A SHORT NOTE ON THE DIVISIBILITY OF CLASS NUMBERS OF REAL QUADRATIC FIELDS

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ABSTRACT. For any integer $l \geq 1$, let $p_1, p_2, \ldots, p_{l+2}$ be distinct prime numbers ≥ 5 . For all real numbers X > 1, we let $N_{3,l}(X)$ denote the number of real quadratic fields K whose absolute discriminant $d_K \leq X$ and d_K is divisible by $(p_1 \ldots p_{l+2})$ together with the class number h_K of K divisible by $2^l \cdot 3$. Then, in this short note, by following the method in [3], we prove that $N_{3,l}(X) \gg X^{\frac{7}{8}}$ for all large enough X's.

1. INTRODUCTION

The problem of the divisibility of class numbers of number fields has been of immense interest to number theorists for quite a long time. Many mathematicians have studied the divisibility problem of class number for quadratic fields.

Nagell [9] showed that there exist infinitely many imaginary quadratic fields whose class numbers are divisible by a given positive integer g. Later, Ankeny and Chowla [1] also proved the same result. The analogous result for real quadratic fields had been proved by Weinberger [12], Yamamoto [13] and many others.

Apart from the qualitative results, a great deal of work has also been done towards the quantitative versions. For a positive integer g, we let $N_g^+(X)$ (respectively, $N_g^-(X)$) denote the number of real (respectively, imaginary) quadratic fields K whose absolute value of the discriminant is $d_K \leq X$ and the class number h_K is divisible by g. Then the general problem is to find lower bounds for the magnitude of $N_g^+(X)$ (respectively, $N_g^-(X)$) as $X \to \infty$. M. Ram Murty [10] proved that, for any integer $g \geq 3$, the inequalities $N_g^-(X) \gg X^{\frac{1}{2} + \frac{1}{g}}$ and $N_g^+(X) \gg X^{\frac{1}{2g}}$ hold. The behavior of $N_g^+(X)$ and $N_g^-(X)$ have also been studied in [8], [11] and [14].

The case g = 3 has been studied in [3], [4], [5], [6], [7] and in many other papers. Byeon [2] studied the case g = 5 and g = 7 and in both the cases, he showed that $N_g^+(X) \gg X^{\frac{1}{2}}$, which is an improvement over the main result of [10].

In this short note, we shall study the following related problem. For a given natural number $l \geq 1$, we fix l + 2 distinct prime numbers ≥ 5 , say, p_1, \ldots, p_{l+2} and let $g \geq 3$, $g \neq p_i$ for all *i*, be a given odd integer. We let $N_{g,l}(X)$ denote the number of real quadratic fields *K* whose absolute discriminant $d_K \leq X$, h_K is divisible by *g* and d_K is divisible by $(p_1 \ldots p_{l+2})$. This, in turn, implies that 2^l divides h_K . Then the general problem is to find the magnitude of $N_{g,l}(X)$.

In this short note, by adopting the method of [3], we prove the following theorem.

Theorem 1. We have, $N_{3,l}(X) \gg X^{\frac{7}{8}}$ for all large enough real numbers X.

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2. Preliminaries

In [7], Kishi and Miyake gave a complete classification of quadratic fields K whose class number h_K is divisible by 3 as follows:

Lemma 1. Let $g(T) = T^3 - uwT - u^2 \in \mathbb{Z}[T]$ be a polynomial with integer coefficients u and w such that gcd(u, w) = 1, $d = 4uw^3 - 27u^2$ is not a perfect square in \mathbb{Z} and one of the following conditions holds:

- (i) $3 \nmid w$
- (ii) $3 \mid w, uw \not\equiv 3 \pmod{9}$, and $u \equiv w \pm 1 \pmod{9}$
- (iii) $3 \mid w, uw \equiv 3 \pmod{9}$, and $u \equiv w \pm 1 \pmod{27}$

If g(T) is irreducible over \mathbb{Q} , then the roots of the polynomial g(T) generate an unramified cyclic cubic extension L over $K := \mathbb{Q}(\sqrt{d})$ (which in turn implies, by Class Field Theory, that 3 divides h_K). Conversely, suppose K is a quadratic field over \mathbb{Q} with 3 dividing the class number h_K . If L is an unramified cyclic cubic extension over K, then L is obtained by adjoining the roots of g(T) in K for some suitable choices of u and w.

Using Lemma 1, Byeon and Koh [3] proved the following result.

Lemma 2. Let m and n be two relatively prime positive integers satisfying $m \equiv 1 \pmod{18}$ and $n \equiv 1 \pmod{54}$. If the polynomial $f(T) = T^3 - 3mT - 2n$ is irreducible over \mathbb{Q} , then the class number of the quadratic field $\mathbb{Q}(\sqrt{3(m^3 - n^2)})$ is divisible by 3.

In [11], Soundararajan proved the following result (see also [3]).

Lemma 3. Let X be a large positive real number and $T = X^{\frac{1}{16}}$. Also, let $M = \frac{T^{\frac{4}{3}}X^{\frac{1}{3}}}{2}$ and $N = \frac{TX^{\frac{1}{2}}}{2^4}$. If N(X) denotes the number of positive square-free integers $d \leq X$ with at least one integer solution (m, n, t) to the equation

(1)
$$m^3 - n^2 = 27t^2d$$

satisfying $T < t \le 2T, M < m \le 2M, N < n \le 2N, gcd(m,t) = gcd(m,n) = gcd(t,6) = 1, m \equiv 19 \pmod{18 \cdot 6}$ and $n \equiv 55 \pmod{54 \cdot 6}$, then, we have,

$$N(X) \simeq \frac{MN}{T} + o(MT^{\frac{2}{3}}X^{\frac{1}{3}}) \gg X^{7/8}.$$

The following result was proved in [4] which provides a lower bound of the number of irreducible cubic polynomials with bounded coefficients.

Lemma 4. Let M and N be two positive real numbers. Let

$$\mathcal{S} = \{ f(T) = T^3 + mT + n \in \mathbb{Z}[T] : |m| \le M, |n| \le N, f(T) \text{ is irreducible over } \mathbb{Q} \\ and \ D(f) = -(4m^3 + 27n^2) \text{ is not a perfect square} \}$$

be a subset of $\mathbb{Z}[T]$. Then $|\mathcal{S}| \gg MN$.

3. Proof of Theorem 1

Let $l \ge 1$ be an integer and let $g = 2^l \cdot 3$. Let K be a real quadratic field over \mathbb{Q} and its class number is h_K . Note that $h_K \equiv 0 \pmod{g}$ if and only if $h_K \equiv 0 \pmod{2^l}$ and $h_K \equiv 0 \pmod{3}$.

Claim 1. The number of quadratic field $\mathbb{Q}(\sqrt{d})$ satisfying $d \leq X$, d is divisible by $p_1 p_2 \dots p_{l+2}$ and 3 divides h_K is $\gg X^{\frac{7}{8}}$.

For each i = 1, 2, ..., l + 2, let a_i and b_i be integers such that

(2)
$$3a_i - 2b_i \not\equiv 0 \pmod{p_i}.$$

Then, consider the simultaneous congruences

$$X \equiv 19 \pmod{18 \cdot 6}$$
$$X \equiv 1 + a_i p_i \pmod{p_i^2},$$

for all i = 1, 2, ..., l + 2. Then, by the Chinese Reminder Theorem, there is a unique integer solution m modulo $18 \cdot 6 \prod_{i=1}^{l+2} p_i^2$. Thus, the number of such integers $m \leq X$ is $((1 + o(1)) X / \left(18 \cdot 6 \prod_{i=1}^{l+2} p_i^2\right) \text{ as } X \to \infty$. Let $N_1(X)$ be the set of all such integers $m \leq X$. Similarly, we consider the simultaneous congruences

$$X \equiv 55 \pmod{54 \cdot 6}$$
$$X \equiv 1 + b_i p_i \pmod{p_i^2},$$

for all i = 1, 2, ..., l + 2. Then, by the Chinese Reminder Theorem, there is a unique integer solution n modulo $54 \cdot 6 \prod_{i=1}^{l+2} p_i^2$. Thus, the number of such integers $n \leq X$ is $((1+o(1))X/\left(54 \cdot 6 \prod_{i=1}^{l+2} p_i^2\right) \text{ as } X \to \infty$. Let $N_2(X)$ be the set of all such integers $n \leq X$.

Let X be a large positive real number and $T = X^{\frac{1}{16}}$. Also, let

$$M = \frac{T^{\frac{2}{3}}X^{\frac{1}{3}}}{2}$$
 and $N = \frac{TX^{\frac{1}{2}}}{2^4}$.

Now, we shall count the number of tuples (m, n, t) satisfying (1) with $T < t \leq 2T$, $M < m \leq 2M$, $N < n \leq 2N$, gcd(m, t) = gcd(m, n) = gcd(t, 6) = 1, $m \in N_1(X)$, $n \in N_2(X)$ with square-free integer d. Then, by Lemma 3, we see that

$$N(X) \gg X^{7/8}.$$

Now note that for any integers $m \in N_1(X)$ and $n \in N_2(X)$, we see that

$$m^{3} - n^{2} = (a_{i}p_{i} + 1)^{3} - (b_{i}p_{i} + 1)^{2} \equiv 3a_{i}p_{i} + 1 - 2b_{i}p_{i} - 1 \equiv p_{i}(3a_{i} - 2b_{i}) \pmod{p_{i}^{2}},$$

for all i = 1, 2, ..., l + 2. By (2), since $3a_i - 2b_i \not\equiv 0 \pmod{p_i}$ for all i, we see that $m^3 - n^2 \not\equiv 0 \pmod{p_i^2}$ and hence p_i divides the square-free part of $m^3 - n^2$ which is d. Thus, $p_1 p_2 ... p_{l+2}$ divides d for all such d's.

In order finish the proof of Claim 1, by Lemma 2, it is enough to count the number of tuples (m, n, t) satisfying (1) for which $f(X) = T^3 - 3mT - 2n$ is irreducible over \mathbb{Q} . By Lemma 4, the number of such irreducible polynomial f(T) is at least $\gg MN \gg X^{\frac{7}{8}}$, which proves Claim 1.

Now, to finish the proof of the theorem, we see that at least $\gg X^{7/8}$ number of real quadratic fields $K = \mathbb{Q}(\sqrt{d})$ satisfying $h_K \equiv 0 \pmod{3}$ and $\omega(d) \geq l+2$, where $\omega(n)$ denotes the number of distinct prime factors of n. Therefore, by Gauss' theory of genera, we conclude that the class number h_K of corresponding real quadratic field is divisible by 2^l also. Combining this fact with Claim 1, we get the theorem.

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