

NONCOMMUTATIVE FIBRATIONS

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ABSTRACT. We show that faithfully flat smooth extensions of associative unital algebras are reduced flat, and therefore, fit into the Jacobi-Zariski exact sequence in Hochschild homology and cyclic (co)homology even when the algebras are noncommutative or infinite dimensional. We observe that such extensions correspond to étale maps of affine schemes, and we propose a definition for generic noncommutative fibrations using distributive laws and homological properties of the induction and restriction functors. Then we show that Galois fibrations do produce the right exact sequence in homology. We then demonstrate the versatility of our model on a geometro-combinatorial example. For a connected unramified covering of a connected graph $G' \rightarrow G$, we construct a smooth Galois fibration $\mathcal{A}_G \subseteq \mathcal{A}_{G'}$ and calculate the homology of the corresponding local coefficient system.

INTRODUCTION

Based on homological connections between the induction $Ind_{A^e}^{B^e}$ and the restriction $Res_{A^e}^{B^e}$ functors, in this paper we gather further evidence that the Hochschild-Jacobi-Zariski exact sequence

$$(0.1) \quad \cdots \rightarrow HH_{n+1}(B|A) \rightarrow HH_n(A|\mathbb{k}) \rightarrow HH_n(B|\mathbb{k}) \rightarrow HH_n(B|A) \rightarrow \cdots$$

is the long exact sequence of associated to a fibration of ordinary (noncommutative) affine spaces $Spec(B) \rightarrow Spec(A)$ when the extension $A \subseteq B$ is reduced flat. First, we prove in Theorem 1.8 that for faithfully flat extensions reduced flatness is equivalent to A^e -flatness of $\Sigma_{B|A}$ the kernel of the relative multiplication map $B \otimes_A B \rightarrow B$. Then in Theorem 1.10 we obtain a *faithfully flat étale descent* result analogous to [23, Theorem (0.1)] but for all associative unital algebras not just commutative ones: We show that any faithfully flat smooth extension $A \subseteq B$ is reduced flat, and therefore, the geometric fibre of $Spec(B) \rightarrow Spec(A)$ is homologically trivial. The result follows from the fact that now we have the Hochschild-Jacobi-Zariski exact sequence with coefficients (1.1) for faithfully flat smooth extensions, and the fact that the restriction functor already induces the correct isomorphisms in homology by Proposition 1.3.

There is an analogous Jacobi-Zariski exact sequence for extensions of commutative algebras in André-Quillen (co)homology without any further restriction on the extension [17, 1]. However, our (0.1) is exact for commutative and noncommutative algebras alike even when they are not finite dimensional or essentially of finite type. The results in this paper came from an observation that smooth extensions and reduced flat extensions are related in terms of homological properties of their induction and restriction functors: while the multiplication map $Ind_{A^e}^{B^e} A \rightarrow B$ induces a Hochschild cohomological equivalence in degrees higher than 1 for a smooth extension, for a reduced flat extension one gets a Hochschild homological equivalence for the same range for the natural A -bimodule embedding $A \rightarrow Res_{A^e}^{B^e} B$. We refer the reader to Subsection 2.3 for a detailed analysis of these connections.

Based on the results we obtained in Section 1, we propose that a special class of extensions of unital associative algebras that contains the class of Hopf-Galois extensions [20] constitutes an appropriate model for generic smooth noncommutative fibrations. We define a noncommutative (unramified)

fibration as a flat extension $A \subseteq B$ that admits a (bijective) distributive law $[2] \bowtie: C \otimes B \rightarrow B \otimes C$ together with an epimorphism of B -bimodules $can: Ind_{A^e}^{B^e} A \rightarrow B \bowtie C$ that satisfies an invariance condition $A = B^C$. Then in Theorems 2.4 and 2.5 we get the correct fibration sequence with the appropriate fibre for the map $Spec(B) \rightarrow Spec(A)$ for Galois fibrations. Since we formulate our extensions in terms of distributive laws instead of cleft Hopf-Galois extensions, the extensions we consider model generic fibrations, not just principal fibrations. We refer the reader to Section 2 for details.

We demonstrate the versatility of our model on a geometro-combinatorial example. For a connected unramified covering $G' \rightarrow G$ of a connected graph G , we construct an unramified reduced flat extension $\mathcal{A}_G \subseteq \mathcal{A}_{G'}$ of noncommutative algebras in Subsection 3.6. We then show in Theorem 3.8 that for such extensions, we get the right analogue of the long exact sequence of a fibration in cyclic homology. Then we extend our result to local coefficient systems on graphs and their cohomology in Theorem 3.11.

The particular result we obtain in Theorem 3.11, combined with Burghlea's [5], is consistent with [16, Chapter III, Theorem 2.20] and [23, Example 2.2] where one obtains the homology of a Galois coverings of schemes from a Hochschild-Serre hyper-homology spectral sequence in which they combine the group cohomology of the structure group of the fibration and the homology of the base. This consistency indicates that our proposal is sound geometrically. Since Theorem 3.11 is a direct consequence of Theorem 2.5, we also see that for cleft Hopf-Galois extensions the Hochschild homology of such an extension relative to the base is the homology of the underlying Hopf algebra. Hence our proposal is sound algebraically as well.

Plan of the article. We recall the results we need on reduced flat and smooth extensions in Section 1. Our Proposition 1.3 and Proposition 1.5 identify the reduced flat extensions and smooth extensions in terms of homological conditions on the induction and restriction functors. Then we define unramified and Galois fibrations, and discuss connections between various types of extensions and fibrations in Section 2. In Subsection 2.4 we prove our main technical results. First, we show that the relative Hochschild homology of a Galois fibration yields the correct homology of the fibre in Theorem 2.4. Then in Theorem 2.5, we show that for reduced flat Galois fibration, we have the required long exact sequences in Hochschild homology and cyclic (co)homology. We apply our main results to graph extension algebras in Section 3. In Subsection 3.7, we define local coefficient systems on graphs, and finally in Theorem 3.11 we prove that the relative homology of a noncommutative fibration with coefficients in a local system gives us the group homology of the local coefficients.

Notation and conventions. Throughout this article, we are going assume \mathbb{k} is a ground field of characteristic 0. All unadorned tensor products \otimes are taken over \mathbb{k} . All algebras are assumed to be over \mathbb{k} , and all are unital and associative. However, they need not be commutative or finite dimensional. We use $\Sigma_{B|A}$ to denote the kernel of the relative multiplication map $B \otimes_A B \rightarrow B$ for an algebra extension $A \subseteq B$. All modules are assumed to be left modules unless otherwise stated. We use $A\text{-Mod}$ to denote a small category of A -modules. For an algebra A , we use A^e to denote the enveloping algebra $A \otimes A^{op}$. Thus modules over A^e are exactly bimodules over A . We use the homological convention for complexes: all complexes are positively graded and differentials reduce the degree by one. We use Tor^A and Ext_A to denote the derived bifunctors of the tensor product \otimes_A and Hom_A -bifunctors, respectively. We are going to use CB_* to denote the bar complex, and CH_* to denote the Hochschild complex. Also, we use HH_* for the Hochschild homology, and HC_* for the

cyclic homology functors. All graphs are assumed to be undirected and simple, but they need not be finite. In particular, we have no loops on a vertex, and no multiple edges between any two vertices.

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1. REDUCED FLAT AND ALMOST SMOOTH EXTENSIONS

For this section, we assume we have an extension of unital associative algebras $A \subseteq B$ such that B viewed as a left and right A -module is flat.

1.1. Relative Hochschild (co)homology. Given an extension $A \subseteq B$, the relative two sided bar complex $\text{CB}_*(B|A)$ is defined to be the graded B^e -module

$$\text{CB}_n(B|A) = \underbrace{B \otimes_A \cdots \otimes_A B}_{n+2\text{-times}}$$

For every $n \geq 1$, the differentials $d_n: \text{CB}_n(B|A) \rightarrow \text{CB}_{n-1}(B|A)$ are defined as

$$d_n(u_0 \otimes \cdots \otimes u_{n+1}) = \sum_{i=0}^n (-1)^i (\cdots \otimes u_{i-1} \otimes u_i u_{i+1} \otimes u_{i+2} \otimes \cdots)$$

for every homogeneous tensor $u_0 \otimes \cdots \otimes u_{n+1} \in \text{CB}_n(B|A)$, then extended linearly. Since $\text{CB}_*(B|A)$ is a (B^e, A^e) -projective resolution of B as a B^e -module, for any B^e -module X we write the relative Hochschild chain and cochain complexes as

$$\text{CH}_*(B|A, X) := \text{CB}_*(B|A) \otimes_{B^e} X \text{ and } \text{CH}^*(B|A, X) := \text{Hom}_{B^e}(\text{CB}_*(B|A), X)$$

that yield the relevant relative Hochschild homology and cohomology groups $HH_*(B|A, X)$ and $HH^*(B|A, X)$, respectively. In the case $A = \mathbb{k}$, we simply write $\text{CB}_*(B)$ and $HH_*(B)$ instead of $\text{CB}_*(B|\mathbb{k})$ and $HH_*(B|\mathbb{k})$.

1.2. Induction and restriction. We have two related functors:

- (i) Induction $\text{Ind}_{A^e}^{B^e} X := B \otimes_A X \otimes_A B$, and
- (ii) Restriction $\text{Res}_{A^e}^{B^e} Y$ where we view Y as an A -bimodule via the inclusion $A \subseteq B$

for every $X \in A^e\text{-Mod}$ and $Y \in B^e\text{-Mod}$.

Lemma 1.1. *For every $X \in A^e\text{-Mod}$ we have*

$$\text{Tor}_n^{A^e}(\text{Res}_{A^e}^{B^e} B, X) \cong \text{Tor}_n^{B^e}(B, \text{Ind}_{A^e}^{B^e} X)$$

for every $n \geq 0$.

Proof. We observe that the $\text{CB}_*(B)$ is a free resolution of the B -bimodule B . Since we assumed B is a right and left flat A -module, we also have that $\text{CB}_*(B)$ is a flat resolution of the A -bimodule B . Then

$$\begin{aligned} \text{Tor}_n^{A^e}(\text{Res}_{A^e}^{B^e} B, X) &\cong H_n(\text{CB}_*(B) \otimes_{A^e} X) \\ &= H_n(\text{CB}_*(B) \otimes_{B^e} (B \otimes_A X \otimes_A B)) \\ &\cong \text{Tor}_n^{A^e}(B, \text{Ind}_{A^e}^{B^e} X) \end{aligned}$$

as we wanted to prove. □

Remark 1.2. By Lemma 1.1 we have a sequence of natural morphisms

$$\begin{aligned} HH_n(A, Y) &= \mathrm{Tor}_n^{A^e}(A, Y) \xrightarrow{\xi_Y} \\ &\mathrm{Tor}_n^{A^e}(\mathrm{Res}_{A^e}^{B^e} B, Y) \cong \mathrm{Tor}_n^{B^e}(B, \mathrm{Ind}_{A^e}^{B^e} Y) = HH_n(B, \mathrm{Ind}_{A^e}^{B^e} Y) \end{aligned}$$

and

$$\begin{aligned} HH_n(A, \mathrm{Res}_{A^e}^{B^e} X) &= \mathrm{Tor}_n^{A^e}(A, \mathrm{Res}_{A^e}^{B^e} X) \cong \mathrm{Tor}_n^{B^e}(\mathrm{Ind}_{A^e}^{B^e} A, X) \\ &\xrightarrow{\nu_X} \mathrm{Tor}_n^{B^e}(B, X) = HH_n(B, X) \end{aligned}$$

for every $X \in B^e\text{-Mod}$, $Y \in A^e\text{-Mod}$ and $n \geq 0$. In the following subsections, we are going to show that ξ viewed as a natural transformation of functors is an isomorphism when the extension is reduced flat, and ν again viewed as a natural transformation of functors is an isomorphism when the extension is (almost) smooth, both for a certain range of n . Moreover, we are also going to show that when the extension is faithfully flat then the fact that ν is an isomorphism implies so is ξ .

1.3. Almost smooth extensions. For an extension of \mathbb{k} -algebras $A \subseteq B$, we define $\Sigma_{B|A}$ to be the kernel of the relative multiplication map $\mathrm{Ind}_{A^e}^{B^e} A = B \otimes_A B \rightarrow B$ as a morphism of B^e -modules. We call a flat extension (almost) smooth if $\Sigma_{B|A}$ is a projective (resp. flat) B^e -module [19]. Notice that when an extension is smooth then it is also almost smooth.

Proposition 1.3. *A flat extension $A \subseteq B$ is almost smooth if and only if we have $HH_n(B, X) \cong HH_n(A, \mathrm{Res}_{A^e}^{B^e} X)$ for every X and for every $n \geq 2$.*

Proof. The proof follows from the fact that $A \subseteq B$ is almost smooth if and only if we have a sequence of isomorphisms of the form

$$\begin{aligned} HH_n(A, \mathrm{Res}_{A^e}^{B^e} X) &= \mathrm{Tor}_n^{A^e}(A, \mathrm{Res}_{A^e}^{B^e} X) \cong \mathrm{Tor}_n^{B^e}(\mathrm{Ind}_{A^e}^{B^e} A, X) \\ &\cong \mathrm{Tor}_n^{B^e}(B, X) = HH_n(B, X) \end{aligned}$$

for every $n \geq 2$ and $X \in B^e\text{-Mod}$. □

Remark 1.4. There is a version of Proposition 1.3 for smooth extensions that works with Hochschild cohomology instead of homology that says $A \subseteq B$ is smooth if and only if

$$HH^n(B, X) \cong HH^n(A, \mathrm{Res}_{A^e}^{B^e} X)$$

for every $X \in B^e\text{-Mod}$ and $n \geq 1$.

1.4. Reduced flat extensions. We now recall from [12] that we call an extension $A \subseteq B$ as *reduced flat* when the cokernel B/A of the A -bimodule inclusion $A \rightarrow \mathrm{Res}_{A^e}^{B^e} B$ is flat as a A -bimodule. We also observe that reduced flatness of the extension is equivalent to the fact that the Hochschild homology of $H_n(A, X)$ of A with coefficients in any $X \in A^e\text{-Mod}$, and the torsion groups $\mathrm{Tor}_n^{A^e}(\mathrm{Res}_{A^e}^{B^e} B, X)$ are isomorphic for all $n \geq 1$. Combining this result with Lemma 1.1 we get

Proposition 1.5. *A flat extension $A \subseteq B$ is reduced flat if and only if we have natural isomorphisms of the form $HH_n(A, X) \cong HH_n(B, \mathrm{Ind}_{A^e}^{B^e} X)$ for every $X \in A^e\text{-Mod}$ and for every $n \geq 1$.*

We will say that an extension $A \subseteq B$ satisfies *Hochschild-Jacobi-Zariski (resp. cyclic-Jacobi-Zariski) condition* [15, 3.5.5.1] if we have a long exact sequence in Hochschild homology (resp. cyclic homology) of the form

$$(1.1) \quad HH_{n+1}(B|A, X) \rightarrow HH_n(A, \mathrm{Res}_{A^e}^{B^e} X) \rightarrow HH_n(B, X) \rightarrow HH_n(B|A, X)$$

for every $n \geq 1$, and for every $X \in B^e\text{-Mod}$.

Proposition 1.6 ([12, Theorem 4.1 and Theorem 4.2]). *If a flat extension $A \subseteq B$ is reduced flat then the extension satisfies both Hochschild-Jacobi-Zariski and cyclic-Jacobi-Zariski conditions for every $X \in B^e\text{-Mod}$.*

Remark 1.7. Recall that the relative homology groups $HH_n(B|A, X)$ measure the failure of the extension of being smooth since we have both the exact sequence by Proposition 1.3 and Proposition 1.6 for a almost smooth reduced flat extensions $A \subseteq B$. This is rather subtle: almost smoothness does imply relative homology vanishes since absolute B^e -flatness of $\Sigma_{B|A}$ implies that its relative (B, A) -flatness as a B^e -module. However, the converse need not be true in general. The fact that the relative homology vanishes for $n \geq 2$ implies $\Sigma_{B|A}$ is only B^e -flat relative to A . The fact that B is reduced flat over A gives us (1.1), and then we get the isomorphisms $HH_n(A, \text{Res}_{A^e}^{B^e} X) \rightarrow HH_n(B, X)$ for the required range, and then Proposition 1.3 gives us the absolute B^e -flatness.

1.5. Faithfully flat almost smooth extensions.

Theorem 1.8. *Assume B is faithfully flat over A . Then B is reduced flat over A if and only if $\text{Res}_{A^e}^{B^e} \Sigma_{B|A}$ is a flat A -bimodule.*

Proof. We start by observing that there is a natural isomorphism of A^e -modules of the form $\text{Res}_{A^e}^{B^e} \Sigma_{B|A} \cong (\text{Res}_{A^e}^{B^e} B/A) \otimes_A \text{Res}_{A^e}^{B^e} B$ coming from the diagram

$$\begin{array}{ccccccc} & & & & A \otimes_A \text{Res}_{A^e}^{B^e} B & \xrightarrow{\cong} & \text{Res}_{A^e}^{B^e} B \\ & & & & \downarrow & & \parallel \\ 0 & \longrightarrow & \text{Res}_{A^e}^{B^e} \Sigma_{B|A} & \longrightarrow & \text{Res}_{A^e}^{B^e} (B \otimes_A B) & \longrightarrow & \text{Res}_{A^e}^{B^e} B \longrightarrow 0 \end{array}$$

using the Snake's Lemma. Let us drop the use of $\text{Res}_{A^e}^{B^e}$ to simplify the notation. Then we see that the functor $(\cdot) \otimes_{A^e} (B/A \otimes_A B)$ is exact if and only if

$$(\cdot) \otimes_{A^e} (B/A \otimes_A B) \cong B/A \otimes_{A^e} (B \otimes_A \cdot)$$

is exact. Since we assumed B is faithfully flat over A , the flatness of B/A of A^e -module is equivalent to the flatness of $\Sigma_{B|A}$ as a A^e -module. \square

Remark 1.9. One should think of Theorem 1.8 as a *faithfully flat descent* result because the fact that an extension $A \subseteq B$ is reduced flat is equivalent to the fact that the Amitsur complex

$$A \rightarrow B \rightarrow B \otimes_A B \rightarrow B \otimes_A B \otimes_A B \rightarrow \cdots$$

is exact [18], or that the cobar complex of the Sweedler coring [4, Chapter 4, Section 25] is contractible. In fact, the *descent data* for an extension [13, 9] is a specific prescription for a contracting homotopy for the Amitsur complex.

Theorem 1.10. *Every faithfully flat almost smooth extension $A \subseteq B$ is reduced flat. Then there are isomorphisms in Hochschild homology $HH_n(A) \cong HH_n(B)$ for every $n \geq 2$, and in cyclic homology $HC_{n+2}(B|A) \cong HC_n(B|A)$ for every $n \geq 1$.*

Proof. Let us first prove that B is reduced flat over A when the extension is almost smooth. Since $A \subseteq B$ is faithfully flat almost smooth, we have that $\Sigma_{B|A}$ is B^e -flat. By Lazard's Theorem [14], $\Sigma_{B|A}$ is a flat B -bimodule if and only if it is a filtered colimit of free B -bimodules. But free B -bimodules are a subclass of flat A -bimodules since our extension $A \subseteq B$ is flat, and every filtered colimit of flat

A -bimodules is also flat. Thus $Res_{A^e}^{B^e} \Sigma_{B|A}$ is a flat A -bimodule since it was a flat B^e -module. Since B is faithfully flat over A , this is equivalent to B being reduced flat over A by Theorem 1.8. For the second assertion we observe that we have a sequence of isomorphisms

$$HH_n(A) \xrightarrow{\cong} HH_n(A, Res_{A^e}^{B^e} B) \cong HH_n(B, Ind_{A^e}^{B^e} A) \xrightarrow{\cong} HH_n(B)$$

which proves we have the desired isomorphisms for $n \geq 2$. The last assertion follows from Connes' SBI-sequence and the fact that $HH_n(B|A)$ is trivial for $n \geq 2$. See [7, Chapter 3] or [15, Chapter II, Section 2.2]. \square

Example 1.11. Let A be an algebra with Hochschild homological dimension 0. Such algebras are also known as *amenable* [10] in the continuous Hochschild homology context. In that case every A^e -module is also A^e -flat, i.e. all extensions of A are almost smooth. The typical examples are field extensions $k \subseteq A$, group algebras $k[G]$ over a finite group G , or algebra of functions $k(G)$ on a compact group G . Then $\Sigma_{B|A}$ is automatically A^e -flat, and as long as B is faithfully flat over A the extension is also reduced flat by Corollary 1.10. So, all faithfully flat extensions over amenable algebras are reduced flat.

2. FIBRATIONS OF ALGEBRAS

2.1. Transpositions, distributive laws and fibrations. Let C be a unital associative \mathbb{k} -algebra and let B be an ordinary \mathbb{k} -vector space. A morphism of \mathbb{k} -vector spaces $\omega: C \otimes B \rightarrow B \otimes C$ is called a *left transposition* [11] if the following diagram is commutative

$$(2.1) \quad \begin{array}{ccc} C \otimes C \otimes B & \xrightarrow{C \otimes \omega} & C \otimes B \otimes C & \xrightarrow{\omega \otimes C} & B \otimes C \otimes C \\ \mu_C \otimes B \downarrow & & & & \downarrow B \otimes \mu_C \\ C \otimes B & \xrightarrow{\omega} & B \otimes C & & \end{array} \quad \begin{array}{ccc} & B & \\ 1 \otimes B \swarrow & & \searrow B \otimes 1 \\ C \otimes B & \xrightarrow{\omega} & B \otimes C \end{array}$$

Right transpositions are defined similarly so that the inverse of a left transposition, should it exist, would be a right transposition, and vice versa.

Now, assume B and C are two unital associative \mathbb{k} -algebras. A morphism of \mathbb{k} -vector spaces $\bowtie: C \otimes B \rightarrow B \otimes C$ is called a *distributive law* [2] if \bowtie is a left transposition with respect to C and a right transposition with respect to B .

One can easily show that \bowtie is a distributive law if and only if $(\mu_B \otimes \mu_C)(B \otimes \bowtie \otimes C)$ is an associative unital product on $B \otimes C$ whose unit is $1_B \otimes 1_C$. We are going to use $B \bowtie C$ to denote this algebra. The triple (B, C, \bowtie) is also called a *matched pair of algebras* and also *twisted tensor product of algebras* [6].

A right transposition $\omega: C \otimes B \rightarrow B \otimes C$ is called *invariant* if B has a set of algebra generators X such that for every $x \in X$ and $c \in C$ there is another $c' \in C$ such that $\omega(c \otimes x) = x \otimes c'$. Invariant left transpositions are defined similarly.

Let $\omega: C \otimes B \rightarrow B \otimes C$ be a right transposition. A subalgebra $A \subseteq B$ is called *C -invariant* if the transposition ω restricts to an invariant transposition on A . The largest subalgebra of B which is C -invariant with respect to a transposition is denoted by B^C .

An extension of algebras $A \subseteq B$ is called a *fibration* with fibres in an algebra C if

- (i) there is an distributive law $\bowtie: C \otimes B \rightarrow B \otimes C$ with $A \subseteq B^C$, and
- (ii) there is an epimorphism of B^e -modules $can: B \otimes_A B \rightarrow B \bowtie C$.

We call a fibration *unramified* (resp. *étale*) when \bowtie is invertible (resp. injective.) We call a fibration *smooth* if the kernel of *can* is B^e -projective. A fibration is called *separable* if *can* is a split epimorphism of B^e -modules. We call a fibration $A \subseteq B$ with fibres in C as a *Galois fibration* when the canonical map *can* is an isomorphism of B^e -modules.

We would like to emphasize that any fibration $A \subseteq B$ with fibres in C (be it unramified, étale, separable, or smooth) presupposes a distributive law $\bowtie: C \otimes B \rightarrow B \otimes C$ with $A \subseteq B^C$, and an epimorphism of B^e -modules *can*: $B \otimes_A B \rightarrow B \bowtie C$.

2.2. Examples.

Example 2.1. Let H be a Hopf algebra and let B be a H -comodule algebra. In other words, H coacts on B via a coaction $\lambda: B \rightarrow H \otimes B$ such that

$$(ab)_{(-1)} \otimes (ab)_{(0)} = a_{(-1)}b_{(-1)} \otimes a_{(0)}b_{(0)}$$

for every $a, b \in A$ where we use the notation

$$\lambda(b) = b_{(-1)} \otimes b_{(0)}$$

for every $b \in B$. Let $A := B^H$ where

$$B^H = \{b \in B \mid \lambda(b) = b \otimes 1_H\}$$

Then there is an invertible distributive law $\bowtie: H \otimes B \rightarrow B \otimes H$ by

$$\bowtie(h \otimes b) = b_{(0)} \otimes hb_{(-1)}$$

for every $h \in H$ and $b \in B$ and $A \subset B^H$ with respect to this distributive law and there is an B^e -bimodule isomorphism *can*: $B \otimes_A B \rightarrow B \otimes H$ when $A \subseteq B$ is a Hopf-Galois extension over H [20].

Example 2.2. Let L be an algebra, and B be an Ore extension $L[X, \alpha, \delta]$ over L whose derivation part is trivial, i.e. $\delta = 0$. Let us set $B = L[X, \alpha, 0]$ and $A = k[X]$, and we consider the extension $A \subseteq B$. Notice that since B has a basis in monomials of the form uX^n where $u \in L$ and $n \in \mathbb{N}$, we have a k -vector space isomorphism $B \cong L \otimes A$. Then we see that we get a Galois fibration $A \subseteq B$ with fibres in L since there is a natural isomorphism of B -bimodules of the form

$$B \otimes_A B \cong L \otimes B$$

with the following invertible distributive law $\bowtie: B \otimes L \rightarrow L \otimes B$

$$\bowtie(uX^m \otimes v) = u\alpha^m(v) \otimes X^m$$

for every $u, v \in L$ and monomial $uX^m \in B$. Notice that $B^L = A$ and that $B \cong L \bowtie A$. Thus every Ore extension of the form $L[X, \alpha, 0]$ is a Galois fibration $A \subseteq L[X, \alpha, 0]$ with fibres in L .

When we consider the relative multiplication map $B \otimes_A B \rightarrow B$ we see that it reduces to $\mu_L \otimes id_A: L \otimes L \otimes A \rightarrow L \otimes A$ the multiplication map on L tensored with the identity on A . So, the kernel of the relative multiplication map $\Sigma_{B|A}$ is isomorphic $\Sigma_{L|k} \otimes A$. Since A is invariant, as a A -bimodule this is just direct sum of $\dim_k \Sigma_{L|k}$ -many copies of A . Since B is already faithfully flat over A , in order for this extension to be reduced flat we need A to be A^e -flat as well, which we know is not the case. In other words, Ore extensions are Galois fibrations but are not reduced flat extensions.

Example 2.3. Let G be a group acting on an algebra L via algebra automorphisms $\triangleright: k[G] \otimes L \rightarrow L$. We define a distributive law $\bowtie: k[G] \otimes L \rightarrow L \otimes k[G]$ by

$$\bowtie(g \otimes u) = g \triangleright u \otimes g$$

for every $g \in G$ and $u \in L$. We let $B := L \bowtie k[G]$ and $A = k[G]$. As in the case with Ore extension, $B \cong L \otimes A$ as k -vector spaces and

$$B \otimes_A B \cong L \otimes B$$

and with a distributive law $\bowtie: B \otimes L \rightarrow L \otimes B$ defined as

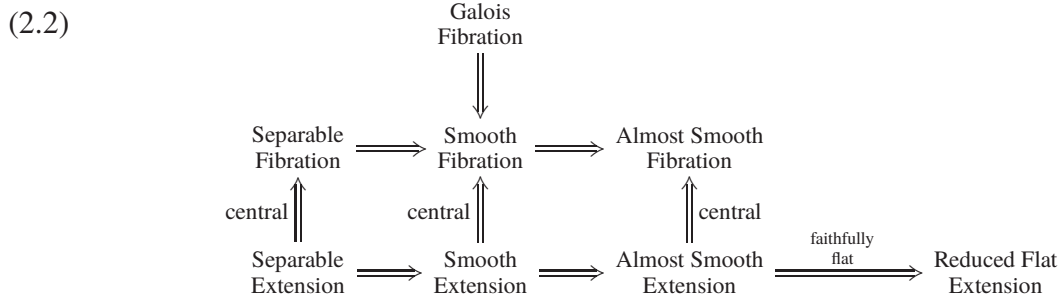
$$\bowtie(ug \otimes v) = u(g \triangleright v) \otimes g$$

for every $ug \in B$ and $v \in L$. We see that $B^L = A$ and we have a Galois fibration $A \subseteq B$ with fibres in L . One can think of this as the more general version of the Ore extension we gave above where $G = \mathbb{N}$.

As for reduced flatness, the same argument above that worked for Ore extensions works here too: the kernel of the relative multiplication map $\Sigma_{B|A}$ is isomorphic to $\Sigma_{L|k} \otimes A$ which is again $\dim_k \Sigma_{L|k}$ copies of A . So, since B is already faithfully flat over A , for this extension to be reduced flat we need A to be A^e -flat. Since A is the group algebra $k[G]$, this means the group has to have trivial homology. In this case any finite group G would work since we assume $\text{char}(k)$ is 0.

In short, all extensions of the form $k[G] \subseteq L \bowtie k[G]$ are reduced flat Galois fibrations if G is finite and $\text{char}(k) = 0$.

2.3. Relationships. Assume $A \subseteq B$ is a flat extension of algebras. Let us depict the relationships between various objects we defined in this paper as follows:



Let us define $\Sigma_{B|A}$ as $\ker(\text{Ind}_{A^e}^{B^e} A \rightarrow A)$, and $\mathcal{U}^{B|A}$ as $\text{coker}(A \rightarrow \text{Res}_{A^e}^{B^e} B)$, and consider the short exact sequences

$$0 \rightarrow \Sigma_{B|A} \rightarrow \text{Ind}_{A^e}^{B^e}(A) \rightarrow B \rightarrow 0 \quad \text{and} \quad 0 \rightarrow A \rightarrow \text{Res}_{A^e}^{B^e}(B) \rightarrow \mathcal{U}^{B|A} \rightarrow 0$$

So, we call an embedding $A \subseteq B$ of unital associative algebras as a

- Reduced flat extension* when $\mathcal{U}^{B|A}$ is a flat A -bimodule. Reduced flatness is equivalent to $\text{Res}_{A^e}^{B^e} \Sigma_{B|A}$ being a flat A -bimodule by Theorem 1.8 when B is faithfully flat over A .
- Smooth extension* when $\Sigma_{B|A}$ is a projective B -bimodule. If in addition A is central in B , by choosing the trivial distributive law $B \otimes \mathbb{k} \rightarrow \mathbb{k} \otimes B$ we can form a smooth fibration.
- Almost smooth extension* when $\Sigma_{B|A}$ is a flat B -bimodule. Notice that when $\Sigma_{B|A}$ is projective, then it is also flat which means smoothness always implies almost smoothness. Since every finitely presented flat B^e -module is projective, in cases where B^e is noetherian almost smoothness and smoothness agree.

- (d) *Smooth fibration* if the kernel of the structure map can is B^e -projective, and *almost smooth* if the kernel is flat as a B^e -module as expected.
- (e) *Separable extension* [15, 1.2.12] when B is a projective B^e -module relative to A which implies $\Sigma_{B|A}$ is projective relative to A . This means the extension is smooth relative to A .
- (f) *Separable fibration* when $B \bowtie C$ is a projective B^e -module relative to A . All separable extensions are separable fibrations over base algebra A .
- (g) *Galois fibration* if the structure map can is an isomorphism. Then one also gets that it is a smooth fibration for free. All Hopf-Galois extensions yield Galois fibrations by definition [20]. In particular, all Hopf-Galois extensions are unramified Galois fibrations.

2.4. Homology of Galois fibrations.

Theorem 2.4. *Assume $A \subseteq B$ is a Galois fibration with fibres in C . Then we have natural isomorphisms in Hochschild homology and cyclic (co)homology of the form (written here only for Hochschild homology)*

$$HH_n(B|A, X) \cong HH_n(C, X/[X, A])$$

for every $X \in B^e\text{-Mod}$ and for all $n \geq 0$.

Proof. In order to calculate $HH_n(B|A, X)$ we are going to use $X \otimes_{B^e} CB_*(B|A)$:

$$\begin{aligned} CH_n(B|A, X) &\cong A \otimes_{A^e} \underbrace{(X \otimes_A B \otimes_A \cdots \otimes_A B)}_{n\text{-times}} \\ &\cong A \otimes_{A^e} \underbrace{(X \otimes_B B \otimes_A \cdots \otimes_A B)}_{n+1\text{-times}} \\ &\cong A \otimes_{A^e} \underbrace{(X \otimes C \otimes \cdots \otimes C)}_{n\text{-times}} \\ &\cong X/[X, A] \otimes \underbrace{(C \otimes \cdots \otimes C)}_{n\text{-times}} \\ &\cong (X/[X, A]) \otimes_{C^e} CB_*(C) \end{aligned}$$

The result follows. □

Theorem 2.5. *Assume $A \subseteq B$ is a reduced flat Galois fibration with fibres in C . Then there are long exact sequences in Hochschild homology*

$$(2.3) \quad \cdots \rightarrow B/[B, A] \otimes HH_{n+1}(C) \rightarrow HH_n(A) \rightarrow HH_n(B) \rightarrow B/[B, A] \otimes HH_n(C) \rightarrow \cdots$$

for $n \geq 1$.

Proof. Since the extension is reduced flat, it follows from [12, Theorem 4.2] that we have the long exact sequence 0.1 for $n \geq 1$. We also have that

$$CH_*(B|A) \cong B/[B, A] \otimes CH_*(C)$$

by Theorem 2.4. The result follows. □

3. HOMOLOGY OF GRAPH COVERINGS

Throughout this section, we assume \mathcal{G} is a groupoid with object set V . We also assume $G = (V, E)$ is a simple graph, i.e. V is a set and E is a subset of multi-subsets of V of size 2.

3.1. Groupoids and their actions. A covariant (resp. contravariant) functor $X: \mathcal{G} \rightarrow \text{SET}$ is called a left (resp. right) \mathcal{G} -set. One can alternatively define a left \mathcal{G} set with the following datum:

- (1) There is a function $i: X \rightarrow V$,
- (2) For every $x \in i^{-1}(c)$ and $b \xleftarrow{g} c \in \mathcal{G}$ there is an element $(b \xleftarrow{g} c) \cdot x \in i^{-1}(b)$ such that
 - (i) $(c \xleftarrow{id} c) \cdot x = x$, and
 - (ii) $(a \xleftarrow{f} b) \cdot ((b \xleftarrow{g} c) \cdot x) = (a \xleftarrow{fg} c) \cdot x$ for every $a \xleftarrow{f} b \in \mathcal{G}$.

One can similarly define a bilateral \mathcal{G} -set M either as a functor of the form $M: \mathcal{G}^e \rightarrow \text{SET}$ where \mathcal{G}^e is the enveloping groupoid $\mathcal{G} \times_V \mathcal{G}^{op}$ with a sequence of conditions similar to the conditions given above.

3.2. Groupoids and distributive laws.

Proposition 3.1. *Assume we have two subgroupoids \mathcal{H} and \mathcal{K} in \mathcal{G} such that the composition in \mathcal{G} induces bijections of the form*

$$(3.1) \quad \bigsqcup_{u \in V} \text{Hom}_{\mathcal{K}}(u, w) \times \text{Hom}_{\mathcal{H}}(v, u) \xrightarrow{\circ} \text{Hom}_{\mathcal{G}}(v, w) \xleftarrow{\circ} \bigsqcup_{u \in V} \text{Hom}_{\mathcal{H}}(u, w) \times \text{Hom}_{\mathcal{K}}(v, u)$$

Then there is an invertible distributive law of the form $\omega: \mathcal{K} \times_V \mathcal{H} \rightarrow \mathcal{H} \times_V \mathcal{K}$.

Proof. Let us denote the inverse of the right leg of the zig-zag of the bijections given in (3.1) by ξ . We define a morphism of groupoids $\omega: \mathcal{K} \times_V \mathcal{H} \rightarrow \mathcal{H} \times_V \mathcal{K}$ is the composition map followed by ξ

$$\mathcal{K} \times_V \mathcal{H} \xrightarrow{\circ} \mathcal{G} \xrightarrow{\xi} \mathcal{H} \times_V \mathcal{K}$$

We need to prove that this map is a left and a right transposition. We consider the diagram

$$\begin{array}{ccc} \mathcal{K} \times_V \mathcal{K} \times_V \mathcal{H} & \xrightarrow{id \times \omega} & \mathcal{K} \times_V \mathcal{H} \times_V \mathcal{K} & \xrightarrow{\omega \times id} & \mathcal{H} \times_V \mathcal{K} \times_V \mathcal{K} \\ \circ \times id \downarrow & & & & \downarrow id \times \circ \\ \mathcal{K} \times_V \mathcal{H} & \xrightarrow{\omega} & & & \mathcal{H} \times_V \mathcal{K} \\ & \searrow \circ & & \swarrow \circ & \\ & & \mathcal{G} & & \end{array}$$

Since the lower triangle is composed of compatible bijections, the upper rectangle must commute. \square

3.3. The free groupoid of a graph. A path in G is a finite sequence of vertices (v_n, \dots, v_0) such that $\{v_{i+1}, v_i\} \in E$ for every $i = 0, \dots, n-1$. A path is called a cycle if it starts and ends at the same vertex. Let $\pi_1(G, x)$ be the free group generated by all cycles on a vertex $x \in V$ subject to the relation

$$(3.2) \quad (v, w, v) = v$$

for every edge $\{v, w\} \in E$. We now let $\pi_1(G)$ be the discrete groupoid defined as the disjoint union $\bigsqcup_{v \in V} \pi_1(G, v)$, and we also define \mathcal{F}_G to be the groupoid on G where given any pair of vertices x and y , the Hom-set $\text{Hom}_{\mathcal{F}_G}(x, y)$ is the set of all paths from x to y subject to the same relation as in Equation (3.2). Notice that in this groupoid the inverse of an arrow (path) is the reverse path.

3.4. The canonical groupoid of a graph. Define a groupoid C_G from G as follows: the set of objects of C_G is the set of vertices V of G . For each $v, w \in V$ the set $\text{Hom}_{C_G}(v, w)$ is empty if and only if there are no paths between v and w in G . If there a path, then the set $\text{Hom}_{C_G}(v, w)$ contains a unique morphism simply denoted by $w \leftarrow v$, or by $p_{w,v}$ whenever it is convenient. Since every arrow in C_G is invertible, this is a groupoid.

Lemma 3.2. *There is an invertible distributive law groupoids of the form*

$$\bowtie: \pi_1(G) \times_V C_G \rightarrow C_G \times_V \pi_1(G)$$

and an isomorphism of groupoids $\mathcal{F}_G \cong C_G \bowtie \pi_1(G)$.

Proof. Let us define an equivalence relation \sim_1 on the morphisms of \mathcal{F}_G as follows: For every $\alpha, \beta \in \text{Hom}_{\mathcal{F}_G}(x, y)$ we write

$$\alpha \sim_1 \beta \text{ if and only if there is one } \gamma \in \pi_1(G, x) \text{ with } \alpha = \beta\gamma$$

We claim the quotient \mathcal{F}_G/\sim_1 is the canonical groupoid C_G . It is clear that the Hom-set $\text{Hom}_{\mathcal{F}_G/\sim_1}(x, y)$ is the set of equivalence classes $\text{Hom}_{\mathcal{F}_G}(x, y)/\sim_1$, and that each quotient set $\text{Hom}_{\mathcal{F}_G}(x, y)/\sim_1$ either contains no elements, or contains exactly one element for every $x, y \in V$. This follows from the fact that given any two paths $\alpha, \beta \in \text{Hom}_{\mathcal{F}_G}(x, y)$ we have $\alpha \sim_1 \beta$ since $\alpha = \beta(\beta^{-1}\alpha)$. So, set-wise the canonical groupoid and our quotient object have the same elements. We must verify that we have an associative composition defined on the equivalence class of morphisms. To this end, let us take $a, a' \in [\alpha] \in \text{Hom}_{\mathcal{F}_G/\sim_1}(y, z)$ and $b, b' \in [\beta] \in \text{Hom}_{\mathcal{F}_G/\sim_1}(x, y)$ with $a' = ac$ and $b' = bd$. We see that $ab = ab'd^{-1}$ which means $ab \sim_1 ab'$. On the other hand $ab' = a'c^{-1}b' = a'b'(cb')^{-1}b'$ which means $ab' \sim a'b'$. Thus we see that the composition $[\alpha][\beta] \in \text{Hom}_{\mathcal{F}_G/\sim_1}(x, z)$ is well-defined. The fact that the composition is associative and has an identity follows immediately. Moreover, we also have a bijection

$$\text{Hom}_{\mathcal{F}_G}(x, y) = \text{Hom}_{\mathcal{F}_G/\sim_1}(x, y) \times \pi_1(G, x)$$

for every $x, y \in V$. On the other hand, we also have a dual equivalence relation \sim_2 defined as

$$\alpha \sim_2 \beta \text{ if and only if there is one } \gamma \in \pi_1(G, y) \text{ with } \gamma\alpha = \beta$$

Notice that we have $\alpha \sim_1 \beta$ if and only if $\alpha \sim_2 \beta$ and we have a bijection of the form

$$\text{Hom}_{\mathcal{F}_G}(x, y) = \pi_1(G, y) \times \text{Hom}_{\mathcal{F}_G/\sim_2}(x, y)$$

for every $x, y \in V$. The the result follows from Proposition 3.1. \square

3.5. Unramified covering of a graph. Our main reference for graph coverings is [8].

If $G = (V, E)$ and $G' = (V', E')$ are two simple graphs, a function $f: V' \rightarrow V$ is called a *map of graphs* if the induced map on the edges restricts to a map of the form $f: E' \rightarrow E$. We are going to represent this category of graphs as \mathbf{GRAPH} .

Definition 3.3. A map of simple graphs $f: G' \rightarrow G$ is called a *covering* if $f: V' \rightarrow V$ is onto. A covering f is called *finite* if $f^{-1}(v)$ is a finite set for every $v \in V$. A finite covering f is called an *n-fold covering* if $|f^{-1}(v)| = n$ for every $v \in V$. A covering f is called *unramified* (resp. *étale*) if for every $\{a, b\} \in E$ there is a bijective (resp. injective) function $\sigma_{b,a}: f^{-1}(a) \rightarrow f^{-1}(b)$ such that $\{x, \sigma_{b,a}(x)\} \in E'$ for every $x \in f^{-1}(a)$.

For the rest of the subsection, assume $f: G' \rightarrow G$ is an unramified covering.

Lemma 3.4. *The vertex set V' of G' is a bilateral \mathcal{F}_G -set and there are bijections of the form $P(G) \times_V V' \cong P(G') \cong V' \times_V P(G)$. Thus we have an invertible transposition between \mathcal{F}_G and V' .*

Proof. We have $f: V' \rightarrow V$ and $s: E \rightarrow V$. Since the covering is étale, there is an injective map $\sigma_{b,a}: f^{-1}(a) \rightarrow f^{-1}(b)$ for every $(a, b) \in E$. Then for every $(x, y) \in E'$ can be written as $(x, \sigma_{b,a}(x))$ where $a = f(x)$ and $b = f(y)$. So, there is a bijection of the form $E' \cong V' \times_V E$. One can extend the bijection to $P(G') \cong V' \times_V P(G)$. The other bijection is obtained similarly. Then the result follows from Proposition 3.1. \square

Proposition 3.5. *There is an isomorphism of groupoids of the form $\mathcal{F}_{G'} \cong V' \times_V \mathcal{F}_G$ where source and target maps for $V' \times_V \mathcal{F}_G$ are defined as*

$$s(x, f(x) \xrightarrow{\alpha} v) = x \quad \text{and} \quad t(x, f(x) \xrightarrow{\alpha} v) = x \cdot \alpha$$

Proof. Follows from Lemma 3.4. \square

3.6. Fibrations of path algebras. For the sake of brevity, we are going to use \mathcal{A}_G to denote the groupoid algebra $\mathbb{k}[\mathcal{F}_G]$ for every graph G . Our main reference for Hochschild and cyclic (co)homology of path algebras is [3].

The following result is well-known. We furnish a proof for the sake of completeness.

Lemma 3.6. *The Hochschild cohomological dimension of \mathcal{A}_G is at most 1 for every graph G .*

Proof. Let F be the disjoint union of edges E of G , and its inverse edges E^{-1} . Then \mathcal{A}_G viewed as a bimodule over itself has a short resolution of the form

$$0 \rightarrow \mathcal{A}_G \otimes_V F \otimes_V \mathcal{A}_G \xrightarrow{\delta} \mathcal{A}_G \otimes_V \mathcal{A}_G \rightarrow 0$$

where

$$\delta(\alpha \otimes f \otimes \beta) = \alpha f \otimes \beta - \alpha \otimes f \beta$$

for every homogeneous element $\alpha \otimes f \otimes \beta$ in $\mathcal{A}_G \otimes_V F \otimes_V \mathcal{A}_G$. \square

For the rest of the subsection, assume $g: G' \rightarrow G$ is a finite unramified connected covering of a connected graph G .

Proposition 3.7. *There is a smooth Galois fibration of groupoid algebras of the form $\mathcal{A}_G \subseteq \mathcal{A}_{G'}$.*

Proof. We embed \mathcal{A}_G to $\mathcal{A}_{G'}$ by sending each idempotent $e \in V$ to $\sum_{f \in g^{-1}(v)} f$ in $\mathcal{A}_{G'}$. This determines a unique map sending each edge $\{x, y\} \in E$ to an element in $\mathcal{A}_{G'}$. Smoothness is forced on the extension by Lemma 3.6 since both algebras have Hochschild cohomological dimension at most 1. \square

Observe that since $\mathcal{A}_{G'}$ is free over \mathcal{A}_G it is faithfully flat, and since the extension was smooth it is also reduced flat by Theorem 1.10.

Theorem 3.8. *The relative Hochschild homology groups $HH_n(\mathcal{A}_{G'}|\mathcal{A}_G, X)$ are trivial for every $\mathcal{A}_{G'}$ -bimodule X and for $n \geq 1$. Thus there is a natural epimorphism $HH_1(\mathcal{A}_G, X) \rightarrow HH_1(\mathcal{A}_{G'}, X)$ for every $\mathcal{A}_{G'}$ -bimodule X , and isomorphism of the form $HC_n(\mathcal{A}_G) \cong HC_n(\mathcal{A}_{G'})$ for every $n \geq 2$.*

Proof. The result follows from Proposition 3.5, Lemma 3.6, Proposition 3.7, Proposition 1.6, and Theorem 2.5. \square

There is an analogous isomorphism $HH_1(\mathcal{A}_G) \cong HH_1(\mathcal{A}_{G'})$ one can get from [3], but the epimorphism in Theorem 3.8 works with Hochschild homology with arbitrary coefficients. Moreover, since the isomorphisms in cyclic cohomology in Theorem 3.8 are obtained by a fibration sequence, one can try to get similar results for path algebras with relations provided we can write an appropriate fibration, i.e. reduced flat extension.

3.7. Local Coefficients on graphs and their homology. Consider our definition of a covering of graphs $f: G' \rightarrow G$ we gave in Definition 3.3. One can think of an unramified covering as a groupoid whose set of objects is the set V of vertices of the base G , and whose morphisms are given by the structure bijections $e_{y,x}: f^{-1}(x) \rightarrow f^{-1}(y)$. Or, one can think of them as *local coefficient systems*:

Definition 3.9. A local coefficient system \mathcal{H} on a graph G is a collection of objects $\{H_x\}_{x \in V}$ in a category (sets, groups, algebras, Hopf algebras etc.) together with a collection of isomorphisms $e_{x,y}: H_x \rightarrow H_y$ for every edge $(x, y) \in E$.

One can see that the fundamental groupoid $\pi_1(G) := \{\pi_1(G, x)\}_{x \in V}$ is a local coefficient system of groups for every graph G . Also, every unramified covering $G' \rightarrow G$ is a local coefficient system of sets on G , by definition. We refer the reader to [22] or [21, pg. 58] for local coefficient systems defined on topological spaces.

Let us assume $f: G' \rightarrow G$ is a local coefficient system of sets on G , i.e. an unramified covering over G . For every $v \in V$, let us define a subgroupoid

$$\mathcal{S}_v = \{\alpha \in \pi_1(G, v) \mid \alpha \triangleright x = x, \text{ for every } x \in f^{-1}(v)\}$$

It is easy to see that if β is a path from a vertex v to another w , then $\beta \mathcal{S}_v \beta^{-1} = \mathcal{S}_w$. Thus, \mathcal{S} is another local coefficient system of groups on G , and more importantly, it is normal in $\pi_1(G)$. Now, we have another local coefficient system of groups $\pi_1(G)/\mathcal{S}$. We call this new system as *the monodromy groupoid* of G' , and denote it by $\mathcal{M}_{G' \downarrow G}$, where the monodromy group on each vertex is denoted by $\mathcal{M}_v := \pi_1(G, v)/\mathcal{S}_v$ for every $v \in V$. One can easily see that we have

Proposition 3.10. *The distributive law in Lemma 3.2 we had for C_G and $\pi_1(G)$ now extends to a distributive law between C_G and \mathcal{S} , and we get $\mathcal{M}_{G' \downarrow G} \cong C_G \bowtie \pi_1(G)/\mathcal{S}$.*

Assume $f: G' \rightarrow G$ is a finite unramified connected covering of a connected graph G . By abuse of notation, let us use $\mathcal{M}_{G' \downarrow G}$ to denote the algebra $\mathbb{k}[\mathcal{M}_{G' \downarrow G}]$ of the covering.

Theorem 3.11. *The extension $C_G \subseteq \mathcal{M}_{G' \downarrow G}$ is reduced flat, and we have isomorphisms in Hochschild and cyclic homologies (written here only for Hochschild homology)*

$$HH_n(\mathcal{M}_{G' \downarrow G}) \cong HH_n(\mathcal{M}_{G' \downarrow G} | C_G) \cong HH_n(\mathcal{M}_v)$$

for every $n \geq 1$ and $v \in V$. And, since we assume $\text{char}(\mathbb{k}) = 0$, we have $HC_n(\mathcal{M}_{G' \downarrow G}) \cong \mathbb{k}$ for all $n \geq 0$.

Proof. We have an unramified smooth Galois fibration $C_G \subseteq \mathcal{M}_{G' \downarrow G}$, and $\mathcal{M}_{G' \downarrow G}/[C_G, \mathcal{M}_{G' \downarrow G}]$ is \mathbb{k} since the cover is assumed to be connected. Thus the extension is also reduced flat by Theorem 1.10. Now, we use Theorem 2.5. For the second assertion, we observe that the group (co)homology $H_*(G, \mathbb{k})$ of a finite group G over a field \mathbb{k} of characteristic 0 is trivial for every $n \geq 1$. Then we use [5]. \square

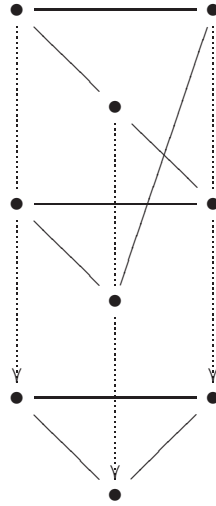


FIGURE 1. $f_{2,3}: C_6 \rightarrow C_3$, the only connected 2-covering of C_3 .

Example 3.12. Let C_n be the cycle graph on n -vertices for $n \geq 3$. Then for every $k \geq 2$, there is a unique connected k -cover $f_{k,n}: C_{kn} \rightarrow C_n$. Figure 1 is a depiction of the only connected 2-covering $f_{2,3}: C_6 \rightarrow C_3$. On each vertex in C_n , the fundamental group is \mathbb{Z} . The stabilizer group S_v of each vertex v in C_{kn} over C_n is k -times the generator in C_n . So, the monodromy group is \mathbb{Z}/k . Then the relative homologies are calculated by Theorem 3.11.

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