STRONG MULTIPLICITY ONE THEOREMS FOR LOCALLY HOMOGENEOUS SPACES OF COMPACT TYPE

EMILIO A. LAURET AND ROBERTO J. MIATELLO

ABSTRACT. Let G be a compact connected semisimple Lie group, let K be a closed subgroup of G, let Γ be a finite subgroup of G, and let τ be a finite dimensional representation of K. For π in the unitary dual $\widehat{G}$ of G, denote by $n_\Gamma(\pi)$ its multiplicity in $L^2(\Gamma\backslash G)$.

We prove a strong multiplicity one theorem in the spirit of Bhagwat and Rajan, for the $n_\Gamma(\pi)$ for π in the set $\widehat{G}_\tau$ of irreducible τ-spherical representations of G. More precisely, for Γ and Γ' finite subgroups of G, we prove that if $n_\Gamma(\pi) = n_{\Gamma'}(\pi)$ for all but finitely many $\pi \in \widehat{G}_\tau$, then Γ and Γ' are τ-representation equivalent, that is, $n_\Gamma(\pi) = n_{\Gamma'}(\pi)$ for all $\pi \in \widehat{G}_\tau$.

Moreover, when $\widehat{G}_\tau$ can be written as a finite union of strings of representations, we prove a finite version of the above result. For any finite subset $\widehat{F}_\tau$ of $\widehat{G}_\tau$ verifying some mild conditions, the values of the $n_\Gamma(\pi)$ for $\pi \in \widehat{F}_\tau$ determine the $n_\Gamma(\pi)$'s for all $\pi \in \widehat{G}_\tau$. In particular, for two finite subgroups Γ and Γ' of G, if $n_\Gamma(\pi) = n_{\Gamma'}(\pi)$ for all $\pi \in \widehat{F}_\tau$ then the equality holds for every $\pi \in \widehat{G}_\tau$. We use algebraic methods involving generating functions and some facts from the representation theory of G.

1. INTRODUCTION

Let G be a Lie group and Γ a discrete cocompact subgroup of G. The right regular representation R_Γ of G on $L^2(\Gamma\backslash G)$ decomposes as a discrete direct sum of unitary irreducible representations (π, V_π) of G occurring with finite multiplicity. That is,

$$(1.1) \qquad L^2(\Gamma\backslash G) \simeq \bigoplus_{\pi \in \widehat{G}} n_\Gamma(\pi)\, V_\pi,$$

with $n_\Gamma(\pi) \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ for any π in the unitary dual $\widehat{G}$ of G.

Now, let K be a compact subgroup of G and let (τ, V_τ) be a finite dimensional representation of K. A unitary representation (π, V_π) of G will be called τ-spherical if $\operatorname{Hom}_K(V_\tau, V_\pi) \neq 0$. Let $\widehat{G}_\tau$ denote the set of τ-spherical irreducible representations of G. The Hilbert space

$$(1.2) \qquad L^2(\Gamma\backslash G)_\tau := \bigoplus_{\pi \in \widehat{G}_\tau} n_\Gamma(\pi)\, V_\pi$$

defines a unitary representation of G.

Two discrete cocompact subgroups Γ and Γ' of G are said to be *representation equivalent* (resp. *τ-representation equivalent*) in G if the representations $L^2(\Gamma\backslash G)$ and $L^2(\Gamma'\backslash G)$ (resp. $L^2(\Gamma\backslash G)_\tau$ and $L^2(\Gamma'\backslash G)_\tau$) are unitarily equivalent, that is, if $n_\Gamma(\pi) = n_{\Gamma'}(\pi)$ for every $\pi \in \widehat{G}$ (resp. $\pi \in \widehat{G}_\tau$).

C. Bhagwat and C.S. Rajan [BR11] studied spectral analogues of the so called *strong multiplicity one theorems*. They showed that if Γ and Γ' are discrete cocompact subgroups in a semisimple Lie group G such that $n_\Gamma(\pi) = n_{\Gamma'}(\pi)$ for all but finitely many $\pi \in \widehat{G}$, then Γ and Γ' are representation equivalent in G. Furthermore, they proved a similar result for 1_K-spherical representations of G, when G/K is a non-compact symmetric space. D. Kelmer [Ke14] obtained refinements of the last result when G/K has real rank one, for any finite-dimensional representation τ of K. He replaced the finite set of exceptions by a possibly infinite set of sufficiently small density. Both Bhagwat-Rajan and Kelmer used analytic methods, the Selberg trace formula as a main tool.

The main goal of this article is to obtain strong multiplicity one type theorems in the case when G is a compact semisimple Lie group. We will use only algebraic methods and facts from the representation theory of compact Lie groups. Furthermore, we will not assume any restriction on K nor that the discrete subgroups Γ and Γ' act freely on G/K, so the quotient spaces $\Gamma\backslash G/K$ and $\Gamma'\backslash G/K$ are compact good orbifolds. In this context, Γ is necessarily finite and, by Frobenius reciprocity, $n_\Gamma(\pi)$ coincides with the dimension of the subspace of Γ-invariant elements of V_π. We will show that under certain conditions on G, K, τ, a finite subset of the multiplicities $n_\Gamma(\pi)$ determines all multiplicities for $\pi \in \widehat{G}_\tau$.

A main tool for us will be the notion of what we shall call a *string* of irreducible representations of G. Fix a maximal torus of G and a positive system in the associated root system. By the highest weight theorem, the irreducible representations of G are in correspondence with the elements in the set of G-integral dominant weights $\mathcal{P}^+(G)$. For $\Lambda \in \mathcal{P}^+(G)$, we denote by π_Λ the irreducible representation of G with highest weight Λ. A string is a sequence of representations of the form $\{\pi_{\Lambda_0+k\omega} : k \in \mathbb{N}_0\}$ with $\omega, \Lambda_0 \in \mathcal{P}^+(G)$. We call ω the direction and Λ_0 the base of the string.

The usefulness of the notion of string of representations is that one can study the multiplicities $n_\Gamma(\pi_{\Lambda_0+k\omega})$ by means of the generating function

$$(1.3) \qquad F_{\omega,\Lambda_0,\Gamma}(z) := \sum_{k\geq 0} n_\Gamma(\pi_{\Lambda_0+k\omega})\, z^k.$$

By using a version of the Weyl character formula, in Proposition 2.2 we prove that $F_{\omega,\Lambda_0,\Gamma}(z)$ is a rational function with denominator $(1 - z^{|\Gamma|})^{|\Phi^+|+1}$, where $|\Phi^+|$ stands for the number of positive roots.

Let q be any positive integer divisible by $|\Gamma|$. As a consequence of the rational form of $F_{\omega,\Lambda_0,\Gamma}(z)$, Proposition 3.1 shows that if a finite subset $\mathcal{A} \subset \mathbb{N}_0$ satisfies that

$$(1.4) \qquad |\mathcal{A} \cap (j + q\mathbb{Z})| \geq |\Phi^+| + 1 \quad \text{for all } 0 \leq j \leq q-1,$$

then the coefficients $n_\Gamma(\pi_{\Lambda_0+k\omega})$ for $k \in \mathcal{A}$ determine $n_\Gamma(\pi_{\Lambda_0+k\omega})$ for all $k \geq 0$. For instance, (1.4) holds when $\mathcal{A}$ contains any interval of length $q(|\Phi^+|+1)$. A bonus in the present context is that the rationality of the generating function allows to obtain an expression for any $n_\Gamma(\pi_{\Lambda_0+k\omega})$ as a linear combination of the $\{n_\Gamma(\pi_{\Lambda_0+k'\omega}) : k' \in \mathcal{A}\}$.

As a corollary of Proposition 3.1 we obtain a strong multiplicity one theorem for strings of representations, valid with a possible infinite set of exceptions of sufficiently small density (see Corollary 3.2). We also prove strong multiplicity one theorems for τ-spherical representations.

Theorem 1.1. *Let G be a compact connected semisimple Lie group, let K be a closed subgroup of G and let τ be a finite dimensional representation of K. If Γ and Γ' are finite subgroups of G such that $n_\Gamma(\pi) = n_{\Gamma'}(\pi)$ for all but finitely many $\pi \in \widehat{G}_\tau$ then Γ and Γ' are τ-representation equivalent in G (i.e. $n_\Gamma(\pi) = n_{\Gamma'}(\pi)$ for all $\pi \in \widehat{G}_\tau$).*

A key point in the proof of the previous theorem is the fact that $\widehat{G}_\tau$ can be always written as a (non-necessarily finite) union of strings (see Lemma 3.3). Under the stronger assumption that $\widehat{G}_\tau$ can be written as a finite union of strings, we obtain the following stronger result.

Theorem 1.2. *Let G be a compact connected semisimple Lie group, let K be a closed subgroup of G, and let τ be a finite dimensional representation of K. Assume there are G-integral dominant weights $\Lambda_{0,i}$ and ω_i for $1 \leq i \leq m$ such that*

$$(1.5) \qquad \widehat{G}_\tau = \bigcup_{i=1}^{m} \{\pi_{\Lambda_{0,i}+k\omega_i} : k \geq 0\}.$$

Given an integer $q > 0$, let $\widehat{F}_\tau$ be any finite subset of $\widehat{G}_\tau$ such that $\mathcal{A}_i := \{k \in \mathbb{N}_0 : \pi_{\Lambda_{0,i}+k\omega_i} \in \widehat{F}_\tau\}$ satisfies (1.4) for each $1 \leq i \leq m$. Then, for any finite subgroup Γ of G with q divisible by $|\Gamma|$, the finite set of multiplicities $n_\Gamma(\pi)$ for $\pi \in \widehat{F}_\tau$ determine the $n_\Gamma(\pi)$ for all $\pi \in \widehat{G}_\tau$. In particular, if Γ and Γ' are finite subgroups of G with q divisible by $|\Gamma|$ and $|\Gamma'|$ such that $n_\Gamma(\pi) = n_{\Gamma'}(\pi)$ for all $\pi \in \widehat{F}_\tau$, then Γ and Γ' are τ-representation equivalent in G.

Remark 1.3. The assumption in the previous theorem holds for Gelfand pairs of rank one, in particular for compact symmetric spaces of real rank one (see Remark 3.4).

A motivation of the authors for the questions treated in this paper are the applications to spectral geometry of locally homogeneous spaces. In fact, Theorem 1.2 is a main tool in [LM18], where the authors study the strong multiplicity one property for the τ-spectrum of a space covered by a compact symmetric space of real rank one and the connection between τ-isospectrality and τ-representation equivalence.

The article is organized as follows. In Section 2 we show that the generating function $F_{\omega,\Lambda_0,\Gamma}(z)$ is a rational function. The strong multiplicity one theorems, including Theorems 1.1 and 1.2, are proved in Section 3.

2. Generating functions of strings

Let G be a compact connected semisimple Lie group, let T be a maximal torus in G and let $W = W(G,T)$ denote the corresponding Weyl group. Let $\Phi = \Phi(G,T)$ be the associated root system and let $\mathcal{P}(G)$ be the lattice of G-integral weights. We fix a system of positive roots $\Phi^+ = \Phi^+(G,T)$ and we let $\mathcal{P}^+(G)$ be the set of dominant G-integral weights. If (π, V_π) is a finite dimensional representation of G, denote by χ_π the character of π, $\chi_\pi(g) = \operatorname{tr} \pi(g)$, for any $g \in G$. If $\Lambda \in \mathcal{P}^+(G)$, let π_Λ be the irreducible representation of G with highest weight Λ.

Definition 2.1. For $\omega, \Lambda_0 \in \mathcal{P}^+(G)$, we call the ordered set $\mathcal{S}(\omega, \Lambda_0) := \{\pi_{\Lambda_0+k\omega} : k \in \mathbb{N}_0\}$ the *string of representations* associated to (ω, Λ_0). The elements ω and Λ_0 will be called the *direction* and the *base* of the string respectively.

Let Γ be a discrete, hence finite, subgroup of G. By Frobenius reciprocity, one has $n_\Gamma(\pi) = \dim V_\pi^\Gamma$ for any $\pi \in \widehat{G}$, where V_π^Γ stands for the subspace of V_π invariant by Γ. In order to study these numbers, we will encode them in a generating function for representations lying in a string. More precisely, given a (ω, Λ_0)-string $\mathcal{S}(\omega, \Lambda_0)$, we define the generating function

$$(2.1) \qquad F_{\omega,\Lambda_0,\Gamma}(z) = \sum_{k \in \mathbb{N}_0} \dim V_{\pi_{\Lambda_k}}^\Gamma \, z^k,$$

where $\Lambda_k = \Lambda_0 + k\omega$ for any $k \geq 0$. From now on we will abbreviate $F_\Gamma(z) = F_{\omega,\Lambda_0,\Gamma}(z)$.

The next result is the main goal of this section. It will be a main tool in the sequel.

Proposition 2.2. *Write* $q = |\Gamma|$. *In the notation above, there exists a complex polynomial* $p(z) = p_{\omega,\Lambda_0,\Gamma}(z)$ *of degree less than* $q(|\Phi^+|+1)$ *such that*

$$F_\Gamma(z) = \frac{p(z)}{(1-z^q)^{|\Phi^+|+1}}.$$

The rest of this section will be devoted to give a proof of this result. One has that

$$(2.2) \qquad F_\Gamma(z) = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\sum_{k\in\mathbb{N}_0}\chi_{\pi_{\Lambda_k}}(\gamma)z^k,$$

since $\dim V_{\pi_{\Lambda_k}}^\Gamma = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\chi_{\pi_{\Lambda_k}}(\gamma)$. Our strategy will be to compute the terms $\chi_{\pi_{\Lambda_k}}(\gamma)$ for each $\gamma\in\Gamma$ by using a version of the Weyl character formula (see [CR15, §2.2]). To state this result we need to introduce some more notation.

For fixed $t\in T$, we let $Z = Z_t = C_G(t)^0$ be the identity component of the centralizer of t in G, that is a compact subgroup of G. Let $\Phi(Z,T)$ be the root system associated to (Z,T), and let $\Phi_Z^+ = \Phi^+ \cap \Phi(Z,T)$, $W^Z = \{\sigma\in W : \sigma^{-1}\Phi_Z^+ \subset \Phi^+\}$, $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$, $\rho_Z = \frac{1}{2}\sum_{\alpha\in\Phi_Z^+}\alpha$, and

$$(2.3) \qquad p_Z(v) = \prod_{\alpha\in\Phi_Z^+}\frac{\langle\alpha, v+\rho_Z\rangle}{\langle\alpha,\rho_Z\rangle}.$$

If $t = \exp(H)\in T$ and μ is a weight, write $t^\mu = e^{\mu(H)}$. Now, the *Weyl character formula* in [CR15, Prop. 2.3] tells us that, for any $\Lambda\in\mathcal{P}^+(G)$ and $t\in T$, one has

$$(2.4) \qquad \chi_{\pi_\Lambda}(t) = \frac{\displaystyle\sum_{\sigma\in W^Z}\varepsilon(\sigma)\, t^{\sigma(\Lambda+\rho)-\rho}\, p_Z(\sigma(\Lambda+\rho)-\rho_Z)}{\displaystyle\prod_{\alpha\in\Phi^+\smallsetminus\Phi_Z^+}(1-t^{-\alpha})}.$$

We now proceed with the proof of Proposition 2.2. For each $\gamma\in\Gamma$ we fix $t_\gamma\in T$, conjugate to γ, and we abbreviate $\Phi_\gamma^+ = \Phi_{Z_\gamma}^+$. By substituting (2.4) in (2.2), we obtain that

$$(2.5) \qquad F_\Gamma(z) = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\Big(\prod_{\alpha\in\Phi^+\smallsetminus\Phi_Z^+}(1-t_\gamma^{-\alpha})^{-1}\Big)\sum_{\sigma\in W^Z}\varepsilon(\sigma)\,c_{\gamma,\sigma}(z),$$

where

$$(2.6) \qquad c_{\gamma,\sigma}(z) = \sum_{k\in\mathbb{N}_0}t_\gamma^{\sigma(\Lambda_k+\rho)-\rho}\,p_Z(\sigma(\Lambda_k+\rho)-\rho_Z)\,z^k.$$

The next goal will be to show that $c_{\gamma,\sigma}(z)$ is a particular rational function.

Lemma 2.3. *In the notation above, we have that*

$$(2.7) \qquad c_{\gamma,\sigma}(z) = \frac{p_{\gamma,\sigma}(z)}{\left(1-t_\gamma^{\sigma(\omega)}z\right)^{|\Phi_\gamma^+|+1}},$$

for some complex polynomial $p_{\gamma,\sigma}(z)$ *of degree* $\leq |\Phi_\gamma^+|$.

Proof. On the one hand,

$$(2.8) \qquad p_Z(\sigma(\Lambda_k + \rho) - \rho_Z) = \prod_{\alpha \in \Phi_Z^+} \frac{\langle \alpha, \sigma(\Lambda_k + \rho) \rangle}{\langle \alpha, \rho_Z \rangle} = \prod_{\alpha \in \Phi_Z^+} \frac{\langle \sigma(\alpha), k\omega + \Lambda_0 + \rho \rangle}{\langle \alpha, \rho_Z \rangle}$$

$$= \left(\prod_{\alpha \in \Phi_Z^+} \frac{1}{\langle \alpha, \rho_Z \rangle} \right) \left(\prod_{\alpha \in \Phi_Z^+} \left(k\langle \sigma(\alpha), \omega \rangle + \langle \sigma(\alpha), \Lambda_0 + \rho \rangle \right) \right),$$

which is a polynomial in k of degree $|\Phi_\gamma^+|$ with complex coefficients. We thus write

$$p_Z(\sigma(\Lambda_k + \rho) - \rho_Z) = \sum_{j=0}^{|\Phi_\gamma^+|} b_j k^j.$$

On the other hand, we have that $t_\gamma^{\sigma(\Lambda_k+\rho)-\rho} = t_\gamma^{\sigma(\Lambda_0+\rho)-\rho} t_\gamma^{k\sigma(\omega)}$. Hence

$$(2.9) \qquad c_{\gamma,\sigma}(z) = t_\gamma^{\sigma(\Lambda_0+\rho)-\rho} \sum_{j=0}^{|\Phi_\gamma^+|} b_j \sum_{k \in \mathbb{N}_0} k^j (t_\gamma^{\sigma(\omega)} z)^k.$$

We claim that for any $0 \le j \le |\Phi_\gamma^+|$,

$$(2.10) \qquad \sum_{k \ge 0} k^j t_\gamma^{k\sigma(\omega)} z^k = \frac{p_j(z)}{\left(1 - t_\gamma^{\sigma(\omega)} z\right)^{|\Phi_\gamma^+|+1}}$$

for some polynomial $p_j(z)$ of degree at most $|\Phi_\gamma^+|$. Once this is done, the lemma clearly follows.

We next prove our claim. The assertion is clear for $j = 0$, by using the geometric series.

Assume now that (2.10) holds for some $j < |\Phi_\gamma^+|$. Since $j!\binom{k+j}{k} = k^j + \sum_{l=0}^{j-1} c_l k^l$ for some $c_l \in \mathbb{Z}$, we obtain that

$$(2.11) \qquad \sum_{k \ge 0} k^j y^k = j! \sum_{k \ge 0} \binom{k+j}{k} y^k - \sum_{l=0}^{j-1} c_l \sum_{k \ge 0} k^l y^k = \frac{j!}{(1-y)^{j+1}} - \sum_{l=0}^{j-1} c_l \sum_{k \ge 0} k^l y^k.$$

Now, since $(1-y)^{-(j+1)} = \frac{(1-y)^{|\Phi_\gamma^+|-j}}{(1-y)^{|\Phi_\gamma^+|+1}}$, by substituting $y = t_\gamma^{\sigma(\omega)} z$, then (2.10) follows by induction. □

Thus, since $t_\gamma^q = \gamma^q = 1$, $t_\gamma^{\sigma(\omega)}$ is a root of unity of order a divisor of q, then Lemma 2.3 yields

$$(2.12) \qquad c_{\gamma,\sigma}(z) = \frac{p_{\gamma,\sigma}(z)}{\left(1 - t_\gamma^{\sigma(\omega)} z\right)^{|\Phi_\gamma^+|+1}} = \frac{p_{\gamma,\sigma}(z)\,(1-z^q)^{|\Phi^+|-|\Phi_\gamma^+|} \displaystyle\prod_{\xi^q=1,\, \xi \ne t_\gamma^{\sigma(\omega)}} (1-\xi z)^{|\Phi_\gamma^+|+1}}{(1-z^q)^{|\Phi^+|+1}}.$$

Since $|\Phi^+| \ge |\Phi_\gamma^+|$ for all $\gamma \in \Gamma$, the degree of the polynomial in the numerator is less than or equal to

$$|\Phi_\gamma^+| + q(|\Phi^+| - |\Phi_\gamma^+|) + (q-1)(|\Phi_\gamma^+|+1) < q(|\Phi^+|+1),$$

hence Proposition 2.2 follows from (2.5) and (2.12).

In the expression for $F_\Gamma(z)$ as a rational function in Proposition 2.2, the numerator $p(z)$ and the denominator $(1-z^q)^{|\Phi^+|+1}$ usually share some factors. We next discuss a simple example illustrating this situation.

Example 2.4. Let $G = \mathrm{SU}(2)$, which has rank one. In this case, $\Phi^+ = \{\alpha := \varepsilon_1 - \varepsilon_2\}$, $\widehat{G} = \{\pi_k := \pi_{\alpha k/2} : k \in \mathbb{N}_0\}$, $\dim V_{\pi_k} = k+1$, and every weight space in V_{π_k} is one-dimensional having the form $\frac{k-2h}{2}\alpha$ for some $0 \le h \le k$. Fix an even positive integer q and set

$$(2.13) \qquad \Gamma = \left\{ \begin{pmatrix} e^{2\pi i h/q} & \\ & e^{-2\pi i h/q} \end{pmatrix} : h \in \mathbb{Z} \right\}.$$

We claim that $\dim V_{\pi_k}^\Gamma = 1 + 2\lfloor \frac{k}{q} \rfloor$ for any $k \ge 0$. Indeed, since Γ acts by scalars in each weight space and these scalars are q-th roots of unity, we have that $\dim V_{\pi_k}^\Gamma$ is given by the number of weights in V_{π_k} of the form $\frac{k-2h}{2}\alpha$ for $0 \le h \le k$ divisible by q. Consequently,

$$(2.14) \qquad F_\Gamma(z) = \sum_{k \ge 0}(1 + \lfloor \tfrac{k}{q} \rfloor) z^k = \sum_{j=0}^{q-1} \sum_{m \ge 0}(1 + \lfloor \tfrac{mq+j}{q} \rfloor) z^{mq+j}$$

$$= \frac{1-z^q}{1-z} \sum_{m \ge 0}(1+2m) z^{qm} = \frac{1+z^q}{(1-z)(1-z^q)} = \frac{1 + z + \cdots + z^{2q-1}}{(1-z^q)^2}.$$

Then $\frac{1-z^{2q}}{1-z}$ and $(1-z^q)^2$ are the numerator and denominator of $F_\Gamma(z)$ respectively, as stated in Proposition 2.2 and the polynomial $\frac{1-z^q}{1-z} = 1 + z + \cdots + z^{q-1}$ is their greatest common divisor.

Kostant [Ko85] computed explicitly $F_\Gamma(z)$ for the (obvious) string $\{\pi_k : k \ge 0\}$ in $\mathrm{SU}(2)$ as in Example 2.4 for every finite subgroup Γ of $\mathrm{SU}(2)$.

3. STRONG MULTIPLICITY ONE THEOREMS

The goal of this section is to prove Theorems 1.1 and 1.2. We first recall the context: G is a compact connected semisimple Lie group, K a closed subgroup of G, and Γ a finite subgroup of G. Furthermore, we fix a maximal torus T in G and a positive system in the associated root system. We denote by Φ^+ the set of positive roots. By Frobenius reciprocity, we have that

$$(3.1) \qquad n_\Gamma(\pi) = \dim \mathrm{Hom}_G(\pi, L^2(\Gamma\backslash G)) = \dim \mathrm{Hom}_\Gamma(1_\Gamma, \pi|_K) = \dim V_\pi^\Gamma$$

for every $\pi \in \widehat{G}$.

The main tool in what follows will be the generating function associated to a string of representations (see (2.1)) and the rational expression obtained in Proposition 2.2.

Proposition 3.1. *Let Γ be a finite subgroup of G, let q be any positive integer divisible by $|\Gamma|$, and let Λ_0 and ω be G-integral dominant weights. If $\mathcal{A} \subset \mathbb{N}_0$ satisfies that*

$$(3.2) \qquad |\mathcal{A} \cap (j + q\mathbb{Z})| \ge |\Phi^+| + 1 \quad \text{for all } 0 \le j \le q-1,$$

then the set of multiplicities $n_\Gamma(\pi_{\Lambda_0+k\omega})$ for $k \in \mathcal{A}$ determine the whole sequence $n_\Gamma(\pi_{\Lambda_0+k\omega})$ for $k \in \mathbb{N}_0$. Moreover, for any $k \in \mathbb{N}_0$, $n_\Gamma(\pi_{\Lambda_0+k\omega})$ can be linearly expressed in terms of the $n_\Gamma(\pi_{\Lambda_h})$ for $h \in \mathcal{A}$. In particular, if Γ and Γ' are finite subgroups of G with q divisible by $|\Gamma|$ and $|\Gamma'|$ such that $n_\Gamma(\pi_{\Lambda_0+k\omega}) = n_{\Gamma'}(\pi_{\Lambda_0+k\omega})$ for all $k \in \mathcal{A}$, then this equality holds for every $k \ge 0$.

Proof. Set $\Lambda_k = \Lambda_0 + k\omega$ for any $k \ge 0$. By Proposition 2.2 and (3.1), there exists a polynomial $p(z) = p_{\omega,\Lambda_0,\Gamma}(z)$ of degree less than $q(|\Phi^+| + 1)$ such that

$$F_\Gamma(z) = \sum_{k \ge 0} n_\Gamma(\pi_{\Lambda_k}) z^k = \frac{p(z)}{(1-z^q)^{|\Phi^+|+1}}.$$

We write $p(z) = \sum_{k=0}^{q(|\Phi^+|+1)-1} b_k z^k$. We first show that one has an expression

$$(3.3) \qquad n_\Gamma(\pi_{\Lambda_{mq+j}}) = \sum_{h=0}^{|\Phi^+|} b_{hq+j} \binom{m-h+|\Phi^+|}{|\Phi^+|}$$

for any $0 \leq j < q$ and for any $m \geq 0$. Here $\binom{m-h+|\Phi^+|}{|\Phi^+|} = 0$ for $h > m$, by convention, so the sum actually runs over $0 \leq h \leq \min(|\Phi^+|, m)$. To show (3.3), since $\frac{1}{(1-y)^{j+1}} = \sum_{k\geq 0} \binom{k+j}{k} y^k$ for any $j \in \mathbb{N}_0$, then

$$F_\Gamma(z) = p(z)(1 - z^q)^{-(|\Phi^+|+1)} = \left(\sum_{j=0}^{q-1} \sum_{h=0}^{|\Phi^+|} b_{hq+j} z^{hq+j} \right) \left(\sum_{k\geq 0} \binom{k+|\Phi^+|}{|\Phi^+|} z^{kq} \right)$$

$$= \sum_{j=0}^{q-1} \sum_{m\geq 0} \left(\sum_{h=0}^{|\Phi^+|} b_{hq+j} \binom{m-h+|\Phi^+|}{|\Phi^+|} \right) z^{mq+j}.$$

We fix $0 \leq j < q$. Since $|\mathcal{A} \cap (j + q\mathbb{Z})| \geq |\Phi^+| + 1$, there are $m_0, \ldots, m_{|\Phi^+|}$ different non-negative integers such that $m_i q + j \in \mathcal{A}$ for all i. The square matrix

$$\left(\binom{m_i - h + |\Phi^+|}{|\Phi^+|} \right)_{i,h=0}^{|\Phi^+|}$$

is clearly non-singular, so the system of $|\Phi^+| + 1$ linear equations with $|\Phi^+| + 1$ unknowns $b_j, \ldots, b_{q|\Phi^+|+j}$,

$$n_\Gamma(\pi_{\Lambda_{m_i q+j}}) = \sum_{h=0}^{|\Phi^+|} \binom{m_i - h + |\Phi^+|}{|\Phi^+|} b_{hq+j} \qquad \text{for } i = 0, \ldots, |\Phi^+|,$$

has a unique solution. Consequently, the coefficients b_{hq+j} for $0 \leq h \leq |\Phi^+|$ can be linearly expressed in terms of the multiplicities $n_\Gamma(\pi_{\Lambda_k})$ for $k \in \mathcal{A} \cap (j + q\mathbb{Z})$.

Since this holds for every j, we conclude that the b_k for every $0 \leq k < q(|\Phi^+| + 1)$ are determined by the $n_\Gamma(\pi_{\Lambda_k})$'s for $k \in \mathcal{A}$, and hence also $p(z)$ as well as $F_\Gamma(z)$, are determined. Therefore all the $\{n_\Gamma(\pi_k) : k \in \mathbb{N}_0\}$ are linearly determined. $\square$

Proposition 3.1 gives a finite strong multiplicity one result for a string of representations. In the next result we show a refinement, by again proving strong multiplicity one, now valid with a possible infinite set of exceptions of sufficiently small density.

Corollary 3.2. *Let Γ, Γ' be finite subgroups of G, let q be the least common multiple between $|\Gamma|$ and $|\Gamma'|$, and let Λ_0 and ω be G-integral dominant weights. If*

$$(3.4) \qquad \limsup_{t\to\infty} \frac{|\{0 \leq k \leq t : n_\Gamma(\pi_{\Lambda_0+k\omega}) \neq n_{\Gamma'}(\pi_{\Lambda_0+k\omega})\}|}{t} < \frac{1}{q},$$

then $n_\Gamma(\pi_{\Lambda_0+k\omega}) = n_{\Gamma'}(\pi_{\Lambda_0+k\omega})$ for all $k \geq 0$. In particular, if $n_\Gamma(\pi_{\Lambda_0+k\omega}) = n_{\Gamma'}(\pi_{\Lambda_0+k\omega})$ for all but finitely many $k \geq 0$, then the equality holds for every $k \geq 0$.

Proof. Set as usual $\Lambda_k = \Lambda_0 + k\omega$ for any $k \geq 0$. We want to show that the subset of $\mathbb{N}_0$ $\mathcal{A} := \{k \in \mathbb{N}_0 : n_\Gamma(\pi_{\Lambda_k}) = n_{\Gamma'}(\pi_{\Lambda_k})\}$ satisfies (3.2). This done, Proposition 3.1 ensures that $n_\Gamma(\pi_{\Lambda_k}) = n_{\Gamma'}(\pi_{\Lambda_k})$ for all $k \geq 0$ as required.

By the assumption (3.4), $\liminf_{t\to\infty} \frac{|\{0 \le k \le t : k \in \mathcal{A}\}|}{t} > \frac{q-1}{q}$. This means that for every $\varepsilon > 0$ there is $r_0 > 0$ such that

$$(3.5) \qquad \frac{|\{k \in \mathcal{A} : k < rq(|\Phi^+|+1)\}|}{rq(|\Phi^+|+1)} > \frac{q-1+\varepsilon}{q} \qquad \text{for all } r \ge r_0,$$

or equivalently,

$$(3.6) \qquad |\{k \in \mathcal{A} : k \le rq(|\Phi^+|+1)\}| > r(q-1+\varepsilon)(|\Phi^+|+1) \qquad \text{for all } r \ge r_0.$$

Let $r \ge r_0$ and $j_0 \in \mathbb{Z}$ satisfying $0 \le j_0 < q$. We have that

$$(3.7) \qquad |\{k \in \mathcal{A} : k < rq(|\Phi^+|+1)\}| = \sum_{j=0}^{q-1} |\{k \in \mathcal{A} : k < rq(|\Phi^+|+1)\} \cap (j + q\mathbb{Z})|$$

$$\le |\mathcal{A} \cap (j_0 + q\mathbb{Z})| + (q-1)r(|\Phi^+|+1).$$

Hence, (3.6) implies that $|\mathcal{A} \cap (j_0 + q\mathbb{Z})| > r\varepsilon(|\Phi^+|+1)$ for every $r \ge r_0$. By taking any $r \ge 1/\varepsilon$ we obtain that $|\mathcal{A} \cap (j_0 + q\mathbb{Z})| > |\Phi^+| + 1$, as required. $\square$

We will need the following useful fact.

Lemma 3.3. *For any $\tau \in \widehat{K}$, the set $\widehat{G}_\tau$ can be written as a union of strings having a common direction ω, that is, $\widehat{G}_\tau = \bigcup_{\Lambda \in \mathcal{Q}_\tau} \{\pi_{\Lambda+k\omega} : k \in \mathbb{N}_0\}$ for some subset $\mathcal{Q}_\tau$ of $\mathcal{P}^+(G)$.*

Proof. The left-regular representation of G on $L^2(G/K)$ decomposes

$$L^2(G/K) \simeq \bigoplus_{\pi \in \widehat{G}_{1_K}} (\dim V_\pi^K)\, V_\pi,$$

where 1_K denotes the trivial representation of K. Let ω be the highest weight of any non-trivial representation in $\widehat{G}_{1_K}$. Then, for any $\pi_\Lambda \in \widehat{G}_\tau$, one has that $\pi_{\Lambda+k\omega} \in \widehat{G}_\tau$ for all $k \ge 0$ (see for instance [Ko04, Thm. 3.9]). That is, $\mathcal{S}(\omega, \Lambda) \subset \widehat{G}_\tau$, and consequently,

$$(3.8) \qquad \widehat{G}_\tau = \bigcup_{\Lambda \in \mathcal{P}^+(G):\, \pi_\Lambda \in \widehat{G}_\tau} \mathcal{S}(\omega, \Lambda),$$

which completes the proof. $\square$

We are now in a position to give the proofs of the main theorems.

Proof of Theorem 1.1. Let q be the least common multiple between $|\Gamma|$ and $|\Gamma'|$. Since the set of possible exceptions to equality of multiplicities in $\widehat{G}_\tau$ is finite, the set of exceptions is also finite in any string associated to (Λ_0, ω). Now, Corollary 3.2 implies that $n_\Gamma(\pi) = n_{\Gamma'}(\pi)$ for every π in each string, thus equality holds for all $\pi \in \widehat{G}_\tau$ by Lemma 3.3. $\square$

Proof of Theorem 1.2. We first recall the assumptions. We have that

$$\widehat{G}_\tau = \bigcup_{i=1}^{m} \{\pi_{\Lambda_{0,i}+k\omega_i} : k \ge 0\},$$

for some G-integral dominant weights $\Lambda_{0,i}$ and ω_i, for $1 \le i \le m$. Furthermore, $n_\Gamma(\pi) = n_{\Gamma'}(\pi)$ for all $\pi \in \widehat{F}_\tau$, where $\widehat{F}_\tau$ is a finite subset of $\widehat{G}_\tau$ such that for each $1 \le i \le m$ the subset $\mathcal{A}_i = \{k \in \mathbb{N}_0 : \pi_{\Lambda_{0,i}+k\omega_i} \in \widehat{F}_\tau\}$ of $\mathbb{N}_0$ satisfies (3.2).

Fix $1 \le i \le m$. From what has been assumed, it follows that $n_\Gamma(\pi_{\Lambda_{0,i}+k\omega_i}) = n_{\Gamma'}(\pi_{\Lambda_{0,i}+k\omega_i})$ for all $k \in \mathcal{A}_i$. Proposition 3.1 forces $n_\Gamma(\pi_{\Lambda_{0,i}+k\omega_i}) = n_{\Gamma'}(\pi_{\Lambda_{0,i}+k\omega_i})$ for all $k \ge 0$. We conclude that $n_\Gamma(\pi) = n_{\Gamma'}(\pi)$ for all $\pi \in \widehat{G}_\tau$, that is, Γ and Γ' are τ-representation equivalent in G. $\square$

Remark 3.4. We end the article by giving some references in the literature of explicit expressions of $\widehat{G}_\tau$ as a union of strings. Explicit descriptions of the set $\widehat{G}_\tau$ have been used for different purposes, which makes the task of providing a complete list of references difficult.

We first assume that G/K is a compact Riemannian symmetric space with G and K connected. Let G'/K be the non-compact dual space, let $G' = KAN$ be an Iwasawa decomposition of G', and let M be the centralizer of A in K. We denote by $\mathfrak{g}', \mathfrak{m}, \mathfrak{a}$ the Lie algebras of G', M, A respectively. Let $\mathfrak{b}$ be a Cartan Subalgebra of $\mathfrak{m}$ such that $\mathfrak{b}_\mathbb{C} \oplus \mathfrak{a}_\mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g}_\mathbb{C}$. Let 1_K denote the trivial representation of K.

The Cartan–Helgason theorem (see for instance [Kn, Thm. 9.70]) ensures that $\pi_\Lambda \in \widehat{G}_{1_K}$ if and only if $\Lambda|_\mathfrak{b} = 0$ and $\langle \Lambda, \alpha \rangle / \langle \alpha, \alpha \rangle \in \mathbb{N}_0$ for every positive restricted root $\alpha \in \Phi^+(\mathfrak{g}', \mathfrak{a})$. This implies that $\widehat{G}_{1_K}$ can be written as a disjoint union of strings of representations with the same direction ω (there are several choices for ω). When G/K has real rank one, $\widehat{G}_{1_K}$ is exactly one string and furthermore, Camporesi [Ca05a, Thm. 2.4] proved (by using [Ko04]) that there is $\Lambda_{\sigma,\tau} \in \mathcal{P}^+(G)$ for each $(\sigma, \tau) \in \widehat{M} \times \widehat{K}$ such that

$$(3.9) \qquad \widehat{G}_\tau = \bigcup_{\sigma \in \widehat{M}:\, \mathrm{Hom}_M(\sigma, \tau|_M) \neq 0} \mathcal{S}(\omega, \Lambda_{\sigma,\tau}),$$

for any $\tau \in \widehat{K}$ (see also [Ca05b]). We note that the union in (3.9) is finite. Formula (3.9) can be seen as a generalization of the Cartan–Helgason theorem for an arbitrary $\tau \in \widehat{K}$.

Heckman and van Pruijssen [HvP16] extended the previous results to Gelfand pairs of rank one. In addition to the compact symmetric spaces of real rank one, these spaces include $(G, K) = (\mathrm{G}_2, \mathrm{SU}(3))$ and $(\mathrm{Spin}(7), \mathrm{G}_2)$.

The authors do not expect there are many other situations where the condition (1.5) in Theorem 1.2 holds than those given in Remark 3.4. We next give two simple instances where this is not the case, i.e. $\widehat{G}_\tau$ cannot be written as a finite union of strings.

Remark 3.5. Let G be any compact connected semisimple Lie group. By setting $K = \{e\}$, one clearly has $\widehat{G}_{1_K} = \widehat{G}$. We claim that, if the rank n of G is at least 2 (i.e. any G aside of $\mathrm{SO}(3)$ and $\mathrm{SU}(2)$), then $\widehat{G}$ cannot be written as a finite union of strings. Let $\varpi_1, \ldots, \varpi_n$ be the fundamental weights of $\Phi^+(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$, thus any element in $\mathcal{P}^+(G)$ is of the form $a_1\varpi_1 + \cdots + a_n\varpi_n$ for some $a_1, \ldots, a_n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

We consider the subset of $\widehat{G}$ given by

$$\mathcal{G} := \widehat{G} \cap \{\pi_{m_1\varpi_1 + m_2\varpi_2} : m_1, m_2 \in \mathbb{N}\},$$

which clearly has infinitely many elements. One has that $\mathcal{S}(\omega, \Lambda_0) \cap \mathcal{G}$ is finite unless $\omega, \Lambda_0 \in \mathbb{N}_0\varpi_1 \oplus \mathbb{N}_0\varpi_2$. It is clear that we cannot cover $\mathcal{G}$ with finitely many strings of the form $\mathcal{S}(\omega, \Lambda_0)$ with $\omega \in \mathbb{N}\varpi_2$ and $\Lambda_0 \in \mathbb{N}_0\varpi_1 \oplus \mathbb{N}_0\varpi_2$. Moreover, given a string $\mathcal{S}(a\varpi_1 + b\varpi_2, c\varpi_1 + d\varpi_2)$ with $a, b, c, d \in \mathbb{N}_0$ and $a > 0$, the highest weight of an irreducible representation in it is of the form

$$k(a\varpi_1 + b\varpi_2) + (c\varpi_1 + d\varpi_2) = (ak + c)\varpi_1 + (bk + d)\varpi_2,$$

thus the quotient

$$\frac{bk + d}{ak + c} \leq \frac{bk + d}{ak} < \frac{b}{a} + d$$

is bounded for all $k \in \mathbb{N}$. Thus, this string cannot reach any element $\pi_{m_1\varpi_1 + m_2\varpi_2} \in \mathcal{G}$ such that $m_2/m_1 > b/a + d$. It follows that $\mathcal{G}$ cannot be covered by finitely many strings.

A similar situation holds for compact symmetric spaces G/K of real rank at least 2. For instance, we set $G = \mathrm{SO}(n)$ and $K = \mathrm{SO}(n-2) \times \mathrm{SO}(2)$ for some $n \geq 5$. Under the standard

choice of $\Phi^+(G, K)$ (e.g. as in [Kn, §C.1]), we get

$$(3.10) \qquad \widehat{G}_{1_K} = \{\pi_{a\varpi_1 + b\varpi_2} : a, b \in \mathbb{N}_0\},$$

where $\varpi_1, \varpi_2, \ldots$ denote the fundamental weights. Now, it is clear that the same argument as above shows that $\widehat{G}_{1_K}$ cannot be written as a finite union of strings.

References

[BR11]   C. Bhagwat, C.S. Rajan. *On a spectral analog of the strong multiplicity one theorem.* Int. Math. Res. Not. IMRN **2011**:18 (2011), 4059–4073. DOI: 10.1093/imrn/rnq243.

[Ca05a] R. Camporesi. *The Helgason Fourier transform for homogeneous vector bundles over compact Riemannian symmetric spaces—the local theory.* J. Funct. Anal. **220**:1 (2005), 97–117. DOI: 10.1016/j.jfa.2004.08.013.

[Ca05b] R. Camporesi. *A generalization of the Cartan–Helgason theorem for Riemannian symmetryc spaces of rank one.* Pacific J. Math. **222**:1 (2005), 1–27. DOI: 10.2140/pjm.2005.222.1.

[CR15]   G. Chenevier, D. Renard. *Level one algebraic cuspforms of classical groups of small rank.* Mem. Amer. Math. Soc. **237**:1121, 2015. DOI: 10.1090/memo/1121.

[HvP16] G. Heckman, M. van Pruijssen. *Matrix valued orthogonal polynomials for Gelfand pairs of rank one.* Tohoku Math. J. (2) **68**:3 (2016), 407–437. DOI: 10.2748/tmj/1474652266.

[Ke14]   D. Kelmer. *A refinement of strong multiplicity one for spectra of hyperbolic manifolds.* Trans. Amer. Math. Soc. **366** (2014), 5925–5961. DOI: 10.1090/S0002-9947-2014-06102-3.

[Kn]   A.W. Knapp. Lie groups beyond an introduction. *Progr. Math.* **140**. Birkhäuser Boston Inc., 2002.

[Ko85]   B. Kostant. *The McKay correspondence, the Coxeter element and representation theory.* Astérisque Numéro Hors Série (The mathematical heritage of Élie Cartan, Lyon) (1985), 209–255.

[Ko04]   B. Kostant. *A branching law for subgroups fixed by an involution and a noncompact analogue of the Borel-Weil theorem.* In *Noncommutative harmonic analysis*, 291–353, *Progr. Math.* **220**. Birkhäuser Boston, Boston MA., 2004.

[LM18]   E.A. Lauret, R.J. Miatello. *Representation equivalence for compact symmetric spaces of real rank one.* arXiv:1812.09606 (2018).

Instituto de Matemática (INMABB), Departamento de Matemática, Universidad Nacional del Sur (UNS)-CONICET, Bahía Blanca B8000CPB, Argentina.

E-mail address: emilio.lauret@uns.edu.ar

CIEM–FaMAF (CONICET), Universidad Nacional de Córdoba, Medina Allende s/n, Ciudad Universitaria, 5000 Córdoba, Argentina.

E-mail address: miatello@famaf.unc.edu.ar