

# Approximate conditions admitted by classes of the Lagrangian $\mathcal{L} = \frac{1}{2}(-u'^2 + u^2) + \epsilon^i G_i(u, u', u'')$

Sameerah Jamal\*

*School of Mathematics and Centre for Differential Equations, Continuum Mechanics and Applications, University of the Witwatersrand, Johannesburg, South Africa*

Nkosingiphile Mnguni

*School of Mathematics and Centre for Differential Equations, Continuum Mechanics and Applications, University of the Witwatersrand, Johannesburg, South Africa*

---

## Abstract

We investigate a class of Lagrangians that admit a type of perturbed harmonic oscillator which occupies a special place in the literature surrounding perturbation theory. We establish explicit and generalized geometric conditions for the symmetry determining equations. The explicit scheme provided can be followed and specialized for any concrete perturbed differential equation possessing the Lagrangian. A systematic solution of the conditions generate nontrivial approximate symmetries and transformations. Detailed cases are discussed to illustrate the relevance of the conditions, namely (a)  $G_1$  as a quadratic polynomial, (b) the Klein-Gordon equation of a particle in the context of Generalized Uncertainty Principle and (c) an orbital equation from an embedded Reissner-Nordström black hole.

*Keywords:* Approximate Symmetries; orbital equation; uncertainty principle.

*2010 MSC:* 22E60; 76M60; 35Q75; 34C20

---



---

\*Corresponding author

*Email addresses:* Sameerah.Jamal@wits.ac.za (Sameerah Jamal), nzmnguni8@gmail.com (Nkosingiphile Mnguni)

## 1. Introduction

A Lie symmetry group forms a robust tool in the analysis of differential equations, primarily because it provides invariant functions which may reduce the order of the equation and lead to the determination of analytic solutions. Differential equations which possess a variational principle or Lagrangian, admit specialized Lie symmetries, called Noether symmetries or divergence symmetries, which in addition to the invariant functions, leave the action invariant. Aside from these classical symmetries, there exist approximate symmetries which are devised from equations, regarded as perturbed equations, that contain some small parameter  $\epsilon$ . Within the literature, among several computational techniques for approximate generators, there are two main formalisms, one proposed by Baikov, Gazizov and Ibragimov [1] and the second was presented by Fushchich and Shtelen [2]. Thereafter, the concept of approximate Noether symmetries and conservation laws emerged [3, 4]. Owing to these developments, many important physical differential equations have been studied successfully, see for instance [5, 6, 7]. Note that unlike exact symmetries, approximate symmetries do not necessarily form a Lie algebra but rather an “approximate Lie algebra” [8].

Two decades ago, a method was devised whereby a known symmetry and its corresponding conservation law of a given partial differential equation can be used to construct a Lagrangian for the equation [9]. However, in the absence of a Lagrangian, there has been significant developments on the derivation of approximate conservation laws. For instance in [10], a method based on partial Lagrangians was introduced to construct approximate conservation laws of approximate Euler-type equations using approximate Noether-type symmetries. In [11], Zhang considered approximate nonlinear self-adjointness for perturbed PDEs and showed how approximate conservation laws, which cannot be obtained by the approximate Noether’s theorem, are constructed. Nevertheless, when a Lagrangian is available, Noether’s work is not only more elegant, but also highly efficient, and will always be the preferred method. As an example of

the advantages of approximate Noether symmetries, over other existing methods previously mentioned, a recent study, by one of the authors, found a geometric connection between the Homothetic algebra of an underlying geometry and the approximated Noether symmetries, that is, if the perturbation terms do not modify the Kinetic energy of regular Lagrangians, approximate symmetries exist if and only if the metric that defines the Kinetic energy, admits a nontrivial Homothetic algebra [12].

The purpose of this paper is three-fold. Firstly, in the following work we stipulate the generalized approximate conditions in the case of a class of perturbed Lagrangians, up to third-order,

$$\mathcal{L}(u, u', u'', \epsilon) = \frac{1}{2}(-u'^2 + u^2) + \epsilon^i G_i(u, u', u''). \quad (1)$$

The Lagrangian defined here has the Latin index  $i$  that is restricted to the values 1, 2 and 3 and  $u$  is a function of  $\phi$ . The above approximate class of Lagrangians and its symmetry generators maintain the specified perturbation order of  $\epsilon$ . To preserve generality we have not made specific assumptions about the  $G_i(u, u', u'')$ . Rather we provide an explicit scheme which can be followed and specialized for any concrete differential equation possessing the Lagrangian (1), whereby one may extract further information using a given  $G_i(u, u', u'')$ . Our next purpose is to use the generalized conditions to find approximate divergence symmetries for several critical cases of interest. Thirdly, the latter will be used to establish the associated approximate first integrals by invoking Noether's theorem.

Before we begin, it is worth mentioning that there are powerful and fully automated software routines to obtain symmetries that are not approximate, commonly referred to as exact symmetries, for example [13, 14, 15]. Eliminating all the perturbed terms in the Lagrangian (1), leads to the derivation of the oscillation equation

$$u'' + u = 0. \quad (2)$$

It is easily seen that this unperturbed equation is maximally symmetric and

admits the 8-dimensional Lie algebra of exact symmetries  $sl(3, R)$  given by

$$\begin{aligned}
X_0^1 &= \partial_\phi, \\
X_0^2 &= \sin(2\phi)\partial_\phi + \cos(2\phi)u\partial_u, \\
X_0^3 &= \cos(2\phi)\partial_\phi - \sin(2\phi)u\partial_u, \\
X_0^4 &= \sin(\phi)\partial_u, \\
X_0^5 &= \cos(\phi)\partial_u, \\
X_0^6 &= u\partial_u, \\
X_0^7 &= u\cos(\phi)\partial_\phi - u^2\sin(\phi)\partial_u, \\
X_0^8 &= u\sin(\phi)\partial_\phi + u^2\cos(\phi)\partial_u.
\end{aligned}$$

In a problem with a small perturbation, one may consider the approximate Lie symmetry approach versus the approximate Noether or variational symmetry approach. We have chosen here the approximate variational approach since we shall find, at our disposal, explicit formulae for the approximate symmetry conditions and conservation laws ensured by Noether's theorem (see Sections 3), whose determination is usually sans the use of algebraic and algorithmic software. Comparatively, the approximate Lie method is tedious and involves extra computations. Thus it is immediate and far more efficient to apply the variational approach. In order to illustrate our main results or derived conditions, some examples are presented in the text. These examples are appropriately chosen, for they are novel in the sense that they have not been subjected to an approximate symmetry investigation. Moreover these examples involve variational principles in a cosmological and relativistic setting. One case explores the approximate symmetries of an orbital equation that arises when a Reissner-Nordström black hole is embedded into a Friedman-Robertson-Walker (FRW) space [16]. To obtain the equation of motion of a planet, it is the norm to rewrite a given metric from the cosmic coordinate system to the Schwarzschild or solar coordinate system and thereafter deduce the geodesic equation. Significantly, such equations have the propensity to show whether or not the orbit of a planet is influenced by the evolution of the universe. As a second case, we investigate the modified Klein-Gordon equation of a spin-0 particle in the Generalized Un-

certainty Principle (GUP) [17, 18, 19, 20, 21]. In general, as detailed below, the modified Klein-Gordon equation is a fourth-order partial differential equation, which we reduce and adapt to possess the perturbed Lagrangian (1). In each case, we state the approximate first integrals corresponding to the approximate Noether symmetries obtained.

The plan of the paper is as follows. In the next section we briefly review the geometric preliminaries surrounding exact and approximate point symmetries of differential equations, with a focus on generators originating from a variational principle. This section also introduces the notation and conventions assumed. The perturbed class of Lagrangians (1) are studied in Section 3, where we show that the approximate symmetry determining equations are generated by a set of generic conditions. In Section 4, we apply the general results of the previous sections to highlight a particular case of  $G_1(u, u', u'')$  that admits an enlarged “group” of approximate Noether symmetry generators. Section 5 describes the case of the modified Klein-Gordon equation of a particle in the GUP while Section 6 draws attention to several orbital equations of interest where the generalized conditions are especially useful. Finally, in section 7 we present our conclusions.

## 2. Point transformations

Our interest lies in point transformations, and for the convenience of the reader we insert the necessary theory pertaining to this analysis. The presentation here is for ordinary differential equations, however most of the theory has been generalized to partial differential equations in the references cited. For the sake of brevity, the summation convention is adopted in this text, in which there is summation over all repeated indices. First, consider a system of second-order ordinary differential equations (unperturbed), where  $t$  is the independent variable and  $x^i$  denotes the dependent variables

$$x''^i = \omega^i(t, x^j, x'^j). \quad (3)$$

An one-parameter point transformation in the space  $\{t, x^j\}$ , has the property of mapping solutions of (3) to itself and satisfies the infinitesimal criterion of invariance

$$X^{[2]}(x''^i - \omega^i) = 0 \quad \text{mod} \quad x''^i - \omega^i = 0,$$

where  $X$  is defined as

$$X = \left. \frac{\partial \bar{t}}{\partial \varepsilon} \right|_{\varepsilon=0} \partial_t + \left. \frac{\partial \bar{x}^i}{\partial \varepsilon} \right|_{\varepsilon=0} \partial_i \quad (4)$$

with  $X^{[2]}$  as the second prolongation of  $X$  in the jet space of variables [22].  $X$  is the generator of the point transformation called a Lie symmetry for the system of differential equations.

On the other hand, if the system (3) follows from the variation of the action integral

$$S = \int \mathcal{L} dt,$$

then the Noether's theorem [23] states that when a (finite) group of transformations leaves the action invariant, i.e.

$$S(t, x^j, \dots) = S(\bar{t}(t, x^j, \varepsilon), \bar{x}^j(t, x^j, \varepsilon), \dots), \quad (5)$$

then a conserved quantity exists. For an unperturbed Lagrangian up-to first-order in derivatives (as is the case in this paper), condition (5) yields

$$X^{[1]}\mathcal{L} + \mathcal{L} \frac{d}{dt} \left( \left. \frac{\partial \bar{t}}{\partial \varepsilon} \right|_{\varepsilon=0} \right) = f' \quad (6)$$

and the corresponding first integral is given by

$$I = \left( x'^j \frac{\partial \mathcal{L}}{\partial x'^j} - \mathcal{L} \right) \left. \frac{\partial \bar{t}}{\partial \varepsilon} \right|_{\varepsilon=0} - \left. \frac{\partial \mathcal{L}}{\partial x'^j} \frac{\partial \bar{x}^j}{\partial \varepsilon} \right|_{\varepsilon=0} + f. \quad (7)$$

In this scenario  $X$  would be called a Noether symmetry which is also a Lie symmetry; however the inverse of this result can be false. We continue with the review of the techniques of finding approximate variational symmetries. For a  $k$ -th order perturbed system of ordinary differential equations

$$E = E_0 + \varepsilon E_1 + \varepsilon^2 E_2 + \dots + \varepsilon^k E_k + O(\varepsilon^{k+1}), \quad (8)$$

corresponding to a Lagrangian, which is perturbed in  $\epsilon$ ,

$$\mathcal{L}(t, x, x'^j, \epsilon) = \mathcal{L}_0(t, x, x'^j) + \epsilon \mathcal{L}_1(t, x, x'^j) + \dots + O(\epsilon^{k+1}), \quad (9)$$

the functional  $\int \mathcal{L} dt$  is invariant under the one-parameter group of transformations with approximate Lie symmetry generator

$$X = X_0 + \epsilon X_1 + \dots + \epsilon^k X_k, \quad (10)$$

up to gauge

$$A = A_0 + \epsilon A_1 + \dots + \epsilon^k A_k, \quad (11)$$

if

$$X\mathcal{L} + \left( D_t \frac{\partial \bar{t}}{\partial \epsilon} \Big|_{\epsilon=0} \right) \mathcal{L} = D_t A, \quad (12)$$

where  $D_t$  is the total derivative operator. In this notation,  $X_0$  is the exact symmetry generator originating from the unperturbed Lagrangian and  $X_1$  the first-order approximate symmetry generator. A perturbed equation always admits the trivial approximate symmetry generator  $\epsilon X_0$ . Also, if  $X = X_0 + \epsilon X_1$  exists with  $X_0 \neq 0$  and  $X_1 \neq kX_0$  ( $k$  an arbitrary constant), then it is called a nontrivial symmetry [24]. These considerations can be generalized to higher-order approximate symmetry generators. An analogous formula for the first-order approximate first integrals can be obtained from Eq. (7) bearing in mind that for first-order, the approximate first integrals are defined by  $I = I_0 + \epsilon I_1$ , where  $I_0$  is the exact first integral and  $I_1$  is the first-order approximate part.

### 3. Third-Order Geometric Conditions

First we obtain the approximate Noether symmetry conditions for the class of Lagrangians (1) by applying the approximate symmetry theory. Then we shall study the approximate Noether symmetries of (1) pertaining to several important problems in the literature. The determination of approximate Noether point symmetries of the Lagrangian (1) consists of two steps: (a) the derivation of the conditions which provide the symmetry determining equations, and (b)

the solution of these determining equations. The first step is precise, however the symmetry conditions which arise can be quite involved. The key point is to express conditions for generic forms of  $G_i(u, u', u'')$ .

As mentioned above, Noether symmetries are just a specialization of Lie symmetries, and thus the  $sl(3, R)$  algebra given above contains the Noether point symmetry generators. The latter comprises of a 5-dimensional Lie algebra  $X_0^{1-5}$  with the corresponding gauge term (the  $c_j$  are constants)

$$A_0 = u^2 \cos(2\phi) c_3 + u^2 \sin(2\phi) c_2 + \sin(\phi) c_5 u - \cos(\phi) c_4 u + c_6.$$

The Noether first integrals corresponding to each  $X_0^h, h = 1, \dots, 5$  are

$$\begin{aligned} I_0^1 &= \frac{1}{2} (u^2 + u'^2), \\ I_0^2 &= \frac{1}{2} (u'^2 - u^2) \sin(2\phi) - uu' \cos(2\phi), \\ I_0^3 &= \frac{1}{2} (u'^2 - u^2) \cos(2\phi) + uu' \sin(2\phi), \\ I_0^4 &= -u' \sin(\phi) + u \cos(\phi), \\ I_0^5 &= -u' \cos(\phi) - u \sin(\phi). \end{aligned}$$

If we include a perturbation up to first-order in  $\epsilon$ , that is, the Lagrangian (1) omits the terms in  $G_2(u, u', u'')$  and  $G_3(u, u', u'')$ , the determination of approximate symmetries takes a particular form. That is, for each term of the Noether condition (12) for the Lagrangian (1) we have the geometric condition

$$\begin{aligned} X\mathcal{L} &= \left( \eta_{1,\phi} + u' \eta_{1,u} - u' \xi_{1,\phi} - (u')^2 \xi_{1,u} \right) (-u') + \eta_{1,u} + \\ &\left( -2 \sin(\phi) \cos(\phi) c_3 u + 2 c_2 u (\cos(\phi))^2 + c_4 \sin(\phi) + c_5 \cos(\phi) - c_2 u \right) G_{1,u} \\ &+ \left( -4 c_2 u \cos(\phi) \sin(\phi) + 2 \sin(\phi) \cos(\phi) c_3 u' - 2 (\cos(\phi))^2 c_2 u' \right) G_{1,u'} \\ &\left( -4 (\cos(\phi))^2 c_3 u - c_5 \sin(\phi) + c_4 \cos(\phi) + c_2 u' + 2 c_3 u \right) G_{1,u''} \\ &+ \left( 8 \sin(\phi) \cos(\phi) c_3 u + 6 \sin(\phi) \cos(\phi) c_3 u'' - 8 c_2 u (\cos(\phi))^2 \right) G_{1,u''} \\ &- \left( 6 (\cos(\phi))^2 c_2 u'' - c_4 \sin(\phi) - c_5 \cos(\phi) + 4 c_2 u + 3 c_2 u'' \right) G_{1,u''}, \end{aligned} \tag{13}$$

$$\left( D_\phi \frac{\partial \bar{\phi}}{\partial \epsilon} \Big|_{\epsilon=0} \right) \mathcal{L} = (2c_2 \cos(2\phi) - 2c_3 \sin(2\phi)) G_1 + (\xi_{1,\phi} + u' \xi_{1,u}) \left( -\frac{(u')^2}{2} + \frac{u^2}{2} \right), \tag{14}$$



$$D_\phi A = A_{1,\phi} + u' A_{1,u}. \quad (15)$$

On the other hand, if the perturbation is up to second-order in  $\epsilon$ , the Lagrangian (1) omits  $G_3(u, u', u'')$ , and in this case the second condition is:

$$\begin{aligned}
X\mathcal{L} = & \left( \eta_{2,\phi} + u' \eta_{2,u} - u' \xi_{2,\phi} - (u')^2 \xi_{2,u} \right) (-u') + \eta_2 u + \eta_1 G_{1,u} \\
& + \left( -u'^2 \xi_{1,u} + u' \eta_{1,u} - u' \xi_{1,\phi} + \eta_{1,\phi} \right) G_{1,u'} + \left( -u'^3 \eta_{1,uu} - 2 u'^2 \xi_{1,u\phi} \right) G_{1,u''} \\
& + \left( \eta_{1,uu} u'^2 - 3 \xi_{1,u} u' u'' + 2 \eta_{1,u\phi} u' \right) G_{1,u''} \\
& - \left( \xi_{1,\phi\phi} u' + \eta_{1,u} u'' - 2 \xi_{1,\phi} u'' + \eta_{1,\phi\phi} \right) G_{1,u''} \\
& + \left( -2 \sin(\phi) \cos(\phi) c_3 u + 2 c_2 u (\cos(\phi))^2 + c_4 \sin(\phi) + c_5 \cos(\phi) - c_2 u \right) G_{2,u} \\
& + \left( -4 c_2 u \cos(\phi) \sin(\phi) + 2 \sin(\phi) \cos(\phi) c_3 u' - 2 (\cos(\phi))^2 c_2 u' \right) G_{2,u'} \\
& - \left( 4 (\cos(\phi))^2 c_3 u - c_5 \sin(\phi) + c_4 \cos(\phi) + c_2 u' + 2 c_3 u \right) G_{2,u'} \\
& + \left( 8 \sin(\phi) \cos(\phi) c_3 u + 6 \sin(\phi) \cos(\phi) c_3 u'' - 8 c_2 u (\cos(\phi))^2 \right) G_{2,u''} \\
& - \left( 6 (\cos(\phi))^2 c_2 u'' - c_4 \sin(\phi) - c_5 \cos(\phi) + 4 c_2 u + 3 c_2 u'' \right) G_{2,u''} \\
& \left( D_\phi \frac{\partial \bar{\phi}}{\partial \epsilon} \Big|_{\epsilon=0} \right) \mathcal{L} = (\xi_{1,\phi} + u' \xi_{1,u}) G_1 + (\xi_{2,\phi} + u' \xi_{2,u}) \left( -\frac{(u')^2}{2} + \frac{u^2}{2} \right) \\
& + (2c_2 \cos(2\phi) - 2c_3 \sin(2\phi)) G_2,
\end{aligned} \quad (16)$$

$$D_\phi A = A_{2,\phi} + u' A_{2,u}. \quad (17)$$

Last but not least, a third-order perturbation in  $\epsilon$  results in the third con-

dition

$$\begin{aligned}
X\mathcal{L} = & \left( \eta_{3,\phi} + u'\eta_{3,u} - u'\xi_{3,\phi} - (u')^2 \xi_{3,u} \right) (-u') + \eta_3 u \\
& + \left( -2 \sin(\phi) \cos(\phi) c_3 u + 2 c_2 u (\cos(\phi))^2 + c_4 \sin(\phi) + c_5 \cos(\phi) - c_2 u \right) G_{3,u} \\
& + \left( -4 c_2 u \cos(\phi) \sin(\phi) + 2 \sin(\phi) \cos(\phi) c_3 u' - 2 (\cos(\phi))^2 c_2 u' \right) G_{3,u'} \\
& - 4 \left( (\cos(\phi))^2 c_3 u - c_5 \sin(\phi) + c_4 \cos(\phi) + c_2 u' + 2 c_3 u \right) G_{3,u''} \\
& + \left( 8 \sin(\phi) \cos(\phi) c_3 u + 6 \sin(\phi) \cos(\phi) c_3 u'' - 8 c_2 u (\cos(\phi))^2 \right) G_{3,u''} \\
& - 6 \left( (\cos(\phi))^2 c_2 u'' - c_4 \sin(\phi) - c_5 \cos(\phi) + 4 c_2 u + 3 c_2 u'' \right) G_{3,u''} \\
& + \eta_1 G_{2,u} + \left( -u'^2 \xi_{1,u} + u' \eta_{1,u} - u' \xi_{1,\phi} + \eta_{1,\phi} \right) G_{2,u'} \\
& + \left( -u'^3 \eta_{1,uu} - 2 u'^2 \xi_{1,u\phi} \right) G_{2,u''} \\
& + \left( \eta_{1,uu} u'^2 - 3 \xi_{1,u} u' u'' + 2 \eta_{1,u\phi} u' \right) G_{2,u''} - \\
& \left( \xi_{1,\phi\phi} u' + \eta_{1,u} u'' - 2 \xi_{1,\phi} u'' + \eta_{1,\phi\phi} \right) G_{2,u''} \\
& + \eta_2 G_{1,u} + \left( -u'^2 \xi_{2,u} + u' \eta_{2,u} - u' \xi_{2,\phi} + \eta_{2,\phi} \right) G_{1,u'} \\
& + \left( -u'^3 \eta_{2,uu} - 2 u'^2 \xi_{2,u\phi} \right) G_{1,u''} \\
& + \left( \eta_{2,uu} u'^2 - 3 \xi_{2,u} u' u'' + 2 \eta_{2,u\phi} u' \right) G_{1,u''} \\
& - \left( \xi_{2,\phi\phi} u' + \eta_{2,u} u'' - 2 \xi_{2,\phi} u'' + \eta_{2,\phi\phi} \right) G_{1,u''},
\end{aligned} \tag{18}$$

$$\begin{aligned}
\left( D_\phi \frac{\partial \bar{\phi}}{\partial \varepsilon} \Big|_{\varepsilon=0} \right) \mathcal{L} = & (\xi_{2,\phi} + u' \xi_{2,u}) G_1 + (2c_2 \cos(2\phi) - 2c_3 \sin(2\phi)) G_3 + \\
& (\xi_{1,\phi} + u' \xi_{1,u}) G_2 + (\xi_{3,\phi} + u' \xi_{3,u}) \left( -\frac{(u')^2}{2} + \frac{u^2}{2} \right),
\end{aligned} \tag{19}$$

$$D_\phi A = A_{3,\phi} + u' A_{3,u}. \tag{20}$$

We remark that the above three conditions must be applied sequentially. The separation and solution of the conditions (13)-(20) gives the approximate coefficients of the Noether point symmetry vectors. To obtain a group classification involving a generic function, in our case  $G_i$ , consists of finding the approximate point symmetries of the given Lagrangian with arbitrary  $G_i$ , and, thereafter to determine all possible and particular cases of  $G_i$  for which the symmetry group can be expanded. Naturally there should be a geometrical or physical motiva-

tion in place for considering such specific cases [25]. In the following sections we proceed with the applications of conditions (13)-(20) in cases of special interest, that is we deal with some equations admitted by the class of Lagrangians (1). Specifically, we study the approximate point symmetries of the modified Klein-Gordon under GUP and secondly, the approximate point symmetries of an orbital equation arising from an embedded Reissner-Nordström black hole. The presentation of results is schematic so not to increase the volume of the paper.

#### 4. A Quadratic Polynomial $G_1(u, u', u'')$

If we allow the conditions (13)-(15) to act as a selection rule for the functional form of  $G_1(u, u', u'')$ , we find some surprising results. In fact the notion of using a symmetry as selection criteria for the free functions or recursion operator of a model, can be traced to many articles (for example [26, 27, 28] or [29, 30], respectively). The specific case in which

$$G_1(u, u', u'') = \frac{1}{2}a_0u^2 + a_1u + a_2,$$

we find the exact symmetry algebra  $X_0^{1-5}$  plus the added approximate generators

$$\begin{aligned}\bar{X}_M^\epsilon &= X_0^2 + \epsilon \left[ (a_0\phi \cos(2\phi) - a_0 \sin(2\phi) - 2 \cos^2(\phi)a_1 + a_1) \partial_\phi - \right. \\ &\quad \left. (\frac{1}{2}a_0u \cos(2\phi) + ua_0\phi \sin(2\phi)) \partial_u \right], \\ \bar{X}_N^\epsilon &= X_0^3 + \epsilon \left[ (-\frac{1}{2}a_0 \cos(2\phi) - \phi a_0 \sin(2\phi)) \partial_\phi - (\phi a_0u \cos(2\phi) + a_1 \sin(2\phi)) \partial_u \right], \\ \bar{X}_O^\epsilon &= X_0^4 + \epsilon \left( \frac{1}{2}a_0\phi \cos(\phi) - \frac{1}{4}a_0 \sin(\phi) \right) \partial_u, \\ \bar{X}_P^\epsilon &= X_0^5 - \epsilon \left( \frac{1}{4}a_0 \cos(\phi) + \frac{1}{2}\phi a_0 \sin(\phi) \right) \partial_u.\end{aligned}$$

Clearly, from the arbitrary constants in  $a_m$ , the  $a_0$  and  $a_1$  should be nonzero to maximize the number of possible approximate generators. Correspondingly,

the first-order approximate gauge term in this case is

$$\begin{aligned}
\bar{A}_1 = & \frac{1}{4} \left( (4c_2\phi + 2c_3)c_6 - 2c_{10} \right) u^2 + 8c_3c_7u + 4c_3a_2 \cos(2\phi) + \\
& \frac{1}{4} \left( (-4c_3a_0\phi + 2c_9)u^2 + 8c_2a_1u + 4c_2a_2 \right) \sin(2\phi) + \\
& \frac{1}{4} \left( (2c_5\phi - c_4)a_0 - 4c_{13} \right) u - 4c_4a_1 \cos(\phi) + \\
& \frac{1}{4} \left( (2c_4\phi + c_5)a_0 + 4c_{12} \right) u + 4c_5c_7 \sin(\phi) + c_{14}
\end{aligned} \tag{21}$$

The first-order approximate first integrals related to  $\bar{X}_{M-P}^\epsilon$  are given by

$$\begin{aligned}
\bar{I}^M = I_0^2 + \epsilon & \left( -2 \sin(\phi) \cos(\phi) a_1 u - a_0 \phi (\cos(\phi))^2 u^2 + a_0 \phi (\cos(\phi))^2 u'^2 + \right. \\
& 1/2 a_0 \phi u^2 - 1/2 a_0 \phi u'^2 - a_0 \sin(\phi) \cos(\phi) u'^2 + u' a_0 u (\cos(\phi))^2 - 1/2 u' a_0 u + \\
& \left. 2 u' u a_0 \phi \sin(\phi) \cos(\phi) - 2 u' (\cos(\phi))^2 a_1 + u' a_1 \right), \\
\bar{I}^N = I_0^3 + \epsilon & \left( -u' \phi a_0 u + 2 u' a_1 \sin(\phi) \cos(\phi) + \right. \\
& \phi a_0 \sin(\phi) \cos(\phi) u^2 - \phi a_0 \sin(\phi) \cos(\phi) u'^2 + 2 u' \phi a_0 u (\cos(\phi))^2 - \\
& 2 (\cos(\phi))^2 a_1 u - 1/2 (\cos(\phi))^2 a_0 u^2 - 1/2 a_0 (\cos(\phi))^2 u'^2 \\
& \left. + 1/4 a_0 u'^2 + 1/4 a_0 u^2 + a_1 u \right), \\
\bar{I}^O = I_0^4 + \epsilon & \left( -1/2 u' a_0 \phi \cos(\phi) + 1/4 u' a_0 \sin(\phi) + 1/4 \cos(\phi) a_0 u + \right. \\
& \left. \cos(\phi) a_1 - 1/2 a_0 \phi u \sin(\phi) \right), \\
\bar{I}^P = I_0^5 + \epsilon & \left( 1/4 u' a_0 \cos(\phi) + 1/2 u' \phi a_0 \sin(\phi) - 1/2 \phi a_0 u \cos(\phi) - \right. \\
& \left. 1/4 \sin(\phi) a_0 u - \sin(\phi) a_1 \right).
\end{aligned}$$

Next, we apply the conditions to some particularly relevant physical choices of  $G_i(u, u', u'')$ .

## 5. The modified Klein-Gordon equation under GUP

The modified Klein-Gordon equation, is a fourth-order partial differential equation

$$\Delta \Psi - 2\beta h^2 \Delta(\Delta \Psi) + V_0 \Psi = 0 \tag{22}$$

where  $V_0 = \left(\frac{mc}{\hbar}\right)^2$ ,  $\Delta$  is the Laplace operator and the terms  $O(\beta^2)$  have been ignored.

The action of the modified Klein-Gordon equation (22) is

$$S = \int dx^4 \sqrt{-g} \mathcal{L}_A(\Psi, D_\sigma \Psi),$$

where the Lagrangian  $\mathcal{L}_A(\Psi, D_\sigma \Psi)$  is given by

$$\mathcal{L}_A = \frac{1}{2} (\sqrt{-g} g^{\mu\nu} D_\mu \Psi D_\nu \Psi - \sqrt{-g} V_0 \Psi^2). \quad (23)$$

Changing variables  $\Psi \equiv u$  and reducing Eq. (22), we have the reduced Klein-Gordon equation with  $V_0 = 1, \epsilon = -2\beta\hbar^2$ , that is a fourth-order ordinary differential equation, which then possesses the Lagrangian Eq. (23) rewritten in the form (1), with

$$G_1(u, u', u'') = -\frac{1}{2}(u'')^2 \quad \text{and} \quad G_2(u, u', u'') = G_3(u, u', u'') = 0.$$

After the application of the conditions (13)-(15) we find a system of five equations after separation of monomials. The resultant symmetries are  $X_0^{1-5}$  and thus the modified Klein-Gordon equation under GUP contains no first-order nontrivial approximate symmetries.

On a side note, in this case a trivial approximate symmetry  $X^\epsilon = \epsilon \partial_t$  would give the approximate obvious first-order first integral  $I = \frac{1}{2}\epsilon (u'^2 + u^2)$ .

## 6. The Radial Orbital equation

The orbital equation or motion equation of a planet, from an embedded Reissner-Nordström black hole is given by

$$u'' + u = \frac{M}{L^2} - \frac{Q^2 u}{L^2} + 3Mu^2 - 2Q^2 u^3 - \frac{H^2}{L^2 u^3}, \quad (24)$$

where  $u \equiv \frac{1}{r}$ , the prime denotes differentiation with respect to  $\phi$  and  $L$  is the angular momentum of the planet. The terms  $3Mu^2$  and  $\frac{H^2}{L^2 u^3}$  come from the general relativity and cosmic expansion effect, respectively. Furthermore, the

term  $\frac{Q^2 u}{L^2}$  and  $-2Q^2 u^2$  are related to charge. The ratio between the term in  $H$  and  $\frac{M}{L^2}$  is  $8 \times 10^{-34}$  for Mercury and  $3.6 \times 10^{-28}$  for Neptune. If we choose

$$\epsilon = 2M \quad \text{and} \quad \kappa \epsilon^2 = Q^2 \quad \text{and} \quad \rho \epsilon^2 = H^2,$$

the Lagrangian corresponding to Eq.(24) is given by the general Lagrangian (1) with

$$G_1(u, u', u'') = \left( -\frac{u}{2L^2} - \frac{u^3}{2} \right) \quad \text{and} \quad G_2(u, u', u'') = \left( \frac{\kappa u^2}{2L^2} + \frac{2\kappa u^4}{4} - \frac{\rho}{2L^2} u^{-2} \right). \quad (25)$$

The first step is to retain the term in  $G_1(u, u', u'')$  from Eq.(24). Consequently, the conditions (13)-(15) provide a system of four equations that solve to give the first-order approximate Noether symmetry generators given by

$$\begin{aligned} X_1^\epsilon &= X_0^4 + \epsilon (2 \sin(\phi) \partial_\phi + u \cos(\phi) \partial_u), \\ X_2^\epsilon &= X_0^5 - \epsilon (2 \cos(\phi) \partial_\phi - u \sin(\phi) \partial_u). \end{aligned}$$

Correspondingly, the first-order approximate gauge term in this case is

$$\begin{aligned} A_1 &= \frac{1}{2L^2} ( -\cos(2\phi) c_7 u^2 L^2 + \sin(2\phi) c_6 u^2 L^2 \\ &\quad + ((-c_4 u^2 - 2c_9 u) L^2 + c_4) \cos(\phi) ) \\ &\quad + \frac{1}{2L^2} ( ((c_5 u^2 + 2c_{10} u) L^2 - c_5) \sin(\phi) + 2c_{11} L^2 ) \end{aligned}$$

The first-order approximate first integrals related to  $X_{1-2}^\epsilon$  are given by

$$\begin{aligned} I_1^1 &= I_0^4 + \epsilon \left( \sin(\phi) u'^2 - \cos(\phi) u u' + \frac{1}{2} \frac{(L^2 u^2 + 1)}{L^2} \sin(\phi) \right), \\ I_1^2 &= I_0^5 - \epsilon \left( u'^2 \cos(\phi) + u u' \sin(\phi) + \frac{1}{2} \frac{L^2 u^2 + 1}{L^2} \cos(\phi) \right). \end{aligned}$$

In the second approximation, we retain the quadratic  $\epsilon$  terms, that is the  $G_1(u, u', u'')$  and  $G_2(u, u', u'')$  defined for Eq.(24). We proceed with the consideration of the conditions (16)-(17) and observe that Eq.(24) possesses no nontrivial second-order approximate symmetry generators, but the first-order approximate symmetry generators are preserved.

## 7. Discussion and Conclusion

In this work we studied the approximate Noetherian point symmetries and first integrals of the class of differential equations which follow from a Lagrangian

perturbed up-to third-order in  $\epsilon$ . We presented new examples where the application of our conditions can be seen. The knowledge of approximate symmetries was used to find the approximate first integrals of the corresponding approximate equations. We believe that this work can be very useful in the study of various differential problems. Indeed numerous equations originate from the generalized Lagrangian (1), such as the orbital equations of perturbed spaces. Conditions (13)-(20), applied to the problems studied in [32, 33, 34], immediately gives the results on approximate symmetries (Table 1), obtained in these works, for orbital equations. To this end, the same explicit conditions facilitate

Table 1: Approximate generators derived using conditions (13)-(20) for several interesting equations

Model or Orbital equation <sup>1</sup>	$G_1(u, u', u'')$ & Approx. Symmetry	$G_2(u, u', u'')$ & Approx. Symmetry	$G_3(u, u', u'')$ & Approx. Symmetry
Schwarzschild	$\left(-\frac{u}{2L^2} - \frac{u^3}{2}\right)$ $X_{1-2}^\epsilon$	0	0
Reissner-Nordström	$\left(-\frac{u}{2L^2} - \frac{u^3}{2}\right)$ $X_{1-2}^\epsilon$	$\left(\frac{ku^2}{2L^2} + \frac{2ku^4}{4}\right)$ $\zeta$	0
Bardeen	$\left(-\frac{u}{2L^2} - \frac{u^3}{2}\right)$ $X_{1-2}^\epsilon$	0	$\left(\frac{3ku^5}{4} + \frac{3ku^3}{4L^2}\right)$ $\zeta$

the identification of approximate symmetry “groups” of other critical orbital equations. By way of examples, we list the orbital equations of the charged rotating Bañados, Teitelboim and Zanelli metric or the Kerr, Kerr-Newman or Kerr-Newman-AdS spaces, all of which can be rewritten with a perturbation.

#### *Acknowledgments*

SJ acknowledges financial support from the National Research Foundation of South Africa (99279). NM acknowledges support from Standard Bank in partnership with Studytrust. We are grateful to Dr. Andronikos Paliathanasis

<sup>1</sup>The speed of light is  $c = 1$ ,  $k$  is a nonzero constant and  $\zeta$  denotes that no nontrivial approximate symmetry exists.

for his useful comments.

## 8. References

### References

- [1] V.A Baikov, R.K. Gazizov and N.H. Ibragimov, Approximate symmetries of equations with a small parameter, *Mat. Sb.* 136 (1988) 435-450 (English Transl. in *Math. USSR Sb.* 64 (1989) 427-441).
- [2] W.I. Fushchich and W.M. Shtelen, On approximate symmetry and approximate solutions of the non-linear wave equation with a small parameter, *J. Phys. A: Math. Gen.* 22 (1989) 887-890.
- [3] T. Feroze and A.H. Kara, Group theoretic methods for approximate invariants and Lagrangians for some classes of  $y'' + \epsilon F(t)y' + y = f(y, y')$ , *Int. J. Non-Linear Mech.* 37 (2002) 275-280.
- [4] A.G. Johnpillai and A.H. Kara, Variational Formulation of Approximate Symmetries and Conservation Laws, *Int. J. Theor. Phys.* 40 (2001) 1501-1509.
- [5] V.A. Baikov, Approximate symmetries of the van der Pol equation, *Differential Equations* 30(10) (1994) 1820-1822.
- [6] M. Pakdemirli, M. Yürüsoy, T. Dolapci, Comparison of Approximate Symmetry Methods for Differential Equations, *Acta Applic. Math.* 80 (2004) 243-271.
- [7] U. Camci, Approximate Noether gauge symmetries of the Bardeen model, *Eur. Phys. J. C* 74 (2014) 3201.
- [8] R.K Gazizov, Lie Algebras of Approximate Symmetries, *J. Nonlinear Math. Phys.*, 3 (1996) 96-101.
- [9] N.H. Ibragimov, A.H. Kara and F.M. Mahomed, Lie-Backlund and Noether Symmetries with Applications, *Nonl. Dyn.* 15 (1998) 115-136.



- [10] A.G. Johnpillai, A.H. Kara and F.M. Mahomed, Approximate Noether-type symmetries and conservation laws via partial Lagrangians for PDEs with a small parameter, *J. Comput. Appl. Math.* 223 (2009) 508-518.
- [11] Z.Y. Zhang, Approximate nonlinear self-adjointness and approximate conservation laws, *J. Phys. A: Math. Theor.* 46 (2013) 155203.
- [12] A. Paliathanasis and S. Jamal, Approximate Noether symmetries and collineations for regular perturbative Lagrangians, *J. Geom. Phys.* 124 (2018) 300-310.
- [13] B. Champagne, W. Hereman, P. Winternitz, The computer calculation of Lie point symmetries of large systems of differential equations, *Comp. Phys. Commun.* 66 (1991) 319-340.
- [14] G. Baumann, *Symmetry Analysis of Differential Equations with Mathematica*, Springer, New York, 2000.
- [15] S. Dimas, D. Tsubelis, SYM: A new symmetry-finding package for Mathematica in *Group Analysis of Differential Equations*, University of Cyprus, Nicosia, Cyprus (2005).
- [16] C.J. Gao, S.N. Zhan, Reissner-Nordström metric in the Friedman-Robertson-Walker universe, *Phys. Lett. B* 595 (2004) 28-35.
- [17] M. Maggiore, A generalized uncertainty principle in quantum gravity, *Phys. Lett. B* 304 (1993) 65-69.
- [18] A. Kempf, Non-pointlike particles in harmonic oscillators, *J. Phys. A Math. Gen.*, 30, (1997) 2093.
- [19] S. Das and E.C. Vagenas, Universality of Quantum Gravity Corrections, *Phys. Rev. Lett.* 101 (2008) 221301.
- [20] S.K. Moayedi, M.R. Setare and H. Moayeri, Quantum Gravitational Corrections to the Real Klein-Gordon Field in the Presence of a Minimal Length, *Int. J. Theor. Phys.* 49 (2010) 2080.

- [21] A. Paliathanasis, S. Pan, S. Pramanik, Scalar field cosmology modified by the Generalized Uncertainty Principle, *Class. Quant. Grav.* 32 (24) (2015) 245006.
- [22] H. Stephani, *Differential Equations: Their Solutions using Symmetry*, Cambridge University Press, 1989.
- [23] E. Noether, *Nachr. d. König. Gesellsch. d. Wiss. zu Göttingen, Math-phys. Klasse* 235 (1918).
- [24] V. Baikov, R.K. Gazizov, N.H. Ibragimov and F.M. Mahomed, Closed orbits and their stable symmetries. *J. Math. Phys.* 35 (1994) 6525-6535.
- [25] P.J. Olver, *Application of Lie Groups to Differential Equations*, Springer, New York, 1993.
- [26] N. Dimakis, A. Giacomini, S. Jamal, G. Leon and A. Paliathanasis, Noether symmetries and stability of ideal gas solutions in Galileon cosmology, *Phys. Rev. D* 95 (2017) 064031.
- [27] S. Capozziello, E. Piedipalumbo, C. Rubano, P. Scudellaro, Noether symmetry approach in phantom quintessence cosmology *Phys.Rev. D* 80 (2009) 104030.
- [28] S. Jamal, A group theoretical application of  $SO(4,1)$  in the de Sitter universe, *Gen. Rel. Grav.* 49 (88) (2017), DOI 10.1007/s10714-017-2253-4.
- [29] P.J. Olver, Evolution equations possessing infinitely many symmetries, *J. Math. Phys.* 18(6) (1977) 1212-1215.
- [30] S. Jamal, A. Mathebula, Generalized Symmetries and Recursive Operators of Some Diffusive Equations, *Bull. Malays. Math. Sci. Soc.*, DOI 10.1007/s40840-017-0510-z.
- [31] G.C. McVittie, The mass-particle in an expanding universe, *Mon. Not. R. Astron. Soc.* 93 (1933) 325-329.

- [32] M. Sharif and S. Waheed, Energy of Bardeen Model Using Approximate Symmetry Method, *Phys. Scr.*, 83 (2011) 015014.
- [33] A.H. Kara, F. M. Mahomed and A. Qadir, Approximate symmetries and conservation laws of the geodesic equations for the Schwarzschild metric, *Nonl. Dyn.* 51 (2008) 183-188.
- [34] I. Hussain, F. M. Mahomed and A. Qadir, Second-Order Approximate Symmetries of the Geodesic Equations for the Reissner-Nordström Metric and Re-Scaling of Energy of a Test Particle, *SIGMA* 3(115) (2007) 1-9.