $\sigma > 0$ and $(\sigma^{1/2}, Q(\theta))$ is a pair of eigenvalue and eigenfunction of (2.1).

It is known from [4] that for $N \geq 3$, the eigenvalues of the equation (2.1) are given by

$$\lambda_k = k(N + k - 2), \quad k \geq 0, \ \ k \in \mathbb{N} \tag{2.3}$$

with multiplicity

$$m_k = \frac{(N - 2 + 2k)(N - 3 + k)!}{k!(N - 2)!}. \tag{2.4}$$

Then Lemma 2.1 implies that the eigenvalues of the equation (2.2) are λ_k^2 with the same multiplicity. In particular, we have

$$\lambda_0 = 0,\ \ m_0 = 1,\ \ Q_1^0(\theta) \equiv \frac{1}{\sqrt{|S^{N-1}|}},$$

$$\lambda_1 = N-1,\ \ m_1 = N,\ \ Q_j^1(\theta) = \frac{x_j|_{S^{N-1}}}{\|x_j|_{S^{N-1}}\|_{L^2}},\ \ 1 \le j \le N\ (:= m_1),$$

$$\lambda_2 = 2N.$$

Therefore, if $w \in H^2(S^{N-1})$ is orthogonal to Q_1^0, i.e. $\overline{w} = 0$, we have

$$\int_{S^{N-1}} |\nabla_\theta w|^2 d\theta \ge (N-1) \int_{S^{N-1}} w^2 d\theta,$$

and

$$\int_{S^{N-1}} |\Delta_\theta w|^2 d\theta \ge (N-1)^2 \int_{S^{N-1}} w^2 d\theta.$$

The boot-strap argument implies that for $1 \le j \le m_k$,

$$\max_{S^{N-1}} |Q_j^k(\theta)| \le D_k, \quad \max_{S^{N-1}} |(Q_j^k)_\theta(\theta)| \le E_k, \tag{2.5}$$

where

$$D_k := C(1 + \lambda_k + \lambda_k^2 + \ldots + \lambda_k^\tau), \quad E_k := C(1 + \lambda_k + \lambda_k^2 + \ldots + \lambda_k^{\tau_1}) \tag{2.6}$$

with $C > 0$ being independent of k and $\tau \ge 1$, $\tau_1 \ge 1$ being positive integers such that $2\tau > N-1$, $2\tau_1 > N$.

In Sections 2-5, we assume that $u \in C^4(\mathbb{R}^N)$ is a positive entire solution of (1.1) with (1.6). Introducing the Kelvin-type transformation:

$$v(y) = |x|^{-\alpha} u(x) - L, \quad y = \frac{x}{|x|^2}, \ \ r = |x| > 0, \tag{2.7}$$

we see that $u(x) = u(r, \theta)$, $v(y) = v(s, \theta)$ with $s = |y| = r^{-1}$ and

$$\begin{aligned}\Delta_x^2 u \ &= \ \Big[\partial_r^4 + 2(N-1)r^{-1}\partial_r^3 + (N-1)(N-3)r^{-2}\partial_r^2 - (N-1)(N-3)r^{-3}\partial_r \\ &\quad +(8-2N)r^{-4}\Delta_\theta + (2N-6)r^{-3}\Delta_\theta\partial_r + 2r^{-2}\Delta_\theta\partial_r^2 + r^{-4}\Delta_\theta^2\Big]u,\end{aligned}$$

with the notations $\partial_r = \frac{\partial}{\partial r}$ and $\partial_r^m = \frac{\partial^m}{\partial r^m}$ for $2 \le m \le 4$. Direct calculations imply that

$$\begin{aligned}&\partial_s^4 v - 2(N-7+2\alpha)s^{-1}\partial_s^3 v + (N^2 + 6\alpha N + 6\alpha^2 - 16N - 36\alpha + 51)s^{-2}\partial_s^2 v \\ &\quad -(N-5+2\alpha)(2N\alpha + 2\alpha^2 - 3N - 10\alpha + 9)s^{-3}\partial_s v \\ &\quad +\alpha(\alpha-2)(N+\alpha-2)(N+\alpha-4)s^{-4}(v+L) - 2(N-5+2\alpha)s^{-3}\Delta_\theta(\partial_s v) \\ &\quad +2(N\alpha + \alpha^2 - N - 4\alpha + 4)s^{-4}\Delta_\theta v + 2s^{-2}\Delta(\partial_s^2 v) + s^{-4}\Delta_\theta^2 v \\ &= -r^{8-\alpha}\Delta_x^2 u \ = -r^{8-\alpha}u^{-p} \ = \ -s^{-8+\alpha(p+1)}(v+L)^{-p}.\end{aligned}$$

Since $L = [\alpha(2-\alpha)(N-2+\alpha)(N-4+\alpha)]^{-\frac{1}{p+1}}$ and $\alpha = \frac{4}{p+1}$, we have

(2.8)
$$\begin{aligned}\partial_s^4 v &- 2(N-7+2\alpha)s^{-1}\partial_s^3 v + (N^2+6\alpha N+6\alpha^2-16N-36\alpha+51)s^{-2}\partial_s^2 v\\ &-(N-5+2\alpha)(2N\alpha+2\alpha^2-3N-10\alpha+9)s^{-3}\partial_s v\\ &-2(N-5+2\alpha)s^{-3}\Delta_\theta(\partial_s v)+2(N\alpha+\alpha^2-N-4\alpha+4)s^{-4}\Delta_\theta v\\ &+2s^{-2}\Delta_\theta(\partial_s^2 v)+s^{-4}\Delta_\theta^2 v-(p+1)s^{-4}L^{-(p+1)}v+s^{-4}f(v)=0,\end{aligned}$$

where $f(t) = (t+L)^{-p} - L^{-p} + pL^{-(p+1)}t = O(t^2)$ for t near 0. Note that $f(t)$ is real analytic at $t=0$ and satisfies $f(0) = f'(0) = 0$, $f''(0) = p(p+1)L^{-(p+2)} > 0$. Therefore, the study of the behavior of u near $|x| = \infty$ is converted to the study of the behavior of v of the equation (2.8) near $|y| = 0$.

Lemma 2.2. *Let $u \in C^4(\mathbb{R}^N)$ be a positive entire solution of* (1.1) *and let v be given in* (2.7). *Suppose that*

$$|x|^{-\alpha}u(x) - L \to 0 \quad \text{as } |x| \to \infty. \tag{2.9}$$

Then for any integer $\ell \ge 0$ there exist constants $M = M(u) > 0$, $s^* = s^*(u) > 0$ such that

$$\lim_{|y|\to 0} v(y) = 0, \quad |\nabla^\ell v(y)| \le \frac{M}{s^\ell} \quad \text{for } s = |y| \le s^*. \tag{2.10}$$

Proof. The estimates in (2.10) follow from (2.9) by standard elliptic theory. □

By Lemma 2.2, we are reduced to study solutions of (2.8) satisfying (2.10). Therefore, we will assume that (2.10) holds in Sections 2-5.

Define

$$w(s,\theta) = v(s,\theta) - \overline{v}(s), \tag{2.11}$$

where

$$\overline{v}(s) = \frac{1}{\omega_{N-1}}\int_{S^{N-1}} v(s,\theta)d\theta, \quad \omega_{N-1} = |S^{N-1}|.$$

Lemma 2.3. *Let v be a solution of* (2.8). *Then $\overline{v}$ and w satisfy*

$$\begin{aligned}&\partial_s^4\overline{v} - 2(N-7+2\alpha)s^{-1}\partial_s^3\overline{v}\\ &\quad + (N^2+6\alpha N+6\alpha^2-16N-36\alpha+51)s^{-2}\partial_s^2\overline{v}\\ &\quad - (N-5+2\alpha)(2N\alpha+2\alpha^2-3N-10\alpha+9)s^{-3}\partial_s\overline{v}\\ &\quad - (p+1)s^{-4}L^{-(p+1)}\overline{v} + s^{-4}\overline{f(v)} = 0\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}
&\partial_s^4 w - 2(N-7+2\alpha)s^{-1}\partial_s^3 w \\
&\quad + (N^2+6\alpha N+6\alpha^2-16N-36\alpha+51)s^{-2}\partial_s^2 w \\
&\quad - (N-5+2\alpha)(2N\alpha+2\alpha^2-3N-10\alpha+9)s^{-3}\partial_s w \\
&\quad + 2(N\alpha+\alpha^2-N-4\alpha+4)s^{-4}\Delta_\theta w - 2(N-5+2\alpha)s^{-3}\Delta_\theta(\partial_s w) \\
&\quad + 2s^{-2}\Delta_\theta(\partial_s^2 w) + s^{-4}\Delta_\theta^2 w - (p+1)s^{-4}L^{-(p+1)}w + s^{-4}g(w) = 0,
\end{aligned} \tag{2.13}$$

respectively, where

$$g(w) := f(v) - \overline{f(v)} = f'(\xi(s,\theta))w(s,\theta) - \overline{f'(\xi(s,\theta))w(s,\theta)}$$

and $\xi(s,\theta)$ is between $v(s,\theta)$ and $\overline{v}(s)$.

Proof. Since

$$\overline{\Delta_\theta v} = \frac{1}{\omega_{N-1}}\int_{S^{N-1}} \Delta_\theta v(s,\theta)d\theta = 0,$$

direct calculations derive (2.12) and (2.13). Moreover, we have

$$\begin{aligned}
g(w) &= f(v) - \overline{f(v)} = f(v) - f(\overline{v}) - (\overline{f(v) - f(\overline{v})}) \\
&= f'(\xi(s,\theta))w(s,\theta) - \overline{f'(\xi(s,\theta))w(s,\theta)}
\end{aligned}$$

for some $\xi(s,\theta)$ between $v(s,\theta)$ and $\overline{v}(s)$. Where

$$f'(\xi(s,\theta)) = pL^{-(p+1)} - p[\xi(s,\theta)+L]^{-(p+1)} \geq 0. \tag{2.14}$$

If we define

$$\zeta(s) := \max_{\theta\in S^{N-1}} f'(\xi(s,\theta)),$$

we see that $\zeta(s) \to 0$ as $s \to 0$. □

To end this section, we notice that since $w(s,\cdot) \in H^2(S^{N-1}) \subset L^2(S^{N-1})$ and $\overline{w} = 0$,

$$w(s,\theta) = \sum_{k=1}^{\infty}\sum_{j=1}^{m_k} w_j^k(s)Q_j^k(\theta), \tag{2.15}$$

where $\{Q_1^0(\theta), Q_1^1(\theta), \ldots, Q_{m_1}^1(\theta), Q_1^2(\theta), Q_2^2(\theta), \ldots, Q_{m_2}^2(\theta), Q_1^3(\theta), \ldots\}$ is the standard normalized basis of $H^2(S^{N-1})$, i.e., $\int_{S^{N-1}} Q_l^i(\theta)Q_m^j(\theta)d\theta = 0$ if $i \neq j$ or $l \neq m$, $\|Q_j^i\|_{L^2(S^{N-1})} = 1$ which is consisted by all the eigenfunctions of the operator $-\Delta_{S^{N-1}}$ or $\Delta_{S^{N-1}}^2$ in $H^2(S^{N-1})$. Note that $\{Q_1^1(\theta), \ldots, Q_N^1(\theta)\}$ is the basis of the eigenspace H_1 of $\Delta_{S^{N-1}}^2$ corresponding to the eigenvalue $(N-1)^2$.

3. A priori estimate of $W(s)$ for s near 0

In this section, we establish some fundamental estimates of $W(s)$ for s near 0, where $W(s)$ is defined by

$$W(s) = \Big(\int_{S^{N-1}} w^2(s,\theta) d\theta \Big)^{\frac{1}{2}}. \tag{3.1}$$

We will see that the Lipschitz continuity of $W(s)$ at the origin is crucial in proving the expansion of u near ∞, which can be used to obtain the symmetry of u by the moving-plane method.

Proposition 3.1. *For $N \geq 3$, there exist $0 < s_0 < \min\{1, s^*\}$ (s^* is given in Lemma 2.2), $0 < \hat{\beta} < 1$ and $C > 0$ independent of s such that for $s \in (0, s_0)$,*

$$W(s) \leq \begin{cases} Cs, & \text{for } N \text{ and } p \text{ satisfying (1.5) or } p = 7 \text{ and } N = 4 \text{ with (1.8)}, \\ Cs^{\hat{\beta}}, & \text{for } p \in (\frac{N+2}{6-N}, p^*) \text{ and } N = 3, 4, 5, \end{cases} \tag{3.2}$$

where p^* is given by (1.7).

In fact, $\hat{\beta} = |\beta_3^{(1)}| = |5 - N - 2\alpha| \in (0,1)$ is given by (3.35) below when $p \in (\frac{N+2}{6-N}, p^*)$ and $N = 3, 4, 5$.)

Proof. Let $Q_j^k(\theta)$ ($1 \leq j \leq m_k, k = 1, 2, \ldots$) be an eigenfunction of $-\Delta_{S^{N-1}}$ corresponding to $\lambda_k = k(N + k - 2)$. From Lemma 2.3, we see that $w_j^k(s)$ satisfies the equation

$$\begin{aligned} &(w_j^k)^{(4)}(s) - 2(N - 7 + 2\alpha)s^{-1}(w_j^k)^{(3)}(s) \\ &\quad + \big(N^2 + 6\alpha N + 6\alpha^2 - 16N - 36\alpha + 51 - 2\lambda_k\big)s^{-2}(w_j^k)''(s) \\ &\quad - \big[(N - 5 + 2\alpha)(2N\alpha + 2\alpha^2 - 3N - 10\alpha + 9) - 2(N - 5 + 2\alpha)\lambda_k\big]s^{-3}(w_j^k)'(s) \\ &\quad + \big[\lambda_k^2 - 2\big(N\alpha + \alpha^2 - N - 4\alpha + 4\big)\lambda_k - (p+1)L^{-(p+1)}\big]s^{-4}w_j^k = s^{-4}g_j^k(s), \end{aligned} \tag{3.3}$$

where

$$g_j^k(s) = \int_{S^{N-1}} f'(\xi(s,\theta))w(s,\theta)Q_j^k(\theta)d\theta,$$

which can be controlled by $|g_j^k(s)| \leq \zeta(s)W(s)$, here $\zeta(s) \to 0$ and $W(s) \to 0$ as $s \to 0$.

Note that $g_j^k(s)$ and $w_j^k(s)$ are Fourier's coefficients of $f'(\xi)w(s,\theta)$ and $w(s,\theta)$ respectively. Moreover,

$$\|f'(\xi)w(s,\theta)\|_{L^2(S^{N-1})} \leq \zeta(s)\|w(s,\theta)\|_{L^2(S^{N-1})} = o_s(1)\|w\|_{L^2(S^{N-1})} \tag{3.4}$$

and $W(s) = [\sum_{k=1}^{\infty}\sum_{j=1}^{m_k}(w_j^k(s))^2]^{\frac{1}{2}}$. Therefore, for any (j,k) fixed and s sufficiently small, to estimate $W(s)$, we only need to assume

$$|g_j^k(s)| = o_s(1)|w_j^k(s)|. \tag{3.5}$$

In fact, from (3.4), the expression of $w(s,\theta)$ given by (2.15) and

$$f'(\xi)w(s,\theta)=\sum_{k=1}^{\infty}\sum_{j=1}^{m_k} g_j^k(s)Q_j^k(\theta),$$

we see that

$$\sum_{k=1}^{\infty}\sum_{j=1}^{m_k}(g_j^k(s))^2=o_s(1)\sum_{k=1}^{\infty}\sum_{j=1}^{m_k}(w_j^k(s))^2.$$

Therefore, there are two cases:

$$(i)\quad |g_j^k(s)|=o_s(1)|w_j^k(s)|;\qquad (ii)\quad |g_j^k(s)|\neq o_s(1)|w_j^k(s)|.$$

For any fixed $s\in(0,s^*)$, denote

$$G_s=\{(j,k):1\le j\le m_k, k\ge 1 \text{ such that (i) holds}\},$$

$$B_s=\{(j,k):1\le j\le m_k, k\ge 1 \text{ such that (ii) holds}\}.$$

We claim that there exists $C>0$ independent of j,k and s such that for any $s\in(0,s^{**})$ and any $(j,k)\in B_s$,

$$|g_j^k(s)|\ge C|w_j^k(s)|, \tag{3.6}$$

where $0<s^{**}\le s^*$ (s^* is given in Lemma 2.2). Suppose not, there exists $c_n\to 0$, $s_n\to 0$ as $n\to\infty$ and $(j_n,k_n)\in B_{s_n}$ such that

$$|g_{j_n}^{k_n}(s_n)|\le c_n|w_{j_n}^{k_n}(s_n)|.$$

This implies

$$|g_{j_n}^{k_n}(s_n)|=o_{s_n}(1)|w_{j_n}^{k_n}(s_n)| \quad \text{for } n \text{ large enough},$$

which contradicts $(j_n,k_n)\in B_{s_n}$. Therefore, for any $s\in(0,s^{**})$,

$$\sum_{(j,k)\in B}|w_j^k(s)|^2\le C^{-2}\sum_{(j,k)\in B}|g_j^k(s)|^2\le o_s(1)\sum_{k=1}^{\infty}\sum_{j=1}^{m_k}|w_j^k(s)|^2.$$

Therefore, without loss of generality, we assume that (3.5) holds for $0<s\le s^{**}$, any $k\ge 1$ and $1\le j\le m_k$.

Let $t=-\ln s$, $z_j^k(t)=w_j^k(s)$. Then $z_j^k(t)$ satisfies the equation

$$\begin{aligned}&(z_j^k)^{(4)}(t)+2(N-4+2\alpha)(z_j^k)^{(3)}(t)\\&\quad+\big[N^2+6\alpha N+6\alpha^2-10N-24\alpha+20-2\lambda_k\big](z_j^k)''(t)\\&\quad+2(N-4+2\alpha)(N\alpha-N-4\alpha+\alpha^2+2-\lambda_k)(z_j^k)'(t)\\&\quad+\big[\lambda_k^2-2(N\alpha+\alpha^2-N-4\alpha+4)\lambda_k-(p+1)L^{-(p+1)}\big]z_j^k(t)=\tilde g_j^k(t),\end{aligned} \tag{3.7}$$

where $\tilde g_j^k(t)=g_j^k(e^{-t})$. We also know from $\zeta(s)$ and $g_j^k(s)$ that

$$|\tilde g_j^k(t)|\le\tilde\zeta(t)\tilde W(t), \tag{3.8}$$

where

$$\tilde\zeta(t):=\zeta(e^{-t})\to 0,\quad \text{and}\ \ \tilde W(t):=W(e^{-t})\to 0 \quad \text{as } t\to\infty. \tag{3.9}$$

The corresponding characteristic polynomial of (3.7) is

$$\begin{aligned}\beta^4 &+ 2(N-4+2\alpha)\beta^3 + \big(N^2 + 6\alpha N + 6\alpha^2 - 10N - 24\alpha + 20 \\ &-2\lambda_k\big)\beta^2 + 2(N-4+2\alpha)(N\alpha - N - 4\alpha + \alpha^2 + 2 - \lambda_k)\beta \\ &+\lambda_k^2 - 2(N\alpha + \alpha^2 - N - 4\alpha + 4)\lambda_k - (p+1)L^{-(p+1)} = 0\end{aligned} \tag{3.10}$$

and using the MATLAB, the four roots of (3.10) are given by

$$\begin{cases} \beta_1^{(k)} = \frac{1}{2}\Big(4 - N - 2\alpha + \sqrt{4 + (N-2+2k)^2 + 4\sqrt{\rho_k}}\Big), \\ \beta_2^{(k)} = \frac{1}{2}\Big(4 - N - 2\alpha - \sqrt{4 + (N-2+2k)^2 + 4\sqrt{\rho_k}}\Big), \\ \beta_3^{(k)} = \frac{1}{2}\Big(4 - N - 2\alpha + \sqrt{4 + (N-2+2k)^2 - 4\sqrt{\rho_k}}\Big), \\ \beta_4^{(k)} = \frac{1}{2}\Big(4 - N - 2\alpha - \sqrt{4 + (N-2+2k)^2 - 4\sqrt{\rho_k}}\Big), \end{cases} \tag{3.11}$$

where

$$\rho_k = (N-2+2k)^2 + p\alpha(2-\alpha)(N-2+\alpha)(N-4+\alpha). \tag{3.12}$$

We first analyze the four roots $\beta_j^{(k)}, j = 1,2,3,4$ for $k = 1,2,\ldots$. Note that

$$\alpha \in (1,2), \ \text{ for } N = 3,\ 1 < p < 3; \quad \alpha \in (0,2), \ \text{ for } N \geq 4,\ p > 1. \tag{3.13}$$

Then, we see from (3.12) that $\rho_k > 0$ for $k = 1,2,\ldots$. Set

$$T_k := [4 + (N-2+2k)^2]^2 - 16\rho_k.$$

Noticing that $p\alpha = 4 - \alpha$, we have

$$(2-\alpha)(N+\alpha-4) \leq (N-2)^2, \quad (4-\alpha)(N+\alpha-2) \leq (N+2)^2.$$

Thus

$$\begin{aligned} T_k &= [(N-2+2k)^2 - 4]^2 - 16(4-\alpha)(2-\alpha)(N+\alpha-2)(N+\alpha-4) \\ &\geq [N^2-4]^2 - (N-2)^2(N+2)^2 \geq 0 \end{aligned}$$

for $N = 3$ and $p \in (1,3)$; $N \geq 4$ and $p \geq 1$ and any $k = 1,2,\ldots$. This indicates that $\beta_j^{(k)}$ are real numbers for $k = 1,2,\ldots$. Therefore, taking into account the expressions in (3.11) of $\beta_j^{(k)}$, we have demonstrated the following statement.

Claim 1. For any $k \geq 1$; $N = 3$ and $p \in (1,3)$; $N \geq 4$ and $p > 1$, the roots $\beta_j^{(k)}$ $(j = 1,2,3,4)$ are real numbers. Moreover,

$$\beta_2^{(k)} < \beta_4^{(k)} \leq \beta_3^{(k)} < \beta_1^{(k)}.$$

Remark 3.2. For $N = 3$ and $p \in (1,3)$; $N \geq 4$ and $p > 1$, noticing that

$$4 - 2\alpha - N < 0 \quad \text{and} \quad 4 + (N+2k-2)^2 - (4-2\alpha-N)^2 > 0, \tag{3.14}$$

we find from (3.11) that

$$\beta_2^{(k)} < \beta_4^{(k)} < 0 < \beta_1^{(k)} \quad \text{for any } k \geq 1. \tag{3.15}$$

We now determine the sign of $\beta_3^{(k)}$ for any $k \geq 2$.

Claim 2. For any $k \geq 2$, $\beta_3^{(k)} > 0$ when $N = 3$ and $p \in (1, 3)$; $N \geq 4$ and $p > 1$.

In fact, by (3.14), we see that $\beta_3^{(k)} > 0$ is equivalent to

$$\hat{T}_k := \left[4 + (N + 2k - 2)^2 - (4 - 2\alpha - N)^2\right]^2 - 16\rho_k > 0,$$

for any $k \geq 2$ when $N = 3$ and $p \in (1, 3)$; $N \geq 4$ and $p > 1$. Writing $\hat{T}_k$ as

$$\begin{aligned}\hat{T}_k &= 16(k - \alpha)(k + 2 - \alpha)(N + \alpha - 2 + k)(N + \alpha - 4 + k) \\ &\quad -16(2 - \alpha)(4 - \alpha)(N + \alpha - 2)(N + \alpha - 4).\end{aligned}$$

Therefore Claim 2 follows since we have

$$\begin{aligned} &k - \alpha \geq 2 - \alpha > 0, \qquad && N + \alpha - 2 + k \geq N + \alpha - 2 > 0, \\ &k + 2 - \alpha \geq 4 - \alpha > 0, && N + \alpha - 4 + k \geq N + \alpha - 4 > 0. \end{aligned}$$

For the root $\beta_4^{(k)}$, we have the following assertion.

Claim 3. For any $k \geq 2$, $\beta_4^{(k)} < -1$ for $N = 3$ and $p \in (1, 3)$; $N \geq 4$ and $p > 1$.

By the expression of $\beta_4^{(k)}$, we have

$$\beta_4^{(k)} + 1 = \frac{1}{2}\left[6 - N - 2\alpha - \sqrt{4 + (N + 2k - 2)^2 - 4\sqrt{\rho_k}}\,\right]. \tag{3.16}$$

Obviously, $\beta_4^{(k)} + 1 < 0$ when $N \geq 6$ and $p > 1$, and even in the case of $N \in \{3, 4, 5\}$ and $6 - N - 2\alpha \leq 0$, i.e. $p \in (1, \frac{N+2}{6-N}]$. On the other cases, we see that $\beta_4^{(k)} + 1 < 0$ is equivalent to

$$0 < 6 - N - 2\alpha < \sqrt{4 + (N + 2k - 2)^2 - 4\sqrt{\rho_k}}.$$

So to obtain our claim, it's sufficient to show that

$$\tilde{T}_k := \left[4 + (N + 2k - 2)^2 - (6 - N - 2\alpha)^2\right]^2 - 16\rho_k > 0.$$

Since

$$\begin{aligned}\tilde{T}_k &= 16(k + 1 - \alpha)(k + 3 - \alpha)(N + \alpha + k - 3)(N + \alpha + k - 5) \\ &\quad -16(2 - \alpha)(4 - \alpha)(N + \alpha - 2)(N + \alpha - 4).\end{aligned}$$

We find again $\tilde{T}_k > 0$ for any $k \geq 2$ and $p \in (\frac{5}{3}, 3)$ with $N = 3$; $p > \frac{N+2}{6-N}$ with $N = 4, 5$ from the facts

$$\begin{aligned} &k + 1 - \alpha \geq 2 - \alpha > 0, \qquad && N + \alpha + k - 3 \geq N + \alpha - 2 > 0, \\ &k + 3 - \alpha \geq 4 - \alpha > 0, && N + \alpha + k - 5 \geq N + \alpha - 4 > 0. \end{aligned}$$

Consequently, the Claim 3 is derived from all these arguments.

Remark 3.3. It follows from Claims 1-3 that for $N = 3$ and $p \in (1, 3)$; $N \geq 4$ and $p > 1$; any $k \geq 2$,

$$\beta_2^{(k)} < \beta_4^{(k)} < -1 < 0 < \beta_3^{(k)} < \beta_1^{(k)}. \tag{3.17}$$

Moreover, we deduce from the expressions of $\beta_j^{(k)}$ that

$$\beta_2^{(k+1)} < \beta_2^{(k)} < 0, \qquad \beta_4^{(k+1)} < \beta_4^{(k)} < -1,$$
$$\beta_3^{(k+1)} > \beta_3^{(k)} > 0, \qquad \beta_1^{(k+1)} > \beta_1^{(k)} > 0$$

and

$$\begin{aligned} \beta_2^{(k+1)} - \beta_2^{(k)} &\to -1 \quad \text{as } k \to \infty, \\ \beta_4^{(k+1)} - \beta_4^{(k)} &\to -1 \quad \text{as } k \to \infty, \\ \beta_3^{(k+1)} - \beta_3^{(k)} &\to 1 \quad \text{as } k \to \infty. \end{aligned}$$

Now we investigate the details of $\beta_j^{(1)},\ j = 1, 2, 3, 4$. Recalling that p^* is given by (1.7), we have:

Claim 4. The following inequalities hold for $k = 1$:

$$\begin{aligned} &\beta_2^{(1)} < \beta_4^{(1)} \le \beta_3^{(1)} = -1 < 0 < \beta_1^{(1)}, && \text{for } \begin{cases} p \in (1, \frac{N+2}{6-N}], & N = 3, 4, 5; \\ p \in (1, \infty), & N \ge 6; \end{cases} \\ &\beta_2^{(1)} < \beta_4^{(1)} = -1 < \beta_3^{(1)} < 0 < \beta_1^{(1)}, && \text{for } p \in (\tfrac{N+2}{6-N}, p^*) \text{ and } N = 3, 4, 5; \\ &\beta_2^{(1)} < \beta_4^{(1)} = -1 < \beta_3^{(1)} = 0 < \beta_1^{(1)}, && \text{for } p = 7 \text{ and } N = 4; \\ &\beta_2^{(1)} < \beta_4^{(1)} = -1 < 0 < \beta_3^{(1)} < \beta_1^{(1)}, && \text{for } p \in (7, \infty) \text{ and } N = 4. \end{aligned}$$

For $k = 1$, from the expressions of

$$\rho_1 = N^2 + (4 - \alpha)(2 - \alpha)(N - 2 + \alpha)(N - 4 + \alpha),$$

a direct calculation shows that

$$\sqrt{4 + (N)^2 - 4\sqrt{\rho_1}} = |N - 6 + 2\alpha|$$

and therefore

$$\begin{aligned} \beta_3^{(1)} &= \frac{1}{2}\Big[4 - N - 2\alpha - |N - 6 + 2\alpha|\Big], \\ \beta_4^{(1)} &= \frac{1}{2}\Big[4 - N - 2\alpha + |N - 6 + 2\alpha|\Big]. \end{aligned}$$

As before, we have obviously $N - 6 + 2\alpha > 0$ when $N \ge 6$ or $N \in \{3, 4, 5\}$ and $p \in (1, \frac{N+2}{6-N})$, which implies

$$\beta_4^{(1)} = 5 - N - 2\alpha < -1 = \beta_3^{(1)}.$$

This combining with Claim 1 and (3.15) yields that

$$\beta_2^{(1)} < \beta_4^{(1)} < \beta_3^{(1)} = -1 < 0 < \beta_1^{(1)}.$$

We obtain also for $p \in [\frac{N+2}{6-N}, p^*)$ and $N = 3$, 4 or 5,

$$\beta_4^{(1)} = -1 \le 5 - N - 2\alpha = \beta_3^{(1)} < 0;$$

and when $N = 4$, $p = p^* = \frac{N+3}{5-N} = 7$,

$$\beta_4^{(1)} = -1 < 0 = \beta_3^{(1)};$$

when $N=4$, $p>\frac{N+3}{5-N}=7$,

$$\beta_4^{(1)}=-1<0<5-N-2\alpha=\beta_3^{(1)}<1.$$

Combining with Claim 1 and Remark 3.2, we prove that Claim 4 holds.

We continue the proof of Proposition 3.1.

For any $k\geq 2$, from the equation satisfied by z_j^k and the ODE theory, we see that, for any $T>-\ln s^{**}$, there exist constants $A_{j,i}^k$, B_i^k $(i=1,2,3,4)$ such that for $t>T$,

$$z_j^k(t)=\sum_{i=1}^{4}\Big[A_{j,i}^k e^{\beta_i^{(k)}t}+B_i^k\int_T^t e^{\beta_i^{(k)}(t-s)}\tilde{g}_j^k(s)ds\Big], \tag{3.18}$$

where $A_{j,i}^k$ $(i=1,2,3,4)$ depend on T and $\beta_i^{(k)}$, but B_i^k $(i=1,2,3,4)$ depend only on $\beta_i^{(k)}$. More precisely, the detailed calculations show that

$$A_{j,1}^k=\frac{F_{j,1}^k(T)}{(\beta_1^{(k)}-\beta_2^{(k)})(\beta_1^{(k)}-\beta_3^{(k)})(\beta_1^{(k)}-\beta_4^{(k)})}e^{-\beta_1^{(k)}T},$$

$$A_{j,2}^k=\Big[\frac{F_{j,1}^k(T)}{(\beta_2^{(k)}-\beta_1^{(k)})(\beta_2^{(k)}-\beta_3^{(k)})(\beta_2^{(k)}-\beta_4^{(k)})}+\frac{F_{j,2}^k(T)}{(\beta_2^{(k)}-\beta_3^{(k)})(\beta_2^{(k)}-\beta_4^{(k)})}\Big]e^{-\beta_2^{(k)}T},$$

$$\begin{aligned}A_{j,3}^k\ =\ &\Big[\frac{F_{j,1}^k(T)}{(\beta_3^{(k)}-\beta_1^{(k)})(\beta_3^{(k)}-\beta_2^{(k)})(\beta_3^{(k)}-\beta_4^{(k)})}\\&+\frac{F_{j,2}^k(T)}{(\beta_3^{(k)}-\beta_2^{(k)})(\beta_3^{(k)}-\beta_4^{(k)})}+\frac{F_{j,3}^k(T)}{(\beta_3^{(k)}-\beta_4^{(k)})}\Big]e^{-\beta_3^{(k)}T},\end{aligned}$$

$$\begin{aligned}A_{j,4}^k\ =\ &\Big[\frac{F_{j,1}^k(T)}{(\beta_4^{(k)}-\beta_1^{(k)})(\beta_4^{(k)}-\beta_2^{(k)})(\beta_4^{(k)}-\beta_3^{(k)})}\\&+\frac{F_{j,2}^k(T)}{(\beta_4^{(k)}-\beta_2^{(k)})(\beta_4^{(k)}-\beta_3^{(k)})}+\frac{F_{j,3}^k(T)}{(\beta_4^{(k)}-\beta_3^{(k)})}+z_j^k(T)\Big]e^{-\beta_4^{(k)}T},\end{aligned}$$

where

$$F_{j,1}^k(T)=(\partial_t-\beta_2^{(k)})(\partial_t-\beta_3^{(k)})(\partial_t-\beta_4^{(k)})z_j^k(T),$$
$$F_{j,2}^k(T)=(\partial_t-\beta_3^{(k)})(\partial_t-\beta_4^{(k)})z_j^k(T),$$
$$F_{j,3}^k(T)=(\partial_t-\beta_4^{(k)})z_j^k(T)$$

and

$$B_i^k=\prod_{j\neq i}\frac{1}{\beta_i^{(k)}-\beta_j^{(k)}},\quad \forall i\in\{1,2,3,4\}.$$

Since $w(s,\cdot)\to 0$ as $s\to 0^+$, we have $z_j^k(t)\to 0$ as $t\to\infty$. Moreover, $\tilde{g}_j^k(t)\to 0$ as $t\to\infty$. It follows from $\beta_1^{(k)}>\beta_3^{(k)}>0$ for $k\geq 2$; $N=3$ and $p\in(1,3)$; $N\geq 4$ and $p>1$ that

$$\int_t^\infty e^{\beta_1^{(k)}(t-s)}\tilde{g}_j^k(s)ds\to 0,\quad \int_t^\infty e^{\beta_3^{(k)}(t-s)}\tilde{g}_j^k(s)ds\to 0\ \text{ as } t\to\infty.$$

By means of $\int_T^t = \int_T^\infty - \int_t^\infty$, we rewrite $z_j^k(t)$ in the following form:

$$\begin{aligned} z_j^k(t) &= M_{j,1}^k e^{\beta_1^{(k)} t} + M_{j,3}^k e^{\beta_3^{(k)} t} + A_{j,2}^k e^{\beta_2^{(k)} t} + A_{j,4}^k e^{\beta_4^{(k)} t} \\ &\quad - B_1^k \int_t^\infty e^{\beta_1^{(k)}(t-s)} \tilde{g}_j^k(s) ds - B_3^k \int_t^\infty e^{\beta_3^{(k)}(t-s)} \tilde{g}_j^k(s) ds \\ &\quad + B_2^k \int_T^t e^{\beta_2^{(k)}(t-s)} \tilde{g}_j^k(s) ds + B_4^k \int_T^t e^{\beta_4^{(k)}(t-s)} \tilde{g}_j^k(s) ds, \end{aligned}$$

where

$$M_{j,1}^k = A_{j,1}^k + B_1^k \int_T^\infty e^{-\tau \beta_1^{(k)}} \tilde{g}_j^k(\tau) d\tau, \quad M_{j,3}^k = A_{j,3}^k + B_3^k \int_T^\infty e^{-\tau \beta_3^{(k)}} \tilde{g}_j^k(\tau) d\tau.$$

The fact that $z_j^k(t) \to 0$ as $t \to \infty$ implies $M_{j,1}^k = M_{j,3}^k = 0$. Therefore,

$$\begin{aligned} z_j^k(t) &= A_{j,2}^k e^{\beta_2^{(k)} T} e^{\beta_2^{(k)}(t-T)} + A_{j,4}^k e^{\beta_4^{(k)} T} e^{\beta_4^{(k)}(t-T)} \\ &\quad - B_1^k \int_t^\infty e^{\beta_1^{(k)}(t-\tau)} \tilde{g}_j^k(\tau) d\tau - B_3^k \int_t^\infty e^{\beta_3^{(k)}(t-\tau)} \tilde{g}_j^k(\tau) d\tau \\ &\quad + B_2^k \int_T^t e^{\beta_2^{(k)}(t-\tau)} \tilde{g}_j^k(\tau) d\tau + B_4^k \int_T^t e^{\beta_4^{(k)}(t-\tau)} \tilde{g}_j^k(\tau) d\tau. \end{aligned} \tag{3.19}$$

We now establish the estimate of $z_j^k(t)$ with $k \geq 1$, $1 \leq j \leq m_1$. We start with $k \geq 2$, $1 \leq j \leq m_k$ and claim that

$$|z_j^k(t)| = O(k e^{\beta_4^{(k)}(t-T)}) \tag{3.20}$$

for $t > T$. For any fixed (k, j), if $z_j^k(t) \equiv 0$, this is trivial. Assume that $z_j^k(t) \not\equiv 0$ for $t \in [T, \infty)$ in the following, it is known from (3.5) that

$$|\tilde{g}_j^k(t)| = o_t(1) |z_j^k(t)| \quad \text{for } t \in (T, \infty). \tag{3.21}$$

It follows from Lemma 2.2 and (3.19) that, for $t \in (T, \infty)$,

$$\begin{aligned} |z_j^k(t)| &\leq O\Big(k e^{\beta_4^{(k)}(t-T)}\Big) + C \int_T^t e^{\beta_4^{(k)}(t-\tau)} o_\tau(1) |z_j^k(\tau)| d\tau \\ &\quad + C \int_t^\infty e^{\beta_3^{(k)}(t-\tau)} o_\tau(1) |z_j^k(\tau)| d\tau. \end{aligned} \tag{3.22}$$

Note that

$$e^{\beta_1^{(k)}(t-\tau)} \leq e^{\beta_3^{(k)}(t-\tau)}, \quad \text{and} \quad e^{\beta_2^{(k)}(t-\tau)} \leq e^{\beta_4^{(k)}(t-\tau)} \quad \text{for } \tau \leq t.$$

Note also that for $\ell = 1, 3$ and any fixed $t > T$,

$$\Big| \int_t^\infty e^{\beta_\ell^{(k)}(t-\tau)} \tilde{g}_j^k(\tau) d\tau \Big| \leq \int_t^\infty e^{\beta_\ell^{(k)}(t-\tau)} o_\tau(1) |z_j^k(\tau)| d\tau$$

and for $\ell = 2, 4$,

$$\Big| \int_T^t e^{\beta_\ell^{(k)}(t-\tau)} \tilde{g}_j^k(\tau) d\tau \Big| \leq \int_T^t e^{\beta_\ell^{(k)}(t-\tau)} o_\tau(1) |z_j^k(\tau)| d\tau.$$

It follows from (3.22) and arguments similar to those in [11] that

$$|z_j^k(t)| = O(ke^{\beta_4^{(k)}(t-T)}) \tag{3.23}$$

for $t \in (T,\infty)$. This implies that our claim (3.20) holds for $z_j^k(t) \not\equiv 0$. Therefore, our claim (3.20) holds.

We now establish the estimate of $z_j^1(t)$ with $1 \le j \le m_1$.

We first consider the estimate for $N = 4$, which can be split to four cases: (i) $p \in (1,3]$; (ii) $p \in (3,7)$; (iii) $p \in (7,\infty)$ and (iv) $p = 7$.

For the case (i), it is known from Claim 4 that $\beta_2^{(1)} < \beta_4^{(1)} \le -1 = \beta_3^{(1)} < 0 < \beta_1^{(1)}$. The fact $z_j^1(t) \to 0$ as $t \to \infty$ implies that $z_j^1(t)$ can be written in the form

$$\begin{aligned} z_j^1(t) &= A_{j,2}^1 e^{\beta_2^{(1)}t} + A_{j,3}^1 e^{-t} + A_{j,4}^1 e^{\beta_4^{(1)}t} \\ &\quad -B_1^1 \int_t^\infty e^{\beta_1^{(1)}(t-s)} \tilde{g}_j^1(s)ds + B_2^1 \int_T^t e^{\beta_2^{(1)}(t-s)} \tilde{g}_j^1(s)ds \\ &\quad +B_3^1 \int_T^t e^{-(t-s)} \tilde{g}_j^1(s)ds + B_4^1 \int_T^t e^{\beta_4^{(1)}(t-s)} \tilde{g}_j^1(s)ds. \end{aligned}$$

Arguments similar to those in the proof of (3.20) imply that, for $1 \le j \le m_1$ and $t > T$,

$$|z_j^1(t)| = O(e^{-(t-T)}). \tag{3.24}$$

For the case (ii), we see from Claim 4 that $\beta_2^{(1)} < \beta_4^{(1)} = -1 < \beta_3^{(1)} < 0 < \beta_1^{(1)}$. Therefore,

$$\begin{aligned} z_j^1(t) &= A_{j,2}^1 e^{\beta_2^{(1)}t} + A_{j,3}^1 e^{\beta_3^{(1)}t} + A_{j,4}^1 e^{-t} \\ &\quad -B_1^1 \int_t^\infty e^{\beta_1^{(1)}(t-s)} \tilde{g}_j^1(s)ds + B_2^1 \int_T^t e^{\beta_2^{(1)}(t-s)} \tilde{g}_j^1(s)ds \\ &\quad +B_3^1 \int_T^t e^{\beta_3^{(1)}(t-s)} \tilde{g}_j^1(s)ds + B_4^1 \int_T^t e^{-(t-s)} \tilde{g}_j^1(s)ds. \end{aligned}$$

Similarly, we have that, for $1 \le j \le m_1$ and $t > T$,

$$|z_j^1(t)| = O(e^{\beta_3^{(1)}(t-T)}). \tag{3.25}$$

For the case (iii), Claim 4 shows us that $\beta_2^{(1)} < \beta_4^{(1)} = -1 < 0 < \beta_3^{(1)} < \beta_1^{(1)}$. Then,

$$\begin{aligned} z_j^1(t) &= A_{j,2}^1 e^{\beta_2^{(1)}t} + A_{j,4}^1 e^{-t} \\ &\quad -B_1^1 \int_t^\infty e^{\beta_1^{(1)}(t-s)} \tilde{g}_j^1(s)ds - B_3^1 \int_t^\infty e^{\beta_3^{(1)}(t-s)} \tilde{g}_j^1(s)ds \\ &\quad +B_2^1 \int_T^t e^{\beta_2^{(1)}(t-s)} \tilde{g}_j^1(s)ds + B_4^1 \int_T^t e^{-(t-s)} \tilde{g}_j^1(s)ds. \end{aligned}$$

By the method analogous to that used above, for $1 \le j \le m_1$ and $t > T$, we get

$$|z_j^1(t)| = O(e^{-(t-T)}). \tag{3.26}$$

For the case (iv), we know that $\beta_2^{(1)} < \beta_4^{(1)} = -1 < \beta_3^{(1)} = 0 < \beta_1^{(1)}$. Then

$$\begin{aligned} z_j^1(t) &= A_{j,2}^1 e^{\beta_2^{(1)} t} + A_{j,4}^1 e^{-t} \\ &\quad -B_1^1 \int_t^\infty e^{\beta_1^{(1)}(t-s)} \tilde{g}_j^1(s)ds - B_3^1 \int_t^\infty \tilde{g}_j^1(s)ds \\ &\quad +B_2^1 \int_T^t e^{\beta_2^{(1)}(t-s)} \tilde{g}_j^1(s)ds + B_4^1 \int_T^t e^{-(t-s)} \tilde{g}_j^1(s)ds. \end{aligned}$$

Similarly, we have that, for $1 \le j \le m_1$ and $t > T$,

$$|z_j^1(t)| = O(e^{-(t-T)}) + C \int_t^\infty |o_s(1) z_j^1(s)| ds, \tag{3.27}$$

where $C > 0$ which depends only on B_1^1 and B_3^1 but independent of T. Let

$$K(t) = \int_t^\infty |z_j^1(s)| ds.$$

Then we can obtain that $K(t)$ is bounded provided that the condition in (1.8) holds. In fact, it follows from (1.8) that there is $0 < \varepsilon_0 < \frac{1}{10}$ such that, for s near 0,

$$|w(s,\theta)|^2 \le C s^{2\varepsilon_0}.$$

Consequently,

$$\begin{aligned} K(t) &= \int_0^s \zeta^{-1} |w_1(\zeta)| d\zeta \le \int_0^s \zeta^{-1} \Big(\int_{S^{N-1}} w^2(\zeta,\theta) d\theta \Big)^{\frac{1}{2}} d\zeta \\ &\le C \int_0^s \zeta^{\varepsilon_0 - 1} d\zeta = \frac{C}{\varepsilon_0} e^{-\varepsilon_0 t} < \infty, \end{aligned}$$

which implies that for any $0 < \epsilon < \varepsilon_0 / C$,

$$\lim_{t\to\infty} e^{C\epsilon t} K(t) = 0.$$

On the other hand, it follow from the definition of $K(t)$ that for t sufficiently large,

$$-K'(t) = |z_j^1(t)| \le O(e^{-(t-T)}) + C\epsilon K(t).$$

We can easily see that

$$K(t) = O(e^{-(t-T)}).$$

This and (3.27) imply

$$|z_j^1(t)| = O(e^{-(t-T)}). \tag{3.28}$$

Now, let $\hat{W}(t) = \Sigma_{k=1}^\infty \Sigma_{j=1}^{m_k} |z_j^k(t)|$. Then $\tilde{W}(t) = \Big(\Sigma_{k=1}^\infty \Sigma_{j=1}^{m_k} (z_j^k)^2(t) \Big)^{1/2} \le \hat{W}(t)$. For the cases (i), (iii) and (iv), we see from (3.20), (3.24), (3.26), (3.27) that

$$\tilde{W}(t) \le \hat{W}(t) \le O\Big(e^{-t}\Big) + O\Big(\sum_{k=2}^\infty k m_k e^{\beta_4^{(k)}(t-T)} \Big). \tag{3.29}$$

Let $T^* = 10T$. We obtain that, for $t > T^*$,

$$\sum_{k=2}^{\infty} k m_k e^{\beta_4^{(k)}(t-T)} = O(e^{\beta_4^{(2)}(t-T)}). \tag{3.30}$$

To see (3.30), we notice that, for any $t > T^*$ (we may enlarge T^*),

$$\lim_{k\to\infty} \frac{(k+1)m_{k+1}e^{\beta_4^{(k+1)}(t-T)}}{km_k e^{\beta_4^{(k)}(t-T)}} = e^{-(t-T)} \lim_{k\to\infty} \frac{(k+1)m_{k+1}}{km_k} = e^{-(t-T)} < \frac{1}{2}.$$

Since $\beta_4^{(2)} < -1$, we easily have that, for $t > T_*$,

$$\tilde{W}(t) = O(e^{-t}). \tag{3.31}$$

Let $s_0 = e^{-T^*}$. We see from (3.31) that there exists $C > 0$ such that,

$$W(s) \le Cs \qquad \text{for } 0 < s < s_0. \tag{3.32}$$

Arguments similar to the above imply that we can obtain

$$\tilde{W}(t) = O\left(e^{\beta_3^{(1)}t}\right) \tag{3.33}$$

for the case (ii). Note that $\beta_3^{(1)} \in (-1, 0)$ in this case.

For $N = 3$ or $N \ge 5$, processing the same procedure as above, we can obtain

$$\tilde{W}(t) = \begin{cases} O(e^{-t}), & \text{for } \begin{cases} p \in (1, \frac{N+2}{6-N}] & \text{when } N = 3 \text{ or } 5, \\ p \in (1,3] \cup (7,\infty) & \text{when } N = 4, \\ p = 7, & \text{when } N = 4 \text{ with (1.8)}, \\ p \in (1,\infty) & \text{when } N \ge 6; \end{cases} \\ O(e^{\beta_3^{(1)}t}), & \text{for } p \in (\frac{N+2}{6-N}, p^*) \qquad \text{when } N = 3,\ 4 \text{ or } 5, \end{cases} \tag{3.34}$$

where p^* is given by (1.7), $\beta_3^{(1)} = 5 - N - 2\alpha \in (-1, 0)$ when $p \in (\frac{N+2}{6-N}, p^*)$ and $N = 3$, 4 or 5. Choosing

$$\hat{\beta} = |\beta_3^{(1)}| = N + 2\alpha - 5, \quad p \in (\tfrac{N+2}{6-N}, p^*) \text{ and } N = 3,\ 4 \text{ or } 5, \tag{3.35}$$

we see that $0 < \hat{\beta} < 1$ in this case. Since $W(s) = \tilde{W}(t)$ and $t = -\ln s$, we obtain the conclusions of Proposition 3.1 from (3.34). This completes the proof of Proposition 3.1. □

4. Estimates for $\overline{v}(s)$, $v(s,\theta)$ near $s = 0$ and expansions of $u(r,\theta)$ near $r = \infty$

This section is devoted to establish some estimates for $\overline{v}(s)$ and $v(s,\theta)$ near $s = 0$ which enable us to obtain expansions of positive entire solutions $u(r,\theta)$ of (1.1) at $r = \infty$.

We begin our analysis by recalling the equation satisfied by $\overline{v}(s)$. From Lemma 2.3, we see that

$$\begin{aligned}
&\overline{v}^{(4)} - 2(N-7+2\alpha)s^{-1}\overline{v}^{(3)} + \big(N^2+6\alpha N+6\alpha^2-16N-36\alpha+51\big)s^{-2}\overline{v}'' \\
&\quad -(N-5+2\alpha)(2N\alpha+2\alpha^2-3N-10\alpha+9)s^{-3}\overline{v}' - (p+1)s^{-4}L^{-(p+1)}\overline{v} \\
&= s^{-4}\Big[f(\overline{v}) - \overline{f(v)}\Big] - s^{-4}f(\overline{v})
\end{aligned}$$

and

$$\begin{aligned}
&\Big|f(\overline{v}) - \overline{f(v)}\Big| \le \frac{1}{\omega_{N-1}}\int_{S^{N-1}} |f(v)-f(\overline{v})|d\theta \\
&\le o_s\Big[\Big(\int_{S^{N-1}} w^2\Big)^{\frac{1}{2}}\Big] \\
&= \begin{cases} o(s), & \text{for } N \text{ and } p \text{ satisfying (1.5) or } p=7,\ N=4 \text{ with (1.8)}, \\ o(s^{\hat{\beta}}), & \text{for } p\in(\frac{N+2}{6-N},p^*) \text{ and } N=3,\ 4 \text{ or } 5, \end{cases}
\end{aligned}$$

where $\hat{\beta}$ is given in (3.35).

Let $t = -\ln s$ and $\overline{z}(t) = \overline{v}(s)$. Then $\overline{z}(t)$ satisfies

$$\begin{aligned}
&\overline{z}^{(4)} + 2(N+2\alpha-4)\overline{z}^{(3)} + (N^2+6N\alpha+6\alpha^2-10N-24\alpha+20)\overline{z}'' \\
&\quad +2(N+2\alpha-4)(N\alpha+\alpha^2-N-4\alpha+2)\overline{z}' - (p+1)L^{-(p+1)}\overline{z} \\
&= \begin{cases} -f(\overline{z}) + o_t(1)e^{-t}, & \text{for } N \text{ and } p \text{ satisfying (1.5)} \\ & \qquad \text{or } p=7 \text{ and } N=4 \text{ with (1.8)}, \\ -f(\overline{z}) + o_t(1)e^{-\hat{\beta}t}, & \text{for } p\in(\frac{N+2}{6-N},p^*) \text{ and } N=3,\ 4 \text{ or } 5. \end{cases}
\end{aligned} \tag{4.1}$$

The corresponding characteristic polynomial of (4.1) is

$$\begin{aligned}
&\beta^4 + 2(N+2\alpha-4)\beta^3 + (N^2+6N\alpha+6\alpha^2-10N-24\alpha+20)\beta^2 \\
&\quad +2(N+2\alpha-4)(N\alpha+\alpha^2-N-4\alpha+2)\beta - (p+1)L^{-(p+1)} = 0.
\end{aligned} \tag{4.2}$$

Comparing (4.2) with (3.10), it is easy to see that the four roots of (4.2) are given by $\beta_j^{(0)}$ corresponding to $\lambda_0 = 0$ for $j = 1,2,3,4$, are given in (3.11). Denote

$$\beta_j = \beta_j^{(0)} \quad \text{for } j = 1,2,3,4. \tag{4.3}$$

From the expression of β_j, we have that, for $N = 3$ and $p\in(1,3)$; $N\ge 4$ and $p>1$,

$$\beta_1, \beta_2 \in \mathbb{R} \quad \text{and} \quad \beta_2 < 2-N-\alpha < -1 < 0 < \beta_1. \tag{4.4}$$

As to the roots β_3, β_4, we have:

Claim 5. When $N = 3$ and $p \in (1, 3)$; $N \geq 4$ and $p > 1$, the following estimates for β_3 and β_4 hold:

$$\beta_3, \beta_4 \in \mathbb{R},\ \beta_4 \leq \beta_3, \quad \beta_3 \begin{cases} \leq -1, & \text{for } \begin{cases} p \in (1, p_3^1], & N = 3; \\ p \in (1, p_c], & N \in [5, 12]; \\ p \in (1, \infty), & N \geq 13; \end{cases} \\ \in (-1, 0), & \text{for } \quad p \in [p_3^2, 3), \quad N = 3; \end{cases}$$

$$\beta_{3,4} = \ell \pm qi \notin \mathbb{R}, \quad \ell \begin{cases} \leq -1, & \text{for } \begin{cases} p \in (p_3^1, \frac{5}{3}], & N = 3; \\ p \in (1, 3], & N = 4; \\ p \in (p_c, 7], & N = 5; \\ p \in (p_c, \infty), & N \in [6, 12]; \end{cases} \\ \in (-1, 0), & \text{for } \begin{cases} p \in (\frac{5}{3}, p_3^2), & N = 3; \\ p \in (3, \infty), & N = 4; \\ p \in (7, \infty), & N = 5. \end{cases} \end{cases}$$

Where p_c, p_3^1, p_3^2 are given in (4.11) and (4.5) below.

We now introduce the function

$$\hbar(p, N) := [4 + (N-2)^2]^2 - 16\rho_0,$$

where $\rho_0 = \rho_k|_{k=0}$ is given in (3.12). For $N = 3$, solving equation $\hbar(p, 3) = 0$, we obtain four foots:

$$\text{(4.5)} \qquad \begin{aligned} p_3^1 &= \frac{5 - \sqrt{13 - 3\sqrt{17}}}{3 + \sqrt{13 - 3\sqrt{17}}}, & p_3^2 &= \frac{5 + \sqrt{13 - 3\sqrt{17}}}{3 - \sqrt{13 - 3\sqrt{17}}}, \\ p_3^3 &= \frac{5 + \sqrt{13 + 3\sqrt{17}}}{3 - \sqrt{13 + 3\sqrt{17}}}, & p_3^4 &= \frac{5 - \sqrt{13 + 3\sqrt{17}}}{3 + \sqrt{13 + 3\sqrt{17}}}. \end{aligned}$$

It is easy to check that $p_3^3 < p_3^4 < 1 < p_3^1 < p_3^2 < 3$. A simple calculation shows $\hbar(1; 3) = 9 > 0$. So, we deduce that

$$\hbar(p, 3) \begin{cases} \geq 0, & \text{for } \quad p \in (1, p_3^1] \cup [p_3^2, 3), \\ < 0, & \text{for } \quad p \in (p_3^1, p_3^2). \end{cases}$$

This implies that $\beta_3, \beta_4 \in \mathbb{R}$ for $p \in (1, p_3^1] \cup [p_3^2, 3)$ and $\beta_3, \beta_4 \notin \mathbb{R}$ for $p \in (p_3^1, p_3^2)$.

For $N = 3$ and $p \in (p_3^1, p_3^2)$, we have

$$\text{(4.6)} \qquad \Re(\beta_3) = \Re(\beta_4) = \frac{1}{2} - \alpha \begin{cases} \in (-\frac{3}{2}, -1] & \text{for } p \in (p_3^1, \frac{5}{3}], \\ \in (-1, -\frac{1}{2}) & \text{for } p \in (\frac{5}{3}, p_3^2). \end{cases}$$

For $N = 3$ and $p \in (1, p_3^1] \cup [p_3^2, 3)$, we see from the representations of β_3 and β_4 that $\beta_4 < \beta_3$. Moreover,

$$\text{(4.7)} \qquad \beta_4 \leq \beta_3 < -1, \quad \text{for } N = 3 \text{ and } p \in (1, p_3^1];$$

$$\text{(4.8)} \qquad \beta_4 \leq \beta_3 \in (-1, 0), \quad \text{for } N = 3 \text{ and } p \in [p_3^2, 3).$$

To see (4.7), we have

$$\beta_3 + 1 = \frac{1}{2}\Big[3 - 2\alpha + \sqrt{5 - 4\sqrt{1 + p\alpha(2-\alpha)(\alpha^2 - 1)}}\Big]. \tag{4.9}$$

Note that $3 - 2\alpha < 0$ and $5 - (3-2\alpha)^2 > 0$ for $p \in (1, p_3^1]$. Then

$$[5 - (3-2\alpha)^2]^2 - 16[1 + p\alpha(2-\alpha)(\alpha^2-1)] = \frac{128(p-1)(p-3)}{(p+1)^2} < 0.$$

This implies $\beta_3 + 1 < 0$ and thus (4.7) holds. To see (4.8), we notice that $3 - 2\alpha > 0$ for $p \in [p_3^2, 3)$. It follows from (4.9) that $\beta_3 + 1 > 0$, i.e. $\beta_3 > -1$. We also know that, for $p \in [p_3^2, 3)$, $1 - 2\alpha < 0$, $5 - (1-2\alpha)^2 > 0$ and

$$[5 - (1-2\alpha)^2]^2 - 16[1 + p\alpha(2-\alpha)(\alpha^2-1)] = \frac{128(p+5)(p-1)(p-3)}{(p+1)^3} < 0.$$

These imply that $\beta_3 < 0$. Therefore, (4.8) holds.

For $N = 4$ and $p > 1$, we have $\hbar(p,4) = -\frac{1024p(p+3)(p-1)}{(p+1)^4} < 0$. This implies that $\beta_3, \beta_4 \notin \mathbb{R}$. At this time,

$$\Re(\beta_3) = \Re(\beta_4) = -\alpha \begin{cases} \le -1 & \text{for } p \in (1,3], \\ \in (-1, 0) & \text{for } p \in (3, \infty). \end{cases} \tag{4.10}$$

For $5 \le N \le 12$ and $p > 1$, a direct calculations imply that the equation $\hbar(p,N) = 0$ has only one root p_c in $(1,\infty)$ and

$$p_c = \frac{N + 2 - \sqrt{4 + N^2 - 4\sqrt{N^2 + H_N}}}{6 - N + \sqrt{4 + N^2 - 4\sqrt{N^2 + H_N}}}, \quad \text{with} \quad H_N = (N(N-4)/4)^2. \tag{4.11}$$

Moreover,

$$\begin{aligned} &\hbar(1,N) = N^2(N-4)^2 > 0, \\ &\hbar(p,N)|_{p=\infty} = (N-4)(N^2 - 144) + 16(N-20) < 0. \end{aligned}$$

Hence, when $5 \le N \le 12$,

$$\hbar(p,N) \begin{cases} > 0, & \text{for } p \in (1, p_c), \\ = 0, & \text{for } p = p_c, \\ < 0, & \text{for } p \in (p_c, \infty), \end{cases}$$

which implies that $\beta_3, \beta_4 \in \mathbb{R}$ for $p \in (1, p_c]$; $\beta_3|_{p=p_c} = \beta_4|_{p=p_c} = 2 - \frac{4}{p_c+1} - \frac{N}{2} < -1$ and $\beta_3, \beta_4 \notin \mathbb{R}$ for $p \in (p_c, \infty)$.

When $5 \le N \le 12$ and $p \in (p_c, \infty)$, $\beta_{3,4} := \ell \pm qi$ and it is easy to find that

$$\ell = \Re(\beta_{3,4}) = 2 - \alpha - \frac{N}{2} \begin{cases} \le -1 & \text{for } p \in (p_c, \infty),\ N \in [6,12], \\ & \quad p \in (p_c, 7], \quad N = 5; \\ \in (-1, -\frac{1}{2}) & \text{for } p > 7,\ N = 5. \end{cases} \tag{4.12}$$

When $5 \le N \le 12$ and $p \in (1, p_c]$, we have

$$\beta_3 + 1 = \frac{1}{2}\Big[6 - 2\alpha - N + \sqrt{4 + (N-2)^2 - 4\sqrt{\rho_0}}\Big].$$

Note that, in this case, $6-2\alpha-N<0$,

$$4+(N-2)^2-(6-2\alpha-N)^2=4+4(2-\alpha)(N-4+\alpha)>0,$$

and

$$\begin{aligned}&[4+(N-2)^2-(6-2\alpha-N)^2]^2-16\rho_0\\ &\quad=-\frac{16}{(p+1)^2}(N-1)[(4N-19)p^2+(p^2+2p-3)N+10p-3]<0.\end{aligned}\tag{4.13}$$

So, we conclude that $\beta_3\le\beta_4<-1$ for $5\le N\le 12$ and $p\in(1,p_c]$.

When $N\ge 13$ and $p>1$, a simple calculation shows that the equation $\frac{\partial\hbar}{\partial p}=0$ has no any solution in $(1,\infty)$, which implies that $\beta_3,\beta_4\in\mathbb{R}$. From the form of β_3+1 and (4.13), we obtain that $\beta_4\le\beta_3<-1$.

Our claim 5 follows from the above discussions.

In view of (4.1), the ODE theory and arguments similar to those in section 3 imply that, for $N=4$ and $p>1$; $N\in[5,12]$ and $p\in(p_c,\infty)$,

$$\begin{aligned}\overline{z}(t)&=M_1e^{\beta_1t}+A_2e^{\beta_2t}+A_3e^{\ell t}\cos(qt)+A_4e^{\ell t}\sin(qt)\\ &\quad-B_1\int_t^\infty e^{\beta_1(t-s)}h(s,\overline{z}(s))ds+B_3\int_T^t e^{\ell(t-s)}\cos[q(t-s)]h(s,\overline{z}(s))ds\\ &\quad+B_2\int_T^t e^{\beta_2(t-s)}h(s,\overline{z}(s))ds+B_4\int_T^t e^{\ell(t-s)}\sin[q(t-s)]h(s,\overline{z}(s))ds;\end{aligned}\tag{4.14}$$

for $N=3$ and $p\in(1,p_3^1]\cup[p_3^2,3)$; $N\in[5,12]$ and $p\in(1,p_c]$; $N\ge 13$ and $p>1$,

$$\begin{aligned}\overline{z}(t)&=M_1e^{\beta_1t}+A_2e^{\beta_2t}+A_3e^{\beta_3t}+A_4e^{\beta_4t}\\ &\quad-B_1\int_t^\infty e^{\beta_1(t-s)}h(s,\overline{z}(s))ds+B_2\int_T^t e^{\beta_2(t-s)}h(s,\overline{z}(s))ds\\ &\quad+B_3\int_T^t e^{\beta_3(t-s)}h(s,\overline{z}(s))ds+B_4\int_T^t e^{\beta_4(t-s)}h(s,\overline{z}(s))ds,\end{aligned}\tag{4.15}$$

where ℓ and q are given in Claim 5,

$$h(t,\overline{z}(t))=\begin{cases}O(\overline{z}^2)+o_t(1)e^{-t}, & \text{for } N \text{ and } p \text{ satisfying (1.5)}\\ & \quad\text{or } p=\frac{N+3}{N-5} \text{ and } N\ge 6 \text{ with (1.8)},\\ O(\overline{z}^2)+o_t(1)e^{-\hat{\beta}t}, & \text{for } p\in(\frac{N+2}{6-N},p^*),\ N=3,5.\end{cases}$$

Note that $f(\overline{z})=O(\overline{z}^2)$. Since $\overline{z}(t)\to 0$ as $t\to\infty$, we see that, for $N=3$ and $p\in(p_3^1,p_3^2)$; $N=4$ and $p>1$; $N\in[5,12]$ and $p\in(p_c,\infty)$,

$$|\overline{z}(t)|\le O(e^{\ell t})+C\int_t^\infty e^{\beta_1(t-s)}|h(s,\overline{z}(s))|ds+C\int_T^t e^{\ell(t-s)}|h(s,\overline{z}(s))|ds,$$

and, for $N = 3$ and $p \in (1, p_3^1] \cup [p_3^2, 3)$; $N \in [5, 12]$ and $p \in (1, p_c]$; $N \geq 13$ and $p > 1$,

$$|\overline{z}(t)| \leq O(e^{\beta_3 t}) + C\int_t^{\infty} e^{\beta_1(t-s)}|h(s, \overline{z}(s))|ds + C\int_T^t e^{\beta_3(t-s)}|h(s, \overline{z}(s))|ds,$$

where $C > 0$ is independent of T. Note that we have also used the fact $\beta_2 < \ell$. Arguments similar to those in the proof of Proposition 3.1 imply that

$$(4.16)\quad |\overline{z}(t)| = O(e^{\ell t}), \quad \text{i.e.} \quad |\overline{v}(s)| = O(s^{-\ell}), \qquad \text{for} \begin{cases} p \in (p_3^1, p_3^2), & N = 3; \\ p \in (1, \infty), & N = 4; \\ p \in (p_c, \infty), & N \in [5, 12] \end{cases}$$

and

$$(4.17)\quad \begin{aligned} &|\overline{z}(t)| = O(e^{\beta_3 t}), \\ &\text{i.e.} \ |\overline{v}(s)| = O(s^{-\beta_3}), \end{aligned} \qquad \text{for} \begin{cases} p \in (1, p_3^1] \cup [p_3^2, 3), & N = 3; \\ p \in (1, p_c], & N \in [5, 12]; \\ p \in (1, \infty), & N \in [13, \infty). \end{cases}$$

The fact $v(s, \theta) = \overline{v}(s) + w(s, \theta)$, Proposition 3.1, (4.16), (4.17) and Claim 5 yield that

$$(4.18)\qquad |v(s, \theta)| = \begin{cases} O(s), & \text{for } p \in (1, \frac{N+2}{6-N}], N = 3, 4, 5; \\ & \quad p \in (1, \infty), N \geq 6; \\ O(s^{\hat{\beta}}), & \text{for } p \in (\frac{N+2}{6-N}, p^*), N = 3, 4, 5; \\ O(s^{-\ell}), & \text{for } p \in (7, \infty), N = 4; \\ & \quad p = 7, N = 4 \text{ with } (1.8). \end{cases}$$

where $\ell = 2 - \alpha - \frac{N}{2} \in (-1, 0)$ is given in Claim 5 and $\hat{\beta} = |\beta_3^{(1)}| = 5 - N - 2\alpha \in (0, 1)$ for $p \in (\frac{N+2}{6-N}, p^*)$ and $N = 3, 4, 5$, which is given in (3.35). We have also used the facts $-1 < \ell < \beta_3^{(1)} < 0$ for $p \in (\frac{N+2}{6-N}, p^*)$ and $N = 3, 4, 5$; $-1 < \beta_3 < \beta_3^{(1)} < 0$ for $p \in [p_3^2, 3)$ and $N = 3$. Moreover, since

$$|f(v) - f(\overline{v})| = p\big|(\xi + L)^{-(p+1)} - L^{-(p+1)}\big||w| = O(|\xi|)|w|,$$

where $\xi(s, \theta) = \gamma w(s, \theta) + (1 - \gamma)\overline{v}(s)$ with $\gamma \in (0, 1)$, the estimate similar to (4.18) yields that

$$(4.19)\qquad |\xi(s, \theta)| = \begin{cases} O(s), & \text{for } p \in (1, \frac{N+2}{6-N}], N = 3, 4, 5; \\ & \quad p \in (1, \infty), N \geq 6; \\ O(s^{\hat{\beta}}), & \text{for } p \in (\frac{N+2}{6-N}, p^*), N = 3, 4, 5; \\ O(s^{-\ell}), & \text{for } p \in (7, \infty), N = 4; \\ & \quad p = 7, N = 4 \text{ with } (1.8). \end{cases}$$

Therefore

$$
(4.20)\qquad |f(v)-f(\overline{v})| = \begin{cases} O(s^2), & \text{for } p\in(1,\frac{N+2}{6-N}], N=3,4,5; \\ & \quad p\in(1,\infty), N\geq 6; \\ O(s^{2\hat{\beta}}), & \text{for } p\in(\frac{N+2}{6-N},p^*), N=3,4,5; \\ O(s^{1-\ell}), & \text{for } p\in(7,\infty), N=4; \\ & \quad p=7, N=4 \text{ with } (1.8). \end{cases}
$$

Consequently, we have the following lemma.

Lemma 4.1. *Let v be a solution to (2.8). Then there exists $M=M(v)>0$ such that for $p\in(1,\frac{N+2}{6-N}]$ and $N=3,\ 4,\ 5$; $p\in(1,\infty)$ and $N\geq 6$,*

$$
(4.21)\qquad |\overline{v}(s)|\leq Ms,\quad |\overline{v}'(s)|\leq M,\quad |\overline{v}''(s)|\leq Ms^{-1}
$$

and

$$
(4.22)\qquad \int_{S^{N-1}} v^2(s,\theta)d\theta\leq Ms^2;
$$

For $p\in(\frac{N+2}{6-N},p^*)$ and $N=3,4,5$,

$$
(4.23)\qquad |\overline{v}(s)|\leq Ms^{\hat{\beta}},\quad |\overline{v}'(s)|\leq Ms^{\hat{\beta}-1},\quad |\overline{v}''(s)|\leq Ms^{\hat{\beta}-2}
$$

and

$$
(4.24)\qquad \int_{S^{N-1}} v^2(s,\theta)d\theta\leq Ms^{2\hat{\beta}},\qquad \hat{\beta}=N+2\alpha-5\in(0,1);
$$

For $p\in(7,\infty)$ and $N=4$; $p=7$ and $N=4$ with (1.8),

$$
(4.25)\qquad |\overline{v}(s)|\leq Ms^{-\ell},\quad |\overline{v}'(s)|\leq Ms^{-(1+\ell)},\quad |\overline{v}''(s)|\leq Ms^{-(2+\ell)},
$$

and

$$
(4.26)\qquad \int_{S^{N-1}} v^2(s,\theta)d\theta\leq Ms^{-2\ell}.
$$

where $\ell=2-\alpha-\frac{N}{2}<0$.

Proof. Proof of this lemma is similar to that of Lemma 4.1 in [11]. We omit the details here. □

Proposition 4.2. *Suppose that $\kappa\geq 0$ is an integer and v is a solution of (2.8). Then there exist $0<s_0<1$ and $M=M(v,\kappa)>0$ (independent of s) such that for $p\in(1,\frac{N+2}{6-N}]$ and $N=3,\ 4,\ 5$; $p\in(1,\infty)$ and $N\geq 6$,*

$$
(4.27)\qquad \max_{|y|=s}|D^\kappa v(y)|\leq Ms^{1-\kappa}.
$$

For $p\in(\frac{N+2}{6-N},p^*)$ and $N=3,4,5$,

$$
(4.28)\qquad \max_{|y|=s}|D^\kappa v(y)|\leq Ms^{\hat{\beta}-\kappa}.
$$

For $p\in(7,\infty)$ and $N=4$; $p=7$ and $N=4$ with (1.8),

$$
(4.29)\qquad \max_{|y|=s}|D^\kappa v(y)|\leq Ms^{-\ell-\kappa}.
$$

Proof. We only show (4.27). The proofs of (4.28) and (4.29) are similar. We first obtain (4.27) for the case of $\kappa = 0$. If we define $z(t,\theta) = w(s,\theta) = \Sigma_{k=1}^{\infty}\Sigma_{j=1}^{m_k} w_j^k(s)Q_j^k(\theta)$, we see that

$$\max_{\theta\in S^{N-1}} |z(t,\theta)| \le \sum_{k=1}^{\infty}\sum_{j=1}^{m_k} |z_j^k(t)| \max_{\theta\in S^{N-1}} |Q_j^k(\theta)| \le \sum_{k=1}^{\infty}\sum_{j=1}^{m_k} D_k|z_j^k(t)|,$$

where D_k is given in (2.5). Arguments similar to those in the proof of Proposition 3.1 imply that there exist $C > 0$ independent of t and $T^* \gg 1$ such that, for $t \ge T^*$,

$$\sum_{k=1}^{\infty}\sum_{j=1}^{m_k} D_k|z_j^k(t)| = O\Big(\sum_{k=2}^{\infty} km_kD_k e^{\beta_4^{(k)}(t-T)}\Big) + O\Big(e^{-(t-T)}\Big) \le Ce^{-t}$$

(note that $\lim_{k\to\infty} \frac{(k+1)m_{k+1}D_{k+1}}{km_kD_k} = 1$) and hence

$$\max_{\theta\in S^{N-1}} |z(t,\theta)| \le Ce^{-t},$$

$$\max_{\theta\in S^{N-1}} |w(s,\theta)| \le Cs \tag{4.30}$$

for $0 < s < s_0 := e^{-T^*}$. Therefore, (4.27) with $\tau = 0$ can be obtained from (4.30) and the fact that $v(s,\theta) = w(s,\theta) + \overline{v}(s)$.

To obtain (4.27) completely, it is enough to show (4.27) for $\kappa = 1$. The other cases are essentially the same by differentiating $w(s,\theta)$. We only need to show $|\nabla w(y)| \le C$. Since $|\nabla w|^2 = w_s^2 + \frac{1}{s^2}|w_\theta|^2$, we need to present the estimates of w_s^2 and $|w_\theta|^2$. We see that $w_s(s,\theta) = \Sigma_{k=1}^{\infty}\Sigma_{j=1}^{m_k}(w_j^k)'(s)Q_j^k(\theta)$, then

$$\max_{\theta\in S^{N-1}} |w_s(s,\theta)| \le \sum_{k=1}^{\infty}\sum_{j=1}^{m_k} D_k|(w_j^k)'(s)|. \tag{4.31}$$

For each $\lambda_k = k(N+k-2)$ and $1 \le j \le m_k$, we see from the expression of $z_j^k(t)$ in (3.19) and $(w_j^k)'(s) = -(z_j^k)'(t)e^t$ $(t = -\ln s)$ that for $0 < s < s_0$,

$$D_k|(w_j^k)'(s)| \le \tilde{M}_k s^{-(\beta_4^{(k)}+1)} \quad \text{for } k \ge 2$$

and

$$D_1|(w_j^1)'(s)| \le \tilde{M}_1.$$

These and (4.31) imply that there is $M_1 = M_1(v, s_0) > 0$ independent of s such that, for $s \in (0, s_0)$,

$$\max_{\theta\in S^{N-1}} |w_s(s,\theta)| \le M_1. \tag{4.32}$$

Note that $\beta_4^{(k)} + 1 < 0$ for $k \ge 2$. Since $|w_\theta(s,\theta)| \le \Sigma_{k=1}^{\infty}\Sigma_{j=1}^{m_k}|w_j^k(s)||(Q_j^k)_\theta|$, we also obtain that there exists $M_2 = M_2(v, s_0) > 0$ independent of s such that for $s \in (0, s_0)$,

$$\max_{\theta\in S^{N-1}} |w_\theta(s,\theta)| \le M_2 s. \tag{4.33}$$

(Again by $\lim_{k\to\infty} \frac{(k+1)m_{k+1}E_{k+1}}{km_kE_k} = 1$.) Therefore, for $s \in (0, s_0)$,

$$\max_{|y|=s} |\nabla w(y)|^2 = \max_{|y|=s} \left[w_s^2 + \frac{1}{s^2}|w_\theta|^2 \right] \le \hat{M}, \tag{4.34}$$

where $\hat{M} = M_1^2 + M_2^2$. Together with $(4.21)_2$, we see that (4.27) holds for $\kappa = 1$. This completes the proof of this proposition. □

To study the properties of v, we introduce a new function

$$\tilde{w}(s,\theta) = \frac{w(s,\theta)}{s}. \tag{4.35}$$

It follows from the above arguments that

$$|\tilde{w}(s,\theta)| = O(1), \quad \text{for} \quad \begin{cases} p \in \begin{cases} (1, \frac{N+2}{6-N}] & \text{when } N = 3 \text{ or } 5, \\ (1,3] \cup (7,\infty) & \text{when } N = 4, \\ (1,\infty) & \text{when } N \ge 6, \end{cases} \\ p = 7 \text{ and } N = 4 \text{ with (1.8)} \end{cases} \tag{4.36}$$

For $p \in (\frac{N+2}{6-N}, p^*)$, $N = 3$, 4 or 5, from (1.9) we have

$$|v(s,\theta)| = o(s^{\hat{\beta}}) \text{ for } s \text{ near } 0,$$

where $\hat{\beta} = |\beta_3^{(1)}| = N + 2\alpha - 5 \in (0,1)$. There holds in this case

$$|\overline{v}(s)| = o(s^{\hat{\beta}}), \quad |w(s,\theta)| = o(s^{\hat{\beta}}) \text{ for } s \text{ near } 0. \tag{4.37}$$

Arguments similar to those in the proof of Proposition 3.1 imply that

$$W(s) = O(s^{|\beta_4^{(1)}|}) \text{ for } s \text{ near } 0.$$

Since $\beta_4^{(1)} = -1$ for $p \in (\frac{N+2}{6-N}, p^*)$ and $N = 3$, 4, 5, we see that $|\tilde{w}(s,\theta)| = O(1)$ also holds when $p \in (\frac{N+2}{6-N}, p^*)$ and $N = 3$, 4, 5 with (1.9).

Taking account of equation (2.13), we see that $\tilde{w}(s,\theta)$ satisfies the equation:

(4.38)
$$\begin{aligned}
&\partial_s^4 \tilde{w} + 2(9 - 2\alpha - N)s^{-1}\partial_s^3 \tilde{w} \\
&\quad + (N^2 + 6\alpha N + 6\alpha^2 - 22N - 48\alpha + 93)s^{-2}\partial_s^2 \tilde{w} \\
&\quad - (N + 2\alpha - 7)(2\alpha N + 2\alpha^2 - 14\alpha - 5N + 21)s^{-3}\partial_s \tilde{w} \\
&\quad - \left[(N + 2\alpha - 5)(2N\alpha + 2\alpha^2 - 3N - 10\alpha + 9) + (p+1)L^{-(p+1)}\right]s^{-4}\tilde{w} \\
&\quad + 2(N\alpha + \alpha^2 - 6\alpha - 2N + 9)s^{-4}\Delta_\theta \tilde{w} + 2(7 - 2\alpha - N)s^{-3}\Delta_\theta(\partial_s \tilde{w}) \\
&\quad + 2s^{-2}\Delta_\theta(\partial_s^2 \tilde{w}) + s^{-4}\Delta_\theta^2 \tilde{w} \\
&= s^{-5}[\overline{f(v)} - f(v)] = s^{-4}[\overline{f'(\xi(s,\theta))\tilde{w}} - f'(\xi(s,\theta))\tilde{w}].
\end{aligned}$$

Now we write

$$\tilde{w}(s,\theta)=\sum_{k=1}^{\infty}\sum_{j=1}^{m_k}\tilde{w}_j^k(s)Q_j^k(\theta), \tag{4.39}$$

where $\tilde{w}_j^k(s)=s^{-1}w_j^k(s)$. It is clear that $\overline{\tilde{w}}=0$. Then, $\tilde{w}_j^k(s)$ satisfies

$$\begin{aligned}(\tilde{w}_j^k)^{(4)}&+2(9-2\alpha-N)s^{-1}(\tilde{w}_j^k)^{(3)}\\&+(N^2+6\alpha N+6\alpha^2-22N-48\alpha+93-2\lambda_k)s^{-2}(\tilde{w}_j^k)''\\&-(N+2\alpha-7)(2\alpha N+2\alpha^2-14\alpha-5N+21-2\lambda_k)s^{-3}(\tilde{w}_j^k)'\\&+\big[(5-N-2\alpha)(2N\alpha+2\alpha^2-3N-10\alpha+9)-(p+1)L^{-(p+1)}\\&\quad-2(N\alpha+\alpha^2-2N-6\alpha+9)\lambda_k+\lambda_k^2\big]s^{-4}\tilde{w}_j^k\\&=s^{-4}o_s(1)\tilde{w}_j^k,\end{aligned} \tag{4.40}$$

where $\lambda_k=k(N+k-2)$ and

$$o_s(1)=\begin{cases}O(s), & \text{for } p\in(1,\frac{N+2}{6-N}], N=3,4,5;\\ & \quad p\in(1,\infty), N\geq 6;\\ o(s^{\hat{\beta}}), & \text{for } p\in(\frac{N+2}{6-N},p^*), N=3,4,5 \text{ with (1.9)};\\ O(s^{-\ell}), & \text{for } p\in(7,\infty), N=4;\\ & \quad p=7, N=4 \text{ with (1.8)}.\end{cases} \tag{4.41}$$

Let $t=-\ln s$ and $\tilde{z}_j^k(t)=\tilde{w}_j^k(s)$. Then $\tilde{z}_j^k(t)$ satisfies the following equation:

$$\begin{aligned}(\tilde{z}_j^k)^{(4)}&+2(N-6+2\alpha)(\tilde{z}_j^k)^{(3)}+(N^2+6\alpha N+6\alpha^2-16N-36\alpha+50\\&-2\lambda_k)(\tilde{z}_j^k)''+2(N+2\alpha-6)(N\alpha+\alpha^2-6\alpha-2N+7-\lambda_k)(\tilde{z}_j^k)'\\&+\big[(5-N-2\alpha)(2N\alpha+2\alpha^2-3N-10\alpha+9)-(p+1)L^{-(p+1)}\\&\quad-2(N\alpha+\alpha^2-2N-6\alpha+9)\lambda_k+\lambda_k^2\big]\tilde{z}_j^k\\&=o_t(1)\tilde{z}_j^k,\end{aligned} \tag{4.42}$$

where

$$o_t(1)=\begin{cases}O(e^{-t}), & \text{for } p\in(1,\frac{N+2}{6-N}], N=3,4,5;\\ & \quad p\in(1,\infty), N\geq 6;\\ o(e^{-\hat{\beta}t}), & \text{for } p\in(\frac{N+2}{6-N},p^*), N=3,4,5 \text{ with (1.9)};\\ O(e^{\ell t}), & \text{for } p\in(7,\infty), N=4;\\ & \quad p=7, N=4 \text{ with (1.8)}.\end{cases} \tag{4.43}$$

The corresponding polynomial of (4.42) is

$$\begin{aligned}\tilde{\beta}^4&+2(N-6+2\alpha)\tilde{\beta}^3+(N^2+6\alpha N+6\alpha^2-16N-36\alpha+50-2\lambda_k)\tilde{\beta}^2\\&+2(N+2\alpha-6)(N\alpha+\alpha^2-6\alpha-2N+7-\lambda_k)\tilde{\beta}\\&+(5-N-2\alpha)(2N\alpha+2\alpha^2-3N-10\alpha+9)-(p+1)L^{-(p+1)}\\&\quad-2(N\alpha+\alpha^2-2N-6\alpha+9)\lambda_k+\lambda_k^2=0.\end{aligned} \tag{4.44}$$

Solving this equation, we obtain four roots:

$$\tilde{\beta}_j^{(k)} = \beta_j^{(k)} + 1, \quad j = 1, 2, 3, 4,$$

where $\beta_j^{(k)}$ is given in (3.11). It follows from Claims 1-4 in the proof of Proposition 3.1 that

$$\tilde{\beta}_2^{(k)} < \tilde{\beta}_4^{(k)} < 0 < \tilde{\beta}_3^{(k)} < \tilde{\beta}_1^{(k)} \text{ for } k \geq 2;\ N = 3 \text{ and } p \in (1, 3);\ N \geq 4 \text{ and } p > 1$$

and

$$\begin{aligned} &\tilde{\beta}_2^{(1)} < \tilde{\beta}_4^{(1)} = 0 < \tilde{\beta}_3^{(1)} < \tilde{\beta}_1^{(1)}, \quad && \text{for } N = 3, 5,\ p \in (\tfrac{N+2}{6-N}, p^*); \\ & && N = 4,\ p \in (3, \infty); \\ &\tilde{\beta}_2^{(1)} < \tilde{\beta}_4^{(1)} \leq \tilde{\beta}_3^{(1)} = 0 < \tilde{\beta}_1^{(1)}, \quad && \text{for } N \geq 6,\ p \in (1, \infty); \\ & && N = 3, 4, 5,\ p \in (1, \tfrac{N+2}{6-N}]. \end{aligned}$$

From (3.23) and Claim 3, we see that for $k \geq 2$,

$$\lim_{s \to 0} \tilde{w}_j^k(s) = 0, \ \text{ for } N = 3 \text{ and } p \in (1, 3);\ N \geq 4 \text{ and } p > 1. \tag{4.45}$$

Moreover, (4.36) and (4.37) imply that $|\tilde{w}_j^1(s)|$ $(1 \leq j \leq m_1)$ is bounded for s near 0, that is, $|\tilde{z}_j^1(t)|$ is bounded for t near ∞, provided that N and p satisfy (1.5) or $N = 4$ and $p = 7$ with (1.8); $N = 3, 4, 5$ and $p \in (\frac{N+2}{6-N}, p^*)$ with (1.9). It follows from (4.42) that, for t sufficiently large,

$$\begin{aligned} \tilde{z}_j^1(t) = {} & C_j^1 + A_{j,2}^1 e^{\tilde{\beta}_2^{(1)} t} + A_{j,4}^1 e^{\tilde{\beta}_4^{(1)} t} \\ & - B_1^1 \int_t^\infty e^{\tilde{\beta}_1^{(1)}(t-s)} O(e^{-s}) \tilde{z}_j^1(s) ds + B_2^1 \int_T^t e^{\tilde{\beta}_2^{(1)}(t-s)} O(e^{-s}) \tilde{z}_j^1(s) ds \\ & + B_3^1 \int_T^t O(e^{-s}) \tilde{z}_j^1(s) ds + B_4^1 \int_T^t e^{\tilde{\beta}_4^{(1)}(t-s)} O(e^{-s}) \tilde{z}_j^1(s) ds \end{aligned} \tag{4.46}$$

for $N \geq 6$ and $p \in (1, \infty)$; $N = 3, 4, 5$ and $p \in (1, \frac{N+2}{6-N}]$;

$$\begin{aligned} \tilde{z}_j^1(t) = {} & C_j^1 + A_{j,2}^1 e^{\tilde{\beta}_2^{(1)} t} \\ & - B_1^1 \int_t^\infty e^{\tilde{\beta}_1^{(1)}(t-s)} O(e^{\ell s}) \tilde{z}_j^1(s) ds - B_3^1 \int_t^\infty e^{\tilde{\beta}_3^{(1)}(t-s)} O(e^{\ell s}) \tilde{z}_j^1(s) ds \\ & + B_2^1 \int_T^t e^{\tilde{\beta}_2^{(1)}(t-s)} O(e^{\ell s}) \tilde{z}_j^1(s) ds + B_4 \int_T^t O(e^{\ell s}) \tilde{z}_j^1(s) ds \end{aligned} \tag{4.47}$$

for $N = 4$ and $p > 7$; $N = 4$ and $p = 7$ with (1.8);

$$\begin{aligned} \tilde{z}_j^1(t) = {} & C_j^1 + A_{j,2}^1 e^{\tilde{\beta}_2^{(1)} t} \\ & - B_1^1 \int_t^\infty e^{\tilde{\beta}_1^{(1)}(t-s)} o(e^{-\hat{\beta} s}) \tilde{z}_j^1(s) ds - B_3^1 \int_t^\infty e^{\tilde{\beta}_3^{(1)}(t-s)} o(e^{-\hat{\beta} s}) \tilde{z}_j^1(s) ds \\ & + B_2^1 \int_T^t e^{\tilde{\beta}_2^{(1)}(t-s)} o(e^{-\hat{\beta} s}) \tilde{z}_j^1(s) ds + B_4^1 \int_T^t o(e^{-\hat{\beta} s}) \tilde{z}_j^1(s) ds \end{aligned} \tag{4.48}$$

for $N = 3, 4, 5$ and $p \in (\frac{N+2}{6-N}, p^*)$ with (1.9). These imply $\tilde{w}_j^1(s) \to C_j$ (a constant, maybe 0) as $s \to 0$. Recalling that $Q_1^1(\theta), \ldots, Q_{m_1}^1(\theta)$ are the eigenfunctions corresponding to $\lambda_1 = N - 1$, we have that

$$\tilde{w}(s, \theta) \to V(\theta) \ \text{ as } s \to 0, \tag{4.49}$$

for N and p satisfy (1.5); $N = 4$ and $p = 7$ with (1.8); $N = 3, 4, 5$ and $p \in (\frac{N+2}{6-N}, p^*)$ with (1.9). Here $V(\theta)$ is 0 or one of the first eigenfunctions of $-\Delta$ on S^{N-1}, i.e.

$$\Delta_\theta V + (N-1)V = 0, \quad \overline{V} = 0.$$

Moreover, it is known from Lemma 8.1 of [25] that

$$V(\theta) = \theta \cdot x_0 \tag{4.50}$$

for some $x_0 \in \mathbb{R}^N$ fixed and $\theta = \frac{x}{|x|} \in S^{N-1}$.

Combining what have been discussed above with Lemma 4.1 and Proposition 4.2, we have established the following asymptotic expansions near $y = 0$ for solutions of (2.8).

Theorem 4.3. *Let v be a solution of (2.8) and $\tilde{w}$ be given by (4.35). Suppose that N and p satisfy (1.5); $N = 4$ and $p = 7$ with (1.8); $N = 3, 4, 5$ and $p \in (\frac{N+2}{6-N}, p^*)$ with (1.9). Then $v(y) = \overline{v}(s) + s\tilde{w}(s, \theta)$ where $\overline{v}(s)$, $\tilde{w}(s, \theta)$ have the following properties and*

(i) $\overline{v}$ satisfies

$$|\overline{v}(s)| = O(s), \ \ |\overline{v}'(s)| = O(1), \ \ |\overline{v}''(s)| = O(s^{-1})$$

for $N = 3, 4, 5$ and $p \in (1, \frac{N+2}{6-N}]$; $N \geq 6$ and $p \in (1, \infty)$;

$$|\overline{v}(s)| = O(s^{-\ell}), \ \ |\overline{v}'(s)| = O(s^{-(\ell+1)}), \ \ |\overline{v}''(s)| = O(s^{-(\ell+2)})$$

for $N = 4$ and $p \in (7, \infty)$; $N = 4$ and $p = 7$ with (1.8);

$$|\overline{v}(s)| = o(s^{\hat{\beta}}), \ \ |\overline{v}'(s)| = o(s^{\hat{\beta}-1}), \ \ |\overline{v}''(s)| = o(s^{\hat{\beta}-2})$$

for $N = 3, 4, 5$ and $p \in (\frac{N+2}{6-N}, p^*)$ with (1.9).

(ii) For any nonnegative integers κ and κ_1, there exists a positive constant $M = M(v, \kappa, \kappa_1)$ such that

$$|s^\kappa D_\theta^{\kappa_1} D_s^\kappa \tilde{w}(s, \theta)| \leq M, \quad y \in \mathbf{B}_{s_0} := \{y : \ |y| < s_0\}, \ \ y \neq 0.$$

Moreover, $\tilde{w}$ satisfies

$$\lim_{s \to 0} \tilde{w}(s, \theta) = V(\theta) \quad \text{uniformly in } C^\kappa(S^{N-1}), \tag{4.51}$$

where $V(\theta)$ is 0 or one of the first eigenfunctions of $-\Delta_\theta$ on S^{N-1}.

Using transformation (2.7) and arguments similar to those in the proof of Theorem 5.1 of [11], we obtain immediately from Theorem 4.3 that the asymptotic expansions for positive entire solutions of (1.1) at ∞.

Theorem 4.4. *Let N and p satisfy* (1.5)*; $N = 4$ and $p = 7$ with* (1.8)*; $N = 3, 4, 5$ and $p \in (\frac{N+2}{6-N}, p^*)$ with* (1.9)*. Assume that u is a positive entire solution of* (1.1) *with* (1.6)*. Then $(u, -\Delta u)$ admits the expansion:*

$$
\begin{cases} u(x) = r^{\alpha}\Big[L + \varphi(r) + \dfrac{\psi(r,\theta)}{r}\Big], \\ w(x) := -\Delta u(x) = -r^{\alpha-2}\Big[L\alpha(N+\alpha-2) + \varphi_1(r) + \dfrac{\psi_1(r,\theta)}{r}\Big], \end{cases} \tag{4.52}
$$

where

$$
\begin{cases} \varphi_1(r) = r^2\varphi'' + (N+2\alpha-1)r\varphi' + \alpha(N+\alpha-2)\varphi, \\ \psi_1(r,\theta) = r^2\psi_{rr} + (N+2\alpha-3)r\psi_r + (\alpha-1)(N+\alpha-3)\psi + r^{-\alpha}\Delta_\theta\psi. \end{cases} \tag{4.53}
$$

Furthermore, the following properties for $\varphi, \psi, \varphi_1, \psi_1$ are satisfied:

(i) $\varphi(r) = r^{-\alpha}\overline{u}(r) - L$, and there exist R_0 ($:= s_0^{-1}$) and a constant $M = M(u) > 0$ such that, for $N = 3, 4, 5$ and $p \in (1, \frac{N+2}{6-N}]$; $N \geq 6$ and $p \in (1, \infty)$,

$$
|\varphi(r)| \leq Mr^{-1}, \quad |\varphi'(r)| \leq Mr^{-2}, \quad |\varphi''(r)| \leq Mr^{-3} \quad \textit{for } r > R_0, \tag{4.54}
$$

$$
|\varphi_1'(r)| \leq Mr^{-1} \quad \textit{for } r > R_0. \tag{4.55}
$$

For $N = 4$ and $p \in (7, \infty)$; $N = 4$ and $p = 7$ with (1.8)*,*

$$
|\varphi(r)| \leq Mr^{\ell}, \quad |\varphi'(r)| \leq Mr^{\ell-1}, \quad |\varphi''(r)| \leq Mr^{\ell-2} \quad \textit{for } r > R_0, \tag{4.56}
$$

$$
|\varphi_1'(r)| \leq Mr^{\ell} \quad \textit{for } r > R_0. \tag{4.57}
$$

For $N = 3, 4, 5$ and $p \in (\frac{N+2}{6-N}, p^*)$ with (1.9)*,*

$$
|\varphi(r)| = o(r^{-\hat{\beta}}), \quad |\varphi'(r)| = o(r^{-\hat{\beta}-1}), \quad |\varphi''(r)| = o(r^{-\hat{\beta}-2}) \quad \textit{for } r > R_0, \tag{4.58}
$$

$$
|\varphi_1'(r)| = o(r^{-\hat{\beta}}) \quad \textit{for } r > R_0. \tag{4.59}
$$

(ii) Let κ and κ_1 be two non-negative integers. Then there exists a positive constant $M = M(u, \kappa, \kappa_1)$ such that

$$
|r^{\kappa} D_\theta^{\kappa_1} D_r^{\kappa}\psi(r,\theta)| \leq M, \quad |\psi_1(r,\theta)| \leq M \quad \textit{for } r > R_0. \tag{4.60}
$$

(iii) Let κ be a non-negative integer. Then $\psi(r,\theta)$ tends to $V(\theta)$ uniformly in $C^{\kappa}(S^{N-1})$ as $r \to \infty$, where $V(\theta)$ is given by (4.50).

5. Proofs of Theorems 1.1-1.3

In this section, we present the proofs of Theorems 1.1-1.3 by using the well known moving plane method.

For $\gamma \in \mathbb{R}$, define the hyperplane:

$$
\Upsilon_\gamma = \{x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N \mid x_1 = \gamma\}.
$$

For any $x \in \mathbb{R}^N$, denote the reflection point of x about Υ_γ by x^γ, i.e.

$$x^\gamma = (2\gamma - x_1, x_2, \cdots, x_N).$$

We have the following lemma by using Theorem 4.4.

Lemma 5.1. *Assume that N and p satisfy* (1.5)*; $N = 4$ and $p = 7$ with* (1.8)*; $N = 3, 4, 5$ and $p \in (\frac{N+2}{6-N}, p^*)$ with* (1.9)*. Let u be a positive entire solution of* (1.1) *satisfying* (1.6), (1.8) *and* (1.9) *respectively. Then,*

(i) if $\gamma^j \in \mathbb{R} \to \gamma$ and $\{x^j\} \to \infty$ with $x_1^j < \gamma^j$, then

$$\lim_{j\to\infty} \frac{|x^j|^{2-\alpha}}{\gamma^j - x_1^j} \left[u(x^j) - u((x^j)^\gamma)\right] = -2\alpha L\gamma - 2(x_0)_1, \tag{5.1}$$

where $(x_0)_1$ is the first component of x_0 given in (4.50).

(ii) Denote $\gamma_0 = -\frac{(x_0)_1}{\alpha L}$. Then there exists a constant $M = M(u) > 0$ such that

$$\frac{\partial u}{\partial x_1}(x) \geq 0, \tag{5.2}$$

if $x_1 \geq \gamma_0 + 1$ and $|x| \geq M$.

Proof. For $N = 3$, 4, 5 and $p \in (1, \frac{N+2}{6-N}]$; $N \geq 6$ and $p > 1$, the proof of this lemma is similar to that of Lemma 6.2 of [15]. For $N = 4$ and $p > 7$; $N = 4$ and $p = 7$ with (1.8); $N = 3, 4, 5$ and $p \in (\frac{N+2}{6-N}, p^*)$ with (1.9), we can obtain the conclusions from the decay rates of $\varphi(r)$, $\varphi_1(r)$, $\psi(r,\theta)$ and $\psi_1(r,\theta)$ in Theorem 4.4. In fact, we only need to replace the estimate:

$$\frac{1}{|x^j|^{\alpha-2}(\gamma - x_1^j)}\left[\xi(|x^j|)|x^j|^\alpha - \xi(|(x^{j^\gamma}|)|x^{j^\gamma}|^\alpha\right] = O(|x^j|^{-1}) \to 0 \text{ as } j \to \infty$$

in the proof of Lemma 6.2 in [15], by

$$\begin{aligned}&\frac{1}{|x^j|^{\alpha-2}(\gamma - x_1^j)}\left[\varphi(|x^j|)|x^j|^\alpha - \varphi(|(x^j)^\gamma|)|(x^j)^\gamma|^\alpha\right]\\ &= \begin{cases} O(|x^j|^\ell) \to 0 & \text{as } j \to \infty \text{ for } N = 4,\ p > 7;\ N = 4,\ p = 7 \text{ with (1.8)};\\ O(|x^j|^{-\hat{\beta}}) \to 0 & \text{as } j \to \infty \text{ for } N = 3, 4, 5,\ p \in (\frac{N+2}{6-N}, p^*) \text{ with (1.9)}, \end{cases}\end{aligned}$$

here we have used (4.56) and (4.58). This completes the proof of this lemma. □

Assume $w(x) = -\Delta u(x)$ and rewrite (1.1) in the following form:

$$\begin{cases} -\Delta u = w & \text{in } \mathbb{R}^N,\\ -\Delta w = -u^{-p} & \text{in } \mathbb{R}^N. \end{cases} \tag{5.3}$$

Let us recall Lemma 4.2 in [23] due to Troy. We obtain readily that

Lemma 5.2. *Let $\gamma \in \mathbb{R}$ and u be a positive entire solution of* (1.1)*. Suppose that*

$$u(x) \leq u(x^\gamma),\ \ u(x) \not\equiv u(x^\gamma),\ \ w(x) \leq w(x^\gamma) \quad \textit{if } x_1 < \gamma.$$

Then

$$u(x) < u(x^\gamma),\ \ w(x) < w(x^\gamma) \quad \textit{if } x_1 < \gamma \tag{5.4}$$

and

$$\frac{\partial u}{\partial x_1}(x) > 0, \quad \frac{\partial w}{\partial x_1}(x) > 0, \quad \text{on } \Upsilon_\gamma, \tag{5.5}$$

where x^γ is the reflection point of x with respect to Υ_γ.

As a consequence of Lemma 5.2, we have the following result.

Lemma 5.3. *Let $\gamma \in \mathbb{R}$, N and p satisfy* (1.5)*; $N = 4$ and $p = 7$ with* (1.8)*; $N = 3, 4, 5$ and $p \in (\frac{N+2}{6-N}, p^*)$ with* (1.9)*. Let u be a positive entire solution of* (1.1) *satisfying* (1.6)*,* (1.8) *and* (1.9) *respectively. If*

$$u(x) \le u(x^\gamma), \ \ u(x) \not\equiv u(x^\gamma) \quad \text{for } x_1 < \gamma,$$

then

$$u(x) < u(x^\gamma), \quad w(x) < w(x^\gamma) \ \ \text{for } x_1 < \gamma. \tag{5.6}$$

Proof. Since $u(x) \le u(x^\gamma)$, $u(x) \not\equiv u(x^\gamma)$ for $x_1 < \gamma$, we deduce that

$$\Delta[w(x) - w(x^\gamma)] = u^{-p}(x) - u^{-p}(x^\gamma) \ge 0 \quad \text{if } x_1 < \gamma.$$

It follows from (4.52)-(4.60) that

$$w(x) - w(x^\gamma) \to 0 \quad \text{as } |x| \to \infty.$$

Moreover, $w(x) = w(x^\gamma)$ on Υ_γ. The maximum principle yields

$$w(x) - w(x^\gamma) \le 0 \quad \text{if } x_1 < \gamma.$$

It follows from Lemma 5.2 that our conclusions in (5.6) hold. □

Proofs of Theorems 1.1, 1.2 and 1.3

We first show the sufficiency of these theorems. The main idea of the proof is similar to those in [11, 25]. We claim that there exists $\gamma' > 0$ such that

$$u(x) < u(x^\gamma), \quad w(x) < w(x^\gamma) \quad \text{for } \gamma \ge \gamma' \text{ and } x_1 < \gamma. \tag{5.7}$$

Suppose for contradiction that (5.7) does not hold. Then by Lemma 5.3, there exist two sequences $\{\gamma^j\} \to \infty$ and $\{x^j\}$ with $x^j < \gamma^j$ such that

$$u(x^j) \ge u(y^j), \quad y^j = (x^j)^{\gamma^j}, \quad j = 1, 2, \ldots. \tag{5.8}$$

Thanks to y^j tends to ∞, we see that $u(y^j)$ tends to infinity. In turn $|x^j| \to \infty$. By Lemma 5.1, we must have

$$x_1^j \le \gamma_0 + 1 = -\frac{(x_0)_1}{\alpha L} + 1 \quad \text{for } j \text{ large enough.}$$

Thus, it follows that, for any $\gamma_1 > \gamma_0 + 1$,

$$u(x^j) \ge u(y^j) \ge u((x^j)^{\gamma_1}) \quad \text{for } j \text{ large,}$$

since $(x^j)_1^{\gamma_1^j} \gg (x^j)_1^{\gamma_1}$ for j large and $u(x) \to \infty$ as $|x| \to \infty$. On the other hand, using Lemma 5.1 again, we conclude that

$$0 \le \frac{|x^j|^{2+\alpha}}{\gamma_1 - x_1^j}\Big[u(x^j) - u((x^j)^{\gamma_1})\Big] \to -2\alpha L\gamma_1 - 2(x_0)_1 < 0,$$

since $x_1^j < \gamma_1$. This is a contradiction and (5.7) follows.

The rest of the proof is same as that of Theorem 1.1 in [11] and [25] for the sufficiency of Theorems 1.1, 1.2 and 1.3. We omit them here.

We now show the necessity of Theorems 1.1, 1.2 and 1.3. Without loss of generality, we assume $x_* = 0$. Then, the necessity of Theorem 1.1 follows from Proposition 8 of [5]. To show the necessity of Theorems 1.2 and 1.3, we first show a lemma, which describes the behavior of the unique minimal positive radial entire solution of (1.1) at ∞.

Lemma 5.4. *Assume $N = 3$ and $p \in (1,3)$; $N \ge 4$ and $p \in (1,\infty)$. Let $u \in C^4(\mathbb{R}^N)$ be the minimal positive radial entire solution of* (1.1)*. Then as $r = |x| \to \infty$, there holds:*

$$u(r) = Lr^{\alpha} + \begin{cases} O(r^{2-\frac{N}{2}}) & \text{for } N = 3 \text{ and } p \in (p_3^1, p_3^2);\ N \in [4,12] \text{ and } p \in (p_c,\infty); \\ O(r^{-1+\alpha}) & \text{for } N \in [5,12] \text{ and } p \in (1,p_c);\ N \ge 13 \text{ and } p \in (1,\infty); \\ O(r^{\beta_3+\alpha}) & \text{for } N = 3 \text{ and } p \in [p_3^2, 3), \end{cases}$$

where $\beta_3 = \frac{1}{2}\Big(1 - 2\alpha + \sqrt{5 - 4\sqrt{1 + p\alpha(2-\alpha)(\alpha^2-1)}}\Big)$ *given by* (4.3).

Proof. In radial coordinate $r = |x|$, (1.1) can be written to: for $r \in (0,\infty)$,

$$u^{(4)} + \frac{2(N-1)}{r}u''' + \frac{(N-1)(N-3)}{r^2}u'' - \frac{(N-1)(N-3)}{r^3}u' = -u^{-p}.$$

For the minimal positive radial entire solution $u(r)$ of (1.1), we know from [5] that it satisfies (1.4).

Inspired by [6, 7, 8, 12, 16, 24], we introduce the Emden-Fowler transformation

$$r = e^t, \quad m(t) = e^{-\alpha t}u(e^t) - L, \quad t \in \mathbb{R}.$$

Under this transformation, (1.1) becomes to

$$\begin{aligned} &m^{(4)} + 2(N+2\alpha-4)m''' + (N^2 + 6N\alpha + 6\alpha^2 - 10N - 24\alpha + 20)m'' \\ &+2(N+2\alpha-4)(N\alpha + \alpha^2 - N - 4\alpha + 2)m' - (p+1)L^{-(p+1)}m + g(m) = 0, \end{aligned} \tag{5.9}$$

where $g(m) = (m+L)^{-p} - L^{-p} + pL^{-(p+1)}m$. Note that (1.4) indicates

$$\lim_{t\to\infty} m(t) = 0,$$

so for $|t|$ large enough, $g(m) = O(m^2)$. Comparing (4.1) with (5.9), we find that they have the same characteristic polynomial (4.2) and the eigenvalues β_j $(j = 1,2,3,4)$ given in Section 4. Taking account of the properties of β_j given in (4.4) and Claim 5, we obtain the presentations of $m(t)$, which are similar to (4.14) and (4.15) in

Section 4 except that $h(s,\bar{z}(s))$ is replaced by $g(w)$. The same arguments imply that

(5.10)

$$|m(t)| = \begin{cases} O(e^{\ell t}) & \text{for } p \in (p_3^1, p_3^2) \text{ and } N=3;\ p\in(1,\infty) \text{ and } N=4; \\ & \quad p\in(p_c,\infty) \text{ and } N\in[5,12]; \\ O(e^{\beta_3 t}) & \text{for } p\in(1,p_3^1]\cup[p_3^2,3) \text{ and } N=3;\ p\in(1,p_c] \text{ and } N\in[5,12]; \\ & \quad p\in(1,\infty) \text{ and } N\in[13,\infty). \end{cases}$$

Note that $\ell = 2-\alpha-\frac{N}{2}$ and $\beta_3 = \frac{1}{2}\left(1-2\alpha+\sqrt{5-4\sqrt{1+p\alpha(2-\alpha)(\alpha^2-1)}}\right)$ when $N=3$. We obtain our desired results by using (4.4) and Claim 5 again. □

We continue to show the necessity of Theorems 1.2 and 1.3. It follows from Lemma 5.4 that if u is the minimal positive radial entire solution of (1.1), then, for r sufficiently large, there holds

(5.11)

$$r^{-\alpha}u(r)-L = \begin{cases} O(r^{\ell}), & \text{for } p\in(\frac{5}{3},p_3^2) \text{ and } N=3;\ p\in(3,7] \text{ and } N=4; \\ & \quad p\in(7,\infty) \text{ and } N=5; \\ O(r^{\beta_3}), & \text{for } p\in[p_3^2,3) \text{ and } N=3. \end{cases}$$

On the other hand, we can easily check that, for $p\in[p_3^2,3)$ and $N=3$,

$$\beta_3 < -\hat{\beta} = 5-3-2\alpha < 0.$$

For $(\frac{5}{3},p_3^2)$ and $N=3$; $p\in(3,7)$ and $N=4$; $p\in(7,\infty)$ and $N=5$,

$$\ell = 2-\frac{N}{2}-\alpha < 5-N-2\alpha = -\hat{\beta} < 0$$

and for $p=7$ and $N=4$,

$$\ell = 2-\frac{N}{2}-\alpha = -\frac{1}{2} < -\epsilon_0,$$

where ϵ_0 is given in Theorem 1.2. It follows from (5.11) that for r sufficiently large,

$$r^{-\alpha}u(r)-L = \begin{cases} o(r^{5-N-2\alpha}), & \text{for } p\in(\frac{5}{3},3) \text{ and } N=3;\ p\in(3,7) \text{ and } N=4; \\ & \quad p\in(7,\infty) \text{ and } N=5; \\ o(r^{-\epsilon_0}), & \text{for } p=7 \text{ and } N=4. \end{cases}$$

This completes the proof of the necessity of Theorems 1.2 and 1.3 and then completes the proof of Theorems 1.1, 1.2 and 1.3. □

6. Proof of Theorem 1.4

In this section, we present the proof of Theorem 1.4. To do this, we first obtain the asymptotic behavior of a non-minimal positive radial entire solution of (1.1). We know from [5] that when $N=3$ and $1<p<3$; $N\geq 4$ and $p>1$, for any fixed $a > 0$ and $\infty > b > \tilde{b}$, (1.3) admits a unique non-minimal positive radial entire solution $u_{a,b}(r)$ such that

$$r^{-2}u_{a,b}(r) \in (A_1, A_2) \text{ for } r \text{ sufficiently large,}$$

where $0 < A_1 < A_2 < \infty$.

The following proposition presents the asymptotic behavior of $u_{a,b}(r)$ at $r = \infty$.

Proposition 6.1. *There exists $d > 0$ (d depends on a and b) such that, for r near $+\infty$,*

$$\Delta u_{a,b}(r) = \begin{cases} d + O(r^{-\min\{N-2,2(p-1)\}}), & \text{if } p \neq \frac{N}{2}, \\ d + O(r^{-(N-2)} \ln r), & \text{if } p = \frac{N}{2}, \end{cases} \tag{6.1}$$

(6.2)

$$r^{-2}u_{a,b}(r) = \begin{cases} \frac{d}{2N} + O(r^{-\kappa}), & \text{if } p \neq \frac{N}{2} \text{ and } \min\{N-2, 2(p-1)\} \neq 2; \\ \frac{d}{2N} + O\left(r^{-\kappa} \ln r\right), & \text{if } p \neq \frac{N}{2} \text{ and } \min\{N-2, 2(p-1)\} = 2; \\ \frac{d}{2N} + O\left(r^{-2}\right), & \text{if } p = \frac{N}{2} \text{ and } N \geq 5; \\ \frac{d}{2N} + O\left(r^{-1} \ln r\right), & \text{if } p = \frac{3}{2} \text{ and } N = 3; \\ \frac{d}{2N} + O\left(r^{-2}(\ln r)^2\right), & \text{if } p = 2 \text{ and } N = 4, \end{cases}$$

where $\kappa = \min\{2, N-2, 2(p-1)\}$.

Proof. We first show

$$\Delta u_{a,b}(r) \to d, \;\; r^{-2}u_{a,b}(r) \to \frac{d}{2N} \text{ as } r \to \infty. \tag{6.3}$$

It is easily seen from the equation in (1.3) that $\Delta u_{a,b}(r)$ is decreasing in $(0, \infty)$. Therefore, there are three cases for $\Delta u_{a,b}(r)$:
(i) $\Delta u_{a,b}(r) \to -e < 0$ (e may be $+\infty$) as $r \to \infty$,
(ii) $\Delta u_{a,b}(r) \to 0$ as $r \to \infty$,
(iii) $\Delta u_{a,b}(r) \to d > 0$ as $r \to \infty$.
We show that the cases (i) and (ii) do not happen. Since

$$r^{-2}u_{a,b}(r) \in (A_1, A_2) \text{ for } r \text{ sufficiently large,}$$

we have that

$$\underline{\lim}_{r\to\infty} r^{-2}u_{a,b}(r) \geq A_1 > 0. \tag{6.4}$$

If (i) occurs, we see that for any small $\epsilon > 0$, there is an $R = R(\epsilon) > 1$ such that

$$\Delta u_{a,b}(r) < -e + \epsilon \text{ for } r > R. \tag{6.5}$$

(We may assume $0 < e < \infty$. If $e = \infty$, we can choose any $0 < e_1 < \infty$ such that (6.5) holds.) This implies

$$r^{N-1}u'_{a,b}(r) - R^{N-1}u'_{a,b}(R) \leq \frac{(-e+\epsilon)}{N}(r^N - R^N),$$

and

$$u'_{a,b}(r) \leq \frac{R^{N-1}}{r^{N-1}}u'_{a,b}(R) + \frac{(-e+\epsilon)}{N}(r - R^N r^{1-N}).$$

Therefore,

$$\begin{aligned} u_{a,b}(r) &\le u_{a,b}(R) + \frac{R^{N-1}u'_{a,b}(R)}{2-N}(r^{2-N} - R^{2-N}) \\ &\quad + \frac{(-e+\epsilon)}{2N}(r^2 - R^2) + \frac{(-e+\epsilon)R^N}{N(N-2)}(r^{2-N} - R^{2-N}). \end{aligned}$$

This implies

$$\overline{\lim}_{r\to\infty} r^{-2} u_{a,b}(r) \le -\frac{e}{2N} < 0$$

by sending ϵ to 0. This contradicts to (6.4).

If (ii) occurs, arguments similar to those in the proof of case (i) imply that

$$\overline{\lim}_{r\to\infty} r^{-2} u_{a,b}(r) \le 0.$$

This also contradicts to (6.4).

Therefore, case (iii) occurs. Clearly using the arguments similar to those in the proof of case (i), we can prove that

$$\lim_{r\to\infty} r^{-2} u_{a,b}(r) = \frac{d}{2N},$$

and then the limits in (6.3) hold.

To prove the identities in (6.1), we define $v(r) = \Delta u(r) - d$. We omit a, b from $u_{a,b}$ in the following. Then $v(r) \to 0$ as $r \to \infty$ and $v(r)$ satisfies the equation $\Delta v(r) = \Delta^2 u(r) = -u^{-p}$. It follows from (6.3) that, for r near $+\infty$,

$$\Delta v(r) = O(r^{-2p}).$$

This implies that

$$r^2 v''(r) + (N-1) r v'(r) = O(r^{2(1-p)}).$$

Making the transformations:

$$w(t) = v(r), \quad t = \ln r,$$

we have that, for t near ∞, $w(t)$ satisfies the equation:

$$w''(t) + (N-2) w'(t) = O(e^{2(1-p)t}).$$

The ODE theory implies that for $T \gg 1$ sufficiently large and $t > T$,

$$\begin{aligned} w(t) &= M_1 + A_2 e^{(2-N)t} - B_1 \int_t^\infty O(e^{2(1-p)t}) ds \\ &\quad + B_2 \int_T^t e^{(2-N)(t-s)} O(e^{2(1-p)s}) ds. \end{aligned}$$

Note that B_1 and B_2 are independent of T. Since $w(t) \to 0$ as $t \to \infty$, we have that $M_1 = 0$ and we easily see that

$$w(t) = \begin{cases} O(e^{-\min\{N-2,\, 2(p-1)\}t}), & \text{if } p \ne \frac{N}{2}, \\ O(te^{-(N-2)t}), & \text{if } p = \frac{N}{2}. \end{cases}$$

This implies that the identities in (6.1) hold. To see the identities in (6.2), we define $\varrho(r) = r^{-2}u(r) - \frac{d}{2N}$. Then $\varrho(r) \to 0$ as $r \to \infty$ and $\varrho(r)$ satisfies the equation

$$r^2\varrho'' + (N+3)r\varrho' + 2N\varrho = \Delta u(r) - d = \begin{cases} O\left(r^{-\min\{N-2,\,2(p-1)\}}\right), & \text{if } p \neq \frac{N}{2}, \\ O\left(r^{-(N-2)}\ln r\right), & \text{if } p = \frac{N}{2}. \end{cases}$$

Making the transformations:

$$z(t) = \varrho(r), \quad t = \ln r,$$

we have that, for t near ∞, $z(t)$ satisfies the equation

$$z''(t) + (N+2)z'(t) + 2Nz(t) = \begin{cases} O\left(e^{-\min\{N-2,\,2(p-1)\}t}\right), & \text{if } p \neq \frac{N}{2}, \\ O\left(te^{-(N-2)t}\right), & \text{if } p = \frac{N}{2}. \end{cases}$$

Arguments similar to those in the proof of (6.1) imply that for t near $+\infty$,

$$z(t) = \begin{cases} O(e^{-\kappa t}), & \text{if } p \neq \frac{N}{2} \text{ and } \min\{N-2, 2(p-1)\} \neq 2, \\ O\left(te^{-\kappa t}\right), & \text{if } p \neq \frac{N}{2} \text{ and } \min\{N-2, 2(p-1)\} = 2, \\ O\left(e^{-2t}\right), & \text{if } p = \frac{N}{2} \text{ and } N \geq 5, \\ O\left(te^{-t}\right), & \text{if } p = \frac{3}{2} \text{ and } N = 3, \\ O\left(t^2e^{-2t}\right), & \text{if } p = \frac{4}{2} \text{ and } N = 4, \end{cases}$$

where $\kappa = \min\{2, N-2, 2(p-1)\}$. This implies that the identities in (6.2) hold. Since $u(r) = r^2\varrho(r) + \frac{d}{2N}r^2$, we have that

$$\Delta(r^2\varrho(r)) = \Delta u(r) - d > 0 \ \text{ for } r \in (0, \infty).$$

If we define $\omega(r) := r^2\varrho(r)$, we see that $\omega'(0) = 0$ and hence

$$\omega'(r) > 0 \ \text{ for } r \in (0, \infty).$$

The proof of this proposition is completed. □

Remark 6.2. We can easily see that for any fixed $a > 0$, $d := d(a, b) > 0$ for $b \in (\tilde{b}, \infty)$ is an increasing function of b with

$$\lim_{b \to \tilde{b}^+} d(a, b) = 0.$$

We also know that

$$\lim_{b \to \infty} d(a, b) = \infty.$$

Proof of Theorem 1.4.

Without loss of generality, we assume $x_* = 0$ in Theorem 1.4. The necessity follows from Proposition 6.1.

To prove the sufficiency of Theorem 1.4, we need to know more information on the asymptotic behavior of an entire solution $u \in C^4(\mathbb{R}^N)$ of (1.1) satisfying (1.10). The main idea is similar to that of the proof of the sufficiency of Theorem 1.1.

Let $u \in C^4(\mathbb{R}^N)$ be an entire solution of (1.1). We introduce the Kelvin-type transformation:

$$v(y) = |x|^{-2}u(x) - D, \quad y = \frac{x}{r^2}, \ \ r = |x| > 0, \ \ D > 0. \tag{6.6}$$

Then $v(y) = v(s,\theta)$ with $s = |y| = r^{-1}$ satisfies $v(s,\theta) \to 0$ as $s \to 0$ and the equation:

$$\begin{aligned}
&\partial_s^4 v - 2(N-3)s^{-1}\partial_s^3 v + (N-1)(N-3)s^{-2}\partial_s^2 v - (N-1)(N-3)s^{-3}\partial_s v \\
&\quad +2Ns^{-4}\Delta_\theta v - 2(N-1)s^{-3}\Delta_\theta(\partial_s v) + 2s^{-2}\Delta_\theta(\partial_s^2 v) + s^{-4}\Delta_\theta^2 v \\
&\quad +s^{-6+2p}(v+D)^{-p} = 0. \qquad (6.7)
\end{aligned}$$

Define

$$w(s,\theta) = v(s,\theta) - \overline{v}(s),$$

where

$$\overline{v}(s) = \frac{1}{|S^{N-1}|}\int_{S^{N-1}} v(s,\theta)d\theta.$$

Then $\overline{v}$ and w respectively satisfy

$$\begin{aligned}
&\overline{v}_s^{(4)} - 2(N-3)s^{-1}\overline{v}_{sss} + (N-1)(N-3)s^{-2}\overline{v}_{ss} - (N-1)(N-3)s^{-3}\overline{v}_s \\
&\quad +s^{-6+2p}\overline{(v+D)^{-p}} = 0 \qquad (6.8)
\end{aligned}$$

and

$$\begin{aligned}
&\partial_s^4 w - 2(N-3)s^{-1}\partial_s^3 w + (N-1)(N-3)s^{-2}\partial_s^2 w - (N-1)(N-3)s^{-3}\partial_s w \\
&\quad +2Ns^{-4}\Delta_\theta w - 2(N-1)s^{-3}\Delta_\theta(\partial_s w) + 2s^{-2}\Delta_\theta(\partial_s^2 w) + s^{-4}\Delta_\theta^2 w \\
&\quad -s^{-4}g(w) = 0, \qquad (6.9)
\end{aligned}$$

where

$$\begin{aligned}
g(w) &= s^{2p-2}(v+D)^{-p} - s^{2p-2}\overline{(v+D)^{-p}} \\
&= s^{2(p-1)}\Big[\Big((v+D)^{-p} - (\overline{v}+D)^{-p}\Big) + \overline{\Big((\overline{v}+D)^{-p} - (v+D)^{-p}\Big)}\Big] \\
&= -ps^{2(p-1)}\Big[(\xi(s,\theta)+D)^{-(p+1)}w(s,\theta) - \overline{(\xi(s,\theta)+D)^{-(p+1)}w(s,\theta)}\Big]
\end{aligned}$$

and $\xi(s,\theta)$ is between $v(s,\theta)$ and $\overline{v}(s)$. If we define

$$\zeta(s) = \max_{\theta\in S^{N-1}} |-ps^{2(p-1)}(\xi(s,\theta)+D)^{-(p+1)}| = ps^{2(p-1)}\max_{\theta\in S^{N-1}} |(\xi(s,\theta)+D)^{-(p+1)}|,$$

we see that

$$\zeta(s) = O(s^{2(p-1)}) \text{ for } s \text{ near } 0. \qquad (6.10)$$

Note that $\xi(s,\theta) \to 0$ as $s \to 0$.

Since $\overline{w}(s) = 0$, we have the expansion:

$$w(s,\theta) = \sum_{i=1}^{\infty}\sum_{j=1}^{m_i} w_j^i(s)Q_j^i(\theta),$$

where $\{Q_1^1(\theta), Q_2^1(\theta), \ldots, Q_{m_1}^1(\theta), Q_1^2(\theta), \ldots, Q_{m_2}^2(\theta), Q_1^3(\theta), \ldots\}$ is given in Section 2. We also see that $w_j^i(s)$ with $1 \le j \le m_i$ satisfies the equation

$$\begin{aligned}
&(w_j^i)^{(4)} - 2(N-3)s^{-1}(w_j^i)_{sss} + [(N-1)(N-3) - 2\lambda_i]s^{-2}(w_j^i)_{ss} \\
&\quad -(N-1)[N-3-2\lambda_k]s^{-3}(w_j^i)_s - (2N\lambda_i - \lambda_i^2)s^{-4}w_j^i \\
&\quad = s^{-4}\tilde{g}_j^i(s), \qquad (6.11)
\end{aligned}$$

where $\lambda_i = i(N+i-2)$, $i = 0, 1, 2, \ldots$ are the eigenvalues of the equation $-\Delta_{S^{N-1}} Q = \lambda Q$ given by (2.3) and

$$\tilde{g}_j^i(s) = \int_{S^{N-1}} g(w) Q_j^i(\theta) d\theta = -p \int_{S^{N-1}} s^{2(p-1)} (\xi(s,\theta) + D)^{-(p+1)} w(s,\theta) Q_j^i(\theta) d\theta.$$

We see that

$$|\tilde{g}_j^i(s)| \leq C\zeta(s) W(s) = O(s^{2(p-1)}) W(s) \quad \text{for } s \text{ near } 0,$$

where $W(s) = \Big(\int_{S^{N-1}} w^2(s,\theta) d\theta \Big)^{\frac{1}{2}}$.

Similar to Proposition 3.1, we have the following result.

Proposition 6.3. *For $N = 3$ and $1 < p < 3$; $N \geq 4$ and $p > 1$, there exist a sufficiently small $0 < s_0 < \frac{1}{10}$ and $C > 0$ independent of s such that for $s \in (0, s_0)$,*

$$W(s) \leq Cs. \tag{6.12}$$

Proof. Let $t = -\ln s$, $z_j^i(t) = w_j^i(s)$. Then $z_j^i(t)$ satisfies the equation

$$\begin{aligned} &(z_j^i)^{(4)} + 2N(z_j^i)_{ttt} + (N^2 + 2N - 4 - 2\lambda_i)(z_j^i)_{tt} + 2N(N - 2 - \lambda_i)(z_j^i)_t \\ &-\lambda_i(2N - \lambda_i) z_j^i = f_j^i(t), \end{aligned} \tag{6.13}$$

where $f_j^i(t) = \tilde{g}_j^i(e^{-t})$. The corresponding polynomial of (6.13) is

$$\nu^4 + 2N\nu^3 + (N^2 + 2N - 4 - 2\lambda_i)\nu^2 + 2N(N - 2 - \lambda_i)\nu - \lambda_i(2N - \lambda_i) = 0. \tag{6.14}$$

Using the Matlab, we obtain four roots of (6.14):

$$\begin{aligned} &\nu_1^{(i)} = \tfrac{1}{2}\Big(2 - N + \sqrt{(N-2)^2 + 4\lambda_i}\Big), \qquad \nu_2^{(i)} = \tfrac{1}{2}\Big(2 - N - \sqrt{(N-2)^2 + 4\lambda_i}\Big), \\ &\nu_3^{(i)} = \tfrac{1}{2}\left(-2 - N + \sqrt{(N-2)^2 + 4\lambda_i}\right), \quad \nu_4^{(i)} = \tfrac{1}{2}\left(-2 - N - \sqrt{(N-2)^2 + 4\lambda_i}\right). \end{aligned} \tag{6.15}$$

Therefore, we have

$$\nu_1^{(i)} = i, \ \ \nu_2^{(i)} = 2 - N - i, \ \ \nu_3^{(i)} = i - 2, \ \ \nu_4^{(i)} = -N - i. \tag{6.16}$$

We easily see that

$$\nu_4^{(i)} < \nu_2^{(i)} < \nu_3^{(i)} < \nu_1^{(i)}.$$

For $i = 1$,

$$\nu_1^{(1)} = 1, \ \ \nu_2^{(1)} = 1 - N, \ \ \nu_3^{(1)} = -1, \ \ \nu_4^{(1)} = -N - 1$$

and

$$\nu_4^{(1)} < \nu_2^{(1)} < \nu_3^{(1)} = -1 < 0 < \nu_1^{(1)}.$$

For $i = 2$,

$$\nu_1^{(2)} = 2, \ \ \nu_2^{(2)} = -N, \ \ \nu_3^{(2)} = 0, \ \ \nu_4^{(2)} = -N - 2$$

and

$$\nu_4^{(2)} < \nu_2^{(2)} < -1 < \nu_3^{(2)} = 0 < \nu_1^{(2)}.$$

For $i \geq 3$, we see that

$$\nu_4^{(i)} < \nu_2^{(i)} < -1 < 0 < \nu_3^{(i)} < \nu_1^{(i)}.$$

For $i \geq 3$, we see from (6.13) and ODE theory that for any $T \gg 1$, there are constants $A^i_{j,k}$, $B^i_{j,k}$ $(k = 1, 2, 3, 4)$ such that, for $t > T$,

$$z^i_j(t) = \sum_{k=1}^{4} \Big[A^i_{j,k} e^{\nu^{(i)}_k t} + B^i_k \int_T^t e^{\nu^{(i)}_k (t-\tau)} f^i_j(\tau) d\tau \Big],$$

where each $A^i_{j,k}$ depends on T and $\nu^{(i)}_k$, but each B^i_k depends only on $\nu^{(i)}_k$. Therefore,

$$\begin{aligned} z^i_j(t) \; = \; & M^i_{j,1} e^{\nu^{(i)}_1 t} + M^i_{j,3} e^{\nu^{(i)}_3 t} + A^i_{j,2} e^{\nu^{(i)}_2 t} + A^i_{j,4} e^{\nu^{(i)}_4 t} \\ & - B^i_1 \int_t^\infty e^{\nu^{(i)}_1 (t-\tau)} f^i_j(\tau) d\tau - B^i_3 \int_t^\infty e^{\nu^{(i)}_3 (t-\tau)} f^i_j(\tau) d\tau \\ & + B^i_2 \int_T^t e^{\nu^{(i)}_2 (t-\tau)} f^i_j(\tau) d\tau + B^i_4 \int_T^t e^{\nu^{(i)}_4 (t-\tau)} f^i_j(\tau) d\tau \end{aligned} \tag{6.17}$$

by using that $\int_T^t = \int_T^\infty - \int_t^\infty$. Note that

$$\int_t^\infty e^{\nu^{(i)}_1 (t-\tau)} f^i_j(\tau) d\tau \to 0, \quad \int_t^\infty e^{\nu^{(i)}_3 (t-\tau)} f^i_j(\tau) d\tau \to 0 \text{ as } t \to \infty. \tag{6.18}$$

Moreover, the facts that $\nu^{(i)}_4 < \nu^{(i)}_2 < 0$ and $t - \tau > 0$ for $\tau \in (T, t)$ imply that

$$\int_T^t e^{\nu^{(i)}_4 (t-\tau)} |f(\tau)| d\tau \leq \int_T^t e^{\nu^{(i)}_2 (t-\tau)} |f(\tau)| d\tau.$$

The facts that $0 < \nu^{(i)}_3 < \nu^{(i)}_1$ and $t - \tau < 0$ for $\tau \in (t, \infty)$ imply that

$$\int_t^\infty e^{\nu^{(i)}_1 (t-\tau)} |f^i_j(\tau)| d\tau \leq \int_t^\infty e^{\nu^{(i)}_3 (t-\tau)} |f^i_j(\tau)| d\tau.$$

Therefore, since $z^i_j(t) \to 0$ as $t \to \infty$, we see that $M^i_{j,1} = M^i_{j,3} = 0$ and there is $C > 0$ depending only on B^i_k $(k = 1, 2, 3, 4)$ but independent of T such that

$$|z^i_j(t)| \leq O(e^{\nu^{(i)}_2 t}) + C \int_T^t e^{\nu^{(i)}_2 (t-\tau)} |f^i_j(\tau)| d\tau + C \int_t^\infty e^{\nu^{(i)}_3 (t-\tau)} |f^i_j(\tau)| d\tau.$$

Arguments similar to those in the proof of (3.20) imply that

$$|z^i_j(t)| = O(i e^{\nu^{(i)}_2 (t-T)}) \tag{6.19}$$

for $t > T$ and $i \geq 3$, $1 \leq j \leq m_i$.

For $i = 1$, it is known that $\nu_4^{(1)} < \nu_2^{(1)} < \nu_3^{(1)} = -1 < 0 < \nu_1^{(1)} = 1$. The fact that $z_j^1(t) \to 0$ as $t \to \infty$ implies that $z_j^1(t)$ can be written in the form

$$\begin{aligned} z_j^1(t) =& A_{j,2}^1 e^{\nu_2^{(1)}t} + A_{j,3}^1 e^{\nu_3^{(1)}t} + A_{j,4}^1 e^{\nu_4^{(1)}t} \\ & - B_1^1 \int_t^\infty e^{\nu_1^{(1)}(t-\tau)} f_j^1(\tau)d\tau + B_3^1 \int_T^t e^{\nu_3^{(1)}(t-\tau)} f_j^1(\tau)d\tau \\ & + B_2^1 \int_T^t e^{\nu_2^{(1)}(t-\tau)} f_j^1(\tau)d\tau + B_4^1 \int_T^t e^{\nu_4^{(1)}(t-\tau)} f_j^1(\tau)d\tau. \end{aligned}$$

Arguments similar to those in the proof of (6.19) imply that

$$|z_j^1(t)| = O(e^{-(t-T)}) \tag{6.20}$$

for $t > T$ and $1 \le j \le m_1$.

For $i = 2$, it is known that $\nu_4^{(2)} < \nu_2^{(2)} < -1 < \nu_3^{(2)} = 0 < \nu_1^{(2)}$. The fact that $z_j^2(t) \to 0$ as $t \to \infty$ implies that $z_j^2(t)$ can be written in the form

$$\begin{aligned} z_j^2(t) &= A_{j,2}^2 e^{\nu_2^{(2)}t} + A_{j,4}^2 e^{\nu_4^{(2)}t} \\ & \quad -B_1^2 \int_t^\infty e^{\nu_1^{(2)}(t-\tau)} f_j^2(\tau)d\tau - B_3^2 \int_t^\infty f_j^2(\tau)d\tau \\ & \quad +B_2^2 \int_T^t e^{\nu_2^{(2)}(t-\tau)} f_j^2(\tau)d\tau + B_4^2 \int_T^t e^{\nu_4^{(2)}(t-\tau)} f_j^2(\tau)d\tau. \end{aligned}$$

Similarly, noting (6.10) we have

$$|z_j^2(t)| = O(e^{\nu_2^{(2)}(t-T)}) \; (= O(e^{-N(t-T)})) \tag{6.21}$$

for $t > T$ and $1 \le j \le m_2$. Therefore, if we set $Z(t) = W(s)$ with $t = -\ln s$, arguments similar to those in the proof of (3.31) imply that

$$Z(t) = O(e^{-t}) \tag{6.22}$$

for $t > T_*$ and $T^* = 10T$.

Let $s_0 = e^{-T_*}$. We see from (6.22) that there exists $C > 0$ such that for $0 < s < s_0$,

$$W(s) \le Cs. \tag{6.23}$$

This completes the proof of this proposition. □

Lemma 6.4. *Let v be a solution of (6.7). Then there exist constant $0 < s_0 < \frac{1}{10}$ and $M = M(v) > 0$ such that for $N = 3$ and $1 < p < 3$; $N \ge 4$ and $p > 1$; $s \in (0, s_0)$,*

(6.24)

$$\begin{cases} |\overline{v}(s)| \le Ms, & |\overline{v}'(s)| \le M, & |\overline{v}''(s)| \le Ms^{-1} & \text{for } p > \frac{3}{2}, \\ |\overline{v}(s)| \le Ms^{1-\epsilon}, & |\overline{v}'(s)| \le Ms^{-\epsilon}, & |\overline{v}''(s)| \le Ms^{-1-\epsilon} & \text{for } p = \frac{3}{2}, \\ |\overline{v}(s)| \le Ms^{2(p-1)}, & |\overline{v}'(s)| \le Ms^{2(p-1)-1}, & |\overline{v}''(s)| \le Ms^{2(p-1)-2} & \text{for } 1 < p < \frac{3}{2}, \end{cases}$$

where $0 < \epsilon < \frac{1}{100}$ *is sufficiently small, and*

$$\int_{S^{N-1}} v^2(s,\theta)d\theta \le \begin{cases} Ms^2 & \text{for } p > \frac{3}{2}, \\ Ms^{2(1-\epsilon)} & \text{for } p = \frac{3}{2}, \\ Ms^{4(p-1)} & \text{for } 1 < p < \frac{3}{2}. \end{cases} \tag{6.25}$$

Proof. We recall that $\overline{v}(s)$ satisfies the equation

$$\begin{aligned} &\overline{v}_s^{(4)} - 2(N-3)s^{-1}\overline{v}_{sss} + (N-1)(N-3)s^{-2}\overline{v}_{ss} - (N-1)(N-3)s^{-3}\overline{v}_s \\ &\quad - s^{-4}h(\overline{v}) = s^{-4}[\overline{h(v)} - h(\overline{v})], \end{aligned}$$

where $h(v) = s^{2(p-1)}(v+D)^{-p}$ and

$$|\overline{h(v)} - h(\bar{v})| \le \frac{1}{\omega_{N-1}} \int_{S^{N-1}} |h(v) - h(\bar{v})| d\theta \le o(W(s)) = o(s).$$

Let $\overline{z}(t) = \overline{v}(s)$, $t = -\ln s$. Then $\overline{z}(t)$ satisfies the equation

$$\overline{z}^{(4)} + 2N(\overline{z})_{ttt} + (N^2+2N-4)(\overline{z})_{tt} + 2N(N-2)(\overline{z})_t = h(\bar{z}) + o(e^{-t}), \tag{6.26}$$

Note that $h(\bar{z}) = s^{2(p-1)}(\bar{z}+D)^{-p} = O(e^{-(2(p-1))t})$ for t near ∞. The corresponding polynomial of (6.26) is

$$\nu^4 + 2N\nu^3 + (N^2+2N-4)\nu^2 + 2N(N-2)\nu = 0. \tag{6.27}$$

The four roots of (6.27) are:

$$\nu_1^{(0)} = 0, \ \ \nu_2^{(0)} = 2-N, \ \ \nu_3^{(0)} = -2, \ \ \nu_4^{(0)} = -N. \tag{6.28}$$

The ODE theory implies

$$\begin{aligned} \overline{z}(t) &= M_1 + A_2e^{-2t} + A_3e^{-(N-2)t} + A_4e^{-Nt} \\ &\quad - B_1\int_t^\infty \overline{f}(\tau)d\tau + B_2\int_T^t e^{-2(t-\tau)}\overline{f}(\tau)d\tau \\ &\quad + B_3\int_T^t e^{-(N-2)(t-\tau)}\overline{f}(\tau)d\tau + B_4\int_T^t e^{-N(t-\tau)}\overline{f}(\tau)d\tau, \end{aligned} \tag{6.29}$$

where $\overline{f}(t) = h(\overline{z}(t)) + o(e^{-t})$. The fact that $\overline{z}(t) \to 0$ as $t \to \infty$ implies that $M_1 = 0$. Notice that $\overline{f}(t) = O(e^{-\min\{2(p-1),1\}t})$, we see from (6.29) that there exists $T \gg 1$ such that for $t > T$,

$$|\overline{z}(t)| = \begin{cases} O(e^{-t}), & \text{for } p > \frac{3}{2}, \\ O(e^{-(1-\epsilon)t}), & \text{for } p = \frac{3}{2} \text{ and sufficiently small } 0 < \epsilon < \frac{1}{100}, \\ O(e^{-2(p-1)t}), & \text{for } 1 < p < \frac{3}{2}. \end{cases} \tag{6.30}$$

(Note that when $N = 3$ and $p = \frac{3}{2}$, the term $|\int_T^t e^{-(N-2)(t-\tau)}O(e^{-2(p-1)\tau})d\tau| \le O(e^{-t}\ln t)$.) This implies that $(6.24)_1$ holds. Differentiating (6.29) with respect to t once and twice respectively and noticing $\overline{v}'(s) = -\overline{z}'(t)e^t$ and $\overline{v}''(s) = [\overline{z}''(t) + \overline{z}'(t)]e^{2t}$, we easily see that $(6.24)_2$ and $(6.24)_3$ hold. Note that $v(s,\theta) = w(s,\theta) + \overline{v}(s)$, we obtain (6.25). This completes the proof of this lemma. □

Lemma 6.5. *Let $\tau \geq 0$ be an integer and let v be a solution of* (6.7). *Then there exist $0 < s_0 < \frac{1}{10}$ and $M = M(v, \tau, s_0) > 0$ such that for $s \in (0, s_0)$,*

$$\max_{|y|=s} |D^\tau v(y)| \leq \begin{cases} Ms^{1-\tau} & \text{for } p > \frac{3}{2}, \\ Ms^{1-\epsilon-\tau} & \text{for } p = \frac{3}{2}, \\ Ms^{2(p-1)-\tau} & \text{for } 1 < p < \frac{3}{2}. \end{cases} \tag{6.31}$$

Proof. Similar to the proof of Proposition 4.2. □

Let

$$\tilde{w}(s,\theta) = \frac{w(s,\theta)}{s}.$$

Then $\tilde{w}(s,\theta)$ satisfies the equation

$$\begin{aligned} &\partial_s^4 \tilde{w} - 2(N-5)s^{-1}\partial_s^3\tilde{w} + (N-3)(N-7)s^{-2}\partial_s^2\tilde{w} + (N-1)(N-3)s^{-3}\partial_s\tilde{w} \\ &\quad -(N-1)(N-3)s^{-4}\tilde{w} + 2s^{-4}\Delta_\theta\tilde{w} - 2(N-3)s^{-3}\Delta_\theta(\partial_s\tilde{w}) \\ &\quad +2s^{-2}\Delta_\theta(\partial_s^2\tilde{w}) + s^{-4}\Delta_\theta^2\tilde{w} - s^{-4}g(\tilde{w}) = 0, \end{aligned} \tag{6.32}$$

where

$$g(\tilde{w}) = -ps^{2(p-1)}[(\xi(s,\theta)+D)^{-(p+1)}\tilde{w} - \overline{(\xi(s,\theta)+D)^{-(p+1)}\tilde{w}}],$$

where $\xi(s,\theta)$ is between $v(s,\theta)$ and $\bar{v}(s,\theta)$. We also have

$$\tilde{w}(s,\theta) = \sum_{i=1}^{\infty}\sum_{j=1}^{m_i} \tilde{w}_j^i(s)Q_j^i(\theta), \quad \tilde{w}_j^i(s) = \frac{w_j^i(s)}{s}.$$

Then, $\tilde{w}_j^i(s)$ satisfies the equation:

$$\begin{aligned} &(\tilde{w}_j^i)^{(4)} - 2(N-5)s^{-1}(\tilde{w}_j^i)_{sss} + [(N-3)(N-7) - 2\lambda_i]s^{-2}(\tilde{w}_j^i)_{ss} \\ &\quad +[(N-1)(N-3) + (2N-6)\lambda_i]s^{-3}(\tilde{w}_j^i)_s \\ &\quad +[-(N-1)(N-3) - 2\lambda_i + \lambda_i^2]s^{-4}\tilde{w}_j^i = s^{-4}\hat{g}_j^i(s), \end{aligned} \tag{6.33}$$

where $\hat{g}_j^i(s) = \int_{S^{N-1}} g(\tilde{w})Q_j^i(\theta)d\theta$. We also know that

$$|\hat{g}_j^i(s)| \leq O(s^{2(p-1)})\tilde{W}(s),$$

where $\tilde{W}(s) = (\int_{S^{N-1}} |\tilde{w}(s,\theta)|^2 d\theta)^{1/2}$.

Let $\tilde{z}_j^i(t) = \tilde{w}_j^i(s)$, $t = -\ln s$, $\tilde{Z}(t) = \tilde{W}(s)$. We see that $\tilde{z}_j^i(t)$ satisfies the equation (for t near ∞):

$$\begin{aligned} &(\tilde{z}_j^i)^{(4)} + 2(N-2)(\tilde{z}_j^i)_{ttt} + (N^2 - 4N + 2 - 2\lambda_i)(\tilde{z}_j^i)_{tt} \\ &\quad -2[N-2+(N-2)\lambda_i](\tilde{z}_j^i)_t + [-(N-1)(N-3) - 2\lambda_i + \lambda_i^2]\tilde{z}_j^i \\ &\quad = \tilde{g}_j^i(t), \end{aligned} \tag{6.34}$$

where $\tilde{g}_j^i(t) = \hat{g}_j^i(s)$ and $|\tilde{g}_j^i(t)| \leq O(e^{-2(p-1)t})\tilde{Z}(t)$. Since $\tilde{Z}(t) = e^t Z(t) = O(1)$ (see Proposition 6.3), we see that $|\tilde{g}_j^i(t)| = O(e^{-2(p-1)t})$. The corresponding polynomial of (6.34) is

$$\begin{aligned} &\nu^4 + 2(N-2)\nu^3 + (N^2 - 4N + 2 - 2\lambda_i)\nu^2 - 2[N-2+(N-2)\lambda_i]\nu \\ &\quad +[-(N-1)(N-3) - 2\lambda_i + \lambda_i^2] = 0, \end{aligned} \tag{6.35}$$

which has four roots:

$$\tilde{\nu}_k^{(i)} = \nu_k^{(i)} + 1, \quad k = 1, 2, 3, 4,$$

i.e.,

$$\tilde{\nu}_1^{(i)} = i + 1, \;\; \tilde{\nu}_2^{(i)} = 3 - N - i, \;\; \tilde{\nu}_3^{(i)} = i - 1, \;\; \tilde{\nu}_4^{(i)} = 1 - N - i. \tag{6.36}$$

Since for each (i, j), $|\tilde{z}_j^i(t)|$ is bounded, arguments similar to those in the proof of Proposition 6.3 imply that $\Sigma_{i=2}^{\infty}\Sigma_{j=1}^{m_i}|\tilde{z}_j^i(t)| \to 0$ as $t \to \infty$. We see that, for $i = 1$, the four roots are given by

$$\tilde{\nu}_1^{(1)} = 2, \;\; \tilde{\nu}_2^{(1)} = 2 - N, \;\; \tilde{\nu}_3^{(1)} = 0, \;\; \tilde{\nu}_4^{(1)} = -N.$$

Thus

$$\tilde{\nu}_4^{(1)} < \tilde{\nu}_2^{(1)} < -1 < 0 = \tilde{\nu}_3^{(1)} < \tilde{\nu}_1^{(1)}$$

and

$$\begin{aligned} \tilde{z}_j^1(t) \;\; = \;\; & C + A_{j,2}^1 e^{\tilde{\nu}_2^{(1)} t} + A_{j,4}^1 e^{\tilde{\nu}_4^{(1)} t} \\ & - B_1^1 \int_t^\infty e^{\tilde{\nu}_1^{(1)}(t-\tau)} O(e^{-2(p-1)t}) d\tau - B_3^1 \int_t^\infty O(e^{-2(p-1)t}) d\tau \\ & + B_2^1 \int_T^t e^{\tilde{\nu}_2^{(1)}(t-\tau)} O(e^{-2(p-1)t}) d\tau + B_4^1 \int_T^t e^{\tilde{\nu}_4^{(1)}(t-\tau)} O(e^{-2(p-1)t}) d\tau. \end{aligned}$$

This implies that $\tilde{z}_j^1(t) \to A_j$ (A_j is a constant, maybe 0) as $t \to \infty$. Since $Q_1^1(\theta), \ldots, Q_{m_1}^1(\theta)$ are the eigenfunctions corresponding to the eigenvalue $\lambda_1 = N-1$, and thus we see that

$$\lim_{s \to 0} \tilde{w}(s, \theta) = V(\theta). \tag{6.37}$$

In conclusion, we have the following theorem.

Theorem 6.6. *Let v be a solution of* (6.7) *and $\tilde{w}$ be given in* (6.32). *Then we have*
(i) $v(y) = \overline{v}(s) + s\tilde{w}(s, \theta)$ satisfies

$$\begin{cases} |\overline{v}(s)| \le Ms, & |\overline{v}'(s)| \le M, & |\overline{v}''(s)| \le Ms^{-1} & \text{for } p > \frac{3}{2}, \\ |\overline{v}(s)| \le Ms^{1-\epsilon}, & |\overline{v}'(s)| \le Ms^{-\epsilon}, & |\overline{v}''(s)| \le Ms^{-1-\epsilon} & \text{for } p = \frac{3}{2}, \\ |\overline{v}(s)| \le Ms^{2(p-1)}, & |\overline{v}'(s)| \le Ms^{2(p-1)-1}, & |\overline{v}''(s)| \le Ms^{2(p-1)-2} & \text{for } 1 < p < \frac{3}{2}. \end{cases}$$

(ii) For any non-negative integers τ and τ_1, there exists $M = M(v, \tau, \tau_1) > 0$ such that

$$|s^\tau D_\theta^{\tau_1} D_s^\tau \tilde{w}(y)| \le M, \;\; y \in B_{s_0}, \;\; y \ne 0, \tag{6.38}$$

where $B_{s_0} = \{y \in \mathbb{R}^N : \; |y| < s_0\}$. Moreover, $\tilde{w}$ satisfies

$$\lim_{s \to 0} \tilde{w}(s, \theta) = V(\theta), \tag{6.39}$$

uniformly in $C^\tau(S^{N-1})$, where $V(\theta)$ is given by (4.50).

We obtain from Theorem 6.6 the asymptotic expansion of $u(x)$ near $|x| = \infty$.

Theorem 6.7. *Let $N = 3$ and $1 < p < 3$; $N \geq 4$ and $p > 1$; u be a solution of* (1.1) *satisfying* (1.10). *Then u admits the expansion:*

$$u(x) = r^2\Big[D + \xi(r) + \frac{\eta(r,\theta)}{r}\Big], \tag{6.40}$$

$$w(x) := -\Delta u(x) = -2ND + \xi_1(r) + \frac{\eta_1(r,\theta)}{r} \tag{6.41}$$

where

$$\xi_1(r) = -[r^2\xi'' + (N+3)r\xi' + 2N\xi],$$
$$\eta_1(r,\theta) = -[r^2\eta_{rr} + (N+1)r\eta_r + (N-1)\eta + \Delta_\theta \eta].$$

Moreover, the following properties are satisfied:

(i) $\xi(r) = r^{-2}\overline{u}(r) - D$ and there exist R_0 $(:= s_0^{-1}) > 0$ and a constant $M = M(u) > 0$ such that, for $r > R_0$,

$$\begin{cases} |\xi(r)| \leq Mr^{-1}, & |\xi'(r)| \leq Mr^{-2}, & |\xi''(r)| \leq Mr^{-3} & \textit{for } p > \frac{3}{2}, \\ |\xi(r)| \leq Mr^{-(1-\epsilon)}, & |\xi'(r)| \leq Mr^{-(2-\epsilon)}, & |\xi''(r)| \leq Mr^{-(3-\epsilon)} & \textit{for } p = \frac{3}{2}, \\ |\xi(r)| \leq Mr^{-2(p-1)}, & |\xi'(r)| \leq Mr^{-2p+1}, & |\xi''(r)| \leq Mr^{-2p} & \textit{for } 1 < p < \frac{3}{2}, \end{cases} \tag{6.42}$$

$$|\xi_1(r)| \leq \begin{cases} Mr^{-1} & \textit{for } p > \frac{3}{2}, \\ Mr^{-(1-\epsilon)} & \textit{for } p = \frac{3}{2}, \\ Mr^{-2(p-1)} & \textit{for } 1 < p < \frac{3}{2}. \end{cases} \tag{6.43}$$

(ii) Let τ and τ_1 be two non-negative integers. Then there exists a positive constant $M := M(u, \tau, \tau_1)$ such that, for $r > R_0$,

$$|r^\tau D_\theta^{\tau_1} D_r^\tau \eta(r,\theta)| \leq M, \tag{6.44}$$

$$|\eta_1(r,\theta)| \leq M. \tag{6.45}$$

(iii) Let τ be a non-negative integer. Then $\eta(r,\theta)$ tends to $V(\theta)$ uniformly in $C^\tau(S^{N-1})$ as $r \to \infty$, where $V(\theta)$ is given by (4.50).

Completion of the proof of Theorem 1.4

We first write (1.1) to a system of equations:

$$\begin{cases} -\Delta u = v, & \text{in } \mathbb{R}^N, \\ -\Delta v = -u^{-p}, & \text{in } \mathbb{R}^N. \end{cases} \tag{6.46}$$

We now start the procedure of moving-plane. As a consequence of the expansions of $u(x)$ in Theorem 6.7, we have the following lemma.

Lemma 6.8. *Let $N = 3$ and $1 < p < 3$; $N \geq 4$ and $p > 1$; u be a solution of* (1.1) *satisfying* (1.10). *Then,*

(i) If $\gamma^j \in \mathbb{R} \to \gamma$ and $\{x^j\} \to \infty$ with $x_1^j < \gamma^j$, then

$$\lim_{j\to\infty} \frac{1}{\gamma^j - x_1^j}\Big[u(x^j) - u((x^j)^\gamma)\Big] = -4D\gamma - 2(x_0)_1, \tag{6.47}$$

where $(x_0)_1$ is the first component of x_0 given in (4.50).

(ii) Define

$$\gamma_0 = -\frac{(x_0)_1}{2D}. \tag{6.48}$$

Then there exists a constant $M = M(u) > 0$ such that

$$\frac{\partial u}{\partial x_1} \geq 0 \ \text{ if } x_1 \geq \gamma_0 + 1 \text{ and } |x| \geq M. \tag{6.49}$$

Proof. To prove (6.47), without loss of generality, we assume that

$$\lim_{j\to\infty} \frac{x_j}{|x_j|} = \overline{\theta} \in S^{N-1}.$$

For simplicity, we also assume that $\gamma^j = \gamma$, $j = 1, 2, \ldots$ since the following arguments work equally well for the sequence $\{\gamma^j\}$. Using the the expansion of u in (6.40), we have

$$\begin{aligned}
\frac{1}{\gamma - x_1^j}\Big[u(x^j) - u((x^j)^\gamma)\Big] &= \frac{1}{\gamma - x_1^j}\Big[D\Big(|x^j|^2 - |(x^j)^\gamma|^2\Big)\Big] \\
&\quad + \frac{1}{\gamma - x_1^j}\Big[|x^j|^2\xi(|x^j|) - |(x^j)^\gamma|^2\xi(|(x^j)^\gamma|)\Big] \\
&\quad + \frac{1}{\gamma - x_1^j}\Big[|x^j|\eta(|x^j|, \theta^j) - |(x^j)^\gamma|\eta(|(x^j)^\gamma|, (\theta^j)^\gamma)\Big] \\
&= I + II + III.
\end{aligned}$$

We have

$$D(|x^j|^2 - |(x^j)^\gamma|^2) = -4D\gamma(\gamma - x_1^j)$$

and hence

$$I = -4D\gamma.$$

We also have that there is β_j between $|x^j|$ and $|(x^j)^\gamma|$ such that

$$|x^j|^2\xi(|x^j|) - |(x^j)^\gamma|^2\xi(|(x^j)^\gamma|) = \Big[2\beta_j\xi(\beta_j) + \beta_j^2\xi'(\beta_j)\Big]\frac{-4\gamma(\gamma - x_1^j)}{|x^j| + |(x^j)^\gamma|},$$

and in turn

$$\begin{aligned}
II &= \frac{1}{\gamma - x_1^j}\Big[2\beta_j\xi(\beta_j) + \beta_j^2\xi'(\beta_j)\Big]\frac{-4\gamma(\gamma - x_1^j)}{|x^j| + |(x^j)^\gamma|} \\
&= \begin{cases} O(|x_j|^{-1}) \to 0, & \text{for } p > \frac{3}{2}, \\ O(|x_j|^{-(1-\epsilon)}) \to 0, & \text{for } p = \frac{3}{2}, \\ O(|x_j|^{-2(p-1)}) \to 0, & \text{for } 1 < p < \frac{3}{2} \end{cases}
\end{aligned}$$

as $j \to \infty$, since $\frac{|(x^j)^\gamma|}{|x^j|} \to 1$ as $j \to \infty$. Here we have used the estimates of $\xi(r)$ and $\xi'(r)$ in (6.42). We now write

$$\begin{aligned} III &= \frac{\eta(|(x^j)^\gamma|, (\theta^j)^\gamma)}{\gamma - x_1^j}\Big[|x^j| - |(x^j)^\gamma|\Big] \\ &\quad + \frac{|x^j|}{\gamma - x_1^j}\Big[\eta(|x^j|, (\theta^j)^\gamma) - \eta(|(x^j)^\gamma|, (\theta^j)^\gamma)\Big] \\ &\quad + \frac{|x^j|}{\gamma - x_1^j}\Big[\eta(|x^j|, \theta^j) - \eta(|x^j|, (\theta^j)^\gamma)\Big] \\ &= III_1 + III_2 + III_3. \end{aligned}$$

As before, by (6.44) and arguments similar to those in the proof of (8.11) in Lemma 5.2 of [25], we obtain that $III_1 = O(|x^j|^{-1}) \to 0$ as $j \to \infty$, $III_2 = O(|x^j|^{-1}) \to 0$ as $j \to \infty$ and $III_3 \to -2(x_0)_1$ as $j \to \infty$. These imply that (6.47) holds.

To prove (6.49), we use (6.47). Suppose that (6.49) is false. Then there exists a sequence $\{x^j\} \to \infty$ such that

$$\frac{\partial u}{\partial x_1}(x^j) < 0, \quad x_1^j \geq \gamma_0 + 1, \quad \forall j \in \mathbb{N}.$$

It follows that there exists a sequence of bounded positive numbers $\{d_j\}$ such that

$$u(x^j) > u(x_{d_j}), \quad x_{d_j} = x^j + (2d_j, 0, \ldots, 0), \quad \forall j \in \mathbb{N}.$$

Let

$$\gamma^j = x_1^j + d_j > x_1^j.$$

We have

$$\text{(6.50)} \qquad \frac{1}{\gamma^j - x_1^j}\Big[u(x^j) - u((x^j)^\gamma)\Big] > 0, \quad \forall j \in \mathbb{N}.$$

There are two possibilities:

$$\liminf_{j\to\infty} \gamma^j < \infty, \quad \lim_{j\to\infty} \gamma^j = \infty.$$

If the first case occurs, we choose a convergent subsequence of $\{\gamma^j\}$ (still denoted by $\{\gamma^j\}$) with the limit $\gamma \geq \gamma_0 + 1$ and apply (6.47) and (6.48) to obtain

$$\lim_{j\to\infty} \frac{1}{\gamma^j - x_1^j}\Big[u(x^j) - u((x^j)^\gamma)\Big] = -4D\gamma - 2(x_0)_1 \leq -4D < 0.$$

This contradicts (6.50). We can derive a contradiction for the second case similarly. The proof is a little variant of the proof of Lemma 8.2 of [25]. Thus, neither the first nor the second case can occur and (6.49) holds. This completes the proof of this lemma. □

To complete the proof of the sufficiency, we use moving-plane arguments of the system of equations (6.46). The proof is exactly the same as the proof of Theorem 1.1. We omit the details here. □

Remark 6.9. We conjecture that the following conclusion holds: *If* $u \in C^4(\mathbb{R}^N)$ *is an entire solution of* (1.1) *with* $N = 3$ *and* $1 < p < 3$ *or* $N \geq 4$ *and* $p > 1$*, then* u *is the minimal radial entire solution of* (1.1) *about some* $x_* \in \mathbb{R}^N$*, if and only if*

$$|x|^{-2}u(x) \to 0 \text{ as } |x| \to \infty. \tag{6.51}$$

This conjecture implies that if u is an entire solution of (1.1) and (6.51) holds for u, then u must have the exact asymptotic behavior at ∞:

$$|x|^{-\alpha}u(x) \to L \text{ as } |x| \to \infty,$$

where α and L are given in (1.2).

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