ON FEEBLY COMPACT SEMITOPOLOGICAL SEMILATTICE $\exp_n \lambda$

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Abstract. We study feebly compact shift-continous topologies on the semilattice $(\exp_n \lambda, \cap)$. It is proved that such T_1-topology is sequentially pracompact if and only if it is $\mathfrak{D}(\omega)$-compact.

We shall follow the terminology of [4, 9, 10, 23]. If X is a topological space and $A \subseteq X$, then by $\operatorname{cl}_X(A)$ and $\operatorname{int}_X(A)$ we denote the closure and the interior of A in X, respectively. By ω we denote the first infinite cardinal and by $\mathbb{N}$ the set of positive integers. By $\mathfrak{D}(\omega)$ and $\mathbb{R}$ we denote an infinite countable discrete space and the real numbers with the usual topology, respectively.

A subset A of a topological space X is called *regular open* if $\operatorname{int}_X(\operatorname{cl}_X(A)) = A$.

We recall that a topological space X is said to be

- *semiregular* if X has a base consisting of regular open subsets;
- *compact* if each open cover of X has a finite subcover;
- *sequentially compact* if each sequence $\{x_n\}_{n\in\mathbb{N}}$ of X has a convergent subsequence in X;
- *countably compact* if each open countable cover of X has a finite subcover;
- *H-closed* if X is a closed subspace of every Hausdorff topological space in which it contained;
- *infra H-closed* provided that any continuous image of X into any first countable Hausdorff space is closed (see [18]);
- *totally countably pracompact* if there exists a dense subset D of the space X such that each sequence of points of the set D has a subsequence with the compact closure in X;
- *sequentially pracompact* if there exists a dense subset D of the space X such that each sequence of points of the set D has a convergent subsequence [15];
- *countably compact at a subset* $A \subseteq X$ if every infinite subset $B \subseteq A$ has an accumulation point x in X;
- *countably pracompact* if there exists a dense subset A in X such that X is countably compact at A;
- *selectively sequentially feebly compact* if for every family $\{U_n \colon n \in \mathbb{N}\}$ of non-empty open subsets of X, one can choose a point $x_n \in U_n$ for every $n \in \mathbb{N}$ in such a way that the sequence $\{x_n \colon n \in \mathbb{N}\}$ has a convergent subsequence ([7]);
- *sequentially feebly compact* if for every family $\{U_n \colon n \in \mathbb{N}\}$ of non-empty open subsets of X, there exists an infinite set $J \subseteq \mathbb{N}$ and a point $x \in X$ such that the set $\{n \in J \colon W \cap U_n = \varnothing\}$ is finite for every open neighborhood W of x (see [8]);
- *selectively feebly compact* for each sequence $\{U_n \colon n \in \mathbb{N}\}$ of non-empty open subsets of X, one can choose a point $x \in X$ and a point $x_n \in U_n$ for each $n \in \mathbb{N}$ such that the set $\{n \in \mathbb{N} \colon x_n \in W\}$ is infinite for every open neighborhood W of x ([7]);
- *feebly compact* (or *lightly compact*) if each locally finite open cover of X is finite [3];
- *d-feebly compact* (or *DFCC*) if every discrete family of open subsets in X is finite (see [21]);
- *pseudocompact* if X is Tychonoff and each continuous real-valued function on X is bounded;
- *Y-compact* for some topological space Y, if $f(X)$ is compact for any continuous map $f \colon X \to Y$.

The following diagram describes relations between the above defined classes of topological spaces.

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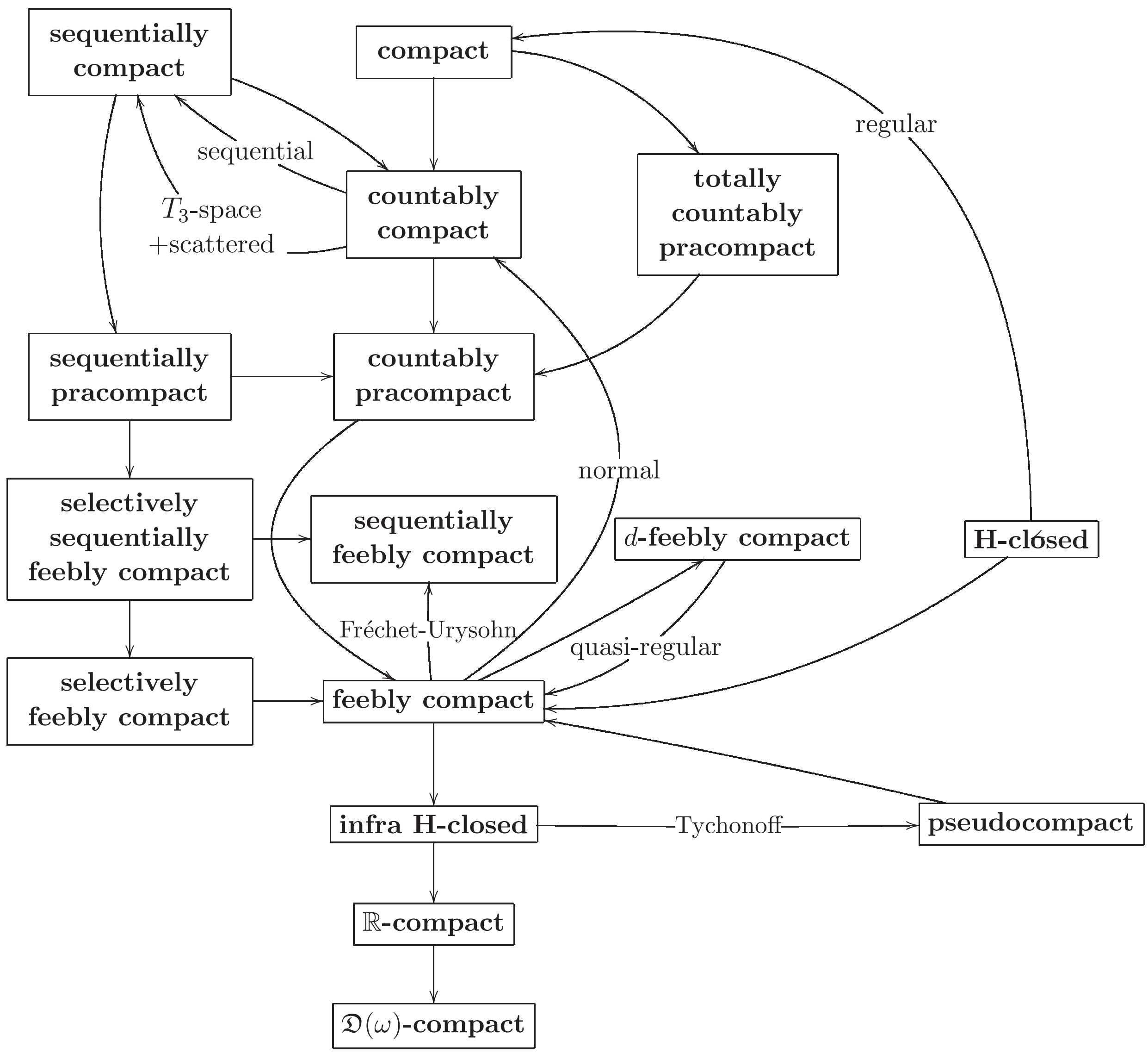

A *semilattice* is a commutative semigroup of idempotents. On a semilattice S there exists a natural partial order: $e \leqslant f$ *if and only if* $ef = fe = e$. For any element e of a semilattice S we put

$$\uparrow e = \{f \in S \colon e \leqslant f\}.$$

A *topological* (*semitopological*) *semilattice* is a topological space together with a continuous (separately continuous) semilattice operation. If S is a semilattice and τ is a topology on S such that (S,τ) is a topological semilattice, then we shall call τ a *semilattice topology* on S, and if τ is a topology on S such that (S,τ) is a semitopological semilattice, then we shall call τ a *shift-continuous topology* on S.

For an arbitrary positive integer n and an arbitrary non-zero cardinal λ we put

$$\exp_n \lambda = \{A \subseteq \lambda \colon |A| \leqslant n\}.$$

It is obvious that for any positive integer n and any non-zero cardinal λ the set $\exp_n \lambda$ with the binary operation $\cap$ is a semilattice. Later in this paper by $\exp_n \lambda$ we shall denote the semilattice $(\exp_n \lambda, \cap)$.

This paper is a continuation of [16] and [17]. In [16] we studied feebly compact semitopological semilattices $\exp_n \lambda$. Therein, all compact semilattice T_1-topologies on $\exp_n \lambda$ were described. In [16] it was proved that for an arbitrary positive integer n and an arbitrary infinite cardinal λ every T_1-semitopological countably compact semilattice $\exp_n \lambda$ is a compact topological semilattice. Also, there we constructed a countably pracompact H-closed quasiregular non-semiregular topology τ_{fc}^2 such that $(\exp_2 \lambda, \tau_{\mathsf{fc}}^2)$ is a semitopological semilattice with the discontinuous semilattice operation and show that for an arbitrary positive integer n and an arbitrary infinite cardinal λ a semiregular feebly compact semitopological semilattice $\exp_n \lambda$ is a compact topological semilattice. In [17] we proved that for any shift-continuous T_1-topology τ on $\exp_n \lambda$ the following conditions are equivalent: (i) τ is countably pracompact; (ii) τ is feebly compact; (iii) τ is d-feebly compact; (iv) $(\exp_n \lambda, \tau)$ is an H-closed space.

In [2] was proved that *every pseudocompact topological group is sequentially feebly compact*. Also, by Corollary 4.6 of [7], the Cantor cube $D^{\mathfrak{c}}$ is selectively sequentially feebly compact. By [9, Theorem 3.10.33], $D^{\mathfrak{c}}$ is not sequentially compact. Therefore, the compact topological group $G = D^{\mathfrak{c}}$ is selectively sequentially feebly compact but not sequentially feebly compact. Also, there exists a dense subgroup of $\mathbb{Z}_2^{\mathfrak{c}}$, where $\mathbb{Z}_2^{\mathfrak{c}}$ is the $\mathfrak{c}$-power of the cyclic two-elements group, which is selectively pseudocompact but not selectively sequentially pseudocompact [24]. This and our above results of [16] and [17] motivates us to investigate selective (sequential) feeble compactness of the semilattice $\exp_n \lambda$ as a semitopological semigroup.

Namely, we show that a shift-continuous T_1-semitopological semilattice $\exp_n \lambda$ is sequentially countably pracompact if and only if it is $\mathfrak{D}(\omega)$-compact.

Lemma 1. *Let n be any positive integer and λ be any infinite cardinal. Then the set of isolated points of a T_1-semitopological semilattice $\exp_n \lambda$ is dense in it.*

Proof. Fix an arbitrary non-empty open subset U of $\exp_n \lambda$. There exists $y \in \exp_n \lambda$ such that ${\uparrow}y \cap U = \{y\}$. By Proposition 1$(iii)$ from [16], ${\uparrow}y$ is an open-and-closed subset of $\exp_n \lambda$ and hence y is an isolated point in $\exp_n \lambda$. ☐

A family of non-empty sets $\{A_i : i \in \mathscr{I}\}$ is called a *Δ-system* (a *sunflower* or a *Δ-family*) if the pairwise intersections of the members are the same, i.e., $A_i \cap A_j = S$ for some set S (for $i \neq j$ in $\mathscr{I}$) [20]. The following statement is well known as the *Sunflower Lemma* or the *Lemma about a Δ-system* (see [20, p. 107]).

Lemma 2. *Every infinite family of n-element sets $(n < \omega)$ contains an infinite Δ-subfamily.*

Proposition 1. *Let n be any positive integer and λ be any infinite cardinal. Then every feebly compact T_1-semitopological semilattice $\exp_n \lambda$ is sequentially pracompact.*

Proof. Suppose to the contrary that there exists a feebly compact T_1-semitopological semilattice $\exp_n \lambda$ which is not sequentially pracompact. Then every dense subset D of $\exp_n \lambda$ contains a sequence of points from D which has no a convergent subsequence.

By Proposition 1 of [17] the subset $\exp_n \lambda \setminus \exp_{n-1} \lambda$ is dense in $\exp_n \lambda$ and by Proposition 1(ii) of [16] every point of the set $\exp_n \lambda \setminus \exp_{n-1} \lambda$ is isolated in $\exp_n \lambda$. Then the set $\exp_n \lambda \setminus \exp_{n-1} \lambda$ contains an infinite sequence of points $\{x_p : p \in \mathbb{N}\}$ which has not a convergent subsequence. By Lemma 2 the sequence $\{x_p : p \in \mathbb{N}\}$ contains an infinite Δ-subfamily, that is an infinite subsequence $\{x_{p_i} : i \in \mathbb{N}\}$ such that there exists $x \in \exp_n \lambda$ such that $x_{p_i} \cap x_{p_j} = x$ for any distinct $i, j \in \mathbb{N}$.

Suppose that $x = 0$ is the zero of the semilattice $\exp_n \lambda$. Since the sequence $\{x_{p_i} : i \in \mathbb{N}\}$ is an infinite Δ-subfamily, the intersection $\{x_{p_i} : i \in \mathbb{N}\} \cap {\uparrow}y$ contains at most one set for every non-zero element $y \in \exp_n \lambda$. Thus $\exp_n \lambda$ contains an infinite locally finite family of open non-empty subsets which contradicts the feeble compactness of $\exp_n \lambda$.

If x is a non-zero element of the semilattice $\exp_n \lambda$ then by Proposition 1(ii) of [16], ${\uparrow}x$ is an open-and-closed subspace of $\exp_n \lambda$, and hence by Theorem 14 from [3] the space ${\uparrow}x$ is feebly compact. We observe that x is zero of the semilattice ${\uparrow}x$, which contradicts so similarly the previous part of the proof. We obtain a contradiction. ☐

Proposition 2. *Let n be an arbitrary positive integer and λ be an arbitrary infinite cardinal. Then every feebly compact T_1-semitopological semilattice $\exp_n \lambda$ is totally countably pracompact.*

Proof. We put $D = \exp_n \lambda \setminus \exp_{n-1} \lambda$. By Proposition 1 of [17] the subset D is dense in $\exp_n \lambda$ and by Proposition 1(ii) of [16] every point of the set D is isolated in $\exp_n \lambda$. Fix an arbitrary sequence $\{x_p \colon p \in \mathbb{N}\}$ of points of D. By Lemma 2 the sequence $\{x_p \colon p \in \mathbb{N}\}$ contains an infinite Δ-subfamily.

Suppose that $x = 0$ is the zero of the semilattice $\exp_n \lambda$. Since the sequence $\{x_{p_i} \colon i \in \mathbb{N}\}$ is an infinite Δ-subfamily, the intersection $\{x_{p_i} \colon i \in \mathbb{N}\} \cap {\uparrow}y$ contains at most one point of the sequence for every non-zero element $y \in \exp_n \lambda$. By Proposition 1(ii) of [16] for every point $a \in \exp_n \lambda \setminus \{0\}$ there exists an open neighbourhood $U(a)$ of a in $\exp_n \lambda$ such that $U(a) \subseteq {\uparrow}a$ and hence our assumption implies that zero 0 is a unique accumulation point of the sequence $\{x_{p_i} \colon i \in \mathbb{N}\}$. Since by Lemma 1 from [16] for an arbitrary open neighbourhood $W(0)$ of zero 0 in $\exp_n \lambda$ there exist finitely many non-zero elements $y_1, \ldots, y_k \in \exp_n \lambda$ such that

$$\left(\exp_n \lambda \setminus \exp_{n-1} \lambda\right) \subseteq W(0) \cup {\uparrow}y_1 \cup \cdots \cup {\uparrow}y_k,$$

we get that $\operatorname{cl}_{\exp_n \lambda}(\{x_{p_i} \colon i \in \mathbb{N}\}) = \{0\} \cup \{x_{p_i} \colon i \in \mathbb{N}\}$ is a compact subset of $\exp_n \lambda$.

If x is a non-zero element of the semilattice $\exp_n \lambda$ then by Proposition 1(ii) of [16], ${\uparrow}x$ is an open-and-closed subspace of $\exp_n \lambda$, and hence by Theorem 14 of [3] the space ${\uparrow}x$ is feebly compact. Then x is zero of the semilattice ${\uparrow}x$ and by the previous part of the proof we have that $\operatorname{cl}_{\exp_n \lambda}(\{x_{p_i} \colon i \in \mathbb{N}\}) = \{x\} \cup \{x_{p_i} \colon i \in \mathbb{N}\}$ is a compact subset of $\exp_n \lambda$. □

Proposition 3. *Let n be any positive integer and λ be any infinite cardinal. Then every $\mathfrak{D}(\omega)$-compact T_1-semitopological semilattice $\exp_n \lambda$ is feebly compact.*

Proof. Suppose to the contrary that there exists a $\mathfrak{D}(\omega)$-compact T_1-semitopological semilattice $\exp_n \lambda$ which is not feebly compact. Then there exists an infinite locally finite family $\mathscr{U} = \{U_i\}$ of open non-empty subsets of $\exp_n \lambda$. Without loss of generality we may assume that the family $\mathscr{U} = \{U_i\}$ is countable, i.e., $\mathscr{U} = \{U_i \colon i \in \mathbb{N}\}$. Lemma 1 implies that for every $U_i \in \mathscr{U}$ there exists $a_i \in U_i$ such that $\mathscr{U}^* = \{\{a_i\} \colon i \in \mathbb{N}\}$ is a family of isolated points of $\exp_n \lambda$. Since the family $\mathscr{U}$ is locally finite, without loss of generality we may assume that $a_i \neq a_j$ for distinct $i, j \in \mathbb{N}$. The family $\mathscr{U}^*$ is locally finite as a refinement of a locally finite family $\mathscr{U}$. Since $\exp_n \lambda, \tau$ is a T_1-space, $\bigcup \mathscr{U}^*$ is a closed subset in $\exp_n \lambda$ and hence the map $f \colon \exp_n \lambda \to \mathbb{N}_{\mathfrak{d}}$, where $\mathbb{N}_{\mathfrak{d}}$ is the set of positive integers with the discrete topology, defined by the formula

$$f(b) = \begin{cases} 1, & \text{if } b \in \exp_n \lambda \setminus \bigcup \mathscr{U}^*; \\ i+1, & \text{if } b = a_i \text{ for some } i \in \mathbb{N}, \end{cases}$$

is continuous. This contradicts $\mathfrak{D}(\omega)$-compactness of the space $\exp_n \lambda$, because every two infinite countable discrete spaces are homeomorphic. □

We summarise our results in the following theorem.

Theorem 1. *Let n be any positive integer and λ be any infinite cardinal. Then for any T_1-semitopological semilattice $\exp_n \lambda$ the following conditions are equivalent:*

- (i) $\exp_n \lambda$ *is sequentially pracompact;*
- (ii) $\exp_n \lambda$ *is totally countably pracompact;*
- (iii) $\exp_n \lambda$ *is feebly compact;*
- (iv) $\exp_n \lambda$ *is $\mathfrak{D}(\omega)$-compact.*

Proof. Implications $(i) \Rightarrow (iii)$, $(ii) \Rightarrow (iii)$ and $(iii) \Rightarrow (iv)$ are trivial. The corresponding their converse implications $(iii) \Rightarrow (i)$, $(iii) \Rightarrow (ii)$ and $(iv) \Rightarrow (iii)$ follow from Propositions 1, 2 and 3, respectively. □

It is well known that the (Tychonoff) product of pseudocompact spaces is not necessarily pseudocompact (see [9, Section 3.10]). On the other hand Comfort and Ross in [6] proved that a Tychonoff product of an arbitrary family of pseudocompact topological groups is a pseudocompact topological group. Ravsky in [22] generalized the Comfort–Ross Theorem and proved that a Tychonoff product of an arbitrary non-empty family of feebly compact paratopological groups is feebly compact.

Also, a counterpart of the Comfort–Ross Theorem for pseudocompact primitive topological inverse semigroups and primitive inverse semiregular feebly compact semitopological semigroups with closed maximal subgroups were proved in [11] and [14], respectively.

Since a Tychonoff product of H-closed spaces is H-closed (see [5, Theorem 3] or [9, 3.12.5 (d)]) Theorem 1 implies a counterpart of the Comfort–Ross Theorem for feebly compact semitopological semilattices $\exp_n \lambda$:

Corollary 1. *Let $\left\{\exp_{n_i} \lambda_i \colon i \in \mathscr{I}\right\}$ be a family of non-empty feebly compact T_1-semitopological semilattices and $n_i \in \mathbb{N}$ for all $i \in \mathscr{I}$. Then the Tychonoff product $\prod \left\{\exp_{n_i} \lambda_i \colon i \in \mathscr{I}\right\}$ is feebly compact.*

Definition 1. If $\{X_i \colon i \in \mathscr{I}\}$ is a family of sets, $X = \prod \{X_i \colon i \in \mathscr{I}\}$ is their Cartesian product and p is a point in X, then the subset

$$\Sigma(p, X) = \{x \in X \colon \ |\{i \in \mathscr{I} \colon x(i) \neq p(i)\}| \leqslant \omega\}$$

of X is called the Σ-*product* of $\{X_i \colon i \in \mathscr{I}\}$ with the basis point $p \in X$. In the case when $\{X_i \colon i \in \mathscr{I}\}$ is a family of topological spaces we assume that $\Sigma(p, X)$ is a subspace of the Tychonoff product $X = \prod \{X_i \colon i \in \mathscr{I}\}$.

It is obvious that if $\{X_i \colon i \in \mathscr{I}\}$ is a family of semilattices then $X = \prod \{X_i \colon i \in \mathscr{I}\}$ is a semilattice as well. Moreover $\Sigma(p, X)$ is a subsemilattice of X for arbitrary $p \in X$. Then Theorem 1 and Proposition 2.2 of [15] imply the following corollary.

Corollary 2. *Let $\left\{\exp_{n_i} \lambda_i \colon i \in \mathscr{I}\right\}$ be a family of non-empty feebly compact T_1-semitopological semilattices and $n_i \in \mathbb{N}$ for all $i \in \mathscr{I}$. Then for every point p of the product $X = \prod \left\{\exp_{n_i} \lambda_i \colon i \in \mathscr{I}\right\}$ the Σ-product $\Sigma(p, X)$ is feebly compact.*

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