A SHORT NOTE ON THE DIVISIBILITY OF CLASS NUMBERS OF REAL QUADRATIC FIELDS

JAITRA CHATTOPADHYAY

ABSTRACT. For any integer $l \geq 1$, let $p_1, p_2, \ldots, p_{l+2}$ be distinct prime numbers ≥ 5 . For all real numbers $X > 1$, we let $N_{3,l}(X)$ denote the number of real quadratic fields K whose absolute discriminant $d_K \leq X$ and d_K is divisible by $(p_1 \dots p_{l+2})$ together with the class number h_K of K divisible by $2^l \cdot 3$. Then, in this short note, by following the method in [\[3\]](#page-3-0), we prove that $N_{3,l}(X) \gg X^{\frac{7}{8}}$ for all large enough X's.

1. Introduction

The problem of the divisibility of class numbers of number fields has been of immense interest to number theorists for quite a long time. Many mathematicians have studied the divisibility problem of class number for quadratic fields.

Nagell [\[9\]](#page-3-1) showed that there exist infinitely many imaginary quadratic fields whose class numbers are divisible by a given positive integer g. Later, Ankeny and Chowla $[1]$ also proved the same result. The analogous result for real quadartic fields had been proved by Weinberger [\[12\]](#page-3-3), Yamamoto [\[13\]](#page-3-4) and many others.

Apart from the qualitative results, a great deal of work has also been done towards the quantitative versions. For a positive integer g, we let $N_g^+(X)$ (respectively, $N_g^-(X)$) denote the number of real (respectively, imaginary) quadratic fields K whose absolute value of the discriminant is $d_K \leq X$ and the class number h_K is divisible by g. Then the general problem is to find lower bounds for the magnitude of $N_g^+(X)$ (respectively, $N_g^-(X)$ as $X \to \infty$. M. Ram Murty [\[10\]](#page-3-5) proved that, for any integer $g \geq 3$, the inequalities $N_g^-(X) \gg X^{\frac{1}{2}+\frac{1}{g}}$ and $N_g^+(X) \gg X^{\frac{1}{2g}}$ hold. The behavior of $N_g^+(X)$ and $N_g^-(X)$ have also been studied in [\[8\]](#page-3-6), [\[11\]](#page-3-7) and [\[14\]](#page-3-8).

The case $g = 3$ has been studied in [\[3\]](#page-3-0), [\[4\]](#page-3-9), [\[5\]](#page-3-10), [\[6\]](#page-3-11), [\[7\]](#page-3-12) and in many other papers. Byeon [\[2\]](#page-3-13) studied the case $g = 5$ and $g = 7$ and in both the cases, he showed that $N_g^+(X) \gg X^{\frac{1}{2}}$, which is an improvement over the main result of [\[10\]](#page-3-5).

In this short note, we shall study the following related problem. For a given natural number $l \geq 1$, we fix $l + 2$ distinct prime numbers ≥ 5 , say, p_1, \ldots, p_{l+2} and let $g \geq 3$, $g \neq p_i$ for all i, be a given odd integer. We let $N_{g,l}(X)$ denote the number of real quadratic fields K whose absolute discriminant $d_K \leq X$, h_K is divisible by g and d_K is divisible by $(p_1 \ldots p_{l+2})$. This, in turn, implies that 2^l divides h_K . Then the general problem is to find the magnitude of $N_{q,l}(X)$.

In this short note, by adopting the method of [\[3\]](#page-3-0), we prove the following theorem.

Theorem 1. We have, $N_{3,l}(X) \gg X^{\frac{7}{8}}$ for all large enough real numbers X.

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2. Preliminaries

In $[7]$, Kishi and Miyake gave a complete classification of quadratic fields K whose class number h_K is divisible by 3 as follows:

Lemma 1. Let $g(T) = T^3 - uwT - u^2 \in \mathbb{Z}[T]$ be a polynomial with integer coefficients u and w such that $gcd(u, w) = 1$, $d = 4uw^3 - 27u^2$ is not a perfect square in Z and one of the following conditions holds:

- (i) 3 \forall w
- (ii) $3 | w, uw \not\equiv 3 \pmod{9}$, and $u \equiv w \pm 1 \pmod{9}$
- (iii) $3 | w, uw \equiv 3 \pmod{9}$, and $u \equiv w \pm 1 \pmod{27}$

If $g(T)$ is irreducible over Q, then the roots of the polynomial $g(T)$ generate an unramified cyclic cubic extension L over $K := \mathbb{Q}(\sqrt{d})$ (which in turn implies, by Class Field Theory, that 3 divides h_K). Conversely, suppose K is a quadratic field over Q with 3 dividing the class number h_K . If L is an unramified cyclic cubic extension over K, then L is obtained by adjoining the roots of $q(T)$ in K for some suitable choices of u and w.

Using Lemma [1,](#page-1-0) Byeon and Koh [\[3\]](#page-3-0) proved the following result.

Lemma 2. Let m and n be two relatively prime positive integers satisfying $m \equiv 1$ (mod 18) and $n \equiv 1 \pmod{54}$. If the polynomial $f(T) = T^3 - 3mT - 2n$ is irreducible over Q, then the class number of the quadratic field $\mathbb{Q}(\sqrt{3(m^3-n^2)})$ is divisible by 3.

In [\[11\]](#page-3-7), Soundararajan proved the following result (see also [\[3\]](#page-3-0)).

Lemma 3. Let X be a large positive real number and $T = X^{\frac{1}{16}}$. Also, let $M = \frac{T^{\frac{2}{3}}X^{\frac{1}{3}}}{2}$ 2 and $N = \frac{T X^{\frac{1}{2}}}{2^4}$ $\frac{X^2}{2^4}$. If $N(X)$ denotes the number of positive square-free integers $d \leq X$ with at least one integer solution (m, n, t) to the equation

(1)
$$
m^3 - n^2 = 27t^2d
$$

satisfying $T < t \leq 2T$, $M < m \leq 2M$, $N < n \leq 2N$, $gcd(m, t) = gcd(m, n) = gcd(t, 6)$ $1, m \equiv 19 \pmod{18 \cdot 6}$ and $n \equiv 55 \pmod{54 \cdot 6}$, then, we have,

$$
N(X) \approx \frac{MN}{T} + o(MT^{\frac{2}{3}}X^{\frac{1}{3}}) \gg X^{7/8}.
$$

The following result was proved in [\[4\]](#page-3-9) which provides a lower bound of the number of irreducible cubic polynomials with bounded coefficients.

Lemma 4. Let M and N be two positive real numbers. Let

$$
S = \left\{ f(T) = T^3 + mT + n \in \mathbb{Z}[T] : |m| \le M, |n| \le N, f(T) \text{ is irreducible over } \mathbb{Q} \right\}
$$

and $D(f) = -(4m^3 + 27n^2)$ is not a perfect square}

be a subset of $\mathbb{Z}[T]$. Then $|\mathcal{S}| \gg MN$.

3. Proof of Theorem [1](#page-0-0)

Let $l \geq 1$ be an integer and let $g = 2^l \cdot 3$. Let K be a real quadratic field over Q and its class number is h_K . Note that $h_K \equiv 0 \pmod{g}$ if and only if $h_K \equiv 0 \pmod{2^l}$ and $h_K \equiv 0 \pmod{3}$.

Claim 1. The number of quadratic field $\mathbb{Q}(\sqrt{d})$ satisfying $d \leq X$, d is divisible by $p_1p_2 \ldots p_{l+2}$ and 3 divides h_K is $\gg X^{\frac{7}{8}}$.

For each $i = 1, 2, \ldots, l + 2$, let a_i and b_i be integers such that

$$
(2) \t 3a_i - 2b_i \not\equiv 0 \pmod{p_i}.
$$

Then, consider the simultaneous congruences

$$
X \equiv 19 \pmod{18 \cdot 6}
$$

$$
X \equiv 1 + a_i p_i \pmod{p_i^2},
$$

for all $i = 1, 2, \ldots, l + 2$. Then, by the Chinese Reminder Theorem, there is a unique integer solution m modulo $18 \cdot 6 \prod$ $l+2$ $i=1$ p_i^2 ². Thus, the number of such integers $m \leq X$ is $((1+o(1)) X / \left(18 \cdot 6 \prod^{l+2} \right)$ $l+2$ $i=1$ p_i^2 i \setminus as $X \to \infty$. Let $N_1(X)$ be the set of all such integers $m \leq X$. Similarly, we consider the simultaneous congruences

$$
X \equiv 55 \pmod{54 \cdot 6}
$$

$$
X \equiv 1 + b_i p_i \pmod{p_i^2},
$$

for all $i = 1, 2, \ldots, l + 2$. Then, by the Chinese Reminder Theorem, there is a unique integer solution n modulo $54 \cdot 6 \prod$ $l+2$ $i=1$ p_i^2 ². Thus, the number of such integers $n \leq X$ is $((1+o(1)) X / \left(54 \cdot 6 \prod^{l+2} \right)$ $l+2$ $i=1$ p_i^2 i \setminus as $X \to \infty$. Let $N_2(X)$ be the set of all such integers $n \leq X$.

Let X be a large positive real number and $T = X^{\frac{1}{16}}$. Also, let

$$
M = \frac{T^{\frac{2}{3}} X^{\frac{1}{3}}}{2}
$$
 and
$$
N = \frac{TX^{\frac{1}{2}}}{2^4}.
$$

Now, we shall count the number of tuples (m, n, t) satisfying [\(1\)](#page-1-1) with $T < t \leq 2T$, $M < m < 2M$, $N < n < 2N$, $gcd(m, t) = gcd(m, n) = gcd(t, 6) = 1$, $m \in N_1(X)$, $n \in N_2(X)$ with square-free integer d. Then, by Lemma [3,](#page-1-2) we see that

$$
(3) \t\t N(X) \gg X^{7/8}.
$$

Now note that for any integers $m \in N_1(X)$ and $n \in N_2(X)$, we see that

$$
m^{3} - n^{2} = (a_{i}p_{i} + 1)^{3} - (b_{i}p_{i} + 1)^{2} \equiv 3a_{i}p_{i} + 1 - 2b_{i}p_{i} - 1 \equiv p_{i}(3a_{i} - 2b_{i}) \pmod{p_{i}^{2}},
$$

for all $i = 1, 2, \ldots, l + 2$. By [\(2\)](#page-2-0), since $3a_i - 2b_i \not\equiv 0 \pmod{p_i}$ for all i, we see that $m^3 - n^2 \not\equiv 0 \pmod{p_i^2}$ ²/_i) and hence p_i divides the square-free part of $m^3 - n^2$ which is d. Thus, $p_1p_2 \ldots p_{l+2}$ divides d for all such d's.

In order finish the proof of Claim 1, by Lemma [2,](#page-1-3) it is enough to count the number of tuples (m, n, t) satisfying [\(1\)](#page-1-1) for which $f(X) = T^3 - 3mT - 2n$ is irreducible over \mathbb{Q} . By Lemma [4,](#page-1-4) the number of such irreducible polynomial $f(T)$ is at least $\gg MN \gg X^{\frac{7}{8}}$, which proves Claim 1.

Now, to finish the proof of the theorem, we see that at least $\gg X^{7/8}$ number of real quadratic fields $K = \mathbb{Q}(\sqrt{d})$ satisfying $h_K \equiv 0 \pmod{3}$ and $\omega(d) \ge l + 2$, where $\omega(n)$ denotes the number of distinct prime factors of n . Therefore, by Gauss' theory of genera, we conclude that the class number h_K of corresponding real quadratic field is divisible by 2^l also. Combining this fact with Claim 1, we get the theorem.

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Harish-Chandra Research Institute, HBNI, Chhatnag Road, Jhunsi, Allahabad - 211019, INDIA

E-mail address, Jaitra Chattopadhyay: jaitrachattopadhyay@hri.res.in