

Spectral characterization of the complete graph removing a path of small length*

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Abstract

A graph G is said to be *determined by its spectrum* if any graph having the same spectrum as G is isomorphic to G . Let $K_n \setminus P_\ell$ be the graph obtained from K_n by removing edges of P_ℓ , where P_ℓ is a path of length $\ell - 1$ which is a subgraph of a complete graph K_n . Cámara and Haemers [11] conjectured that $K_n \setminus P_\ell$ is determined by its adjacency spectrum for every $2 \leq \ell \leq n$. In this paper we show that the conjecture is true for $7 \leq \ell \leq 9$.

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1 Introduction

All graphs considered in this paper are undirected, finite and simple graphs. For some notations and terminologies in graph spectra, see [1].

Let $G = (V, E)$ be a graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \dots, e_m\}$. Let $A(G)$ be the $(0,1)$ -adjacency matrix of G , the *characteristic polynomial* of G is defined as $P_G(\lambda) = \det(\lambda I - A(G))$. The *spectrum* of G consists of all the eigenvalues of G (including the multiplicities). Two graphs are *cospectral* if they share the same adjacency spectrum. A graph G is said to be *determined by its spectrum* (DS for short) if any graph having the same spectrum as G is necessarily isomorphic to G .

The spectrum of a graph encodes useful combinatorial information about the given graph. A fundamental question in the theory of graph spectra is “Which graphs are DS?”. The problem dates back to more than 60 years ago and originates from Chemistry. It has received a lot of attention from researcher in recent years.

However, it turns out that proving a graph to be DS is generally a very hard problem. Up to now, very few classes of graphs with very special structures have been proved to be DS. Usually it is case that the graphs shown to be DS have very few edges, such as the

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T-shape trees [5], the ∞ -graphs [6], the lollipop graphs [7], the θ -graphs [8], the graphs with index at most $\sqrt{2 + \sqrt{5}}$ [9], and the pineapple graphs [10], to just name a few. For dense graphs, it is usually quite difficult to show them to be DS, for example, the complement of the path \bar{P}_n was shown to be DS in [4], but the proof is much more involved than the proof that the path P_n is DS. For some excellent surveys of this topic, we refer the reader to van Dam and Haemers [2, 3] and the references therein.

In [11], Cámara and Haemers investigated, among others, when a complete graph with some edges deleted is DS. Denoted by P_ℓ a path of length $\ell - 1$ and K_n the complete graph on n vertices. Denoted by $K_n \setminus P_\ell$ the graph obtained from K_n by removing the edges of the path P_ℓ . The authors proposed the following

Conjecture 1 (Cámara and Haemers [11]). $K_n \setminus P_\ell$ is DS for every $2 \leq \ell \leq n$.

It was shown in [11] that Conjecture 1 is true for $\ell \leq 6$. And for $n = \ell$, Conjecture 1 is true and it is the main result from [4]. In this paper we show that Conjecture 1 is true for $7 \leq \ell \leq 9$. Thus we have the following

Theorem 1.1. *The graph $K_n \setminus P_\ell$ is DS for $7 \leq \ell \leq 9$.*

The proof of the above theorem is based on some eigenvalue properties of the graph $K_n \setminus P_\ell$, and a detailed classification of all of its possible cospectral mates.

The rest of the paper is organized as follows. In the next section, we will give some important lemmas that will be needed in the sequel. In Section 3, we present the proof of Theorem 1.1. Conclusions and some further research problems are given in Section 4.

2 Some lemmas

In this section, we will present some lemmas which are needed in the proof of the main result. First we give some known results about the spectra of graphs.

Lemma 2.1 (van Dam and Haemers [2]). *The following properties of a graph G can be deduced from the adjacency spectrum:*

- (i) *The number of vertices.*
- (ii) *The number of edges.*
- (iii) *The number of closed walks of any fixed length.*

Let $N_G(H)$ be the number of subgraphs (not necessarily induced) of a graph G which are isomorphic to H and let $N_G(i)$ be the number of closed walks of length i in G . Let $N'_H(i)$ be the number of closed walks of H of length i which contain all the edges of H and let $S_i(G)$ be the set consisting of all the connected subgraph H of G such that $N'_H(i) \neq 0$. It is easy to see that $N_G(i)$ can be expressed by

$$N_G(i) = \sum_{H \in S_i(G)} N_G(H) N'_H(i).$$

Lemma 2.2 (Omid [13]). *The number of closed walks of length of 2, 3, 4 and 5 of a graph G are given as follows:*

- (i) $N_G(2) = 2m, N_G(3) = 6N_G(K_3)$;
- (ii) $N_G(4) = 2m + 4N_G(P_3) + 8N_G(C_4), N_G(5) = 30N_G(K_3) + 10N_G(C_5) + 10N_G(G_a)$.

Where m is the number of edges of G and graph G_a denotes the graph obtained from a triangle by adding a pendent edge to one of its vertices.

The following lemma is useful which gives the number of triangles of the complement of a graph G in terms of that of G and the numbers of 4-walks and 5-walks of G .

Lemma 2.3 (Doob and Haemers [4]). *Let G be a graph with n vertices, m edges, t triangles, and degree sequence d_1, d_2, \dots, d_n . Let \bar{t} be the number of triangles in the complement of G . Then*

$$\bar{t} = \binom{n}{3} - (n-1)m + \frac{1}{2} \sum_{i=1}^n d_i^2 - t.$$

Lemma 2.4 (Cámara and Haemers [11]). *The number of 4-walks in the complement of a graph G only depends on the number of vertices and edges of G , and the number of different subgraphs (not necessarily induced) in G isomorphic to $P_3, K_2 \cup K_2, P_4$ and C_4 . More precisely, if these numbers are n, m, m_1, m_2, m_3 , and m_4 , and $W_n = (n-1)^4 + n-1$ is the number of 4-walks in K_n , then the number of 4-walks in the complement of G equals*

$$W_n - (8n^2 - 32n + 34)m + (8n - 20)m_1 + 16m_2 - 8m_3 + 8m_4,$$

where

$$\begin{aligned} n &:= |V(G)|, m := |E(G)|, m_1 := N_G(P_3), \\ m_2 &:= N_G(K_2 \cup K_2), m_3 := N_G(P_4), m_4 := N_G(C_4). \end{aligned}$$

For closed walks of length 5-walks things become more complicated. We have the following lemma.

Lemma 2.5. *The number of 5-walks in the complement of a graph G only depends on the number of vertices and edges of G , and the number of different subgraphs (not necessarily induced) in G which are isomorphic to $P_3, K_2 \cup K_2, P_4, K_3, P_3 \cup K_2, K_{1,3}, P_5, G_a$ and C_5 . More precisely, let these numbers be $n, m, m_1, m_2, m_3, s_1, s_2, s_3, s_4, s_5$ and s_6 , and let $W_n = 30\binom{n}{3} + 120\binom{n}{5} + 30(n-3)\binom{n}{3}$ be the number of 5-walks in K_n . Then the number of 5-walks in the complement of G equals*

$$\begin{aligned} W_n - (10n^3 - 50n^2 + 90n - 60)m + (10n^2 - 20n)m_1 + (40n - 120)m_2 \\ - (10n - 20)m_3 - (30n - 60)s_1 - 20s_2 - 30s_3 + 10s_4 + 10s_5 - 10s_6, \end{aligned}$$

where

$$\begin{aligned} s_1 &:= N_G(K_3), s_2 := N_G(P_3 \cup K_2), s_3 := N_G(K_{1,3}), \\ s_4 &:= N_G(P_5), s_5 := N_G(G_a), s_6 := N_G(C_5). \end{aligned}$$

Proof. The result is a consequence of the inclusion-exclusion principle. Assume that G and K_n have the same vertex set. Let E be the edge set of G . For a subset $F \subset E$. Let

W_F denote the set of 5-walks in K_n containing all edges of F . Then the total number of 5-walks in K_n that contain at least one edge from E equals

$$\left| \bigcup_{|F| \geq 1} W_F \right| = \sum_{|F|=1} |W_F| - \sum_{|F|=2} |W_F| + \sum_{|F|=3} |W_F| - \sum_{|F|=4} |W_F| + \sum_{|F|=5} |W_F|.$$

If $|F| = 1$, then $|W_F| = 10(n-2)(n-3)(n-4) + 40(n-2)(n-3) + 30(n-2)$. If $|F| = 2$, then $|W_F|$ depends on the mutual position of the two edges. If they have a vertex in common, then $|W_F| = 10(n-3)(n-4) + 50(n-3) + 30$, and if the two edges are independent then $|W_F| = 40(n-4) + 40$. If $|F| = 3$, then the three edges are $P_4, K_3, P_3 \cup K_2, K_{1,3}$ in G and if the three edges are a path P_4 then $|W_F| = 10(n-4) + 20$, if the three edges are a triangle K_3 then $|W_F| = 30(n-3) + 30$, if the three edges are $P_3 \cup K_2$, then $|W_F| = 20$, if the three edges are a star $K_{1,3}$ then $|W_F| = 30$. If $|F| = 4$, then the four edges are a path or a triangle with a pendant edge to one vertex. If the four edges are a path, then $|W_F| = 10$ and if the four edges are a triangle with a pendant edge then $|W_F| = 10$. Suppose $|F| = 5$, then the edges are a cycle of length 5 in G . Each of them leads to 10 distinct 5-walks, so $|W_F| = 10$. \square

Suppose $K_n \setminus P_\ell$ and $K_n \setminus H$ are cospectral. Here and below we define that m_i (resp. m'_i), s_j (resp. s'_j) to be the number of $P_3, K_2 \cup K_2, P_4, C_4, K_3, P_3 \cup K_2, K_{1,3}, P_5, G_a, C_5$ in graph P_ℓ (resp. graph H) for $i = 1, 2, 3, 4, j = 1, 2, 3, 4, 5, 6$.

Lemma 2.6. *The pair of graphs $K_n \setminus P_\ell$ and $K_n \setminus (C_4 \cup P_{\ell-4})$ are not cospectral for any $\ell \geq 7$.*

Proof. For graph P_ℓ we can directly compute that

$$m_1 = \ell - 2, m_2 = \frac{(\ell-2)(\ell-3)}{2}, m_3 = \ell - 3, m_4 = 0.$$

And for graph $C_4 \cup P_b$ we have

$$m'_1 = \ell - 2, m'_2 = \frac{(\ell-2)(\ell-3)}{2}, m'_3 = \ell - 3, m'_4 = 1.$$

According to Lemma 2.4, it follows that the pair of graphs in the lemma can be distinguished by the number of 4-walks. \square

Lemma 2.7. *The pair of graphs $K_n \setminus P_\ell$ and $K_n \setminus (aK_{1,3} \cup P_b)$ are not cospectral for any $a \geq 1, b \geq 2$, where $3a + b = \ell$.*

Proof. For graph P_ℓ we can directly compute that

$$m_1 = \ell - 2, m_2 = \frac{(\ell-2)(\ell-3)}{2}, m_3 = \ell - 3, m_4 = 0.$$

And for graph $aK_{1,3} \cup P_2$ and $aK_{1,3} \cup P_3$ we have

$$m'_1 = \ell - 2, m'_2 = \frac{(\ell-2)(\ell-3)}{2}, m'_3 = 0, m'_4 = 0.$$

And for graph $aK_{1,3} \cup P_b$ ($b \geq 4$) we have

$$m'_1 = \ell - 2, m'_2 = \frac{(\ell - 2)(\ell - 3)}{2}, m'_3 = b - 3, m'_4 = 0.$$

By use of Lemma 2.4 it follows straightforwardly that they can be distinguished by the number of 4-walks. \square

Lemma 2.8. *The pair of graphs $K_n \setminus P_\ell$ and $K_n \setminus (P_a \cup T_{b,c,d})$ are not cospectral for any $a \geq 2, b \geq 1, c \geq 1, d \geq 1$, where $a + b + c + d = \ell$ ($\ell \geq 6$).*

Proof. For graph P_ℓ we can directly compute that

$$m_1 = \ell - 2, m_2 = \frac{(\ell - 2)(\ell - 3)}{2}, m_3 = \ell - 3, m_4 = 0,$$

$$s_2 = (\ell - 3)(\ell - 4), s_4 = \ell - 4, s_1 = s_3 = s_5 = s_6 = 0.$$

And for graph $P_a \cup T_{b,c,d}$, we can also compute the corresponding number of subgraphs; see Table 1 below:

Table 1: The number of subgraphs in $P_a \cup T_{b,c,d}$

(a, b, c, d)	m'_1	m'_2	m'_3	m'_4	s'_1	s'_2	s'_3	s'_4	s'_5	s'_6
$a \geq 4, b = 1, c = 1, d = 1$	$\ell - 2$	$\frac{(\ell - 2)(\ell - 3)}{2}$	$\ell - 6$	0						
$a \geq 3, b = 1, c = 1, d \geq 2$	$\ell - 2$	$\frac{(\ell - 2)(\ell - 3)}{2}$	$\ell - 5$	0						
$a = 2, b = 1, c = 1, d \geq 2$	$\ell - 2$	$\frac{(\ell - 2)(\ell - 3)}{2}$	$\ell - 4$	0						
$a \geq 3, b = 1, c \geq 2, d \geq 2$	$\ell - 2$	$\frac{(\ell - 2)(\ell - 3)}{2}$	$\ell - 4$	0						
$a = 2, b \geq 2, c \geq 2, d \geq 2$	$\ell - 2$	$\frac{(\ell - 2)(\ell - 3)}{2}$	$\ell - 2$	0						
$a = 2, b = 1, c = 2, d = 2$	$\ell - 2$	$\frac{(\ell - 2)(\ell - 3)}{2}$	$\ell - 3$	0		9	1	1		
$a = 2, b = 1, c = 2, d \geq 3$	$\ell - 2$	$\frac{(\ell - 2)(\ell - 3)}{2}$	$\ell - 3$	0	0	$\ell^2 - 7\ell + 9$	1	$\ell - 5$	0	0
$a = 2, b = 1, c \geq 3, d \geq 3$	$\ell - 2$	$\frac{(\ell - 2)(\ell - 3)}{2}$	$\ell - 3$	0	0	$\ell^2 - 7\ell + 9$	1	$\ell - 4$	0	0
$a = 3, b = 2, c = 2, d = 2$	$\ell - 2$	$\frac{(\ell - 2)(\ell - 3)}{2}$	$\ell - 3$	0	0	$\ell^2 - 7\ell + 9$	1	$\ell - 6$	0	0
$a \geq 4, b = 2, c = 2, d = 2$	$\ell - 2$	$\frac{(\ell - 2)(\ell - 3)}{2}$	$\ell - 3$	0	0	$\ell^2 - 7\ell + 9$	1	$\ell - 7$	0	0
$a = 3, b = 2, c = 2, d \geq 3$	$\ell - 2$	$\frac{(\ell - 2)(\ell - 3)}{2}$	$\ell - 3$	0	0	$\ell^2 - 7\ell + 9$	1	$\ell - 5$	0	0
$a \geq 4, b = 2, c = 2, d \geq 3$	$\ell - 2$	$\frac{(\ell - 2)(\ell - 3)}{2}$	$\ell - 3$	0	0	$\ell^2 - 7\ell + 9$	1	$\ell - 6$	0	0
$a = 3, b = 2, c \geq 3, d \geq 3$	$\ell - 2$	$\frac{(\ell - 2)(\ell - 3)}{2}$	$\ell - 3$	0	0	$\ell^2 - 7\ell + 9$	1	$\ell - 4$	0	0
$a \geq 4, b = 2, c \geq 3, d \geq 3$	$\ell - 2$	$\frac{(\ell - 2)(\ell - 3)}{2}$	$\ell - 3$	0	0	$\ell^2 - 7\ell + 9$	1	$\ell - 5$	0	0
$a = 3, b \geq 3, c \geq 3, d \geq 3$	$\ell - 2$	$\frac{(\ell - 2)(\ell - 3)}{2}$	$\ell - 3$	0	0	$\ell^2 - 7\ell + 9$	1	$\ell - 3$	0	0
$a \geq 4, b \geq 3, c \geq 3, d \geq 3$	$\ell - 2$	$\frac{(\ell - 2)(\ell - 3)}{2}$	$\ell - 3$	0	0	$\ell^2 - 7\ell + 9$	1	$\ell - 4$	0	0

From Table 1 we know that for only the case $a \geq 4, b = 2, c = 2, d = 2$ the pair of graphs in the lemma have the same number of 4-walks and 5-walks. However, the adjacency matrix A of $K_n \setminus P_\ell$ satisfies $\text{rank}(A) \geq n - 1$ whilst the adjacency matrix A' of $K_n \setminus (P_{\ell-6} \cup T_{2,2,2})$ satisfies $\text{rank}(A') \leq n - 2$. Thus the two graphs have different multiplicities for the eigenvalue 0. \square

Lemma 2.9. *Suppose that the pair of graphs $K_n \setminus P_\ell$ and $K_n \setminus H$ are cospectral. Then the number of triangles t' in H must be even.*

Proof. We use the same notations d_i, d'_i, m_i, m'_i and t' as above. If graph $K_n \setminus P_\ell$ and $K_n \setminus H$ are cospectral, then they have the same number of 3-walks. By Lemma 2.3 we have

$$\sum_{i=1}^n d_i'^2 - t' = \sum_{i=1}^n d_i^2.$$

Graphs $K_n \setminus P_\ell$ and $K_n \setminus H$ also have the same number of 4-walks. By Lemma 2.4 we have

$$\begin{aligned} & (8n - 20)m'_1 + 16m'_2 - 8m'_3 + 8m'_4 \\ &= (8n - 20)m_1 + 16m_2 - 8m_3 + 8m_4 \\ &= (8n - 20)m_1 + 8(\ell - 3)^2. \end{aligned}$$

Moreover,

$$m'_1 = \sum_{i=1}^n \binom{d'_i}{2} = \sum_{i=1}^n \binom{d_i}{2} + t' = m_1 + t'.$$

so we have

$$(2n - 5)t' + 4m'_2 - 2m'_3 + 2m'_4 = 2(\ell - 3)^2.$$

As $4m'_2, 2m'_3, 2m'_4$ and $2(\ell - 3)^2$ are all even numbers, $2n - 5$ is an odd number, so the number of triangles t' in H must be even. \square

Let Γ be a graph with $|V(\Gamma)| = \ell$, and $C = (C_{ij})_{\ell \times k}$ be a $(0, 1)$ -matrix. We construct a new graph, denoted by $(\Gamma, C, n - \ell)$, which is obtained from the disjoint union of Γ and k copies of the complete graph $K_{n-\ell}$ by adding some edges according to the following rule: if $C_{ij} = 1$, then each vertex of the j -th complete graph $K_{n-\ell}$ is adjacent to vertices $i \in V(\Gamma)$ and is not adjacent to vertices $i \in V(\Gamma)$ with $C_{ij} = 0$ (for $i = 1, 2, \dots, \ell; j = 1, 2, \dots, k$) (see Fig. 1).

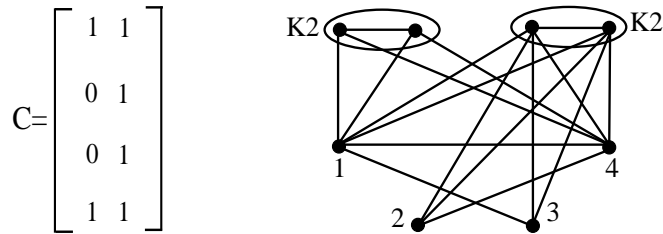


Fig. 1: The graph $(\Gamma, C, n - \ell)$ for $\Gamma = P_4$.

Lemma 2.10 (Jing and Koolen [14]). $\lambda_{\min}(\Gamma, C, n - \ell) \geq \lambda_{\min}(\Gamma, C, n - \ell + 1)$, $\lambda_{\min}(\Gamma, C, n - \ell) \geq \lambda_{\min}(\mathcal{A} - CC^T)$ and $\lim_{n \rightarrow \infty} \lambda_{\min}(\Gamma, C, n - \ell) = \lambda_{\min}(\mathcal{A} - CC^T)$, where $\mathcal{A} = A(\Gamma)$.

A vector $[x_1, \dots, x_n]$ is called *skew palindromic* if $[x_1, \dots, x_n]^T = -[x_n, \dots, x_1]$.

Lemma 2.11 (Doob and Haemers [4]). *Suppose \bar{A} is the adjacency matrix of $K_\ell \setminus P_\ell$, then \bar{A} has $\lfloor \frac{\ell}{2} \rfloor$ known eigenvalues $\bar{\lambda}_i = -1 + 2 \cos \pi \frac{i-1}{\ell+1}$ with skew palindromic eigenvectors ξ_i if $2 \leq i \leq \ell$ and $i \equiv \ell \pmod{2}$.*

Suppose $B = \begin{bmatrix} \bar{A} & J_{\ell \times (n-\ell)} \\ J_{(n-\ell) \times \ell} & J_{(n-\ell) \times (n-\ell)} - I \end{bmatrix}$ is the adjacency matrix of $K_n \setminus P_\ell$. Then B also has $\lfloor \frac{\ell}{2} \rfloor$ known eigenvalues $\bar{\lambda}_i = -1 + 2 \cos \pi \frac{i-1}{\ell+1}$ with skew palindromic eigenvectors $[\xi_i, 0]$ since ξ_i is a skew palindromic eigenvector, orthogonal to the all-one vector $\mathbf{1}$.

Lemma 2.12 (Doob and Haemers [4]). *If $\ell > 2$, then every eigenvalue of $K_\ell \setminus P_\ell$ has multiplicity one.*

The following lemma lies at the heart of the proof of Theorem 1.1.

Lemma 2.13. *Let $n > 2$. Then every eigenvalue of graph $K_n \setminus P_\ell$ has multiplicity one, except for -1 . The multiplicity of -1 is $n - \ell$ if ℓ is odd, and it is $n - \ell - 1$ if ℓ is even. Moreover, we have $\lambda_{\min}(K_n \setminus P_\ell) > -3$.*

Proof. First we give the eigenvectors associated with eigenvalue -1 explicitly. We distinguish the following cases:

Case 1. For $\ell = 2k$, there are $n - \ell - 1$ eigenvectors:

$$\begin{aligned} & \overbrace{(0, 0, \dots, 0, -1, 1, 0, \dots, 0, 0)}^\ell, \\ & (0, 0, \dots, 0, -1, 0, 1, \dots, 0, 0), \\ & \vdots \\ & (0, 0, \dots, 0, -1, 0, 0, \dots, 1, 0), \\ & (0, 0, \dots, 0, -1, 0, 0, \dots, 0, 1). \end{aligned}$$

Case 2. For $\ell = 4k + 1$, there are $n - \ell$ eigenvectors:

$$\begin{aligned} & \overbrace{(-1, 0, 1, 0, -1, 0, 1, 0, \dots, -1, 0, 1, 0, -1, 1, 0, \dots, 0, 0)}^\ell, \\ & (-1, 0, 1, 0, -1, 0, 1, 0, \dots, -1, 0, 1, 0, -1, 0, 1, \dots, 0, 0), \\ & \vdots \\ & (-1, 0, 1, 0, -1, 0, 1, 0, \dots, -1, 0, 1, 0, -1, 0, 0, \dots, 1, 0), \\ & (-1, 0, 1, 0, -1, 0, 1, 0, \dots, -1, 0, 1, 0, -1, 0, 0, \dots, 0, 1). \end{aligned}$$

Case 3. For $\ell = 4k + 3$, there are $n - \ell$ eigenvectors:

$$\begin{aligned} & \overbrace{(-1, 0, 1, 0, -1, 0, 1, 0, \dots, -1, 0, 1, 0, -1, 0, 1, 0, 0, \dots, 0)}^\ell, \\ & \overbrace{(0, 0, \dots, 0, -1, 1, 0, \dots, 0, 0)}^\ell, \\ & (0, 0, \dots, 0, -1, 0, 1, \dots, 0, 0), \\ & \vdots \\ & (0, 0, \dots, 0, -1, 0, 0, \dots, 1, 0), \\ & (0, 0, \dots, 0, -1, 0, 0, \dots, 0, 1). \end{aligned}$$

It follows from Lemma 2.10 that

$$\lambda_{\min}(K_n \setminus P_\ell) \geq \lambda_{\min}(\mathcal{A} - CC^T) = \lambda_{\min}(J - I - A - J) = \lambda_{\min}(-I - A),$$

where A is the adjacency matrix of the path P_ℓ .

The eigenvalues of P_ℓ (that is the eigenvalues of A) are well known to be

$$\lambda_i = 2 \cos \frac{\pi i}{\ell + 1}, i = 1, 2, \dots, \ell.$$

So the smallest eigenvalue of $K_n \setminus P_\ell$ satisfies that $\lambda_{\min}(K_n \setminus P_\ell) > -3$.

It remains to show that every eigenvalue of graph $K_n \setminus P_\ell$ has multiplicity one, except for -1. We prove this assertion by induction on k . First suppose $k = 1$, we shall show that $K_{\ell+1} \setminus P_\ell$ has multiplicity one.

From Lemma 2.12 $K_\ell \setminus P_\ell$ has ℓ different eigenvalues which will be ordered as $\lambda_1 > \lambda_2 > \dots > \lambda_\ell$ where $\lfloor \frac{\ell}{2} \rfloor$ known eigenvalues are given in lemma 2.11. $K_{\ell+1} \setminus P_\ell$ has $\ell + 1$ eigenvalues which will be ordered as $\mu_1 \geq \mu_2 \geq \dots \geq \mu_{\ell+1}$. The eigenvalues of $K_\ell \setminus P_\ell$ interlace those of $K_{\ell+1} \setminus P_\ell$, that is, $\mu_i \geq \lambda_i \geq \mu_{i+1}$ for $i = 1, 2, \dots, \ell$. Next we shall show that $\lfloor \frac{\ell}{2} \rfloor$ known eigenvalues have multiplicity 1. This proves this assertion, since every other eigenvalue lies between two eigenvalues with multiplicity 1.

Suppose $\bar{\lambda}$ is such an eigenvalue. Then, substituting $i = \ell + 2 - 2m$ gives $\bar{\lambda} = -1 - 2 \cos \varphi$ with $\varphi = 2\pi m / (\ell + 1)$ for some integer m , $1 \leq m \leq \ell/2$. Let $x = [x_1, x_2, \dots, x_\ell]^T$, $y_1 = [x_1, x_2, \dots, x_\ell, a]^T$ be an eigenvector for $\bar{\lambda}$. If $a = 0$, y_1 is an eigenvector of $K_\ell \setminus P_\ell$ and the corresponding eigenvalue is $2 \cos \varphi$. Suppose $a \neq 0$, $B_1 y_1 = \bar{\lambda} y_1$ (where $B_1 = \begin{bmatrix} J - I - A & \mathbf{1} \\ \mathbf{1}^T & 0 \end{bmatrix}$ is the adjacency matrix of $K_{\ell+1} \setminus P_\ell$) implies that

$$\begin{aligned} \mathbf{1}^T x &= \bar{\lambda} a, \\ x_2 &= a(\bar{\lambda} + 1) - (\bar{\lambda} + 1)x_1, \\ x_i &= a(\bar{\lambda} + 1) - (\bar{\lambda} + 1)x_{i-1} - x_{i-2}, \text{ for } i = 2, 3, \dots, \ell. \end{aligned}$$

The general solution of this recurrence has the form

$$x_i = \alpha \cos i\varphi + \beta \sin i\varphi + \frac{a(\bar{\lambda} + 1)}{(\bar{\lambda} + 3)}.$$

(Note that $\bar{\lambda} + 3 > 0$). Substituting $x_0 = 0$ gives $\alpha = -\frac{a(\bar{\lambda} + 1)}{(\bar{\lambda} + 3)}$. Moreover, $x_\ell = \alpha \cos \ell\varphi + \beta \sin \ell\varphi + \frac{a(\bar{\lambda} + 1)}{(\bar{\lambda} + 3)} = \alpha \cos \varphi - \beta \sin \varphi + \frac{a(\bar{\lambda} + 1)}{(\bar{\lambda} + 3)}$. Hence

$$x_1 + x_\ell = 2\alpha \cos \varphi + \frac{2a(\bar{\lambda} + 1)}{(\bar{\lambda} + 3)} = a(\bar{\lambda} + 1).$$

Next we equate $\mathbf{1}^T(\bar{A}x)$ to $(\mathbf{1}^T \bar{A})x$ to get $\bar{\lambda}^2 - \ell = (\ell - 3)\bar{\lambda} + (\bar{\lambda} + 1)$. Thus we find $\ell = \bar{\lambda} + 1 - \frac{2}{\bar{\lambda} + 1}$ which cannot be an integer; a contradiction.

Next, suppose $k \geq 2$. By induction, we have $K_{\ell+k} \setminus P_\ell$ has different eigenvalues (except for -1) which will be ordered as $\lambda_1 > \lambda_2 > \dots > -1 = -1 = \dots = -1 > \dots > \lambda_{\ell+k}$ where $\lfloor \frac{\ell}{2} \rfloor$ known eigenvalues are given in lemma 2.11. $K_{\ell+k+1} \setminus P_\ell$ has $\ell + k + 1$ eigenvalues which will be ordered as $\mu_1 \geq \mu_2 \geq \dots \geq \mu_{\ell+k+1}$. The eigenvalues of $K_{\ell+k} \setminus P_\ell$ interlace

those of $K_{\ell+k+1} \setminus P_\ell$, that is, $\mu_i \geq \lambda_i \geq \mu_{i+1}$ for $i = 1, 2, \dots, \ell + k$. Next we shall show that $\lfloor \frac{\ell}{2} \rfloor$ known eigenvalues have multiplicity 1.

Suppose $\bar{\lambda}$ is such an eigenvalue. Let $x = [x_1, x_2, \dots, x_\ell]^T, \gamma = [a_1, a_2, \dots, a_{k+1}]^T$ and $y_2 = [x_1, x_2, \dots, x_\ell, a_1, \dots, a_{k+1}]^T$ be an eigenvector for $\bar{\lambda}$. Actually a_1, a_2, \dots, a_{k+1} must be equal if $\bar{\lambda} \neq -1$.

Let $B_2 = \begin{bmatrix} J_1 - I - A & J_2 \\ J_2^T & J_3 - I \end{bmatrix}$ be the adjacency matrix of $K_{\ell+k+1} \setminus P_\ell$. It follows from $B_2 y_2 = \bar{\lambda} y_2$ that

$$J_2^T x + (J_3 - I)\gamma = \bar{\lambda}\gamma,$$

i.e.,

$$[(\bar{\lambda} + 1)I - J_3]\gamma = J_2^T x.$$

Then we have $(\bar{\lambda} + 1)(a_i - a_j) = 0$ and if $\bar{\lambda} \neq -1$, then a_1, a_2, \dots, a_{k+1} must be equal to each other.

So we can assume $y_2 = [x_1, x_2, \dots, x_\ell, a, \dots, a]^T$. If $a = 0$, then y_2 is an eigenvector of $K_{\ell+k+1} \setminus P_\ell$. Suppose $a \neq 0$. It follows from $B_2 y_2 = \bar{\lambda} y_2$ that

$$\begin{cases} (J_1 - I - A)x + (k+1)a\mathbf{1} = \bar{\lambda}x, \\ \mathbf{1}^T x + ka = \bar{\lambda}a. \end{cases}$$

i.e.,

$$x_i = a(\bar{\lambda} + 1) - (\bar{\lambda} + 1)x_{i-1} - x_{i-2}, \text{ for } i = 2, 3, \dots, \ell.$$

Similarly as before we also get a contradiction. This completes the proof. \square

Remark. As we know that every eigenvalue of $K_\ell \setminus P_{\ell-1}$ has multiplicity one. So all conclusions in [11] hold for the graph $K_\ell \setminus P_{\ell-1}$. So we directly have $K_\ell \setminus P_{\ell-1}$ is DS.

Lemma 2.14. *Suppose $K_n \setminus H$ is cospectral with $K_n \setminus P_\ell$. Then graph H has the following three properties.*

1. *No component of H is a cycle except for C_3 and C_4 .*
2. *Graph H cannot contain the disjoint union of two cycles $C_k \cup C_s$, where C_k and C_s are both induced subgraph of H .*
3. *No two components of H are paths of the same nonzero length except for P_3 .*

Proof. (i) Suppose that there exists one component of H that is a cycle C_a ($a \neq 3, 4$). The eigenvalues of C_a are $2\cos(2\pi i/a)$ for $i = 0, 1, \dots, a-1$. Note that $a-i$ and i give the same value, so almost all eigenvalues have multiplicity 2. Since each cycle has eigenvalue 2 with all-one eigenvector $\mathbf{1}$, every other eigenvectors ξ_i with eigenvalue $2\cos(2\pi i/a)$ are all orthogonal to all-one vector $\mathbf{1}$, for $i = 1, \dots, a-1$. We readily find that $K_n \setminus H$ has eigenvalues $-1 - 2\cos(2\pi i/a)$ with eigenvectors $[\xi_i, o]^T$ for $i = 1, \dots, a-1$, which contradicts Lemma 2.13.

(ii) Suppose H contains an induced subgraph Δ which consists of two disjoint cycles C_k and C_s . Then the complement of Δ has eigenvalue -3 with the eigenvector

$v = \overbrace{[-s, \dots, -s]}^k, \overbrace{[k, \dots, k]}^s$. Eigenvalue interlacing theorem gives that $\bar{\lambda}_{\min}(K_n \setminus P_\ell) \leq -3$; a contradiction.

(iii) Suppose H contains P_k ($k \neq 3$) twice. Then both paths have an eigenvalue λ with a skew palindromic eigenvector. Since a skew palindromic vector is orthogonal to $\mathbf{1}$, graph H has an eigenvalue λ with at least two independent eigenvectors orthogonal to $\mathbf{1}$. This implies that $K_n \setminus H$ has an eigenvalue $-\lambda - 1$ with multiplicity at least two, which contradicts Lemma 2.13. \square

3 The Proof of Theorem 1.1

In this section, we present the proof of Theorem 1.1.

Let $G = P_\ell + (n - \ell)K_1$ (the complement of $K_n \setminus P_\ell$). Apparently, G has n vertices, $\ell - 1$ edges, no triangles, and degree sequence $\{0^{n-\ell}, 1^2, 2^{\ell-2}\}$. Thus, the number of triangles \bar{t} in graph $K_n \setminus P_\ell$ is

$$\bar{t} = \binom{n}{3} - (n-1)(\ell-1) + \frac{1}{2}(4\ell-6).$$

Let $\Gamma = H + (n - V(H))K_1$ (the complement of $K_n \setminus H$), which has n vertices, $\ell - 1$ edges, t' triangles, and degree sequence $\{0^{n-\sum_{i=1}^k x_i}, 1^{x_1}, 2^{x_2}, \dots, k^{x_k}\}$. Then the number of triangles \bar{t}' in graph $K_n \setminus H$ is

$$\bar{t}' = \binom{n}{3} - (n-1)(\ell-1) + \frac{1}{2} \sum_{i=1}^k (i^2 x_i) - t'.$$

Suppose the pair of graphs $K_n \setminus H$ and $K_n \setminus P_\ell$ are cospectral, then they have the same number of triangles. So we have

$$\begin{cases} \sum_{i=1}^k i x_i = 2\ell - 2, \\ \sum_{i=1}^k i^2 x_i = 4\ell - 6 + 2t'. \end{cases} \quad (1)$$

Lemma 3.1. *The graph $K_n \setminus P_7$ is determined by its adjacency spectrum.*

Proof. If $\ell = 7$, then we have $0 \leq t' \leq 4, 0 \leq k \leq 5, 0 \leq x_i \leq \min\{\frac{12}{i}, \frac{22+2t'}{i^2}\}$, with the help of Mathematica software, we find all possible combinations of $\{t', x_1, x_2, x_3, x_4, x_5\}$ that satisfy (1):

$$\begin{aligned} & \{0, 2, 5, 0, 0, 0\}, \{0, 5, 2, 1, 0, 0\}, \{1, 0, 6, 0, 0, 0\}, \{1, 3, 3, 1, 0, 0\}, \{1, 6, 0, 2, 0, 0\}, \\ & \{1, 8, 0, 0, 1, 0\}, \{2, 1, 4, 1, 0, 0\}, \{2, 4, 1, 2, 0, 0\}, \{2, 6, 1, 0, 1, 0\}, \{3, 2, 2, 2, 0, 0\}, \\ & \{3, 4, 2, 0, 1, 0\}, \{4, 0, 3, 2, 0, 0\}, \{4, 2, 3, 0, 1, 0\}, \{4, 3, 0, 3, 0, 0\}, \{4, 5, 0, 1, 1, 0\}. \end{aligned}$$

A set of parameters $\{t', x_1, x_2, x_3, x_4, x_5\}$ is called *graphic* if there exists a graph with the same parameters. Actually not all of these combinations are graphic and for some of

Table 2: All the possible graphs cospectral with $K_n \setminus P_7$

graphic combinations	corresponding graphs
$\{0,2,5,0,0,0\}$	$C_5 \cup P_2, C_4 \cup P_3$
$\{0,5,2,1,0,0\}$	$T_{1,1,3} \cup P_2, T_{1,2,2} \cup P_2, T_{1,1,2} \cup P_3, T_{1,1,1} \cup P_4$
$\{1,3,3,1,0,0\}$	$K_{1,3} \cup C_3, G_a \cup P_3, G_h \cup P_2$

them there may exist more than one graphs; see the Table 2. Here we only give the graphic combinations and corresponding graphs (the related subgraph in the Table see Fig. 2):

By Lemma 2.6, Lemma 2.7 and Lemma 2.9, we have that graph $K_n \setminus P_7$ is determined by its adjacency spectrum. \square

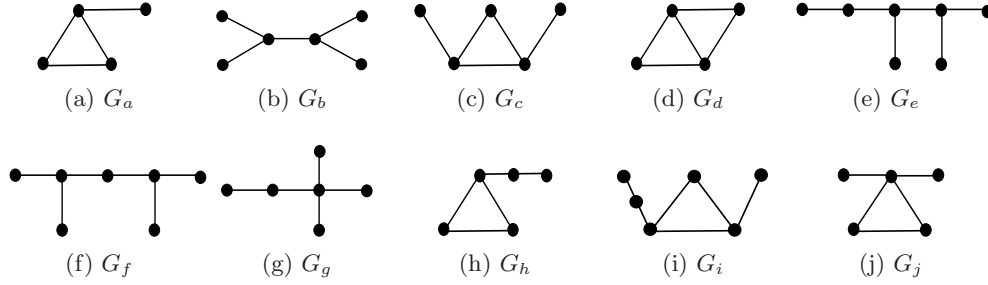


Fig. 2: Subgraphs related to $K_n \setminus P_7, K_n \setminus P_8$, and $K_n \setminus P_9$

Lemma 3.2. *The graph $K_n \setminus P_8$ is determined by its adjacency spectrum.*

Proof. If $\ell = 8$, then we have $0 \leq t' \leq 4, 0 \leq k \leq 5, 0 \leq x_i \leq \min\{\frac{14}{i}, \frac{26+2t'}{i^2}\}$, similarly we find all possible combinations of $\{t', x_1, x_2, x_3, x_4, x_5\}$ that satisfy (1):

$$\begin{aligned} &\{0, 2, 6, 0, 0, 0\}, \{0, 5, 3, 1, 0, 0\}, \{0, 8, 0, 2, 0, 0\}, \{0, 10, 0, 0, 1, 0\}, \{1, 0, 7, 0, 0, 0\}, \\ &\{1, 3, 4, 1, 0, 0\}, \{1, 6, 1, 2, 0, 0\}, \{1, 8, 1, 0, 1, 0\}, \{2, 1, 5, 1, 0, 0\}, \{2, 4, 2, 2, 0, 0\}, \\ &\{2, 6, 2, 0, 1, 0\}, \{3, 2, 3, 2, 0, 0\}, \{3, 4, 3, 0, 1, 0\}, \{3, 5, 0, 3, 0, 0\}, \{3, 7, 0, 1, 1, 0\}, \\ &\{4, 0, 4, 2, 0, 0\}, \{4, 2, 4, 0, 1, 0\}, \{4, 3, 1, 3, 0, 0\}, \{4, 5, 1, 1, 1, 0\}, \{4, 9, 0, 0, 0, 1\}. \end{aligned}$$

Table 3 only gives the graphic combinations and its corresponding graphs (for the related subgraph in the table, see Fig. 2):

By Lemma 2.6, lemma 2.7, Lemma 2.8, Lemma 2.9, Lemma 2.14 all these graphs are not cospectral with graph $K_n \setminus P_8$. So graph $K_n \setminus P_8$ is *DS*. \square

Lemma 3.3. *The graph $K_n \setminus P_9$ is determined by its adjacency spectrum.*

Table 3: possible cospectral graphs with $K_n \setminus P_8$

graphic combinations	corresponding graphs
$\{0,2,6,0,0,0\}$	$C_6 \cup P_2, C_5 \cup P_3, C_4 \cup P_4,$
$\{0,5,3,1,0,0\}$	$T_{1,1,4} \cup P_2, T_{1,2,3} \cup P_2, T_{2,2,2} \cup P_2, T_{1,1,3} \cup P_3, T_{1,2,2} \cup P_3, T_{1,1,2} \cup P_4, T_{1,1,1} \cup P_5$
$\{0,8,0,2,0,0\}$	$G_b \cup 2P_2, 2K_{1,3} \cup P_2$
$\{0,10,0,0,1,0\}$	$K_{1,4} \cup 3P_2$
$\{1,0,7,0,0,0\}$	$C_3 \cup C_4$
$\{1,3,4,1,0,0\}$	$G_a \cup P_4, T_{1,1,2} \cup C_3$
$\{1,6,1,2,0,0\}$	$G_c \cup 2P_2$
$\{2,1,5,1,0,0\}$	$G_a \cup C_3$
$\{2,4,2,2,0,0\}$	$G_d \cup 2P_2$

Proof. If $\ell = 9$, we have $0 \leq t' \leq 5, 0 \leq k \leq 6, 0 \leq x_i \leq \min\{\frac{16}{i}, \frac{30+2t'}{i^2}\}$, we find all possible combinations of $\{t', x_1, x_2, x_3, x_4, x_5, x_6\}$ that satisfy (1):

$\{0, 0, 6, 0, 1, 0, 0\}, \{0, 1, 3, 3, 0, 0, 0\}, \{0, 3, 3, 1, 1, 0, 0\}, \{0, 4, 0, 4, 0, 0, 0\}, \{0, 6, 0, 2, 1, 0, 0\},$
 $\{0, 7, 2, 0, 0, 1, 0\}, \{0, 8, 0, 0, 2, 0, 0\}, \{1, 1, 4, 1, 1, 0, 0\}, \{1, 2, 1, 4, 0, 0, 0\}, \{1, 4, 1, 2, 1, 0, 0\},$
 $\{1, 5, 3, 0, 0, 1, 0\}, \{1, 6, 1, 0, 2, 0, 0\}, \{1, 8, 0, 1, 0, 1, 0\}, \{2, 0, 2, 4, 0, 0, 0\}, \{2, 2, 2, 2, 1, 0, 0\},$
 $\{2, 3, 4, 0, 0, 1, 0\}, \{2, 4, 2, 0, 2, 0, 0\}, \{2, 6, 1, 1, 0, 1, 0\}, \{3, 0, 3, 2, 1, 0, 0\}, \{3, 1, 5, 0, 0, 1, 0\},$
 $\{3, 2, 3, 0, 2, 0, 0\}, \{3, 3, 0, 3, 1, 0, 0\}, \{3, 4, 2, 1, 0, 1, 0\}, \{3, 5, 0, 1, 2, 0, 0\}, \{3, 10, 0, 0, 0, 0, 1\},$
 $\{4, 0, 4, 0, 2, 0, 0\}, \{4, 1, 1, 3, 1, 0, 0\}, \{4, 2, 3, 1, 0, 1, 0\}, \{4, 3, 1, 1, 2, 0, 0\}, \{4, 5, 0, 2, 0, 1, 0\},$
 $\{4, 7, 0, 0, 1, 1, 0\}, \{4, 8, 1, 0, 0, 0, 1\}, \{5, 0, 4, 1, 0, 1, 0\}, \{5, 1, 2, 1, 2, 0, 0\}, \{5, 3, 1, 2, 0, 1, 0\},$
 $\{5, 5, 1, 0, 1, 1, 0\}, \{5, 6, 2, 0, 0, 0, 1\}.$

Table 4 only gives graphic combinations and its corresponding graphs (for the related subgraph in the table, see Fig. 2):

Similarly, by Lemmas 2.6 2.9 and Lemma 2.14, all these graphs are not cospectral with graph $K_n \setminus P_9$, except for graphs $G_b \cup P_2 \cup P_3, T_{1,1,2} \cup K_{1,3} \cup P_2,$ and $G_d \cup P_2 \cup P_3$. However, these three graphs have different number of 4-walks with $K_n \setminus P_9$. So graph $K_n \setminus P_9$ is DS. \square

Finally, we present the proof Theorem 1.1:

Proof. Combining Lemmas 3.1, 3.2 and 3.3, Theorem 1.1 follows immediately. \square

4 Conclusions

In this paper, we have derived some eigenvalue properties for the graph $K_n \setminus P_\ell$, based on which we are able to show that $K_n \setminus P_\ell$ is DS for some small values of ℓ . It is noticed that the multiplicity of the eigenvalue -1 of $K_n \setminus P_\ell$ is larger than one, while it is one for $K_\ell \setminus P_\ell$. So the results in [4] cannot be directly applied in this paper. Also, the proof of Theorem 1.1

Table 4: possible cospectral graphs with $K_n \setminus P_9$

graphic combinations	corresponding graphs
$\{0,2,7,0,0,0,0\}$	$C_7 \cup P_2, C_6 \cup P_3, C_5 \cup P_4, C_4 \cup P_5,$
$\{0,5,4,1,0,0,0\}$	$T_{1,1,5} \cup P_2, T_{1,2,4} \cup P_2, T_{1,3,3} \cup P_2, T_{2,2,3} \cup P_2, T_{1,1,4} \cup P_3, T_{1,2,3} \cup P_3,$ $T_{2,2,2} \cup P_3, T_{1,1,3} \cup P_4, T_{1,2,2} \cup P_4, T_{1,1,2} \cup P_5, T_{1,1,6} \cup P_6$
$\{0,8,1,2,0,0,0\}$	$G_e \cup 2P_2, G_f \cup 2P_2, G_b \cup P_2 \cup P_3, T_{1,1,2} \cup K_{1,3} \cup P_2, 2K_{1,3} \cup P_3$
$\{0,10,1,0,1,0,0\}$	$G_g \cup 3P_2, K_{1,4} \cup 2P_2 \cup P_3$
$\{1,0,8,0,0,0,0\}$	$C_3 \cup C_5$
$\{1,3,5,1,0,0,0\}$	$G_h \cup P_4, G_a \cup P_5$
$\{1,6,2,2,0,0,0\}$	$G_i \cup 2P_2, G_c \cup P_2 \cup P_3$
$\{1,8,2,0,1,0,0,0\}$	$G_j \cup 3P_2$
$\{2,1,6,1,0,0,0\}$	$G_h \cup C_3$
$\{2,4,3,2,0,0,0\}$	$G_d \cup P_2 \cup P_3$

is based on a detailed classification of all the possible cospectral graphs of $K_n \setminus P_\ell$, and it is getting more involved for larger ℓ . Thus, to deal with the general case of Conjecture 1, new tools and insights are needed. This will be investigated in the future.

References

- [1] D.M. Cvetković, M. Doob, H. Sachs, Spectra of Graphs, Academic Press, New York, 1982.
- [2] E.R. van Dam, W.H. Haemers, Which graphs are determined by their spectrum? Linear Algebra Appl., 373 (2003) 241-272.
- [3] E.R. van Dam, W.H. Haemers, Developments on spectral characterizations of graphs, Discrete Mathematics, 309 (2009) 576-586.
- [4] M. Doob, W. H. Haemers, The complement of the path is determined by its spectrum, Linear Algebra Appl., 356 (2002) 57-65.
- [5] W. Wang, C.X. Xu, On the spectral characterization of T-shape trees, Linear Algebra Appl., 414 (2006) 492-501.
- [6] F. Liu, Q. Huang, J. Wang, Q. Liu, The spectral characterization of ∞ -graphs, Linear Algebra Appl., 437 (2012) 1482-1502.
- [7] W. H. Haemers, X. G. Liu, Y. P. Zhang, Spectral characterization of lolipop graphs, Linear Algebra Appl., 428 (2008) 2415-2423.
- [8] F. Ramezani, N. Broojerdian, B. Tayfeh-Rezaie, A note on the spectral characterization of θ -graphs, Linear Algebra Appl., 431 (2009) 626-632.
- [9] N. Ghareghai, G. R. Omid, B. Tayfeh-Rezaie, Spectral characterization of graphs with index at most $\sqrt{2 + \sqrt{5}}$, Linear Algebra Appl., 420 (2007) 483-489.

- [10] H.Topuc, S. Sorgun, W. H. Haemers, On the spectral characterization of pineapple graphs, *Linear Algebra Appl.*, 507 (2016) 267-273.
- [11] M. Cámara, W. H. Haemers, Spectral characterization of almost complete graphs, *Discrete Applied Mathematics*, 176 (2014) 19-23.
- [12] T.Z. Wu, H.P. Zhang, Per-spectral and adjacency spectral characterization of a complete graph removing six edges, *Discrete Applied Mathematics*, 203 (2016) 158-170.
- [13] G.R. Omid, On a signless Laplacian spectral characterization of T-shape trees, *Linear Algebra Appl.*, 431 (2009) 1607-1615.
- [14] H.J. Jing, J. Koolen, On fat Hoffman graphs with smallest eigevalue at least -3, *Ars Mathematica Contemporanea*, 7 (2014) 105-121.