

THE MINKOWSKI FORMULA AND THE QUASI-LOCAL MASS

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ABSTRACT. In this article, we estimate the quasi-local energy with reference to the Minkowski spacetime [16, 17], the anti-de Sitter spacetime [4], or the Schwarzschild spacetime [3]. In each case, the reference spacetime admits a conformal Killing–Yano 2-form which facilitates the application of the Minkowski formula in [15] to estimate the quasi-local energy. As a consequence of the positive mass theorems in [9, 13] and the above estimate, we obtain rigidity theorems which characterize the Minkowski spacetime and the hyperbolic space.

1. INTRODUCTION

In this article, we estimate the quasi-local mass with reference to the Minkowski spacetime [16, 17], the anti-de Sitter spacetime [4], or the Schwarzschild spacetime [3]. In each case, the reference spacetime admits a conformal Killing–Yano 2-form. As a result of this “hidden symmetry”, the classical Minkowski formula is extended to spacelike codimension-two submanifolds in these reference spacetimes by Wang, Wang and Zhang in [15]. Our estimate of the quasi-local energy is based on the $k = 2$ Minkowski formula for surfaces in these reference spacetimes.

In the classical Minkowski formula [11], the mean curvature H and the Gauss curvature K of a surface Σ in \mathbb{R}^3 are related as follows:

$$\int Hd\Sigma = \int K(X \cdot e_3)d\Sigma,$$

where X is the position vector of \mathbb{R}^3 and e_3 is the outward unit normal of the surface Σ . A major application of the classical Minkowski formula is the rigidity of isometric convex surfaces in \mathbb{R}^3 , namely, two convex surfaces Σ_1 and Σ_2 in \mathbb{R}^3 with the same induced metric are the same up to an isometry of \mathbb{R}^3 [5]. The Minkowski formula is used to evaluate the integral of the difference of the mean curvatures.

Let H_1 and H_2 be the mean curvatures, and let h_1 and h_2 be the second fundamental forms of the surfaces Σ_1 and Σ_2 given by the two embeddings X_1, X_2 into \mathbb{R}^3 , respectively.

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It follows from the Minkowski formula that

$$\int (H_1 - H_2) d\Sigma = \int \det(h_1 - h_2)(X_1 \cdot e_3) d\Sigma.$$

Using the Gauss equations of the surfaces, we can conclude that

$$\det(h_1 - h_2) \leq 0.$$

Reversing the order of two surfaces, we obtain $h_1 = h_2$ and thus the two embeddings differ by an isometry of \mathbb{R}^3 . In this article, we use the Minkowski formula in a similar manner to evaluate the quasi-local energy. For the Minkowski reference, this is done in Theorem 3.2. For the anti-de Sitter reference, this is done in Theorem 5.2 and for the Schwarzschild reference, this is done in Theorem 6.2.

Using Theorem 3.2 and Theorem 5.2, we derive upper bounds of the quasi-local energy in terms of the curvature tensor of the physical spacetime. See Theorem 4.2 and Corollary 4.4 for the Minkowski reference and Corollary 5.3 for the anti-de Sitter reference. Combining the upper bound of the quasi-local energy with the positive mass theorems [9, 13], we obtain rigidity theorems which characterize the Minkowski space, the Euclidean 3-space and the hyperbolic space. See Theorem 4.3, Corollary 4.5 and Theorem 5.4, respectively.

2. KILLING–YANO 2-FORM AND THE MINKOWSKI FORMULA

In this section, we review the Minkowski formula of surfaces in 4-dimensional spacetimes admitting a Killing–Yano 2-form [15]. First we recall the definition of the Killing–Yano 2-form.

Definition 2.1. *Let Q be a 2-form in an $(n+1)$ -dimensional pseudo-Riemannian manifold $(\mathfrak{N}, \langle, \rangle)$ with Levi-Civita connection D . Q is called a conformal Killing–Yano 2-form if*

$$(2.1) \quad D_X Q(Y, Z) + D_Y Q(X, Z) = \frac{1}{n} (2\langle X, Y \rangle \langle \xi, Z \rangle - \langle X, Z \rangle \langle \xi, Y \rangle - \langle Y, Z \rangle \langle \xi, X \rangle),$$

where $\xi = \operatorname{div} Q$.

In [15], the Minkowski formulae for higher order mixed mean curvatures are derived for submanifolds in a spacetime admitting a Killing–Yano 2-form. In particular, for a spherical symmetric spacetime \mathfrak{N} with metric

$$-f(r)^2 dt^2 + \frac{dr^2}{f^2(r)} + r^2 dS^2,$$

the 2-form

$$Q = r dr \wedge dt$$

is a conformal Killing–Yano 2-form with

$$\operatorname{div} Q = -3 \frac{\partial}{\partial t}.$$

Let Σ be a 2-surface in \mathfrak{N} . Let $\{e_3, e_4\}$ be a frame of the normal bundle of Σ in \mathfrak{N} . Let h_3 and h_4 be the second fundamental form of Σ in \mathfrak{N} in the direction of e_3 and e_4 , respectively and let

$$\alpha_{e_3}(\cdot) = \langle D_{(\cdot)}e_3, e_4 \rangle$$

be the connection 1-form of the normal bundle determined by the frame $\{e_3, e_4\}$.

Let H_0 be the mean curvature vector of Σ in \mathfrak{N} and J_0 be the reflection of H_0 through the light cone in the normal bundle of Σ . The $(r, s) = (2, 0)$ Minkowski formula [15, Theorem 4.3] is

$$\begin{aligned} & - \int_{\Sigma} \langle J_0, \frac{\partial}{\partial t} \rangle d\Sigma \\ &= \int_{\Sigma} \left\{ 2(\det(h_3) - \det(h_4))Q_{34} + (R^{ab}{}_{a3}Q_{b4} - R^{ab}{}_{a4}Q_{b3}) + [R^{ab}{}_{43} - (d\alpha_{e_3})^{ab}]Q_{ab} \right\} d\Sigma, \end{aligned}$$

where R denote the curvature tensor of the spacetime \mathfrak{N} . This generalized the $k = 2$ Minkowski formula for surfaces in \mathbb{R}^3 .

In particular, for surfaces in the Minkowski space, the curvature tensor vanish and the formula reduces to

$$- \int_{\Sigma} \langle J_0, \frac{\partial}{\partial t} \rangle d\Sigma = \int_{\Sigma} \{ 2(\det(h_3) - \det(h_4))Q(e_3, e_4) - (d\alpha_{e_3})^{ab}Q_{ab} \} d\Sigma.$$

3. THE MINKOWSKI IDENTITY AND THE QUASI-LOCAL ENERGY

In this section, we rewrite the Wang–Yau quasi-local energy [16, 17] using the Minkowski formula for surfaces in $\mathbb{R}^{3,1}$.

Let N be our physical spacetime. Given a spacelike 2-surface Σ in N , let $\{e'_3, e'_4\}$ be a frame of the normal bundle of Σ in N . Let h'_3 and h'_4 be the second fundamental forms of Σ in N in the direction of e'_3 and e'_4 , respectively, and let $\alpha_{e'_3}$ be the connection 1-form

$$\alpha_{e'_3}(\cdot) = \langle \nabla_{(\cdot)}^N e'_3, e'_4 \rangle.$$

Let X be an isometric embedding of Σ into $\mathbb{R}^{3,1}$. Let $\{e_3, e_4\}$ be a frame of the normal bundle of $X(\Sigma)$ in $\mathbb{R}^{3,1}$ and h_3, h_4 and α_{e_3} be the corresponding second fundamental forms and the connection 1-form.

Theorem 3.1. *Given a spacelike 2-surface Σ in N and a frame $\{e'_3, e'_4\}$ of the normal bundle, let X be an isometric embedding of Σ into $\mathbb{R}^{3,1}$. Suppose there is a frame $\{e_3, e_4\}$ of the normal bundle of $X(\Sigma)$ in $\mathbb{R}^{3,1}$ such that*

$$\alpha_{e'_3} = \alpha_{e_3}.$$

Then we have

$$\begin{aligned}
(3.1) \quad & \int \left\{ -\left\langle \frac{\partial}{\partial t}, e_4 \right\rangle (tr h_3 - tr h'_3) + \left\langle \frac{\partial}{\partial t}, e_3 \right\rangle (tr h_4 - tr h'_4) \right\} d\Sigma \\
& = \int [2det(h_3) - 2det(h_4) - tr h_3 tr h'_3 + h_3 \cdot h'_3 + tr h_4 tr h'_4 - h_4 \cdot h'_4] Q_{34} d\Sigma \\
& \quad + \int \{ R^{ab}{}_{a4} Q_{b3} - R^{ab}{}_{a3} Q_{b4} - Q_{bc} \sigma^{cd} [(h_3)_{da} h_4'^{ab} - (h_4)_{da} h_3'^{ab}] \} d\Sigma,
\end{aligned}$$

where R is the curvature tensor for the spacetime N .

Proof. We consider the following two divergence quantities on Σ :

$$\nabla_a \left([tr h_3 \sigma^{ab} - h_3^{ab}] Q_{b4} - [tr(h_4) \sigma^{ab} - h_4^{ab}] Q_{b3} \right)$$

and

$$\nabla_a \left([tr h'_3 \sigma^{ab} - h_3'^{ab}] Q_{b4} - [tr(h'_4) \sigma^{ab} - h_4'^{ab}] Q_{b3} \right).$$

The first divergence quantity is exactly the one considered in [15, Theorem 4.3]. It gives

$$(3.2) \quad \int \left\{ -\left\langle \frac{\partial}{\partial t}, e_4 \right\rangle (tr h_3) + \left\langle \frac{\partial}{\partial t}, e_3 \right\rangle (tr h_4) \right\} d\Sigma = \int [2[det(h_3) - det(h_4)] - (d\zeta)^{ab} Q_{ab}] d\Sigma.$$

where $\zeta = \alpha_{e'_3} = \alpha_{e_3}$.

For the second divergence quantity, following the proof of [15, Theorem 4.3] (see also [14, Theorem 3.3]), we compute

$$\begin{aligned}
(3.3) \quad & \nabla_a (tr(h'_3) \sigma^{ab} - h_3'^{ab}) = -R^{ab}{}_{a3} - \zeta^b tr h'_4 + \zeta_a h_4'^{ab} \\
& \nabla_a (tr(h'_4) \sigma^{ab} - h_4'^{ab}) = -R^{ab}{}_{a4} - \zeta^b tr h'_3 + \zeta_a h_3'^{ab}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
(3.4) \quad & \nabla_a Q_{b4} = (D_a Q)_{b4} - (h_3)_{ab} Q_{34} + Q_{bc} \sigma^{cd} (h_4)_{da} - Q_{b3} \zeta_a \\
& \nabla_a Q_{b3} = (D_a Q)_{b3} + (h_4)_{ab} Q_{43} + Q_{bc} \sigma^{cd} (h_3)_{da} - Q_{b4} \zeta_a.
\end{aligned}$$

Putting (3.3) and (3.4) together, we get

$$\begin{aligned}
(3.5) \quad & \nabla_a \left((tr(h'_3) \sigma^{ab} - h_3'^{ab}) Q_{b4} - (tr(h'_4) \sigma^{ab} - h_4'^{ab}) Q_{b3} \right) \\
& = R^{ab}{}_{a3} Q_{b4} - R^{ab}{}_{a4} Q_{b3} + (tr(h'_3) \sigma^{ab} - h_3'^{ab}) \left((D_a Q)_{b4} - (h_3)_{ab} Q_{34} + Q_{bc} \sigma^{cd} (h_4)_{da} \right) \\
& \quad - (tr(h'_4) \sigma^{ab} - h_4'^{ab}) \left((D_a Q)_{b3} + (h_4)_{ab} Q_{43} + Q_{bc} \sigma^{cd} (h_3)_{da} \right).
\end{aligned}$$

From the definition of conformal Killing-Yano 2-forms, we have

$$\begin{aligned}
& (tr(h'_3) \sigma^{ab} - h_3'^{ab}) (D_a Q)_{b4} = \frac{1}{2} (tr(h'_3) \sigma^{ab} - h_3'^{ab}) ((D_a Q)_{b4} + (D_b Q)_{a4}) \\
& = \left\langle \frac{\partial}{\partial t}, tr(h'_3) e_4 \right\rangle.
\end{aligned}$$

Similarly,

$$(tr(h'_4)\sigma^{ab} - h'^{ab}_4)(D_a Q)_{b3} = \langle \frac{\partial}{\partial t}, tr(h'_4)e_3 \rangle.$$

Collecting terms, we get

$$(3.6) \quad \begin{aligned} & \nabla_a \left([tr h'_3 \sigma^{ab} - h'^{ab}_3] Q_{b4} - [tr(h'_4)\sigma^{ab} - h'^{ab}_4] Q_{b3} \right) \\ & = R^{ab}{}_{a3} Q_{b4} - R^{ab}{}_{a4} Q_{b3} + \langle \frac{\partial}{\partial t}, e_4 \rangle (tr h'_3) - \langle \frac{\partial}{\partial t}, e_3 \rangle (tr h'_4) - (d\zeta)^{ab} Q_{ab} \\ & \quad + [tr h_3 tr h'_3 - h_3 \cdot h'_3 - tr h_4 tr h'_4 + h_4 \cdot h'_4] Q_{34} + Q_{bc} \sigma^{cd} [(h_3)_{da} h'^{ab}_4 - (h_4)_{da} h'^{ab}_3]. \end{aligned}$$

Integrating (3.6) over Σ , we get

$$(3.7) \quad \begin{aligned} & \int \left\{ -\langle \frac{\partial}{\partial t}, e_4 \rangle (tr h'_3) + \langle \frac{\partial}{\partial t}, e_3 \rangle (tr h'_4) \right\} d\Sigma \\ & = \int [tr h_3 tr h'_3 - h_3 \cdot h'_3 - tr h_4 tr h'_4 + h_4 \cdot h'_4] Q_{34} d\Sigma \\ & \quad + \int \{ R^{ab}{}_{a3} Q_{b4} - R^{ab}{}_{a4} Q_{b3} + Q_{bc} \sigma^{cd} [(h_3)_{da} h'^{ab}_4 - (h_4)_{da} h'^{ab}_3] \} d\Sigma. \end{aligned}$$

(3.1) follows from the difference between (3.2) and (3.7). \square

Next, we relate the left hand side of (3.1) to the Wang-Yau quasi-local energy when the frame $\{e'_3, e'_4\}$ and $\{e_3, e_4\}$ are the canonical gauge corresponding to a pair of an isometric embedding X of Σ into $\mathbb{R}^{3,1}$ and a constant future directed timelike unit vector T_0 .

Let $\widehat{\Sigma}$ be the projection of $X(\Sigma)$ onto the orthogonal complement of T_0 . Let \check{e}_3 be the unit outward normal of $\widehat{\Sigma}$ in the orthogonal complement of T_0 . We extend \check{e}_3 along T_0 by parallel translation. Let \check{e}_4 be the unit normal of Σ which is also normal to \check{e}_3 . Let $\{\bar{e}_3, \bar{e}_4\}$ be the unique frame of the normal bundle of Σ in N such that

$$\langle H, \bar{e}_4 \rangle = \langle H_0, \check{e}_4 \rangle.$$

The quasi-local energy of Σ with respect to the pair (X, T_0) is

$$E(\Sigma, X, T_0) = \frac{1}{8\pi} \int [-\langle H_0, \check{e}_3 \rangle \sqrt{1 + |\nabla\tau|^2} - \alpha_{\check{e}_3}(\nabla\tau) + \langle H, \bar{e}_3 \rangle \sqrt{1 + |\nabla\tau|^2} + \alpha_{\bar{e}_3}(\nabla\tau)] d\Sigma.$$

Let h_3, h'_3, h_4 and h'_4 be the second fundamental form in the directions of $\check{e}_3, \bar{e}_3, \check{e}_4$ and \bar{e}_4 , respectively. If $\alpha_{\bar{e}_3} = \alpha_{\check{e}_3}$, we have

$$(3.8) \quad \begin{aligned} & \frac{1}{8\pi} \int [-\langle \frac{\partial}{\partial t}, e_4 \rangle (tr h_3 - tr h'_3) + \langle \frac{\partial}{\partial t}, e_3 \rangle (tr h_4 - tr h'_4)] d\Sigma \\ & = \frac{1}{8\pi} \int [-\langle H_0, \check{e}_3 \rangle \sqrt{1 + |\nabla\tau|^2} + \langle H, \bar{e}_3 \rangle \sqrt{1 + |\nabla\tau|^2}] d\Sigma \\ & = E(\Sigma, X, T_0) \end{aligned}$$

Hence, we have proved the following:

Theorem 3.2. *Given a surface Σ in the spacetime N , suppose we have a pair (X, T_0) of observer such that $\alpha_{\bar{e}_3} = \alpha_{\check{e}_3}$. Then we have*

$$(3.9) \quad \begin{aligned} E(\Sigma, X, T_0) &= \frac{1}{8\pi} \int \{2[\det(h_3) - \det(h_4)] - [\text{tr}h_3\text{tr}h'_3 - h_3 \cdot h'_3 - \text{tr}h_4\text{tr}h'_4 + h_4 \cdot h'_4]Q_{34}\}d\Sigma \\ &\quad + \frac{1}{8\pi} \int \{R^{ab}{}_{a4}Q_{b3} - R^{ab}{}_{a3}Q_{b4} - Q_{bc}\sigma^{cd}[(h_3)_{da}h'_4{}^{ab} - (h_4)_{da}h'_3{}^{ab}]\}d\Sigma \end{aligned}$$

4. UPPER BOUND OF THE LIU-YAU QUASI-LOCAL MASS

In this section, we apply Theorem 3.2 to the Liu-Yau quasi-local mass. For a surface Σ in M , we consider the isometric embedding X of Σ into the orthogonal complement of $T_0 = \frac{\partial}{\partial t}$. The quasi-local energy $E(\Sigma, X, T_0)$ is precisely the Liu-Yau quasi-local mass $m_{LY}(\Sigma)$ of the surface. To apply Theorem 3.2, we assume that $\alpha_H = 0$. Under the assumption, $m_{LY}(\Sigma)$ is a critical point of the Wang-Yau quasi-local energy. We have the following lemma concerning the Liu-Yau mass:

Lemma 4.1. *Suppose $\alpha_H = 0$. The Liu-Yau mass is*

$$(4.1) \quad m_{LY}(\Sigma) = \frac{1}{8\pi} \int \{[2\det(h_3) - \text{tr}h_3\text{tr}h'_3 + h_3 \cdot h'_3](X \cdot e_3) - R^{ab}{}_{a3}(X \cdot e_b)\}d\Sigma.$$

Proof. For the isometric embedding into the orthogonal complement of $\frac{\partial}{\partial t}$, we have $\check{e}_4 = \frac{\partial}{\partial t}$ and

$$\langle H_0, \check{e}_4 \rangle = 0.$$

Hence, $\bar{e}_4 = \frac{J}{|H|}$ and

$$\alpha_{\bar{e}_3} = \alpha_{\check{e}_3} = 0.$$

This verifies the assumption of Theorem 3.2. Moreover, for the isometric embedding into the orthogonal complement of $\frac{\partial}{\partial t}$, $h_4 = 0$. As a result of Theorem 3.2, we get

$$(4.2) \quad m_{LY}(\Sigma) = \frac{1}{8\pi} \int \{[2\det(h_3) - \text{tr}h_3\text{tr}h'_3 + h_3 \cdot h'_3]Q_{34} + R^{ab}{}_{a4}Q_{b3} - R^{ab}{}_{a3}Q_{b4}\}d\Sigma.$$

Finally, we observe that $Q_{ab} = 0$, $Q_{a3} = 0$ and

$$Q_{34} = (X \cdot e_3)$$

$$Q_{b4} = (X \cdot e_b)$$

since $\check{e}_4 = \frac{\partial}{\partial t}$ and $Q = r \frac{\partial}{\partial r} \wedge \frac{\partial}{\partial t}$. □

From Lemma 4.1, we derive the following upper bound for the Liu-Yau quasi-local mass in terms of the curvature tensor of N along Σ .

Theorem 4.2. *Let Σ be a topological sphere in a spacetime N . Let $e'_3 = -\frac{H}{|H|}$ and $e'_4 = \frac{J}{|H|}$. Suppose $\alpha_{e'_3} = 0$ and $R^{ab}{}_{a4} = 0$. Finally, assume the second fundamental form in the*

direction of e'_3 is positive definite and the Gauss curvature of the induced metric on Σ is positive. We have

$$m_{LY}(\Sigma) \leq \frac{1}{8\pi} \int \{2R_{1212}^+(X \cdot e_3) - R_{a3}^{ab}(X \cdot e_b)\} d\Sigma.$$

where $R_{1212}^+ = \max\{R_{1212}, 0\}$.

Proof. We use the Gauss equation

$$K = \det(h'_3) - \det(h'_4) + R_{1212}$$

and the Codazzi equations

$$\begin{aligned} \nabla_a(h'_3)_{bc} - \nabla_b(h'_3)_{ac} &= R_{abc3} + (\alpha_{e'_3})_b(h'_4)_{ac} - (\alpha_{e'_3})_a(h'_4)_{bc} \\ \nabla_a(h'_4)_{bc} - \nabla_b(h'_4)_{ac} &= R_{abc4} + (\alpha_{e'_3})_b(h'_3)_{ac} - (\alpha_{e'_3})_a(h'_3)_{bc} \end{aligned}$$

of the surface Σ in N . By our assumption, $\alpha_{e'_3} = 0$ and $R_{a4}^{ab} = 0$. As $trh'_4 = 0$, the last Codazzi equation gives

$$\nabla^c(h'_4)_{bc} = \nabla^c(h'_4)_{bc} - \nabla_b trh'_4 = 0.$$

This implies $h'_4 = 0$ since it is traceless and symmetric. The Gauss equation simplifies to

$$K = \det(h'_3) + R_{1212}.$$

We estimate $2\det(h_3) - trh_3 trh'_3 + h_3 \cdot h'_3$, keeping in mind that h_3 and h'_3 are both positive definite. At each point, we diagonalize h_3 and assume

$$h_3 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad h'_3 = \begin{pmatrix} a' & c' \\ c' & b' \end{pmatrix}.$$

We compute

$$2\det(h_3) - trh_3 trh'_3 + h_3 \cdot h'_3 = 2ab - a'b - ab'.$$

The Gauss equations of the surface Σ in N and $X(\Sigma)$ in \mathbb{R}^3 read

$$K = ab \quad , \quad K = a'b' - (c')^2 + R_{1212}.$$

We claim that

- (1) For the points where $R_{1212} \leq 0$, we have $2ab - a'b - ab' \leq 0$
- (2) For the points where $R_{1212} > 0$, we have $2ab - a'b - ab' \leq 2R_{1212}$.

The first claim is easy. If $R_{1212} \leq 0$, then $a'b' \geq K$ and

$$a'b + ab' \geq a'b + \frac{K^2}{a'b} \geq 2K.$$

For the second case, we have

$$a'b' > K - R_{1212}.$$

Let $C > 1$ be the constant such that

$$(K - R_{1212})C^2 = K.$$

We have $(Ca')(Cb') \geq K$ and

$$2ab - a'b - ab' \leq 2K - \frac{2K}{C} \leq 2R_{1212}.$$

□

From the above upper bound for the Liu-Yau quasi-local mass, we obtain the following rigidity theorem characterizing the Minkowski spacetime.

Theorem 4.3. *Let Σ be a surface in a spacetime N satisfying the dominant energy condition. Suppose Σ bounds a spacelike hypersurface M . Let $e'_3 = -\frac{H}{|H|}$ and $e'_4 = \frac{J}{|H|}$. Suppose $\alpha_{e'_3} = 0$, $R^{ab}{}_{a4} = 0$, $\nabla_b R^{ab}{}_{a3} = 0$, and $R_{1212} \leq 0$ on Σ . Finally, assume the second fundamental form in the direction of e_3 is positive definite and the Gauss curvature of Σ is positive. Then the domain of dependence of M is isometric to a open set in $\mathbb{R}^{3,1}$.*

Proof. We have

$$(X \cdot e_2)e_2 + (X \cdot e_1)e_1 = \frac{1}{2}\nabla|X|^2.$$

It follows that

$$\int [R^{ab}{}_{a3}(X \cdot e_b)]d\Sigma = -\frac{1}{2} \int [|X|^2 \nabla_b R^{ab}{}_{a3}]d\Sigma = 0.$$

We conclude that

$$m_{LY}(\Sigma) \leq \frac{1}{8\pi} \int [2R_{1212}^+(X \cdot e_3) - R^{ab}{}_{a3}(X \cdot e_b)]d\Sigma = 0.$$

The theorem follows from the positivity and rigidity of the Liu-Yau quasi-local mass [9]. □

As corollaries of the above theorem, we have the following two statements about time-symmetric initial data (M, \bar{g}) . Let \bar{R}_{ijkl} and $\bar{R}ic_{ij}$ be the Riemannian curvature and Ricci curvature tensor of g , respectively. First, we get the following upper bound of the Brown-York quasi-local mass m_{BY} .

Corollary 4.4. *Let Σ be a convex surface in a time-symmetric hypersurface (M, \bar{g}) with positive Gauss curvature. We have the following upper bound for its Brown-York quasi-local mass.*

$$(4.3) \quad \begin{aligned} m_{BY}(\Sigma) &= \frac{1}{8\pi} \int \{[2\det(h_3) - \text{tr}h_3 \text{tr}h'_3 + h_3 \cdot h'_3](X \cdot e_3) - \bar{R}^{ab}{}_{a3}(X \cdot e_b)\}d\Sigma. \\ &\leq \frac{1}{8\pi} \int \{2\bar{R}_{1212}^+(X \cdot e_3)d\Sigma + \frac{1}{16\pi} \int |X|^2 \nabla_b \bar{R}^{ab}{}_{a3}\}d\Sigma. \end{aligned}$$

where $\bar{R}_{1212}^+ = \max\{\bar{R}_{1212}, 0\}$.

Proof. Consider the static spacetime N with the metric

$$g = -dt^2 + \bar{g}$$

We have

$$R_{ijkl} = \bar{R}_{ijkl}$$

and

$$R_{ijk0} = 0.$$

The corollary follows from Lemma 4.1 and Theorem 4.2. \square

We also get the following rigidity theorem:

Theorem 4.5. *Let (M, \bar{g}) be a 3-manifold with boundary Σ . Assume that the scalar curvature of \bar{g} is non-negative and Σ is a convex 2-sphere with positive Gauss curvature. If $\nabla_b \bar{R}^ab_{a3} = 0$ and $\bar{R}_{1212} \leq 0$ on Σ . Then \bar{g} is the flat metric.*

Proof. It follows from Corollary 4.4 that

$$m_{BY}(\Sigma) \leq 0.$$

The theorem follows from the the positivity and rigidity of the Brown-York mass. \square

For asymptotically flat initial data sets, it is known that the limit of the Brown-York-Liu-Yau quasi-local mass of the coordinate spheres recovers the ADM mass of the initial data [7]. In the general relativity literature (see Ashtekar–Hansen [2], Chruściel [6] and Schoen [12]), it is also known that the ADM mass can be computed using the Ricci curvature of the induced metric of the initial data. As a corollary of Lemma 4.1, we obtain a simple proof that the limit of the Brown-York-Liu-Yau quasi-local mass coincides with the ADM mass via the Ricci curvature (see also the earlier proof by Miao–Tam–Xie in [10]).

Definition 4.6. *(M^3, \bar{g}) is an asymptotically flat manifold of order τ if, outside a compact set, M^3 is diffeomorphic to $\mathbb{R}^3 \setminus \{|x| \leq r_0\}$ for some $r_0 > 0$ and under the diffeomorphism, we have*

$$\bar{g}_{ij} - \delta_{ij} = O(|x|^{-\tau}), \partial \bar{g}_{ij} = O(|x|^{-1-\tau}), \partial^2 \bar{g}_{ij} = O(|x|^{-2-\tau})$$

for some $\tau > \frac{1}{2}$. Here ∂ denotes the partial differentiation on \mathbb{R}^3 .

Theorem 4.7. *Suppose we have an asymptotically manifold of order $\alpha > \frac{1}{2}$ and Σ_r be the coordinate spheres of the asymptotically flat coordinates. We have*

$$(4.4) \quad \lim_{r \rightarrow \infty} \int [H_0 - H + (\bar{R}ic - \frac{1}{2} \bar{R}\bar{g})(X, e_3)] d\Sigma_r = 0.$$

Here X denote the position vector of \mathbb{R}^3 which is identified with a vector field along the surface Σ_r on the initial data set via the canonical gauge of the isometric embedding.

Proof. Using (4.3), we have

$$\begin{aligned}
& \int (H_0 - H) d\Sigma_r \\
&= \int \{ [2\det(h_3) - \text{tr}h_3\text{tr}h'_3 + h_3 \cdot h'_3](X \cdot e_3) - \bar{Ric}(e_3, e_a)X \cdot e_a \} d\Sigma_r \\
&= \int \{ \det(h_3 - h'_3)(X \cdot e_3) + [\det(h_3) - \det(h'_3)](X \cdot e_3) - \bar{Ric}(e_3, e_a)X \cdot e_a \} d\Sigma_r \\
&= \int \{ \det(h_3 - h'_3)(X \cdot e_3) + [\frac{\bar{R}}{2} - \bar{Ric}(e_3, e_3)](X \cdot e_3) - \bar{Ric}(e_3, e_a)X \cdot e_a \} d\Sigma_r \\
&= \int \{ \det(h_3 - h'_3)(X \cdot e_3) + [\frac{1}{2}\bar{R}\bar{g} - \bar{Ric}](X, e_3) \} d\Sigma_r
\end{aligned}$$

For an asymptotically flat initial data set of order $\tau > \frac{1}{2}$, it is straight-forward to check that $X \cdot e_3 = O(r)$ and

$$\det(h_3 - h'_3) = O(r^{-2\tau-2}).$$

□

5. QUASI-LOCAL MASS WITH REFERENCE TO THE ANTI-DE SITTER SPACETIME

The anti-de Sitter spacetime,

$$-(1 + r^2)dt^2 + \frac{dr^2}{1 + r^2} + r^2dS^2,$$

also admits the Killing-Yano 2-form $Q = r \frac{\partial}{\partial r} \wedge \frac{\partial}{\partial t}$. In the following, we derive the analogue of Theorem 3.1 with respect to the anti-de Sitter spacetime.

Theorem 5.1. *Given a spacelike 2-surface Σ in N and a frame $\{e'_3, e'_4\}$ of the normal bundle and let X be an isometric embedding of Σ into the Anti-de Sitter spacetime. Suppose there is a frame $\{e_3, e_4\}$ of the normal bundle of $X(\Sigma)$ such that*

$$\alpha_{e'_3} = \alpha_{e_3}.$$

Then we have

$$\begin{aligned}
(5.1) \quad & \int [-\langle \frac{\partial}{\partial t}, e_4 \rangle (\text{tr}h_3 - \text{tr}h'_3) + \langle \frac{\partial}{\partial t}, e_3 \rangle (\text{tr}h_4 - \text{tr}h'_4)] d\Sigma \\
&= \int [2\det(h_3) - 2\det(h_4) - \text{tr}h_3\text{tr}h'_3 + h_3 \cdot h'_3 + \text{tr}h_4\text{tr}h'_4 - h_4 \cdot h'_4] Q_{34} d\Sigma \\
&\quad + \int [R^{ab}{}_{a4}Q_{b3} - R^{ab}{}_{a3}Q_{b4} - Q_{bc}\sigma^{cd}[(h_3)_{da}h'_4{}^{ab} - (h_4)_{da}h'_3{}^{ab}]] d\Sigma
\end{aligned}$$

where R is the curvature tensor for the spacetime N .

Proof. The proof is the same as that of Theorem 3.1. While the anti-de Sitter space is not flat, it is a space form and thus, curvature tensor for the reference space does not show up in the formula. □

We relate the left hand side of equation (5.1) to the quasi-local mass with the reference in the anti-de Sitter spacetime when the frame $\{e'_3, e'_4\}$ and $\{e_3, e_4\}$ are the canonical gauge corresponding to a pair of isometric embedding X of Σ into the anti-de Sitter spacetime and the Killing vector field $T_0 = \frac{\partial}{\partial t}$.

The Killing vector field T_0 generates a one-parameter family of isometries ϕ_t of the anti-de Sitter spacetime. Let C be the image of Σ under the one-parameter family ϕ_t . The intersection of C with the static slice $t = 0$ is $\widehat{\Sigma}$. By a slight abuse of terminology, we refer to $\widehat{\Sigma}$ as the projection of Σ . Let \check{e}_3 be the outward unit normal of $\widehat{\Sigma}$ in the static slice $t = 0$. Consider the pushforward of \check{e}_3 by the one-parameter family ϕ_t , which is denoted by \check{e}_3 again. Let \check{e}_4 be the future directed unit normal of Σ normal to \check{e}_3 and extend it along C in the same manner. Let $\{\bar{e}_3, \bar{e}_4\}$ be the unique frame of the normal bundle of Σ in N such that

$$\langle H, \bar{e}_4 \rangle = \langle H_0, \check{e}_4 \rangle.$$

The quasi-local energy of Σ with respect to the pair (X, T_0) is

$$E(\Sigma, X, T_0) = \frac{1}{8\pi} \int [\langle H_0, \check{e}_3 \rangle \langle \frac{\partial}{\partial t}, \check{e}_4 \rangle + \alpha_{\check{e}_3}(T_0^T) - \langle H, \bar{e}_3 \rangle \langle \frac{\partial}{\partial t}, \check{e}_4 \rangle - \alpha_{\bar{e}_3}(T_0^T)] d\Sigma.$$

Assume again that $\alpha_{\bar{e}_3} = \alpha_{\check{e}_3}$ and use $trh_4 = trh'_4$, we have

$$(5.2) \quad E(\Sigma, X, T_0) = \frac{1}{8\pi} \int [-\langle \frac{\partial}{\partial t}, \check{e}_4 \rangle (trh_3 - trh'_3)] d\Sigma$$

To summarize, we have proved the following:

Theorem 5.2. *Given a surface Σ in the spacetime N , suppose we have an isometric embedding X of Σ into the anti-de Sitter spacetime and the Killing vector field $T_0 = \frac{\partial}{\partial t}$ such that $\alpha_{\bar{e}_3} = \alpha_{\check{e}_3}$, then we have*

$$(5.3) \quad E(\Sigma, X, T_0) = \frac{1}{8\pi} \int [2det(h_3) - 2det(h_4) - trh_3trh'_3 + h_3 \cdot h'_3 + trh_4trh'_4 - h_4 \cdot h'_4] Q_{34} d\Sigma \\ + \frac{1}{8\pi} \int \{R^{ab}{}_{a4} Q_{b3} - R^{ab}{}_{a3} Q_{b4} - Q_{bc} \sigma^{cd} [(h_3)_{da} h_4'^{ab} - (h_4)_{da} h_3'^{ab}]\} d\Sigma$$

In particular, we can rewrite the quasi-local mass with reference in the $t = 0$ slice of the anti-de Sitter spacetime.

Corollary 5.3. *Suppose $\alpha_H = 0$. Let X be the isometric embedding of the surface into the $t = 0$ slice in the anti-de Sitter spacetime. We have*

$$(5.4) \quad E(\Sigma, X, \frac{\partial}{\partial t}) = \frac{1}{8\pi} \int \{[2det(h_3) - trh_3trh'_3 + h_3 \cdot h'_3] (r \frac{\partial}{\partial r} \cdot e_3) - R^{ab}{}_{a3} (r \frac{\partial}{\partial r} \cdot e_b)\} d\Sigma.$$

Proof. For the isometric embedding of the surface into the $t = 0$ slice of the anti-de Sitter spacetime, we have $h_4 = 0$ and $Q_{ab} = Q_{a3} = 0$. Hence (5.4) follows from Theorem 5.2. \square

We get the following rigidity theorem:

Theorem 5.4. *Let (M, \bar{g}) be a 3-manifold with boundary. Assume that the scalar curvature $\bar{R}(g)$ satisfies $\bar{R}(g) \geq -6$, the boundary is convex and the Gauss curvature of the induced metric is bounded from below by -1 . Let \bar{R}_{ijkl} be the curvature tensor of \bar{g} . If $\nabla_b \bar{R}^{ab}{}_{a3} = 0$ and $\bar{R}_{1212} \leq -1$ on the boundary. Then g is the hyperbolic metric.*

Proof. We pick an isometric embedding of Σ into the hyperbolic space such that

$$r \frac{\partial}{\partial r} \cdot e_3 > 0.$$

In view of the form the hyperbolic metric $\frac{dr^2}{1+r^2} + r^2 dS^2$ on the $t = 0$ slice, it is easy to check that $r \frac{\partial}{\partial r}$ is a gradient vector field with the potential $\frac{1}{2}r^2 + \frac{1}{4}r^4$. In particular, $r \frac{\partial}{\partial r} \cdot e_b = \nabla_b (\frac{1}{2}r^2 + \frac{1}{4}r^4)$ on $X(\Sigma)$.

Let N be the spacetime with metric

$$g = -dt^2 + \bar{g}.$$

We have

$$R_{ijkl} = \bar{R}_{ijkl}.$$

We apply Corollary 5.3 to express the quasi-local energy. By our assumption,

$$\int R^{ab}{}_{a3} (r \frac{\partial}{\partial r} \cdot e_b) d\Sigma = 0.$$

As a result, (5.4) implies

$$E(\Sigma, X, \frac{\partial}{\partial t}) = \frac{1}{8\pi} \int [2\det(h_3) - \text{tr}h_3 \text{tr}h'_3 + h_3 \cdot h'_3] (r \frac{\partial}{\partial r} \cdot e_3) d\Sigma.$$

The Gauss equations read

$$\begin{aligned} K &= \bar{R}_{1212} + \det(h'_3) \\ K &= -1 + \det(h_3). \end{aligned}$$

We estimate

$$2\det(h_3) - \text{tr}h_3 \text{tr}h'_3 + h_3 \cdot h'_3$$

as in the proof of Theorem 4.2 and use the assumption that $\bar{R}_{1212} \leq -1$. We get

$$(5.5) \quad E(\Sigma, X, \frac{\partial}{\partial t}) \leq 0.$$

The theorem now follows from the positive mass theorem of [13]. \square

For an asymptotically hyperbolic manifold, we can use Corollary 5.3 to express the limit of the quasi-local mass with reference in the hyperbolic space in terms of the limit of Ricci curvature similar to Theorem 4.6. This gives a new proof of the results proved by Herzlich in [8] and by Miao, Tam and Xie in [10]. Let

$$g_0 = \frac{dr^2}{r^2 + 1} + r^2 dS^2$$

be the hyperbolic metric of the hyperbolic space \mathbb{H}^3 and $V = \sqrt{r^2 + 1}$ be the static potential.

Definition 5.5. (M^3, \bar{g}) is an asymptotically hyperbolic manifold of order τ if, outside a compact set, M^3 is diffeomorphic to $\mathbb{H}^3 \setminus \{|x| \leq r_0\}$ for some $r_0 > 0$. Under the diffeomorphism, we have

$$|\bar{g} - g_0| = O(|x|^{-\tau}), |\partial(\bar{g} - g_0)| = O(|x|^{-1-\tau}), \partial^2(\bar{g} - g_0) = O(|x|^{-2-\tau})$$

for some $\tau > \frac{3}{2}$. Here ∂ denotes the partial differentiation with respect to the coordinate system of \mathbb{H}^3 and the norm is measured with respect to g_0 .

Theorem 5.6. Suppose we have an asymptotically flat initial data set of order $\alpha > \frac{3}{2}$ and Σ_r be the coordinate spheres of the asymptotically flat coordinates. We have

$$(5.6) \quad \lim_{r \rightarrow \infty} \int \{V(H_0 - H) + (\bar{Ric} - \frac{1}{2}(\bar{R} + 2)\bar{g})(X, e_3)\} d\Sigma_r = 0.$$

Proof. The proof is the same as Theorem 4.6 using Corollary 5.3 instead of Corollary 4.4. The resulting formula has $\bar{R} + 2$ instead of \bar{R} due to the curvature of the hyperbolic space when applying the Gauss equation to the image of the isometric embedding. \square

6. QUASI-LOCAL MASS WITH REFERENCE TO THE SCHWARZSCHILD SPACETIME

The Schwarzschild spacetime,

$$-(1 - \frac{2M}{r})dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 dS^2,$$

also admits the Killing-Yano 2-form $Q = r \frac{\partial}{\partial r} \wedge \frac{\partial}{\partial t}$. In the following, we derive the analogue of Theorem 3.1 with respect to the Schwarzschild spacetime.

Theorem 6.1. Given a spacelike 2-surface Σ in N and a frame $\{e'_3, e'_4\}$ of the normal bundle and let X be an isometric embedding of Σ into the Schwarzschild spacetime. Suppose there is a frame $\{e_3, e_4\}$ of the normal bundle of $X(\Sigma)$ such that

$$\alpha_{e'_3} = \alpha_{e_3}.$$

Then we have

$$(6.1) \quad \begin{aligned} & \int \{ -\langle \frac{\partial}{\partial t}, e_4 \rangle (tr h_3 - tr h'_3) + \langle \frac{\partial}{\partial t}, e_3 \rangle (tr h_4 - tr h'_4) \} d\Sigma \\ &= \int \{ [2det(h_3) - 2det(h_4) - tr h_3 tr h'_3 + h_3 \cdot h'_3 + tr h_4 tr h'_4 - h_4 \cdot h'_4] Q_{34} \} d\Sigma \\ & \quad + \int \{ (R^{ab}{}_{a4} - R_s{}^{ab}{}_{a4}) Q_{b3} - (R^{ab}{}_{a3} - R_s{}^{ab}{}_{a3}) Q_{b4} - Q_{bc} \sigma^{cd} [(h_3)_{da} h'_4{}^{ab} - (h_4)_{da} h'_3{}^{ab}] \} d\Sigma \end{aligned}$$

where R and R_s the curvature tensor for the spacetime N and the Schwarzschild spacetime, respectively.

Proof. The proof is the same as that of Theorem 3.1. The corresponding curvature terms appear when we apply the Codazzi equation of the surface in the Schwarzschild spacetime. \square

We relate the left hand side of equation (6.1) to the quasi-local mass with the reference in the Schwarzschild spacetime when the frame $\{e'_3, e'_4\}$ and $\{e_3, e_4\}$ are the canonical gauge corresponding to a pair of isometric embedding X of Σ into the Schwarzschild spacetime and the Killing vector field $T_0 = \frac{\partial}{\partial t}$.

The Killing vector field T_0 generates a one-parameter family of isometries ϕ_t of the Schwarzschild spacetime. Let C be the image of Σ under the one-parameter family ϕ_t . The intersection of C with the static slice $t = 0$ is $\widehat{\Sigma}$. By a slight abuse of terminology, we refer to $\widehat{\Sigma}$ as the projection of Σ . Let \check{e}_3 be the outward unit normal of $\widehat{\Sigma}$ in the static slice $t = 0$. Consider the pushforward of \check{e}_3 by the one-parameter family ϕ_t , which is denoted by \check{e}_3 again. Let \check{e}_4 be the future directed unit normal of Σ normal to \check{e}_3 and extend it along C in the same manner. Let $\{\bar{e}_3, \bar{e}_4\}$ be the unique frame of the normal bundle of Σ in N such that

$$\langle H, \bar{e}_4 \rangle = \langle H_0, \check{e}_4 \rangle.$$

The quasi-local energy of Σ with respect to the pair (X, T_0) is

$$E(\Sigma, X, T_0) = \frac{1}{8\pi} \int [\langle H_0, \check{e}_3 \rangle \langle \frac{\partial}{\partial t}, \check{e}_4 \rangle + \alpha_{\check{e}_3}(T_0^T) - \langle H, \bar{e}_3 \rangle \langle \frac{\partial}{\partial t}, \check{e}_4 \rangle - \alpha_{\bar{e}_3}(T_0^T)] d\Sigma.$$

Assume again that $\alpha_{\bar{e}_3} = \alpha_{\check{e}_3}$ and use $trh_4 = trh'_4$, we have

$$(6.2) \quad E(\Sigma, X, T_0) = \frac{1}{8\pi} \int [-\langle \frac{\partial}{\partial t}, \check{e}_4 \rangle (trh_3 - trh'_3)] d\Sigma$$

To summarize, we have proved the following:

Theorem 6.2. *Given a surface Σ in the spacetime N , suppose we have an isometric embedding X of Σ into the Schwarzschild spacetime and the Killing vector field $T_0 = \frac{\partial}{\partial t}$ such that $\alpha_{\bar{e}_3} = \alpha_{\check{e}_3}$, then we have*

$$(6.3) \quad \begin{aligned} & E(\Sigma, X, T_0) \\ &= \frac{1}{8\pi} \int [2det(h_3) - 2det(h_4) - trh_3trh'_3 + h_3 \cdot h'_3 + trh_4trh'_4 - h_4 \cdot h'_4] Q_{34} d\Sigma \\ &+ \frac{1}{8\pi} \int \{ (R^{ab}{}_{a4} - R^{ab}{}_{s4}) Q_{b3} - (R^{ab}{}_{a3} - R^{ab}{}_{s3}) Q_{b4} - Q_{bc} \sigma^{cd} [(h_3)_{da} h'^{ab}_4 - (h_4)_{da} h'^{ab}_3] \} d\Sigma \end{aligned}$$

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