ON p-PARTS OF BRAUER CHARACTER DEGREES AND p-REGULAR CONJUGACY CLASS SIZES OF FINITE GROUPS

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Abstract. Let G be a finite group, p a prime, and $\mathrm{IBr}_p(G)$ the set of irreducible p-Brauer characters of G. Let $\bar{e}_p(G)$ be the largest integer such that $p^{\bar{e}_p(G)}$ divides $\chi(1)$ for some $\chi \in \mathrm{IBr}_p(G)$. We show that $|G : O_p(G)|_p \leq p^{k\bar{e}_p(G)}$ for an explicitly given constant k. We also study the analogous problem for the p-parts of the conjugacy class sizes of p-regular elements of finite groups.

1. Introduction

It is a classic theme to study how arithmetic conditions on characters of a finite group affect the structure of the group. Some of the most important problems in the representation theory of finite groups deal with character degrees and prime numbers.

Let G be a finite group and P be a Sylow p-subgroup of G; it is reasonable to expect that the p-parts of the degrees of irreducible characters of G somehow restrict the structure of P. The Ito-Michler theorem says that each irreducible ordinary character degree is coprime to p if and only if G has a normal abelian Sylow p-subgroup, which of course implies that $|G : O_p(G)|_p = 1$.

We write $e_p(G)$ to denote the exponent of the largest p-part of the degrees of the irreducible complex characters of G. Moretó [23, Conjecture 4] conjectured that the largest character degree of P is bounded by some function of $e_p(G)$. For the case of solvable groups, the conjecture was proved by Moretó and Wolf [24], and the bounds have been improved by the second author in [32] and [33]. Recently, Lewis, Navarro and Wolf [19] studied the special case when $e_p(G) = 1$, and showed that $|G : O_p(G)|_p \leq p^2$ when G is solvable. For $p > 2$, Lewis, Navarro, Tiep and Tong-Viet [20] also studied the case when $e_p(G) = 1$ for arbitrary finite groups. The conjecture of Moretó was recently settled by Qian and the second author in [34].

It is natural to study the Brauer character degree analogue of the Ito-Michler theorem. This has been investigated by Michler [22] and Manz [21]. They showed that each irreducible Brauer character degree is coprime to p if and only if G has a normal Sylow p-subgroup, i.e., that $|G : O_p(G)|_p = 1$. Thus we would like to ask the following question:

Let $\mathrm{IBr}_p(G)$ be the set of irreducible p-Brauer characters of G, and $\bar{e}_p(G)$ be the largest integer such that $p^{\bar{e}_p(G)}$ divides $\chi(1)$ for some $\chi \in \mathrm{IBr}_p(G)$, then how does $\bar{e}_p(G)$ affect the structure of the Sylow p-subgroup of G? We show the following results as an effort to study this question. This could be viewed as a generalization of the Brauer character degree analogue of the Ito-Michler theorem. We remark that the case $\bar{e}_p(G) = 1$ has been studied in [20].

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Theorem A. *Let G be a finite group and $\bar{e}_p(G)$ be the largest integer such that $p^{\bar{e}_p(G)}$ divides $\chi(1)$ for some $\chi \in \mathrm{IBr}_p(G)$.*

(1) *If $p \geq 5$, then $\log_p |G : O_p(G)|_p \leq 6.5\, \bar{e}_p(G)$.*
(2) *If $p = 3$, then $\log_p |G : O_p(G)|_p \leq 20\, \bar{e}_p(G)$.*
(3) *If $p = 2$, then $\log_p |G : O_p(G)|_p \leq 24\, \bar{e}_p(G)$.*

As conjugacy classes are closely related to the irreducible characters, we could study related questions on conjugacy class sizes. The conjugacy class size analogues of p-Brauer character degrees are obviously the class sizes of the p-regular elements. Similarly to the situation for p-Brauer character degrees, it is also reasonable to expect that the p-parts of the conjugacy class sizes of the p-regular elements somehow restrict the structure of P.

Let $\overline{ecl}_p(G)$ be the largest integer such that $p^{\overline{ecl}_p(G)}$ divides some $|C| \in \mathrm{clsize}_{p'}(G)$; we will show that $|G : O_p(G)|_p$ is also bounded by a function of $\overline{ecl}_p(G)$.

Theorem B. *Let G be a finite group and let p be a prime; let $P \in \mathrm{Syl}_p(G)$. Let $\overline{ecl}_p(G)$ be the largest integer such that $p^{\overline{ecl}_p(G)}$ divides some $|C| \in \mathrm{clsize}_{p'}(G)$.*

(1) *If $p \geq 5$, then $\log_p |G : O_p(G)|_p \leq 6.5\, \overline{ecl}_p(G)$.*
(2) *If $p = 3$, then $\log_p |G : O_p(G)|_p \leq 19\, \overline{ecl}_p(G)$.*
(3) *If $p = 2$, then $\log_p |G : O_p(G)|_p \leq 17\, \overline{ecl}_p(G)$.*

We notice that recently Tong-Viet has done some related work in finding various conditions on Brauer character degrees for a finite group to have a normal Sylow p-subgroup (see [31]).

2. Notation and preliminary results

We first fix some notation:

(1) We use $\mathbf{F}(G)$ to denote the Fitting subgroup of G. Let $\mathbf{F}_i(G)$ be the ith ascending Fitting subgroup of G, i.e., $\mathbf{F}_0(G) = 1$, $\mathbf{F}_1(G) = \mathbf{F}(G)$ and $\mathbf{F}_{i+1}(G)/\mathbf{F}_i(G) = \mathbf{F}(G/\mathbf{F}_i(G))$.
(2) We use $\mathbf{F}^*(G)$ to denote the generalized Fitting subgroup of G.
(3) Let p be a prime number, we say that an element $x \in G$ is p-regular if the order of x is not a multiple of p.
(4) We use $\mathrm{clsize}_{p'}(G)$ to denote the set of conjugacy class sizes of p-regular elements of G.
(5) We use $\mathrm{cl}(G)$ to denote the set of all the conjugacy classes of G, and we use $\mathrm{cl}_{p'}(G)$ to denote the set of all the conjugacy classes of p-regular elements of G.
(6) We denote $\mathrm{cd}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$.
(7) We use $\mathrm{IBr}_p(G)$ to denote the set of all the irreducible p-Brauer characters of G.
(8) Let $\bar{e}_p(G)$ be the largest integer such that $p^{\bar{e}_p(G)}$ divides $\chi(1)$ for some $\chi \in \mathrm{IBr}_p(G)$.
(9) Let $\overline{ecl}_p(G)$ be the largest integer such that $p^{\overline{ecl}_p(G)}$ divides some $|C| \in \mathrm{clsize}_{p'}(G)$.
(10) We use the notation $\mathrm{dl}(G)$ for the derived length of a solvable group G.

(11) If a group G acts on a set Ω and ω is an element in Ω, we will use the notation $\mathbf{C}_G(\omega)$ to denote the stabilizer of the element ω under the action of G. In particular, if λ is an irreducible character of a normal subgroup N of G, then $\mathbf{C}_G(\lambda)$ denotes the inertia group of λ in G. Let Ω_1 be a subset of Ω, we use $\mathrm{Stab}_G\Omega_1$ to denote the stabilizer of Ω_1 under the action of G as a set (consider the induced action of G on $\mathcal{P}(\Omega)$, the power set of Ω).

We need the following results about simple groups.

Lemma 2.1. *Let A act faithfully and coprimely on a non-abelian simple group S. Then A has at least 2 regular orbits on $\mathrm{Irr}(S)$.*

Proof. This is [26, Proposition 2.6]. □

Lemma 2.2. *If G is a non-abelian finite simple group, then $|\mathrm{cd}(G)| \geq 4$.*

Proof. This follows from [13, Theorem 12.15]. □

Lemma 2.3. *If G is a non-abelian finite simple group, then $|cs(G)| \geq 4$.*

Proof. This follows from [15]. □

The main results are proved using an orbit theorem for p-solvable groups. This method provides a unified approach to the Brauer character degree and the p-regular class size version of the problem.

We now state the orbit theorem for p-solvable groups. This result has been proved in [34] but the proof there has some glitch; we take the opportunity to provide a corrected proof here.

Theorem 2.4. *Let $V \trianglelefteq G$, where G/V is p-solvable for an odd prime p, and V is a direct product of isomorphic non-abelian simple groups $S_1, \dots, S_n$. Suppose that G acts transitively on the groups $S_1, \dots, S_n$, and write $O = \bigcap_k N_G(S_k)$. Then there exist nonprincipal v_1, v_2 and $v_3 \in \mathrm{Irr}(V)$ of different degrees such that all Sylow p-subgroups of $\mathbf{C}_G(v_j)$ are contained in O for all $j = 1, 2, 3$.*

Proof. Clearly O is normal in G and G is a transitive permutation group on the set $\{S_1, \dots, S_n\}$ with kernel O. If $n = 1$, then the required result follows by Lemma 2.2. Thus we may assume that $n > 1$. Let $(\Delta_1, \dots, \Delta_m)$ be a system of imprimitivity of G with maximal block-size b. Then $(\Delta_1, \dots, \Delta_m)$ is a partition of $\{S_1, \dots, S_n\}$ and each block Δ_i has size b. Thus

$$1 \leq b < n; bm = n, m \geq 2.$$

Let $\Omega = \{\Delta_1, \dots, \Delta_m\}$. Then G is a primitive permutation group of degree m on the set Ω. Set

$$J_i = \mathrm{Stab}_G(\Delta_i),\ K = \bigcap_{1 \leq i \leq m} J_i,\ V_i = \prod_{S_t \in \Delta_i} S_t, i = 1, \dots, m.$$

Observe that

$$J_i = N_G(V_i),$$

the groups J_i are permutationally equivalent transitive groups of degree b, and that K is a normal subgroup of G and stabilizes each of the blocks Δ_i. In particular, G/K is a primitive group of degree m acting upon the set Ω.

Let us consider $\sigma \in \mathrm{Irr}(V_i)$. We may view σ as a character of V. Note that if σ is nonprincipal, then $\mathbf{C}_G(\sigma) \leq J_i$ because G acts transitively on Ω, and therefore $\mathbf{C}_G(\sigma) = \mathbf{C}_{J_i}(\sigma)$.

Let us consider J_i and the action of J_i on $\mathrm{Irr}(V_i) = \prod_{S_t \in \Delta_i} \mathrm{Irr}(S_t)$. Since G acts transitively on $\{S_1, \ldots, S_n\}$ and acts transitively on Ω, we see that J_i acts transitively on Δ_i. Write

$$O_i = \bigcap_{t \in \Delta_i} N_{J_i}(S_t), i = 1, \ldots, m.$$

Clearly $O = \bigcap_{i=1}^m O_i$. Note that if $S_t \in \Delta_i$, then $N_G(S_t) \leq J_i$ because G acts transitively on Ω. Therefore $N_G(S_t) = N_{J_i}(S_t)$, and this implies that

$$O_i = \bigcap_{S_t \in \Delta_i} N_G(S_t).$$

Since $J_i < G$, by induction there exist nonprincipal θ_i, λ_i, and $\chi_i \in \mathrm{Irr}(V_i)$ of different degrees such that all Sylow p-subgroups of $\mathbf{C}_{J_i}(\theta_i), \mathbf{C}_{J_i}(\lambda_i)$ and $\mathbf{C}_{J_i}(\chi_i)$ are contained in O_i, that is, all Sylow p-subgroups of $\mathbf{C}_G(\theta_i), \mathbf{C}_G(\lambda_i)$, $\mathbf{C}_G(\chi_i)$ are contained in O_i. Clearly we may choose θ_i, $1 \leq i \leq m$, to be G-conjugate, and we can do the same for λ_i and χ_i. We may assume that $\theta_i(1) > \lambda_i(1) > \chi_i(1)$.

We claim that there exist proper subsets Ω_1 and Ω_2 of Ω such that $\Omega = \Omega_1 \cup \Omega_2$, $\Omega_1 \cap \Omega_2 = \emptyset$, and $\mathrm{Stab}_{G/K}(\Omega_1) \cap \mathrm{Stab}_{G/K}(\Omega_2)$ is a p'-group except for a few cases listed below.

(1) $|\Omega| = 8$, $G/K \cong \mathrm{A\Gamma L}(1, 8)$.

(2) $|\Omega| = 9$, $G/K \cong \mathrm{AGL}(2, 3)$ or $G/K \cong \mathrm{ASL}(2, 3)$.

To see the claim, we need to investigate the action of G/K on the power set $\mathcal{P}(\Omega)$ of Ω. Clearly we may assume that p divides $|G/K|$. Note that if $m \geq 5$, then $\mathrm{Alt}(m) \not\leq G/K$ because G/K is p-solvable. Note that if G/K has a regular orbit on $\mathcal{P}(\Omega)$, then there exists a (clearly proper) subset Ω_1 of Ω such that $\mathrm{Stab}_{G/K}(\Omega_1) = 1$, thus Ω_1 and $\Omega_2 = \Omega - \Omega_1$ meet our requirement. Hence we may assume G has no regular orbit on $\mathcal{P}(\Omega)$.

Suppose that G/K is solvable. By Gluck's result about solvable primitive permutations groups [8], we see that there exists a partition Ω_1, Ω_2 of Ω such that $\mathrm{Stab}_{G/K}(\Omega_1) \cap \mathrm{Stab}_{G/K}(\Omega_2)$ is a 2-group, except for the following cases:

(1) $n = 8$, $G/K \cong \mathrm{A\Gamma L}(1, 8)$.

(2) $n = 9$, $G/K \cong \mathrm{AGL}(2, 3)$ or $G/K \cong \mathrm{ASL}(2, 3)$.

Suppose that G/K is nonsolvable. By [30, Theorem 2], G/K is not r-solvable for any prime divisor r of $|G/K|$, we get a contradiction.

We first assume that there exist proper subsets Ω_1 and Ω_2 of Ω such that $\Omega = \Omega_1 \cup \Omega_2$, $\Omega_1 \cap \Omega_2 = \emptyset$, and $\mathrm{Stab}_{G/K}(\Omega_1) \cap \mathrm{Stab}_{G/K}(\Omega_2)$ is a p'-group.

Assume that $\Omega_1 = \{\Delta_1, \dots, \Delta_s\}$, $\Omega_2 = \{\Delta_{s+1}, \dots, \Delta_m\}$. Set

$$v_1 = \prod_{i=1}^{s} \theta_i \cdot \prod_{i=s+1}^{m} \lambda_i, \; v_2 = \prod_{i=1}^{s} \theta_i \cdot \prod_{i=s+1}^{m} \chi_i, \; v_3 = \prod_{i=1}^{s} \lambda_i \cdot \prod_{i=s+1}^{m} \chi_i.$$

Clearly, v_1, v_2 and v_3 have different degrees. Let us investigate $\mathbf{C}_G(v_1)$ and its Sylow p-subgroup P. Since G acts transitively on Ω and thus on $\{V_1, \dots, V_m\}$, we see that $\mathbf{C}_G(v_1) \leq \mathrm{Stab}_G(\Omega_1) \cap \mathrm{Stab}_G(\Omega_2)$. As $(\mathrm{Stab}_G(\Omega_1) \cap \mathrm{Stab}_G(\Omega_2))/K$ is a p'-group by the claim, it forces that

$$P \leq K \cap \mathbf{C}_G(v_1) \cap P = \mathbf{C}_K(v_1) \cap P.$$

Observing that all groups V_i are normal in K, we have

$$\mathbf{C}_K(v_1) = (\bigcap_{i=1}^{s} \mathbf{C}_K(\theta_i)) \cap (\bigcap_{i=s+1}^{m} \mathbf{C}_K(\lambda_i)).$$

We get the required result that

$$P \leq (\bigcap_{i=1}^{s} (\mathbf{C}_K(\theta_i) \cap P)) \cap (\bigcap_{i=s+1}^{m} (\mathbf{C}_K(\lambda_i) \cap P)) \leq \bigcap_{i=1}^{m} O_i = O.$$

Similarly all Sylow p-subgroups of $\mathbf{C}_G(v_2)$ and $\mathbf{C}_G(v_3)$ are contained in O.

We next assume that $n = 8$, and $G/K \cong \mathrm{A\Gamma L}(1, 8)$. We set $\Omega_1 = \{1, 2, 3\}$, $\Omega_2 = \{4, 5, 6\}$, and $\Omega_3 = \{7, 8\}$. We see that $\mathrm{Stab}_{G/K}(\Omega_1) \cap \mathrm{Stab}_{G/K}(\Omega_2) \cap \mathrm{Stab}_{G/K}(\Omega_3) = 1$.

Set

$$v_1 = \prod_{i \in \Omega_1} \theta_i \cdot \prod_{i \in \Omega_2} \lambda_i \cdot \prod_{i \in \Omega_3} \chi_i, \; v_2 = \prod_{i \in \Omega_1} \theta_i \cdot \prod_{i \in \Omega_2} \chi_i \cdot \prod_{i \in \Omega_3} \lambda_i, \; v_3 = \prod_{i \in \Omega_1} \lambda_i \cdot \prod_{i \in \Omega_2} \chi_i \cdot \prod_{i \in \Omega_3} \theta_i.$$

Clearly, v_1, v_2 and v_3 have different degrees. Let us investigate $\mathbf{C}_G(v_1)$ and its Sylow p-subgroup P. Since G acts transitively on Ω and thus on $\{V_1, \dots, V_m\}$, we see that $\mathbf{C}_G(v_1) \leq \mathrm{Stab}_G(\Omega_1) \cap \mathrm{Stab}_G(\Omega_2) \cap \mathrm{Stab}_G(\Omega_3)$. As $(\mathrm{Stab}_G(\Omega_1) \cap \mathrm{Stab}_G(\Omega_2) \cap \mathrm{Stab}_G(\Omega_3))/K$ is a trivial group, it forces that

$$P \leq K \cap \mathbf{C}_G(v_1) \cap P = \mathbf{C}_K(v_1) \cap P.$$

Observing that all groups V_i are normal in K, we have

$$\mathbf{C}_K(v_1) = (\bigcap_{i \in \Omega_1} \mathbf{C}_K(\theta_i)) \cap (\bigcap_{i \in \Omega_2} \mathbf{C}_K(\lambda_i)) \cap (\bigcap_{i \in \Omega_3} \mathbf{C}_K(\chi_i)).$$

We get the required result that

$$P \leq (\bigcap_{i \in \Omega_1} (\mathbf{C}_K(\theta_i) \cap P)) \cap (\bigcap_{i \in \Omega_2} (\mathbf{C}_K(\lambda_i) \cap P) \cap (\bigcap_{i \in \Omega_3} (\mathbf{C}_K(\chi_i) \cap P)) \leq \bigcap_{i=1}^{m} O_i = O.$$

Similarly all Sylow p-subgroups of $\mathbf{C}_G(v_2)$ and $\mathbf{C}_G(v_3)$ are contained in O.

We finally assume that $n = 9$, and $G/K \cong AGL(2, 3)$ or $G/K \cong ASL(2, 3)$. We set $\Omega_1 = \{1, 2, 3, 4\}$, $\Omega_2 = \{5, 6, 7\}$, and $\Omega_3 = \{8, 9\}$. We see that $\mathrm{Stab}_{G/K}(\Omega_1) \cap \mathrm{Stab}_{G/K}(\Omega_2) \cap \mathrm{Stab}_{G/K}(\Omega_3)$ is a 2-group.

Set

$$v_1=\prod_{i\in\Omega_1}\theta_i\cdot\prod_{i\in\Omega_2}\lambda_i\cdot\prod_{i\in\Omega_3}\chi_i,\ v_2=\prod_{i\in\Omega_1}\lambda_i\cdot\prod_{i\in\Omega_2}\theta_i\cdot\prod_{i\in\Omega_3}\chi_i,\ v_3=\prod_{i\in\Omega_1}\lambda_i\cdot\prod_{i\in\Omega_2}\chi_i\cdot\prod_{i\in\Omega_3}\theta_i.$$

Clearly, v_1, v_2 and v_3 have different degrees. Let us investigate $\mathbf{C}_G(v_1)$ and its Sylow p-subgroup P. Since G acts transitively on Ω and thus on $\{V_1,\dots,V_m\}$, we see that $\mathbf{C}_G(v_1)\leq \mathrm{Stab}_G(\Omega_1)\cap\mathrm{Stab}_G(\Omega_2)\cap\mathrm{Stab}_G(\Omega_3)$. As $(\mathrm{Stab}_G(\Omega_1)\cap\mathrm{Stab}_G(\Omega_2)\cap\mathrm{Stab}_G(\Omega_3))/K$ is a 2-group, it forces that

$$P\leq K\cap\mathbf{C}_G(v_1)\cap P=\mathbf{C}_K(v_1)\cap P.$$

Observe that all V_is are normal in K, we have

$$\mathbf{C}_K(v_1)=(\bigcap_{i\in\Omega_1}\mathbf{C}_K(\theta_i))\cap(\bigcap_{i\in\Omega_2}\mathbf{C}_K(\lambda_i))\cap(\bigcap_{i\in\Omega_3}\mathbf{C}_K(\chi_i)).$$

We get the required result that

$$P\leq(\bigcap_{i\in\Omega_1}(\mathbf{C}_K(\theta_i)\cap P))\cap(\bigcap_{i\in\Omega_2}(\mathbf{C}_K(\lambda_i)\cap P)\cap(\bigcap_{i\in\Omega_3}(\mathbf{C}_K(\chi_i)\cap P))\leq\bigcap_{i=1}^m O_i=O.$$

Similarly all Sylow p-subgroups of $\mathbf{C}_G(v_2)$ and $\mathbf{C}_G(v_3)$ are contained in O. □

3. On p-parts of p-Brauer character degrees

It is a fundamental fact in block theory that if an ordinary irreducible character χ is such that $\chi(1)_p=|G|_p$, for a prime p, then its reduction modulo p gives an irreducible Brauer character of the same degree. Hence then $e_p(G)\leq\bar{e}_p(G)$, and the bounds obtained with respect to ordinary characters still hold in the case of p-Brauer characters.

For the solvable case, the problems in this paper have been studied in [25] and certain bounds were obtained; more explicitly, it was shown that for a finite solvable group G with $O_p(G)=1$, $\log_p|G|_p\leq 96\bar{e}_p(G)$ and $\log_p|G|_p)\leq 683\overline{ecl}_p(G)$. We greatly improve those bounds, and we will obtain corresponding results for arbitrary finite groups.

We first note that if N is a normal subgroup of G, then it is easy to see that $\bar{e}_p(G/N)\leq\bar{e}_p(G)$ and $\bar{e}_p(N)\leq\bar{e}_p(G)$. We shall use this fact freely in the following arguments.

The following lemma is due to Martin Isaacs [14].

Lemma 3.1. *Let P be a nontrivial p-group that acts faithfully on a group H, where $|H|$ is not divisible by p. Then there exists an element $x\in H$ such that $|\mathbf{C}_P(x)|\leq|P|^{1/2}$.*

3.1. **The solvable case.**

Theorem 3.2. *Let G be a finite solvable group with $O_p(G)=1$, where $p\geq 5$; set $n=\bar{e}_p(G)$. Then $|G|_p\leq p^{2.5n}$.*

Proof. Let $|G|_p=p^a$. By [33], the group G has a p-block of defect $d\leq\frac{3}{5}a$. Since $a-d\leq n$ (see [13, Section 15]), we obtain $a\leq\frac{5}{2}n$. Hence the claim holds. □

Remark 3.3. For G a group of odd order with $O_p(G) = 1$, Espuelas and Navarro have shown in [5] that there is in fact a p-block of defect $d \leq \lfloor a/2 \rfloor$ (and this bound is best possible). Using the same argument (and notation) as above, we then obtain the better bound $|G|_p \leq p^{2n}$ in Theorem 3.2. Already in [5] the question is posed whether for finite groups with $O_p(G) = 1$ and $p \geq 5$, such p-blocks of small defect always exist; clearly, this would then also give a better bound in Theorem A, for $p \geq 5$. It was already noticed in [5] that for $p = 2$ for example the group $G = \mathfrak{A}_7$ has no 2-block of the desired small defect 1; note that we still have $|G|_2 \leq 2^{2n}$ in this case. However, the example $G = M_{22}$ (discussed later) shows that for $p = 2$ the bound $|G|_2 \leq 2^{2n}$ does not hold in general; there may still be room to improve the bounds given in Theorem A, though.

Theorem 3.4. *Let G be a finite solvable group with $O_p(G) = 1$ and set $n = \bar{e}_p(G)$. Then $|G|_p \leq p^{15n}$ if $p = 2$ or $p = 3$.*

Proof. By Gaschütz's theorem, $G/\mathbf{F}(G)$ acts faithfully and completely reducibly on $\mathrm{Irr}(\mathbf{F}(G)/\Phi(G))$. Since $p \nmid |\mathbf{F}(G)/\Phi(G)|$, $\mathrm{Irr}(\mathbf{F}(G)/\Phi(G)) = \mathrm{IBr}(\mathbf{F}(G)/\Phi(G))$. It follows from [32, Theorem 3.3] that there exists $\lambda \in \mathrm{IBr}(\mathbf{F}(G)/\Phi(G))$ such that $T = \mathbf{C}_G(\lambda) \leq \mathbf{F}_8(G)$.

Let $K_{i+1} = \mathbf{F}_{i+1}(G)/\mathbf{F}_i(G)$ and let $K_{i+1,p}$ be the Sylow p-subgroup of K_{i+1} for all $i \geq 1$. We know that $K_{i+1,p}$ acts faithfully and completely reducibly on $K_i/\Phi(G/\mathbf{F}_{i-1}(G))$. It is clear that we may write $K_i/\Phi(G/\mathbf{F}_{i-1}(G)) = V_{i1} + V_{i2}$ where V_{i1} is the p-part of $K_i/\Phi(G/\mathbf{F}_{i-1}(G))$ and V_{i2} is the p'-part of $K_i/\Phi(G/\mathbf{F}_{i-1}(G))$ for all $i \geq 1$.

We observe that $K_{i+1,p}$ acts faithfully and completely reducibly on $\mathrm{Irr}(V_{i2})$ for all $i \geq 1$. Since $\mathrm{IBr}(V_{i2}) = \mathrm{Irr}(V_{i2})$, we have $|K_{i+1,p}| \leq p^{2n}$ by Lemma 3.1.

Next, we show that $|G : T|_p \leq p^n$.

Take $\chi \in \mathrm{IBr}(G)$ lying over λ. Then $|G : T|_p$ divides $\chi(1)$, which is at most p^n.

We know from before that $|K_{i,p}| \leq p^{2n}$ for $2 \leq i \leq 8$. This implies that $|G|_p \leq (p^{2n})^7 \cdot p^n = p^{15n}$. □

3.2. The p-solvable case.

We now obtain bounds for p-solvable groups and then extend those to arbitrary groups.

Theorem 3.5. *Let G be a p-solvable group for an odd prime p. Assume that G has no nontrivial solvable normal subgroup. Then there exists $\chi \in \mathrm{IBr}_p(G)$ such that $\chi(1)_p \geq \sqrt{|G|_p}$.*

Proof. Since G has no nontrivial solvable normal subgroup, the socle L of G can been written as $L = L_1 \times \cdots \times L_n$, where $L_i = S_{i1} \times \cdots \times S_{it_i}$ is minimal normal in G, and $S_{i1}, \ldots, S_{it_i}$ are isomorphic to a nonabelian simple group S_i.

We observe that since G is p-solvable, $p \nmid |L|$. Thus $\mathrm{IBr}_p(L_i) = \mathrm{Irr}(L_i)$ and $\mathrm{IBr}_p(L) = \mathrm{Irr}(L)$.

Write $O_i = \bigcap_{j=1}^{t_i} N_G(S_{ij})$ and $O = \bigcap_{i=1}^{n} O_i$. Clearly O and all O_i are normal in G, all S_{ij} are normal in O_i and O. Repeatedly using Dedekind's Modular Law, we have that

$$L = \bigcap_{i=1}^{n} \bigcap_{j=1}^{t_i} S_{ij}\mathbf{C}_G(S_{ij}).$$

This implies that

$$O/L = O/\bigcap_{i=1}^{n} \bigcap_{j=1}^{t_i} S_{ij}\mathbf{C}_G(S_{ij}) \lesssim \prod_{i,j} N_G(S_{ij})/(S_{ij}\mathbf{C}_G(S_{ij})) \lesssim \prod_{i,j} \mathrm{Out}(S_{ij}).$$

Since all S_{ij} are p-solvable, $\mathrm{Out}(S_{ij})$ has a normal cyclic Sylow p-subgroup (for example, [20, Lemma 2.3(ii)]). Thus O/L has a normal and abelian Sylow p-subgroup.

By Lemma 2.1, it is easy to find an irreducible character μ of L such that $\mathbf{C}_O(\mu)$ is a p'-group. Hence there exists an irreducible constituent χ_1 of μ^G such that

$$\chi_1(1)_p \geq |O|_p.$$

Also, by Theorem 2.4, we may find $\lambda_i \in \mathrm{IBr}_p(L_i)$ such that $\mathbf{C}_G(\lambda_i) \leq O_i$ for each i. Set $\lambda = \prod_i \lambda_i$ and let χ_2 be an irreducible constituent of λ^G. Since all L_i are normal in G, we have

$$\mathbf{C}_G(\lambda) = \bigcap_i \mathbf{C}_G(\lambda_i) \leq \bigcap_i O_i = O.$$

This implies that

$$\chi_2(1)_p \geq |G/O|_p.$$

Thus there exists $\chi \in \{\chi_1, \chi_2\}$ such that $\chi(1)_p \geq \sqrt{|G|_p}$. □

3.3. **The general case.**

For a group G, let $b(G)$ denote the largest degree of an irreducible character of G.

Lemma 3.6. *Let G be a finite group, $P \in \mathrm{Syl}_p(G)$ and $\bar{P} = P/O_p(G)$; set $n = \bar{e}_p(G)$. Assume that $|G : O_p(G)|_p \leq p^{kn}$. Then $b(\bar{P}) \leq p^{kn/2}$ and $\mathrm{dl}(\bar{P}) \leq 4 + \log_2 n + \log_2 k$.*

Proof. Clearly, $b(\bar{P}) \leq |\bar{P}|^{1/2} \leq p^{kn/2}$.

By [13, Theorem 12.26] and the nilpotency of $\bar{P}$, we have that $\bar{P}$ has an abelian subgroup B of index at most $b(\bar{P})^4$. By [28, Theorem 5.1], we deduce that $\bar{P}$ has a normal abelian subgroup A of index at most $|\bar{P} : B|^2$. Thus, $|\bar{P} : A| \leq |\bar{P} : B|^2 \leq b(\bar{P})^{8s}$, where $b(\bar{P}) = p^s$. By [11, Satz III.2.12], $\mathrm{dl}(\bar{P}/A) \leq 1 + \log_2(8s)$ and so $\mathrm{dl}(\bar{P}) \leq 2 + \log_2(8s) = 5 + \log_2(s)$. Since s is at most $kn/2$, we have $\mathrm{dl}(\bar{P}) \leq 4 + \log_2 n + \log_2 k$. □

Theorem 3.7. *Let G be a finite p-solvable group for an odd prime p, $P \in \mathrm{Syl}_p(G)$ and $\bar{P} = P/O_p(G)$; set $n = \bar{e}_p(G)$. We set $k = 4.5$ if $p \geq 5$, and $k = 17$ if $p = 3$. Then $|G : O_p(G)|_p \leq p^{kn}$, $b(\bar{P}) \leq p^{kn/2}$, and $\mathrm{dl}(\bar{P}) \leq 4 + \log_2 n + \log_2 k$.*

Proof. We first prove the assertion in the case when $p \geq 5$. In view of Lemma 3.6, we only need to show that $|G : O_p(G)|_p \leq p^{4.5n}$.

Let T be the maximal normal solvable subgroup of G. Since $O_p(G) \leq T$, $O_p(T) = O_p(G)$. Since $T \lhd G$, p^{n+1} does not divide $\lambda(1)$ for all $\lambda \in \mathrm{IBr}_p(T)$. Thus by Theorem 3.2, $|T : O_p(G)|_p \leq p^{2.5n}$.

Let $\tilde{G} = G/T$ and $\bar{G} = \tilde{G}/\mathbf{F}^*(\tilde{G})$. It is clear that $\mathbf{F}^*(\tilde{G})$ is a direct product of finite non-abelian simple groups. Since $\tilde{G}$ is p-solvable, $p \nmid |\mathbf{F}^*(\tilde{G})|$.

By Theorem 3.5, $|\bar{G}|_p \leq p^{2n}$, and we are done in this case.

We now consider the case when $p = 3$ and we only need to show that $|G : O_p(G)|_p \leq p^{17n}$ in view of Lemma 3.6. The proof is similar to the previous case when $p \geq 5$ but using Theorem 3.4 instead of Theorem 3.2. □

By the work of [6], and stated explicitly in [20, Lemma 3.1], we have the following result that is used in both the character context as well as the context of conjugacy classes:

Lemma 3.8. *Let S be a finite non-abelian simple group and let p be a prime dividing $|S|$. Then $|S|_p > |\mathrm{Out}(S)|_p$.*

In dealing with the simple groups, we need the following result which completes [31, Theorem 2.5] in that the remaining cases of alternating groups ($\mathfrak{A}_n$ for $n \in \{22, 24, 26\}$) are treated, and it is a slight correction as the exception in the case of $\mathfrak{A}_7$ at $p = 2$ was overlooked.

Theorem 3.9. *Let S be a finite non-abelian simple group, and let p be a prime divisor of $|S|$. Then there exists $\phi \in \mathrm{IBr}_p(S)$ such that*

$$|\mathrm{Aut}(S)|_p < \phi(1)_p^2$$

except in the following cases:

- *$p = 2$, $S = M_{22}$, then $|\mathrm{Aut}(S)|_2 = 2^8$, and $\bar{e}_2(S) = 1$;*
- *$p = 2$, $S = \mathfrak{A}_7$, then $|\mathrm{Aut}(S)|_2 = 2^4$, and $\bar{e}_2(S) = 2$;*
- *$p = 3$, $S = \mathfrak{A}_7$, then $|\mathrm{Aut}(S)|_3 = 3^2$ and $\bar{e}_3(S) = 1$.*

Proof. The precise statements in the listed exceptional cases are checked using the information on Brauer characters provided in tables coming from GAP [7]. If we are not in one of these cases, [31, Theorem 2.5] (in the corrected version, including the exception for $\mathfrak{A}_7$ at $p = 2$) tells us that there are possibly only the cases of $S = \mathfrak{A}_n$ with $n \in \{22, 24, 26\}$ at $p = 2$ where the desired inequality might not hold.

For $n = 22, 24$ and 26, we have $|\mathrm{Aut}(S)|_2 = 2^{19}, 2^{22}$ and 2^{23}, respectively; in these cases, the 2-Brauer character tables are not available, and using a similar argument as in [31] for finding a suitable Brauer character in a 2-block of smallest defect is not strong enough. So we have to use other methods to find $\phi \in \mathrm{IBr}_2(S)$ such that $\phi(1)_2$ is large.

We consider the Specht modules S^λ of $\mathfrak{S}_n$ labelled by the partitions $(10, 7, 4, 1)$ of 22, $(14, 7, 2, 1)$ of 24, and $(14, 7, 4, 1)$ of 26; the 2-powers in the degrees are 2^{13}, 2^{12} and 2^{14}, respectively, by the hook formula. By the Carter criterion [16, 24.9], in all three cases the 2-modular reduction is the corresponding irreducible module D^λ. Restricting these modules to $\mathfrak{A}_n$ gives irreducible modules for $\mathfrak{A}_n$ by Benson's criterion [2]. Hence the 2-powers in the degrees of the corresponding 2-Brauer characters are sufficiently large, as required. □

Corollary 3.10. *Let S be a finite non-abelian simple group, and let p be a prime divisor of $|S|$. Then there exists $\phi \in \mathrm{IBr}_p(S)$ such that $|\mathrm{Aut}(S)|_p < \phi(1)_p^2$ if $p \geq 5$, $|\mathrm{Aut}(S)|_p < \phi(1)_p^3$ if $p = 3$, and $|\mathrm{Aut}(S)|_p < \phi(1)_p^9$ if $p = 2$.*

Proof. This is a direct corollary of Theorem 3.9. □

Hypothesis 3.11. Let p be a prime and let $N = W_1 \times \cdots \times W_s$ be a normal subgroup of a finite group G with the following assumptions: $\mathbf{C}_G(N) = 1$; every W_i, $1 \leq i \leq s$, is a non-abelian simple group of order divisible by p.

Lemma 3.12. *Let G, N, p be as in Hypothesis 3.11. If there exists $\phi_i \in \mathrm{IBr}_p(W_i)$ such that $|\mathrm{Aut}(W_i)|_p < \phi_i(1)_p^k$ for every $i = 1, \dots, s$, then there exists $\phi \in \mathrm{IBr}_p(N)$ such that $|G|_p < \phi(1)_p^k$.*

Proof. The proof is the same as [29, Lemma 2.6]. □

Theorem 3.13. *Let G be a finite group, p be a prime, $P \in \mathrm{Syl}_p(G)$ and $\bar{P} = P/O_p(G)$; set $n = \bar{e}_p(G)$. We set $k = 6.5$ if $p \geq 5$, $k = 20$ if $p = 3$, and $k = 24$ if $p = 2$. Then $|G : O_p(G)|_p \leq p^{kn}$, $b(\bar{P}) \leq p^{kn/2}$, and $\mathrm{dl}(\bar{P}) \leq 4 + \log_2 n + \log_2 k$.*

Proof. Let T be the maximal normal p-solvable subgroup of G. Since $O_p(G) \leq T$, $O_p(T) = O_p(G)$. Since $T \lhd G$, p^{n+1} does not divide $\lambda(1)$, for all $\lambda \in \mathrm{IBr}_p(T)$.

If $p \geq 5$, then $|T : O_p(G)|_p \leq p^{4.5n}$ by Theorem 3.7. If $p = 3$, then $|T : O_p(G)|_p \leq p^{17n}$ by Theorem 3.7. If $p = 2$, then $|T : O_p(G)|_p \leq p^{15n}$ by Theorem 3.4.

We now consider $\bar{G} = G/T$, we know that $\mathbf{F}^*(\bar{G})$ is a direct product of non-abelian simple groups, where p divides the order of each of them.

Since $\bar{G}$ and $\mathbf{F}^*(\bar{G})$ satisfy Hypothesis 3.11, by Lemma 3.12 and Corollary 3.10, we have that $|\bar{G}|_p \leq p^{2n}$ if $p \geq 5$, $|\bar{G}|_p \leq p^{3n}$ if $p = 3$, and $|\bar{G}|_p \leq p^{9n}$ if $p = 2$.

Thus, we have,

(1) $|G : O_p(G)|_p \leq |G : T|_p |T : O_p(G)|_p \leq p^{6.5n}$ if $p \geq 5$.
(2) $|G : O_p(G)|_p \leq |G : T|_p |T : O_p(G)|_p \leq p^{20n}$ if $p = 3$.
(3) $|G : O_p(G)|_p \leq |G : T|_p |T : O_p(G)|_p \leq p^{24n}$ if $p = 2$.

The bounds for $b(\bar{P})$ and $\mathrm{dl}(\bar{P})$ follow from Lemma 3.6. □

4. On p-parts of p-regular conjugacy class sizes

We now start to prove results related to the p-parts of p-regular conjugacy class sizes.

With respect to the p-regular class size version of the problem, we make the following observations. We will use the following results very often in the proofs so we state them here.

Lemma 4.1. *Let N be a normal subgroup of G. Then*

(1) *If $x \in N$, $|x^N|$ divides $|x^G|$.*
(2) *If $x \in G$, $|(xN)^{G/N}|$ divides $|x^G|$.*

Remark 4.2. We first observe that the condition p^k does not divide $|x^G|$ for every p-regular element $x \in G$ is inherited by all the normal subgroups of G and all the quotient groups of G. Since the normal subgroups case easily follows from Lemma 4.1(1), we will just explain for the quotient groups. Let $N \lhd G$, and T be a p-regular class of G/N then we have a p-regular element $xN \in G/N$ such that $T = (xN)^{G/N}$. We may write $x = yz$, where y is a p'-element, z is a p-element and $yz = zy$. Let $H = \langle x \rangle N$, we know that $|H/N|$ is a p' number, and thus $z \in N$. We have $xN = yN$, and $T = (yN)^{G/N}$. We have that $|T| \mid |y^G|$ and the result follows.

Theorem 4.3. *Let G be a solvable group with $O_p(G) = 1$, and let $P \in \mathrm{Syl}_p(G)$. Set $n = \overline{ecl}_p(G)$. Then $|G|_p \leq p^{15n}$ if $p = 2$ or $p = 3$. In particular, $e_p(G) \leq 15n$, $b(P) \leq p^{7.5n}$, and $\mathrm{dl}(P)$ is bounded by a logarithmic function of n.*

Proof. By Gaschütz's theorem, $G/\mathbf{F}(G)$ acts faithfully and completely reducibly on $\mathbf{F}(G)/\Phi(G)$. Since $p \nmid |\mathbf{F}(G)/\Phi(G)|$, every element in $\mathbf{F}(G)/\Phi(G)$ is a p'-element. It follows from [32, Theorem 3.3] that there exists $x \in \mathbf{F}(G)/\Phi(G)$ such that $T = \mathbf{C}_G(x) \leq \mathbf{F}_8(G)$.

Let $K_{i+1} = \mathbf{F}_{i+1}(G)/\mathbf{F}_i(G)$ and let $K_{i+1,p}$ be the Sylow p-subgroup of K_{i+1} for all $i \geq 1$. We know that $K_{i+1,p}$ acts faithfully and completely reducibly on $K_i/\Phi(G/\mathbf{F}_{i-1}(G))$. It is clear that we may write $K_i/\Phi(G/\mathbf{F}_{i-1}(G)) = V_{i1} + V_{i2}$ where V_{i1} is the p-part of $K_i/\Phi(G/\mathbf{F}_{i-1}(G))$ and V_{i2} is the p'-part of $K_i/\Phi(G/\mathbf{F}_{i-1}(G))$ for all $i \geq 1$.

We observe that $K_{i+1,p}$ acts faithfully and completely reducibly on V_{i2} for all $i \geq 1$. Since $p \nmid |V_{i2}|$, every element in V_{i2} is a p'-element. We have $|K_{i+1,p}| \leq p^{2n}$ by Lemma 3.1.

Next, we show that $|G : T|_p \leq p^n$.

We now consider $|x^G|$; clearly $|G : T|_p$ divides $|x^G|$, hence is at most p^n.

We know from before that $|K_{i,p}| \leq p^{2n}$ for $2 \leq i \leq 8$. This implies that $|G|_p \leq (p^{2n})^7 \cdot p^n = p^{15n}$. □

Theorem 4.4. *Let G be a solvable group with $O_p(G) = 1$ where $p \geq 5$ is a prime, and let $P \in \mathrm{Syl}_p(G)$; set $n = \overline{ecl}_p(G)$. Then $|G|_p \leq p^{2.5n}$. In particular, $e_p(G) \leq 2.5n$, $b(P) \leq p^{1.25n}$, and $\mathrm{dl}(P)$ is bounded by a logarithmic function of n.*

Proof. Let $|G|_p = p^a$. By [33], the group G has a p-block of defect $d \leq \frac{3}{5}a$. Now G has a p-regular element $x \in G$ such that $|C_G(x)|_p = p^d$ (see [13, Section 15]). Hence $|x^G|_p = p^{a-d}$, which implies that $a - d \leq n$, and thus $a \leq \frac{5}{2}n$. □

We now state the class size version of Theorem 2.4.

Theorem 4.5. *Let $V \trianglelefteq G$, where G/V is p-solvable for an odd prime p, and V is a direct product of isomorphic non-abelian simple groups $S_1, \ldots, S_n$. Suppose that G acts transitively on the groups $S_1, \ldots, S_n$, and write $O = \bigcap_k N_G(S_k)$. Then there exist nonidentity v_1, v_2 and $v_3 \in \mathrm{cl}(V)$ of different sizes such that all Sylow p-subgroups of $\mathbf{C}_G(v_j)$ are contained in O for all $j = 1, 2, 3$.*

Proof. The proof is similar to the proof of Theorem 2.4 but using Lemma 2.3 instead of Lemma 2.2. □

We now prove the conjugacy class analogues of Theorem 3.5 and Theorem 3.7.

Theorem 4.6. *Let G be a p-solvable group for an odd prime p. Assume that G has no nontrivial solvable normal subgroup. Then there exists $C \in \mathrm{cl}_{p'}(G)$ such that $|C|_p \geq \sqrt{|G|_p}$.*

Proof. The proof is similar to the proof of Theorem 3.5 but using Theorem 4.5 instead of Theorem 2.4. □

Lemma 4.7. *Let S be a finite non-abelian simple group and $p \geq 3$ be a prime divisor of $|S|$, then there exists $C \in \mathrm{cl}_{p'}(S)$ such that $|\mathrm{Aut}(S)|_p < |C|_p^2$.*

Proof. For the simple groups of Lie type and any prime p, or the alternating groups and $p \geq 5$, there is always a p-block of defect 0. Hence there is a p-regular element $x \in G$ such that $|C_G(x)|_p = 1$, and thus $|x^G|_p = |G|_p$. Then the result follows from Lemma 3.8.

Thus one only needs to consider the alternating groups and $p = 3$.

First assume that n is odd. If α is an n-cycle, then $\alpha \in \mathfrak{A}_n$ and $|\mathrm{cl}_{\mathfrak{A}_n}(\alpha)| = \frac{1}{2}(n-1)!$. If β is an $(n-2)$-cycle, then $\beta \in \mathfrak{A}_n$ and $|\mathrm{cl}_{\mathfrak{A}_n}(\beta)| = n!/((n-2)2)$. Now if $3 \nmid n$, then the class of α satisfies the condition. If $3 \mid n$, then $3 \nmid n-2$ and the class of β satisfies the condition.

Now let n be even. If α is an $(n-1)$-cycle, then $\alpha \in \mathfrak{A}_n$ and $|\mathrm{cl}_{\mathfrak{A}_n}(\alpha)| = \frac{1}{2} \cdot \frac{n!}{n-1}$. If β is an $(n-3)$-cycle, then $\beta \in \mathfrak{A}_n$ and $|\mathrm{cl}_{\mathfrak{A}_n}(\beta)| = \frac{n!}{(n-3)\cdot 6}$. Now if $3 \nmid n-1$, then the class of α satisfy the condition. If $3 \mid n-1$, then $3 \nmid n-3$ and the class of β satisfies the condition.

For sporadic groups, the result can be checked by using [4]. □

Given a group G, we write $b^*(G)$ to denote the largest size of the conjugacy classes of G.

Lemma 4.8. *Let G be a finite group, $P \in \mathrm{Syl}_p(G)$ and $\bar{P} = P/O_p(G)$; set $n = \overline{ecl}_p(G)$. Assume that $|G : O_p(G)|_p \leq p^{kn}$. Then $b^*(\bar{P}) \leq p^{kn}$, and $|\bar{P}'| \leq p^{kn(kn+1)/2}$.*

Proof. It is clear that for $x \in \bar{P}$, we have $|x^{\bar{P}}| = |\bar{P} : \mathbf{C}_{\bar{P}}(x)| \leq p^{kn}$.

To obtain the bounds for the order of $\bar{P}'$ it suffices to apply a theorem of Vaughan-Lee [12, Theorem VIII.9.12]. □

Theorem 4.9. *Let G be a finite p-solvable group for an odd prime p, $P \in \mathrm{Syl}_p(G)$, $\bar{P} = P/O_p(G)$; set $n = \overline{ecl}_p(G)$. Then there exists a constant k such that $|G : O_p(G)|_p \leq p^{kn}$, $b^*(\bar{P}) \leq p^{kn}$, and $|\bar{P}'| \leq p^{kn(kn+1)/2}$ where $k = 4.5$ if $p \geq 5$, and $k = 17$ if $p = 3$.*

Proof. This is the class size version of Theorem 3.7, and the proof is similar. We first obtain the bound for $|G : O_p(G)|_p$, and then apply Lemma 4.8 to obtain the other parts. □

Lemma 4.10. *Let G, N, p be as in Hypothesis 3.11. If there exists $C_i \in \mathrm{cl}_{p'}(W_i)$ such that $|\mathrm{Aut}(W_i)|_p < |C_i|_p^k$ for every $i = 1, \ldots, s$, then there exists $C \in \mathrm{cl}_{p'}(N)$ such that $|G|_p < |C|_p^k$.*

Proof. The proof is the same as that of [29, Lemma 2.6]. □

Theorem 4.11. *Let G be a finite group, p a prime, $P \in \mathrm{Syl}_p(G)$ and $\bar{P} = P/O_p(G)$; set $n = \overline{ecl}_p(G)$. We set $k = 6.5$ if $p \geq 5$, $k = 19$ if $p = 3$, and $k = 17$ if $p = 2$. Then $|G : O_p(G)|_p \leq p^{kn}$, $b^*(\bar{P}) \leq p^{kn}$, and $|\bar{P}'| \leq p^{kn(kn+1)/2}$.*

Proof. Let T be the maximal normal p-solvable subgroup of G. Since $O_p(G) \leq T$, $O_p(T) = O_p(G)$. Since $T \lhd G$, p^{n+1} does not divide $|C|$ for all $C \in \mathrm{cl}_{p'}(T)$.

If $p \geq 5$, then $|T : O_p(G)|_p \leq p^{4.5n}$ by Theorem 4.9. If $p = 3$, then $|T : O_p(G)|_p \leq p^{17n}$ by Theorem 4.9. If $p = 2$, then $|T : O_p(G)|_p \leq p^{15n}$ by Theorem 4.3.

We now consider $\bar{G} = G/T$, we know that $\mathbf{F}^*(\bar{G})$ is a direct product of non-abelian simple groups, where p divides the order of each of them.

Since $\bar{G}$ and $\mathbf{F}^*(\bar{G})$ satisfy Hypothesis 3.11, by Lemma 4.10 and Lemma 4.7, we have that $|\bar{G}|_p \leq p^{2n}$.

Thus, we have,

(1) $|G : O_p(G)|_p \leq |G : T|_p |T : O_p(G)|_p \leq p^{6.5n}$ if $p \geq 5$.
(2) $|G : O_p(G)|_p \leq |G : T|_p |T : O_p(G)|_p \leq p^{19n}$ if $p = 3$.
(3) $|G : O_p(G)|_p \leq |G : T|_p |T : O_p(G)|_p \leq p^{17n}$ if $p = 2$.

The bounds for $b^*(\bar{P})$ and $|\bar{P}'|$ follow from Lemma 4.8. □

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