Restricted Interpolation and Lack Thereof in Stit Logic

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Abstract. We consider the propositional logic equipped with *Chellas stit* operators for a finite set of individual agents plus the historical necessity modality. We settle the question of whether such a logic enjoys restricted interpolation property, which requires the existence of an interpolant only in cases where the consequence contains no Chellas stit operators occurring in the premise. We show that if action operators count as logical symbols, then such a logic has restricted interpolation property iff the number of agents does not exceed three. On the other hand, if action operators are considered to be non-logical symbols, the restricted interpolation fails for any number of agents exceeding one. It follows that unrestricted Craig interpolation also fails for almost all versions of stit logic.

Keywords. stit logic, interpolation, Robinson Consistency Property

1 Introduction

The so-called stit logic is the modal logic of actions that uses the locution 'j sees to it that A' (where j is an agent name and A a sentence) as its paradigm of action modality. The very name 'stit' derives from the acronym of this paradigm locution. This logic has been present and explored in the literature on philosophical logic at least since the 1980s. Many of the early defining texts in the stit tradition were authored and coauthored by N. Belnap, and the book [2] is a useful guide to the early steps of this type of research and its attending controversies. However, in [2] N. Belnap comes forward as a proponent of the so-called *achievement* stit operator, whereas the later work in stit logic mainly concentrated around the *Chellas* stit and *deliberative* stit operators.[1] Deliberative stit operator was independently proposed by F. von Kutschera (see, e.g. [13]) and J. Horty (see, e.g., [8]). The present paper follows this line so that the name of stit logic gets applied to the logic of Chellas stit/deliberative stit operator

[1]Chellas stit is named after B. Chellas, who introduced a similar operator in [4]. These two stit operators are interdefinable in the presence of historical necessity modality; therefore, one is inclined to say that they share the same logic. Chellas stit operator is somewhat simpler and often used as the basic one, whereas the deliberative stit is often defined in terms of Chellas stit.

with Chellas stit taken as the basic stit operator, and deliberative stit as the defined one.

Most of the work on stit logic since these early days had a conceptual focus, applying stit semantics to modelling philosophical questions and exploring alternative stit operators which were proposed as improved versions of achievement and deliberative stit in some respect (see, e.g., [3]). More recently emerged the attempts to enrich stit logic with other types of operators, e.g. the ones borrowed from temporal logic (see, e.g., [9]) or justification logic (see, e.g., [10] and [11]). Sometimes these attempts were intertwined with attempts to recast the stit semantics itself so as to make it more suitable for the enrichment in question.

As for the more technical work on stit logic, it mostly concentrated on forging axiomatizations and, to some extent, solving the computational complexity questions. Some of the relatively recent important contributions to this research are e.g. [6] and [1].

One of the standard refinements of completeness results is the Craig Interpolation Property. However, to the best of our knowledge, this direction of research in stit logic has yet to see its first contributions. We hope that our paper will be able to cover this gap at least to some extent. The paper mainly focuses on a restriction of the Craig Interpolation Property which only requires existence of an interpolant if the antecedent shares no agent names with the consequent. However, we show that even this weakened version of interpolation property fails for stit logic if the logic deals with more than three different agents. Of course, the failure of restricted Craig interpolation entails also the failure of the unrestricted interpolation property. Therefore, an easy corollary to the main result of this paper is the failure of unrestricted Craig interpolation in stit logic for any number of agents exceeding three, which yields the negative solution to the problem of Craig interpolation for the vast majority of variants of the basic stit logic.

We now briefly touch upon the structure of the text below. Section 2 defines the version of stit logic at hand in terms of language, semantics, and a strongly complete axiomatization. We also introduce the main notations to be used in the paper and give the precise definition of the Restricted Craig Interpolation Property for stit logic of n agents. The latter property will be the main subject of the two following sections. We are going to show, first, that whenever our version of stit logic has no more than three different agents, it enjoys this property. The proof of this positive part of our main result is given in Section 3. The corresponding negative part, saying that the Restricted Craig Interpolation Property fails for stit logic with more than three agents, is then formulated and proven in Section 4. After that, Section 5 explores the various corollaries of the main result in relation to the following topics: (a) unrestricted Craig interpolation, (b) the Restricted Robinson Consistency Property, and (c) the stronger versions of both unrestricted and restricted interpolation property which treat stit operators as non-logical symbols.

Section 6 sums up the preceding sections and charts some natural continuations for the line of research presented in the paper.

2 Preliminaries

On the basis of a given a finite agent community Ag and a set of propositional variables V, we define the set $\mathcal{L}^{Ag}_V$ of (Ag, V)-stit formulas as follows:

$$A := p \mid A \to A \mid \bot \mid \Box A \mid [j]A,$$

where $p \in V$ and $j \in Ag$. Stit formulas will be denoted by letters A, B, C, D, decorated with sub- and superscripts whenever needed. Formulas of the type $\Box A$ and $[j]A$ are informally read as 'A is (historically) necessary' and 'the agent j sees to it that A', respectively. We reserve $\Diamond A$ and $\langle j\rangle A$ as the notations for the duals of these modalities.

Modalities of the form $[j]$ for $j \in Ag$ are called *action modalities* and will be interpreted as Chellas stit operators for the respective agent j. We will not use deliberative stit operator $[d : j]$ in this paper, but it can be defined on the basis of Chellas stit and historical necessity: $[d : j]A := [j]A \wedge \neg\Box A$. Although Ag is normally assumed to be non-empty, in this paper we will allow for $Ag = \emptyset$ as a border case for the sake of notational convenience. The set $\mathcal{L}^{\emptyset}_V$ is then basically a variant of the language of the logic of historical necessity. This logic is known to coincide with propositional S5 and hence has Craig Interpolation Property.[2] Therefore, even though empty agent communities are allowed by our notation, we will not consider interpolation properties of the languages devoid of action modalities in this paper.

Stit formulas are interpreted over the respective classes of stit models. An (Ag, V)-*stit model* is a structure of the form $\mathfrak{S} = \langle Tree, \leq, Choice, V\rangle$, such that:

- $Tree$ is a non-empty set. Elements of $Tree$ are called *moments*.
- $\leq$ is a partial order on $Tree$ for which a temporal interpretation is assumed.
- $Hist(Tree, \leq)$ is the set of maximal chains in $Tree$ w.r.t. $\leq$. Since $Hist(Tree, \leq)$ is completely determined by $Tree$ and $\leq$, it is not included into the structure of a model as a separate component. Elements of $Hist(Tree, \leq)$ are called *histories*. The set of histories containing a given moment m will be denoted $H^{\mathfrak{S}}_m$. The following set

  $$MH(Tree, \leq) = \{(m, h) \mid m \in Tree,\, h \in H^{\mathfrak{S}}_m\},$$

  called the set of *moment-history pairs*, will be used to evaluate formulas in $\mathcal{L}^{Ag}_V$. Two histories, $h, g \in H^{\mathfrak{S}}_m$ we call *undivided* at $m \in Tree$ and write $h \approx_m g$ iff h and g share some later moment m'. In other words, we stipulate that:

  $$h \approx_m g \Leftrightarrow (h, g \in H^{\mathfrak{S}}_m)\&(\exists m' > m)(h, g \in H^{\mathfrak{S}}_{m'}).$$

- $Choice$ is a function mapping $Tree \times Ag$ into $2^{2^{Hist(Tree,\leq)}}$ in such a way that for any given $j \in Ag$ and $m \in Tree$ we have as $Choice(m, j)$ (to be denoted as $Choice^m_j$ below) a partition of $H^{\mathfrak{S}}_m$. For a given $h \in H^{\mathfrak{S}}_m$ we will denote by $Choice^m_j(h)$ the element of the partition $Choice^m_j$ (otherwise called a *choice cell*) containing h. Intuitively, the idea is that j cannot distinguish by her activity at m between histories that belong to one and the same choice cell.

[2] In fact, propositional S5 even has the stronger Lyndon interpolation property, see e.g [5, Theorem 5.14, p. 140].

- V is an evaluation function, mapping the set V into $2^{MH(Tree,\leq)}$

In what follows, for a given (Ag, V)-stit model $\mathfrak{S} = \langle Tree, \leq, Choice, V\rangle$, we will sometimes use $Hist(\mathfrak{S})$ and $MH(\mathfrak{S})$ to denote $Hist(Tree, \leq)$ and $MH(Tree, \leq)$, respectively.

Additionally, every stit model $\mathfrak{S}$ is required to satisfy the following constraints:

1. **Historical connection**:
$$(\forall m, m_1 \in Tree)(\exists m_2 \in Tree)(m_2 \leq m \,\&\, m_2 \leq m_1) \tag{HC}$$
2. **No backward branching**:
$$(\forall m, m_1, m_2 \in Tree)((m_1 \leq m \,\&\, m_2 \leq m) \Rightarrow (m_1 \leq m_2 \vee m_2 \leq m_1)) \tag{NBB}$$
3. **No choice between undivided histories**:
$$(\forall m \in Tree)(\forall h, h' \in H^{\mathfrak{S}}_m)(h \approx_m h' \Rightarrow Choice^m_j(h) = Choice^m_j(h')) \tag{NCUH}$$
for every $j \in Ag$.
4. **Independence of agents**:
$$(\forall f : Ag \to 2^{H^{\mathfrak{S}}_m})((\forall j \in Ag)(f(j) \in Choice^m_j) \Rightarrow \bigcap_{j \in Ag} f(j) \neq \emptyset) \tag{IA}$$
for every $m \in Tree$.

We omit the motivation for these constraints, referring the reader to the existing literature on stit logic, e.g. [2] and [7]. The inductive definition of the satisfaction relation for the members of $\mathcal{L}^{Ag}_V$ is then as follows:

$$\mathfrak{S}, m, h \models p \Leftrightarrow (m, h) \in V(p);$$
$$\mathfrak{S}, m, h \models [j]A \Leftrightarrow (\forall h' \in Choice^m_j(h))(\mathfrak{S}, m, h' \models A);$$
$$\mathfrak{S}, m, h \models \Box A \Leftrightarrow (\forall h' \in H^{\mathfrak{S}}_m)(\mathfrak{S}, m, h' \models A),$$

with the usual clauses for the Boolean connectives. The notions of satisfaction and validity are also defined in a standard way.

Stit logic, as given above, admits of the following strongly complete axiomatization $\mathbb{S}$ which we borrow from [1].[3] The axiom schemes of $\mathbb{S}$ are as follows:

A full set of axioms for classical propositional logic	(A0)
$S5$ axioms for $\Box$ and $[j]$ for every $j \in Ag$	(A1)
$\Box A \to [j]A$ for every $j \in Ag$	(A2)
$(\Diamond[j_1]A_1 \wedge \ldots \wedge \Diamond[j_n]A_n) \to \Diamond([j_1]A_1 \wedge \ldots \wedge [j_n]A_n)$	(A3)

The assumption is that in (A3) $j_1, \ldots, j_n$ are pairwise different.

[3] The original proof, due to Ming Xu, used a somewhat more expressive language allowing also to describe equality/inequality relations between agents, see e.g. [2, Ch. 17].

In addition to the axioms, $\mathbb{S}$ contains two inference rules:

$$\text{From } A, A \to B \text{ infer } B; \tag{MP}$$
$$\text{From } A \text{ infer } \Box A; \tag{Nec}$$

Provability of A in $\mathbb{S}$ we will denote by $\vdash A$. It is clear that the strong completeness of $\mathbb{S}$ also implies compactness of stit logic for any given finite community Ag of agents and any given set V of propositional variables.

We introduce some further useful notations related to sets of stit formulas. If $\Gamma \subseteq \mathcal{L}^{Ag}_V$, then we let $\Gamma^\Box$ denote the set of all boxed formulas from Γ. Similarly, whenever $j \in Ag$, we use $\Gamma^{[j]}$ to denote the set $\{[j]A \in \mathcal{L}^{Ag}_V \mid [j]A \in \Gamma\}$.

For arbitrary Ag, V, and a set $\Gamma \cup \{A\} \subseteq \mathcal{L}^{Ag}_V$, we extend the notation $\vdash$ to contexts like $\Gamma \vdash A$ to mean that $\vdash (A_1 \wedge \ldots \wedge A_r) \to A$ for some $A_1, \ldots, A_r \in \Gamma$. Then $\Gamma \subseteq \mathcal{L}^{Ag}_V$ is called *inconsistent* iff $\Gamma \vdash \bot$, and *consistent* otherwise. Moreover, $\Gamma \subseteq \mathcal{L}^{Ag}_V$ is (Ag, V)-*maxiconsistent* iff it is consistent and no consistent subset of $\mathcal{L}^{Ag}_V$ properly extends Γ. It can be shown, in the usual way, that an arbitrary $\Gamma \subseteq \mathcal{L}^{Ag}_V$ is (Ag, V)-maxiconsistent iff for every $A \in \mathcal{L}^{Ag}_V$ the set $\Gamma \cap \{A, \neg A\}$ is a singleton. In what follows we will need the following classical lemma about maxiconsistent sets:

Lemma 1. *For any finite Ag and any set of propositional variables V, if $\Gamma \subseteq \mathcal{L}^{Ag}_V$ is consistent but not maxiconsistent, then there is an $A \in \mathcal{L}^{Ag}_V$ such that $\{A, \neg A\} \cap \Gamma = \emptyset$.*

Proof. If $\Gamma \subseteq \mathcal{L}^{Ag}_V$ is consistent but not maxiconsistent, then choose a consistent Ξ such that $\Gamma \subset \Xi \subseteq \mathcal{L}^{Ag}_V$ and choose any $A \in \Xi \setminus \Gamma$. Then $A \notin \Gamma$ by choice of A, and if $\neg A \in \Gamma$, then $\{A, \neg A\} \subseteq \Gamma \cup \{A\} \subseteq \Xi$, which contradicts the consistency of Ξ since, of course, $\vdash (A \wedge \neg A) \to \bot$. Therefore, we must also have $\neg A \notin \Gamma$ so that $\{A, \neg A\} \cap \Gamma = \emptyset$. $\Box$

For a $\Gamma \subseteq \mathcal{L}^{Ag}_V$ we define that:

$$|\Gamma| := \{p \in V \mid p \text{ occurs in } \Gamma\},$$

and:

$$Ag(\Gamma) := \{j \in Ag \mid j \text{ occurs in } \Gamma\},$$

If Γ is a singleton $\{A\}$, then we use the notations $|A|$ and $Ag(A)$ instead of $|\{A\}|$ and $Ag(\{A\})$.

In this paper we will be mainly testing the applicability to stit logic of the following property:

Definition 1. *For a positive integer n, stit logic has the* Restricted n-Craig Interpolation Property *(abbreviated by $(RCIP)_n$) iff for any set of propositional variables V, and all $A, B \in \mathcal{L}^{\{1,\ldots,n\}}_V$, whenever $\vdash A \to B$ and $Ag(A) \cap Ag(B) = \emptyset$, then there exists a $C \in \mathcal{L}^{Ag(A)\cup Ag(B)}_{|A|\cap|B|}$ such that both $\vdash A \to C$ and $\vdash C \to B$.*

3 The case $n \leq 3$

The main result of this section looks as follows:

Theorem 1. *For every $n \leq 3$, stit logic has $(RCIP)_n$.*

We prepare the result by proving several technical lemmas first.

Lemma 2. *The following statements are true:*

1. For every agent index j, $[j]$ is an S5-modality.

2. Let $A, B_1, \dots B_n, C \in \mathcal{L}^{Ag}_V$, let $i_1, \dots, i_n, j \in Ag$ be pairwise different, and let $\vdash (\Box A \wedge [i_1]B_1 \wedge \dots \wedge [i_n]B_n) \to \neg C$. Then also $\vdash (\Box A \wedge \Diamond[i_1]B_1 \wedge \dots \wedge \Diamond[i_n]B_n) \to \neg\Diamond[j]C$.

3. Let $A, B, C \in \mathcal{L}^{Ag}_V$, let $j \in Ag$, and let $\vdash (\Box A \wedge [j]B) \to C$. Then also $\vdash (\Box A \wedge \Diamond[j]B) \to \Diamond[j]C$.

Proof. (Part 1). Immediately by (A1), (Nec), and (A2).

(Part 2). Assume the hypothesis of Part 2 and assume that we have:

$$\vdash (\Box A \wedge [i_1]B_1 \wedge \dots \wedge [i_n]B_n) \to \neg C \tag{1}$$

Then we reason as follows:

$$\vdash (\Box A \wedge (\Diamond[i_1]B_1 \wedge \dots \wedge \Diamond[i_n]B_n \wedge \Diamond[j]C)) \to$$
$$\to (\Box A \wedge \Diamond([i_1]B_1 \wedge \dots \wedge [i_n]B_n \wedge [j]C)) \quad \text{(by (A3))} \tag{2}$$
$$\vdash (\Box A \wedge \Diamond([i_1]B_1 \wedge \dots \wedge [i_n]B_n \wedge [j]C)) \to$$
$$\to (\Box A \wedge \Diamond([i_1]B_1 \wedge \dots \wedge [i_n]B_n \wedge C)) \quad \text{(by (A1))} \tag{3}$$
$$\vdash (\Box A \wedge \Diamond([i_1]B_1 \wedge \dots \wedge [i_n]B_n \wedge C)) \to$$
$$\to \Diamond(\Box A \wedge [i_1]B_1 \wedge \dots \wedge [i_n]B_n \wedge C) \quad (\Box \text{ is S5}) \tag{4}$$
$$\vdash (\Box A \wedge (\Diamond[i_1]B_1 \wedge \dots \wedge \Diamond[i_n]B_n \wedge \Diamond[j]C)) \to$$
$$\to \Diamond(\Box A \wedge [i_1]B_1 \wedge \dots \wedge [i_n]B_n \wedge C) \quad \text{(by (2)-(4))} \tag{5}$$
$$\vdash \Box((\Box A \wedge [i_1]B_1 \wedge \dots \wedge [i_n]B_n) \to \neg C) \quad \text{(by (1) and (Nec))} \tag{6}$$
$$\vdash \neg\Diamond(\Box A \wedge [i_1]B_1 \wedge \dots \wedge [i_n]B_n \wedge C) \quad \text{(by (6) and prop. logic)} \tag{7}$$
$$\vdash \neg(\Box A \wedge (\Diamond[i_1]B_1 \wedge \dots \wedge \Diamond[i_n]B_n \wedge \Diamond[j]C)) \quad \text{(by (5) and (7))} \tag{8}$$

From (8), it follows by propositional logic that $\vdash (\Box A \wedge \Diamond[i_1]B_1 \wedge \dots \wedge \Diamond[i_n]B_n) \to \neg\Diamond[j]C$.

(Part 3). We reason as follows:

$$\vdash (\Box A \wedge [j]B) \to C \quad \text{(premise)} \tag{9}$$
$$\vdash [j]((\Box A \wedge [j]B) \to C) \quad \text{(by (9) and Part 1)} \tag{10}$$
$$\vdash ([j]\Box A \wedge [j]B) \to [j]C \quad \text{(by (10) and Part 1)} \tag{11}$$
$$\vdash \Box A \to \Box\Box A \quad \text{(by (A1))} \tag{12}$$
$$\vdash \Box\Box A \to [j]\Box A \quad \text{(by (A2))} \tag{13}$$
$$\vdash \Box A \to [j]\Box A \quad \text{(by (12) and (13))} \tag{14}$$
$$\vdash (\Box A \wedge [j]B) \to [j]C \quad \text{(by (11) and (14))} \tag{15}$$
$$\vdash (\Box A \wedge \Diamond[j]B) \to \Diamond[j]C \quad \text{(by (15) and S5 properties of } \Box) \tag{16}$$

□

Assume that V is a set of propositional variables and Ag a finite community of agents. A pair (Γ, Δ) of sets of (Ag, V)-stit formulas, is called *inseparable*, iff

$Ag(\Gamma) \cap Ag(\Delta) = \emptyset$, and for no $A \in \mathcal{L}^{Ag(\Gamma)\cup Ag(\Delta)}_{|\Gamma|\cap|\Delta|}$ it is true that both $\Gamma \vdash A$ and $\Delta \vdash \neg A$. Below we basically repeat the classical argument for the proof of the following standard lemma about inseparability:

Lemma 3. *Let (Γ, Δ) be an inseparable pair, and assume that both $|\Gamma|$ and $|\Delta|$ are at most countable.*[4] *Then:*

1. *There exist Γ' and Δ' such that $\Gamma \subseteq \Gamma' \subseteq \mathcal{L}^{Ag(\Gamma)}_{|\Delta|}$, $\Delta \subseteq \Delta' \subseteq \mathcal{L}^{Ag(\Delta)}_{|\Delta|}$, (Γ', Δ') is inseparable, Γ' is $(Ag(\Gamma), |\Gamma|)$-maxiconsistent, and Δ' is $(Ag(\Delta), |\Delta|)$-maxiconsistent.*
2. *If $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$, then (Γ', Δ') is inseparable.*

Proof. (Part 1) We proceed as in the case of classical logic. We first enumerate the formulas in $\mathcal{L}^{Ag(\Gamma)}_{|\Gamma|}$ as $A_0, \ldots, A_s, \ldots$, and the formulas in $\mathcal{L}^{Ag(\Delta)}_{|\Delta|}$ as $B_0, \ldots, B_s, \ldots$,. We then define two increasing sequences of sets of formulas:

$$\Gamma = \Gamma_0 \subseteq \ldots \subseteq \Gamma_s \subseteq \ldots$$

and

$$\Delta = \Delta_0 \subseteq \ldots \subseteq \Delta_s \subseteq \ldots$$

in $\mathcal{L}^{Ag(\Gamma)}_{|\Gamma|}$ and $\mathcal{L}^{Ag(\Delta)}_{|\Delta|}$, respectively. The definition is as follows. Γ_0 and Δ_0 are just Γ and Δ, and whenever Γ_r and Δ_r are defined for an $r \in \omega$, then we set:

$$\Gamma_{r+1} = \begin{cases} \Gamma_r \cup \{A_r\}, & \text{if } (\Gamma_r \cup \{A_r\}, \Delta_r) \text{ is inseparable;} \\ \Gamma_r, & \text{otherwise.} \end{cases}$$

and, further:

$$\Delta_{r+1} = \begin{cases} \Delta_r \cup \{B_r\}, & \text{if } (\Gamma_{r+1}, \Delta_r \cup \{B_r\}) \text{ is inseparable;} \\ \Delta_r, & \text{otherwise.} \end{cases}$$

Claim 1. For every $r \in \omega$, the pairs (Γ_r, Δ_r) and (Γ_{r+1}, Δ_r) are inseparable.

The Claim is proved by induction on r. If $r = 0$ then $(\Gamma_0, \Delta_0) = (\Gamma, \Delta)$ is inseparable by the assumption of the lemma, and the inseparability of (Γ_1, Δ_0) follows by the definition of Γ_1. If $r = s + 1$, then (Γ_{s+1}, Δ_s) is inseparable by the induction hypothesis, whence the inseparability of $(\Gamma_{s+1}, \Delta_{s+1})$ follows by the definition of Δ_{s+1}. From the latter, the inseparability of $(\Gamma_{s+2}, \Delta_{s+1})$ follows by the definition of Γ_{s+2}. Claim 1 is proved.

We now set:

$$\Gamma' := \bigcup_{s\in\omega} \Gamma_s; \qquad \Delta' := \bigcup_{s\in\omega} \Delta_s.$$

We clearly have both:

$$\Gamma \subseteq \Gamma_1 \subseteq \ldots \subseteq \Gamma_s \subseteq \ldots \subseteq \Gamma' \subseteq \mathcal{L}^{Ag(\Gamma)}_{|\Gamma|} \tag{17}$$

and:

$$\Delta \subseteq \Delta_1 \subseteq \ldots \subseteq \Delta_s \subseteq \ldots \subseteq \Delta' \subseteq \mathcal{L}^{Ag(\Delta)}_{|\Delta|} \tag{18}$$

[4]This lemma also holds for uncountable sets of variables but we will not need this more general version in the present paper.

We now show a series of further claims:

Claim 2. The sets Γ', Δ' are consistent

Indeed, if Γ' is inconsistent then $\vdash A_{t_1} \wedge \ldots \wedge A_{t_r} \to \bot$ for some $A_{t_1}, \ldots, A_{t_r}$ in the above enumeration of $\mathcal{L}^{Ag(\Gamma)}_{|\Gamma|}$ such that $A_{t_1}, \ldots, A_{t_r} \in \Gamma'$. Then, by definition of Γ', we must also have $A_{t_1}, \ldots, A_{t_r} \in \Gamma_s$, where $s = max(t_1, \ldots, t_r) + 1$ so that we have $\Gamma_s \vdash \bot$. Of course, we also have $\Delta_s \vdash \neg\bot$, and since $\bot \in \mathcal{L}^{Ag(\Gamma)\cup Ag(\Delta)}_{|\Gamma|\cap|\Delta|}$, it follows that (Γ_s, Δ_s) is separable, a contradiction to Claim 1. Therefore, Γ' is consistent, and the consistency of Δ' is established in a similar way.

Claim 3. The sets Γ', Δ' are $(Ag(\Gamma), |\Gamma|)$-maxiconsistent, and $(Ag(\Delta), |\Delta|)$-maxiconsistent, respectively.

Indeed, if Γ' is not $(Ag(\Gamma), |\Gamma|)$-maxiconsistent, then it follows from Claim 2 and Lemma 1, that there is an $A \in \mathcal{L}^{Ag(\Gamma)}_{|\Gamma|}$ such that $\{A, \neg A\} \cap \Gamma' = \emptyset$. Then we will have $A = A_r$ and $\neg A = A_{r'}$ for some $r, r' \in \omega$ in terms of our enumeration of $\mathcal{L}^{Ag(\Gamma)}_{|\Gamma|}$. Since $A_r, A_{r'} \notin \Gamma'$ we will have, by definition of Γ', that $(\Gamma_r \cup \{A_r\}, \Delta_r)$ and $(\Gamma_{r'} \cup \{A_{r'}\}, \Delta_{r'})$ are separable. This means that there exist some $A^r_1, \ldots, A^r_{t_1} \in \Gamma_r$, $A^{r'}_1, \ldots, A^{r'}_{t_2} \in \Gamma_{r'}$, $B^r_1, \ldots, B^r_{t_3} \in \Delta_r$, $B^{r'}_1, \ldots, B^{r'}_{t_4} \in \Delta_{r'}$, and $C, D \in \mathcal{L}^{Ag(\Gamma)\cup Ag(\Delta)}_{|\Gamma|\cap|\Delta|}$ such that all of the following holds:

$$\vdash (A^r_1 \wedge \ldots \wedge A^r_{t_1} \wedge A) \to C \tag{19}$$

$$\vdash (B^r_1 \wedge \ldots \wedge B^r_{t_3}) \to \neg C \tag{20}$$

$$\vdash (A^{r'}_1 \wedge \ldots \wedge A^{r'}_{t_2} \wedge \neg A) \to D \tag{21}$$

$$\vdash (B^{r'}_1 \wedge \ldots \wedge B^{r'}_{t_4}) \to \neg D \tag{22}$$

We then infer, by propositional logic, that:

$$\vdash (\bigwedge_{s=1}^{t_1} A^r_s \wedge \bigwedge_{s=1}^{t_2} A^{r'}_s) \to (C \vee D) \tag{23}$$

$$\vdash (\bigwedge_{s=1}^{t_3} B^r_s \wedge \bigwedge_{s=1}^{t_4} B^{r'}_s) \to \neg(C \vee D) \tag{24}$$

Now set $r'' := max(r, r')$. By (17) and (18) we know that $\{A^r_s \mid 1 \leq s \leq t_1\} \cup \{A^{r'}_s \mid 1 \leq s \leq t_2\} \subseteq \Gamma_{r''}$ and that $\{B^r_s \mid 1 \leq s \leq t_3\} \cup \{B^{r'}_s \mid 1 \leq s \leq t_4\} \subseteq \Delta_{r''}$. We also clearly have that $C \vee D \in \mathcal{L}^{Ag(\Gamma)\cup Ag(\Delta)}_{|\Gamma|\cap|\Delta|}$. Therefore, it follows from (23) and (24) that $(\Gamma_{r''}, \Delta_{r''})$ is separable, in contradiction to Claim 1. Therefore, Γ' must be $(Ag(\Gamma), |\Gamma|)$-maxiconsistent. Maxiconsistency of Δ' is shown in a similar way.

Claim 4. (Γ', Δ') is inseparable.

Since Γ', Δ' are maxiconsistent, they are closed for finite conjunctions. Therefore, we can assume wlog, that there are $A \in \Gamma'$, $B \in \Delta'$ and $C \in \mathcal{L}^{Ag(\Gamma)\cup Ag(\Delta)}_{|\Gamma|\cap|\Delta|}$ such that all of the following holds:

$$\vdash A \to C \tag{25}$$

$$\vdash B \to \neg C \tag{26}$$

Then let $r, s \in \omega$ be such that $A \in \Gamma_r$ and $B \in \Delta_s$. Setting $t := max(r, s)$, we know that $A \in \Gamma_t$ and $B \in \Delta_t$ whence it follows that (Γ_t, Δ_t) is separable, in contradiction to Claim 1.

Claims 2–4 then imply the first part of the Lemma.

(Part 2). Immediate from the definition of separability. □

Lemma 4. *If* (Γ, Δ) *is separable then for some finite* $\Gamma' \subseteq \Gamma$ *and* $\Delta' \subseteq \Delta$ *the pair* (Γ', Δ') *is also separable.*

Proof. If (Γ, Δ) is separable then for some $A \in \mathcal{L}^{Ag(\Gamma)\cup Ag(\Delta)}_{|\Gamma|\cap|\Delta|}$ it is true that both $\Gamma \vdash A$ and $\Delta \vdash \neg A$. By definition, this means that there are $A_1, \ldots, A_r \in \Gamma$ and $B_1, \ldots, B_s \in \Delta$ such that both $\vdash (A_1 \wedge \ldots \wedge A_r) \to A$ and $\vdash (B_1 \wedge \ldots \wedge B_s) \to \neg A$. Therefore, we can set $\Gamma' := \{A_1, \ldots, A_r\}$ and $\Delta' := \{B_1, \ldots, B_s\}$. □

Next we prove two lemmas which sum up some important facts about inseparability that are peculiar to stit logic:

Lemma 5. *Let* V *be a set of propositional variables, let* $n \leq 3$*, and let* $\Gamma, \Delta \subseteq \mathcal{L}^{\{1,\ldots,n\}}_V$ *be such that* (Γ, Δ) *is inseparable. Moreover, assume that* Γ *is* $(Ag(\Gamma), |\Gamma|)$*-maxiconsistent and* Δ *is* $(Ag(\Delta), |\Delta|)$*-maxiconsistent. Finally, assume that there exist* $\Diamond[j_1]A_1, \ldots, \Diamond[j_r]A_r \in \Gamma$*, and* $\Diamond[i_1]B_1, \ldots, \Diamond[i_s]B_s \in \Delta$ *such that* $j_1, \ldots, j_r \in Ag(\Gamma)$ *are pairwise different and* $i_1, \ldots, i_s \in Ag(\Delta)$ *are pairwise different.*

Then the pair:

$$(\Gamma^{\Box} \cup \{[j_1]A_1, \ldots, [j_r]A_r\}, \Delta^{\Box} \cup \{[i_1]B_1, \ldots, [i_s]B_s\}) \tag{27}$$

is inseparable.

Proof. Assume the hypothesis, and assume, for *reductio*, that (27) is separable. Then, by compactness of stit logic and the S5 properties of $\Box$, there must be $\Box A \in \Gamma$, $\Box B \in \Delta$, and $C \in \mathcal{L}^{Ag(\Gamma)\cup Ag(\Delta)}_{|\Gamma|\cap|\Delta|}$ such that both of the following equations hold:

$$\vdash (\Box A \wedge [j_1]A_1 \wedge \ldots \wedge [j_r]A_r) \to C, \tag{28}$$

and

$$\vdash (\Box B \wedge [i_1]B_1 \wedge \ldots \wedge [i_s]B_s) \to \neg C. \tag{29}$$

Since $Ag(\Gamma) \cap Ag(\Delta) = \emptyset$, all of the agent indices in the united sequence $j_1, \ldots, j_r, i_1, \ldots, i_s$ must be pairwise different and we must have $r + s \leq n$. Therefore, $r + s \in \{0, 1, 2, 3\}$ which gives us our three cases below. Although these cases show many similarities, we consider them separately. In every case we reason by contraposition, showing that the separability of (27) (expressed by (28) and (29)) implies the separability of (Γ, Δ), thus contradicting the initial assumption of the lemma.

Case 1. Let $\{r, s\} = \{1, 2\}$. Assume, wlog, that $r = 2$ and $s = 1$, the other subcase is symmetric. Then, by (28) and (29), there exist i, j and k such that $\{i, j, k\} = \{1, 2, 3\}$, and that both of the following hold:

$$\vdash (\Box A \wedge [i]A_1 \wedge [j]A_2) \to C, \tag{30}$$

and

$$\vdash (\Box B \wedge [k]B_1) \to \neg C. \tag{31}$$

By Lemma 2.2, (30), and propositional logic, we get that:

$$\vdash (\Box A \wedge \Diamond[i]A_1 \wedge \Diamond[j]A_2) \to \neg\Diamond[k]\neg C, \tag{32}$$

On the other hand, by Lemma 2.3 and (31):

$$\vdash (\Box B \wedge \Diamond[k]B_1) \to \Diamond[k]\neg C. \tag{33}$$

Since C, by its choice, is in $\mathcal{L}^{Ag(\Gamma)\cup Ag(\Delta)}_{|\Gamma|\cap|\Delta|}$, we clearly have $\Diamond[k]\neg C \in \mathcal{L}^{Ag(\Gamma)\cup Ag(\Delta)}_{|\Gamma|\cap|\Delta|}$, and we also have, by the initial choice of our formulas, that $\Box A, \Diamond[i]A_1, \Diamond[j]A_2 \in \Gamma$ and $\Box B, \Diamond[k]B_1 \in \Delta$. Therefore, it follows from (32) and (33), that (Γ, Δ) is separable.

Case 2. Let $\{r, s\} = \{1\}$. Then, by (28) and (29), there exist $i, j \in \{1, 2, 3\}$ such that $i \neq j$ and both of the following hold:

$$\vdash (\Box A \wedge [i]A_1) \to C, \tag{34}$$

and

$$\vdash (\Box B \wedge [j]B_1) \to \neg C. \tag{35}$$

By Lemma 2.2 and (34) we get that:

$$\vdash (\Box A \wedge \Diamond[i]A_1) \to \neg\Diamond[j]\neg C, \tag{36}$$

On the other hand, by Lemma 2.3 and (35):

$$\vdash (\Box B \wedge \Diamond[j]B_1) \to \Diamond[j]\neg C. \tag{37}$$

Since C, by its choice, is in $\mathcal{L}^{Ag(\Gamma)\cup Ag(\Delta)}_{|\Gamma|\cap|\Delta|}$, we clearly have $\Diamond[j]\neg C \in \mathcal{L}^{Ag(\Gamma)\cup Ag(\Delta)}_{|\Gamma|\cap|\Delta|}$, and we also have, by the initial choice of our formulas, that $\Box A, \Diamond[i]A_1 \in \Gamma$ and $\Box B, \Diamond[j]B_1 \in \Delta$. Therefore, it follows from (36) and (37), that (Γ, Δ) is again separable, contrary to our assumptions.

Case 3. $0 \in \{r, s\}$. We may assume, wlog, that $s = 0$, the other subcase being symmetric. By (29), we must have then:

$$\vdash \Box B \to \neg C. \tag{38}$$

By S5 properties of $\Box$, we get then:

$$\vdash (\Box A \wedge \Diamond[j_1]A_1 \wedge \ldots \wedge \Diamond[j_r]A_r) \to \Diamond C \qquad \text{(from (28))} \tag{39}$$

$$\vdash \Box B \to \Box\neg C \qquad \text{(from (38))} \tag{40}$$

It follows then, by the choice of the formulas involved, that (Γ, Δ) is separable, contrary to our assumptions.

This exhausts the list of possible cases and thus the Lemma is proved. □

Lemma 6. *Let V be a set of propositional variables, Ag a finite agent community, and let $\Gamma, \Delta \subseteq \mathcal{L}^{Ag}_V$ be such that (Γ, Δ) is inseparable. Moreover, assume that Γ is $(Ag(\Gamma), |\Gamma|)$-maxiconsistent and Δ is $(Ag(\Delta), |\Delta|)$-maxiconsistent. Then:*

1. *If $\neg\Box A_1 \in \Gamma$, then the pair $(\Gamma^{\Box} \cup \{\neg A_1\}, \Delta^{\Box})$ is inseparable.*

2. *If* $\neg\Box B_1 \in \Delta$*, then the pair* $(\Gamma^{\Box}, \Delta^{\Box} \cup \{\neg B_1\})$ *is inseparable.*

3. *If* $\neg[j]A_1 \in \Gamma$*, then the pair* $(\Gamma^{\Box} \cup \Gamma^{[j]} \cup \{\neg A_1\}, \Delta^{\Box})$ *is inseparable.*

4. *If* $\neg[i]B_1 \in \Delta$*, then the pair* $(\Gamma^{\Box}, \Delta^{\Box} \cup \Delta^{[i]} \cup \{\neg B_1\})$ *is inseparable.*

Proof. (Part 1). Assume the hypothesis. If the pair $(\Gamma^{\Box} \cup \{\neg A_1\}, \Delta^{\Box})$ is separable, then, by compactness of stit logic, maxiconsistency of Γ and Δ, and S5 properties of all the modalities in stit logic, there must be $\Box A \in \Gamma$, $\Box B \in \Delta$, and $C \in \mathcal{L}^{Ag(\Gamma)\cup Ag(\Delta)}_{|\Gamma|\cap|\Delta|}$ such that (38) holds together with the following equation:

$$\vdash (\Box A \wedge \neg A_1) \to C. \tag{41}$$

From (41) we infer, using S5 properties of $\Box$:

$$\vdash (\Box A \wedge \Diamond\neg A_1) \to \Diamond C. \tag{42}$$

On the other hand, from (38) we infer (40) arguing as in Case 3 in the proof of Lemma 5 above. Taken together, (40) and (42) show separability of (Γ, Δ), contrary to our assumptions. Therefore, (41) and (38) cannot hold, whence $(\Gamma^{\Box} \cup \{\neg A_1\}, \Delta^{\Box})$ must be inseparable, and we are done.

Part 2 is symmetric to Part 1.

(Part 3). Assume the hypothesis. If the pair $(\Gamma^{\Box} \cup \Gamma^{[j]} \cup \{\neg A_1\}, \Delta^{\Box})$ is separable, then, by compactness of stit logic, maxiconsistency of Γ and Δ, and S5 properties of all the modalities in stit logic, there must be $\Box A, [j]A' \in \Gamma$, $\Box B \in \Delta$, and $C \in \mathcal{L}^{Ag(\Gamma)\cup Ag(\Delta)}_{|\Gamma|\cap|\Delta|}$ such that (38) holds together with the following equation:

$$\vdash (\Box A \wedge [j]A' \wedge \neg A_1) \to C \tag{43}$$

Next we infer:

$\vdash [j]((\Box A \wedge [j]A' \wedge \neg C) \to A_1)$	(by (43), $[j]$ is S5)	(44)
$\vdash ([j]\Box A \wedge [j]A' \wedge [j]\neg C) \to [j]A_1$	(by (44), $[j]$ is S5)	(45)
$\vdash \Box A \to [j]\Box A$	(cf. (14) above)	(46)
$\vdash (\Box A \wedge [j]A' \wedge [j]\neg C) \to [j]A_1$	(by (45) and (46))	(47)
$\vdash (\Box A \wedge [j]A' \wedge \neg[j]A_1) \to \neg[j]\neg C$	(by (47) and prop. logic)	(48)

We also infer (40) from (38), arguing as in Case 3 in the proof of Lemma 5 above. From (40) and (A2) it then follows that:

$$\vdash \Box B \to [j]\neg C \tag{49}$$

Taken together, (48) and (49) show separability of (Γ, Δ), contrary to our assumptions. Therefore, (43) and (38) cannot hold together, whence $(\Gamma^{\Box} \cup \Gamma^{[j]} \cup \{\neg A_1\}, \Delta^{\Box})$ must be inseparable, and we are done.

Part 4 is symmetric to Part 3. ☐

We are now prepared to prove Theorem 1. Assume that $n \leq 3$, assume for *reductio*, that $A, B \in \mathcal{L}^{\{1,\ldots,n\}}_{V}$, and we have $\vdash A \to B$, $Ag(A) \cap Ag(B) = \emptyset$, but for no $C \in \mathcal{L}^{Ag(A)\cup Ag(B)}_{|A|\cap|B|}$ we have both $\vdash A \to C$ and $\vdash C \to B$. This means that the pair $(\{A\}, \{\neg B\})$ is inseparable and can be extended, using Lemma 3, to an inseparable pair (Ξ_0, Ξ_1) such that Ξ_0 is $(Ag(A), |A|)$-maxiconsistent and Ξ_1 is $(Ag(B), |B|)$-maxiconsistent. We now define a $(Ag(A) \cup Ag(B), |A| \cup |B|)$-stit model $\mathfrak{S}$ which we will show to satisfy $\Xi_0 \cup \Xi_1$.

Now we start defining components of $\mathfrak{S} = \langle Tree, \leq, Choice, V\rangle$:

- We first define the set of *standard pairs* as the set of all inseparable pairs (Γ, Δ) such that Γ is $(Ag(A), |A|)$-maxiconsistent, Δ is $(Ag(B), |B|)$-maxiconsistent, and the following condition holds:

$$\Xi_0^{\Box} \subseteq \Gamma \& \Xi_1^{\Box} \subseteq \Delta.$$

  The set of standard pairs is non-empty since (Ξ_0, Ξ_1) is clearly a standard pair.

- We then define $Tree$ as the set of all standard pairs plus a single additional moment $\dagger$.

- $\leq$ is the reflexive closure of the relation $\{(\dagger, (\Gamma, \Delta)) \mid (\Gamma, \Delta)$ is a standard pair $\}$

Immediately we get the following lemma:

Lemma 7. *If (Γ, Δ) is a standard pair then $\Gamma^{\Box} = \Xi_0^{\Box}$ and $\Delta^{\Box} = \Xi_1^{\Box}$.*

Proof. We show that $\Gamma^{\Box} = \Xi_0^{\Box}$, the other part is similar. We have $\Xi_0^{\Box} \subseteq \Gamma$ by the definition of standard pair, whence clearly $\Xi_0^{\Box} \subseteq \Gamma^{\Box}$. In the other direction, assume that $\Box C \in \Gamma$. Since Ξ_0 is $(Ag(A), |A|)$-maxiconsistent, we must have either $\Box C \in \Xi_0$ or $\neg\Box C \in \Xi_0$. In the latter case, by S5 properties of $\Box$ and $(Ag(A), |A|)$-maxiconsistency of Ξ_0 we get that $\Box\neg\Box C \in \Xi_0$. We have established, therefore, that either $\Box C \in \Xi_0^{\Box}$ or $\Box\neg\Box C \in \Xi_0^{\Box}$. However, we cannot have $\Box\neg\Box C \in \Xi_0^{\Box}$, since we know that $\Xi_0^{\Box} \subseteq \Gamma$, and also $\Box C \in \Gamma$. It follows that we must have $\Box C \in \Xi_0^{\Box}$. $\Box$

We pause to reflect on the structure of histories induced by the pair $(Tree, \leq)$. Every such history has the form $h_{(\Gamma,\Delta)} = \{\dagger, (\Gamma, \Delta)\}$. It is clear, moreover, that we have both $H^{\mathfrak{S}}_{\dagger} = Hist(\mathfrak{S})$ and $H_{(\Gamma,\Delta)} = \{h_{(\Gamma,\Delta)}\}$ for every standard pair (Γ, Δ). We then define the choice function for our model in the following way:

- For every $j \in Ag(A)$ and standard pairs (Γ, Δ) and (Γ_0, Δ_0), we define that $h_{(\Gamma_0,\Delta_0)} \in Choice^{\dagger}_j(h_{(\Gamma,\Delta)})$ iff $\Gamma^{[j]} \subseteq \Gamma_0$.

- Similarly, for every $i \in Ag(B)$ and standard pairs (Γ, Δ) and (Γ_0, Δ_0), we define that $h_{(\Gamma_0,\Delta_0)} \in Choice^{\dagger}_j(h_{(\Gamma,\Delta)})$ iff $\Delta^{[i]} \subseteq \Delta_0$.

- For every $j \in Ag(A) \cup Ag(B)$ and every standard pair (Γ, Δ) we set that $Choice^{(\Gamma,\Delta)}_j = \{H_{(\Gamma,\Delta)}\} = \{\{h_{(\Gamma,\Delta)}\}\}$.

- Finally, for a $p \in |A|$, we define that $V(p) = \{(\dagger, (\Gamma, \Delta)) \mid p \in \Gamma\}$; symmetrically, for a $q \in |B|$, we define that $V(q) = \{(\dagger, (\Gamma, \Delta)) \mid q \in \Delta\}$.

First of all, we need to show that we have in fact defined a stit model:

Lemma 8. *The structure $\mathfrak{S} = \langle Tree, \leq, Choice, V\rangle$, as defined above, is a $(Ag(A) \cup Ag(B), |A| \cup |B|)$-stit model.*

Proof. It is obvious that $\leq$ is a forward-branching preorder on the non-empty set $Tree$. The fact that $Choice^m_j$ is a partition of $H^{\mathfrak{S}}_m$ trivially follows from definition, whenever $m \neq \dagger$. If, on the other hand, $m = \dagger$, then this same fact follows from S5 properties of $[j]$ together with the fact that, for every standard pair (Γ, Δ), $Ag(\Gamma) \cap Ag(\Delta) = Ag(A) \cap Ag(B) = \emptyset$.

As for the constraints, (HC) is satisfied since $\dagger$ is the $\leq$-least moment in $Tree$ and (NCUH) is satisfied because there are no undivided histories in $\mathfrak{S}$. We consider (IA). Let $m \in Tree$ and let f be a function on Ag such that $(\forall j \in Ag(A) \cup Ag(B))(f(j) \in Choice^m_j)$. We are going to show that in this case $\bigcap_{j \in Ag(A) \cup Ag(B)} f(j) \neq \emptyset$. If $m \neq \dagger$, then this is obvious, since every agent will have a vacuous choice. We treat the case when $m = \dagger$.

Then, for every $j \in Ag(A) \cup Ag(B)$, we pick an $h_j \in f(j)$ so that $f(j) = Choice^{\dagger}_j(h_j)$. Since $Hist(Tree, \leq) = \{h_{(\Gamma,\Delta)} \mid (\Gamma, \Delta) \text{ is a standard pair}\}$, we can choose, for every $j \in Ag(A) \cup Ag(B)$, a standard pair (Γ_j, Δ_j) such that $h_j = h_{(\Gamma_j,\Delta_j)}$. Together with $f(j) = Choice^{\dagger}_j(h_j)$, this gives us the following equation:

$$(\forall j \in Ag(A) \cup Ag(B))(f(j) = Choice^{\dagger}_j(h_{(\Gamma_j,\Delta_j)})) \tag{50}$$

Now consider the pair:

$$(\Xi^{\Box}_0 \cup \bigcup\{\Gamma^{[j]}_j \mid j \in Ag(A)\}, \Xi^{\Box}_1 \cup \bigcup\{\Delta^{[i]}_i \mid i \in Ag(B)\} \tag{51}$$

We will show that the pair (51) is inseparable. Indeed, suppose otherwise. Then, by Lemma 4, there must be $\Box A_1, \ldots, \Box A_r \in \Xi^{\Box}_0$, $\Box B_1, \ldots, \Box B_{r'} \in \Xi^{\Box}_1$, $[j]A^j_1, \ldots, [j]A^j_{r(j)} \in \Gamma_j$ (for every $j \in Ag(A)$), and $[i]B^i_1, \ldots, [i]B^i_{r(i)} \in \Delta_i$ (for every $i \in Ag(B)$) such that the pair:

$$\begin{aligned}(\{\Box A_1, \ldots, \Box A_r\} \cup \bigcup\{\{[j]A^j_1, \ldots, [j]A^j_{r(j)}\} \mid j \in Ag(A)\},\\ \{\Box B_1, \ldots, \Box B_{r'}\} \cup \bigcup\{\{[i]B^i_1, \ldots, [i]B^i_{r(i)})\} \mid i \in Ag(B)\})\end{aligned} \tag{52}$$

is separable. Now the contraposition of Lemma 3.2 entails that in this case also the pair:

$$\begin{aligned}(\Xi^{\Box}_0 \cup \bigcup\{\{[j]A^j_1, \ldots, [j]A^j_{r(j)}\} \mid j \in Ag(A)\},\\ \Xi^{\Box}_1 \cup \bigcup\{\{[i]B^i_1, \ldots, [i]B^i_{r(i)}\} \mid i \in Ag(B)\})\end{aligned} \tag{53}$$

must be separable. Next, for every $j \in Ag(A)$ and every $i \in Ag(B)$, we set:

$$\alpha_j := A^j_1 \wedge \ldots \wedge A^j_{r(j)}; \qquad \beta_i := B^i_1 \wedge \ldots \wedge B^i_{r(i)}.$$

By Lemma 2.1 and the separability of the pair (53), we know that also the following pair must be separable:

$$(\Xi^{\Box}_0 \cup \{[j]\alpha_j \mid j \in Ag(A)\}, \Xi^{\Box}_1 \cup \{[i]\beta_i \mid i \in Ag(B)\}. \tag{54}$$

For every $j \in Ag(A)$, the formulas $[j]A^j_1, \ldots, [j]A^j_{r(j)}$ were chosen in Γ_j, therefore, it follows from Lemma 2.1 and maxiconsistency of Γ_j that also $[j]\alpha_j \in \Gamma_j$. By S5 properties of $\Box$, this means that also $\Diamond[j]\alpha_j \in \Gamma_j$ so that, by consistency, $\Box\neg[j]\alpha_j \notin \Gamma_j$.

The latter means, by Lemma 7, that $\Box\neg[j]\alpha_j \notin \Xi_0$, therefore, by maxiconsistency, $\Diamond[j]\alpha_j \in \Xi_0$. By a parallel argument, one can also show that, for every $i \in Ag(B)$, $\Diamond[i]\beta_i \in \Xi_1$. Therefore, by Lemma 5, the separability of the pair (54) entails the separability of (Ξ_0, Ξ_1) which contradicts the choice of the latter pair. The obtained contradiction shows that the pair (51) must be inseparable.

Therefore, by Lemma 3.1, the pair (51) can be extended to a pair (Γ_0, Δ_0) such that Γ_0 is $(Ag(A), |A|)$-maxiconsistent and Δ_0 is $(Ag(B), |B|)$-maxiconsistent. By the choice of (51), it is also clear that both $\Xi_0^\Box \subseteq \Gamma_0$ and $\Xi_1^\Box \subseteq \Delta_0$, which means that (Γ_0, Δ_0) is a standard pair. Therefore, we must have $h_{(\Gamma_0,\Delta_0)} \in H_\dagger^\mathfrak{S}$. Now, let $j \in Ag(A)$. Then, by the choice of (51), $\Gamma_j^{[j]} \subseteq \Gamma_0$, whence we get, by (50) and the definition of *Choice*, that $h_{(\Gamma_0,\Delta_0)} \in Choice_j^\dagger(h_{(\Gamma_j,\Delta_j)}) = f(j)$. Similarly, if $i \in Ag(B)$, then, by the choice of (51), $\Delta_i^{[i]} \subseteq \Delta_0$, whence we get, by (50) and the definition of *Choice*, that $h_{(\Gamma_0,\Delta_0)} \in Choice_i^\dagger(h_{(\Gamma_i,\Delta_i)}) = f(i)$. Summing up, we obtain that:

$$h_{(\Gamma_0,\Delta_0)} \in \bigcap_{j \in Ag(A) \cup Ag(B)} f(j) \neq \emptyset,$$

and (IA) is thus satisfied. $\Box$

For the defined model $\mathfrak{S}$, we show the following truth lemma:

Lemma 9. *Let $\mathfrak{S}$ be as defined above, let (Γ, Δ) be a standard pair, let $C \in \mathcal{L}_{|\Gamma|}^{Ag(A)}$, and let $D \in \mathcal{L}_{|\Delta|}^{Ag(B)}$. Then:*

1. $\mathfrak{S}, \dagger, h_{(\Gamma,\Delta)} \models C \Leftrightarrow C \in \Gamma$;
2. $\mathfrak{S}, \dagger, h_{(\Gamma,\Delta)} \models D \Leftrightarrow D \in \Delta$.

Proof. We show Part 1, the other part is similar. The proof proceeds by induction on the construction of C.

Basis. $C = p \in |\Gamma|$. Then:

$$\mathfrak{S}, \dagger, h_{(\Gamma,\Delta)} \models p \Leftrightarrow (\dagger, h_{(\Gamma,\Delta)}) \in V(p) \Leftrightarrow p \in \Gamma,$$

by the definition of V above.

Induction step. The Boolean cases are strightforward. We treat the modal cases:

Case 1. $C = \Box D$. ($\Leftarrow$) Assume that $\Box D \in \Gamma$ and take an arbitrary $g \in H_\dagger^\mathfrak{S}$. We will show that $\mathfrak{S}, \dagger, g \models D$. Indeed, we must have $g = h_{(\Gamma_0,\Delta_0)}$ for an appropriate standard pair (Γ_0, Δ_0). By Lemma 7, we must have $\Gamma^\Box = \Xi_0^\Box = \Gamma_0^\Box$, whence it follows that $\Box D \in \Gamma_0$. By S5 properties of $\Box$ and $(Ag(A), |A|)$-maxiconsistency of Γ_0, it follows further that $D \in \Gamma_0$, whence $\mathfrak{S}, \dagger, g(= h_{(\Gamma_0,\Delta_0)}) \models D$ by induction hypothesis. Since g was chosen in $H_\dagger^\mathfrak{S}$ arbitrarily, it follows that $\mathfrak{S}, \dagger, h_{(\Gamma,\Delta)} \models \Box D$.

($\Rightarrow$). Assume that $\Box D \notin \Gamma$. By $(Ag(A), |A|)$-maxiconsistency of Γ, we must have then that $\neg\Box D \in \Gamma$, which, by Lemma 6.1, means that the pair $(\Gamma^\Box \cup \{\neg D\}, \Delta^\Box)$ must be inseparable. By Lemma 7, we know that also the pair $(\Xi_0^\Box \cup \{\neg D\}, \Xi_1^\Box)$ must be inseparable. We then extend the latter pair, using Lemma 3.1, to a standard pair (Γ_0, Δ_0). It is clear that $D \notin \Gamma_0$, hence, by induction hypothesis, $\mathfrak{S}, \dagger, h_{(\Gamma_0,\Delta_0)} \not\models D$. Since $h_{(\Gamma_0,\Delta_0)} \in H_\dagger^\mathfrak{S}$, this further means that $\mathfrak{S}, \dagger, h_{(\Gamma,\Delta)} \not\models \Box D$, as desired.

Case 2. $C = [j]D$ for some $j \in Ag(A)$. ($\Leftarrow$) Assume that $[j]D \in \Gamma$ and take an arbitrary $g \in Choice^{\dagger}_{j}(h_{(\Gamma,\Delta)})$. We will show that $\mathfrak{S}, \dagger, g \models D$. Indeed, we must have $g = h_{(\Gamma_0,\Delta_0)}$ for an appropriate standard pair (Γ_0, Δ_0). Given that $h_{(\Gamma_0,\Delta_0)} = g \in Choice^{\dagger}_{j}(h_{(\Gamma,\Delta)})$, we must also have, by the definition of *Choice*, that $\Gamma^{[j]} \subseteq \Gamma_0$. Therefore, $[j]D \in \Gamma_0$, and it follows by S5 properties of $[j]$ and $(Ag(A), |A|)$-maxiconsistency of Γ_0, that also $D \in \Gamma_0$ whence $\mathfrak{S}, \dagger, g(= h_{(\Gamma_0,\Delta_0)}) \models D$ by the induction hypothesis. Since g was chosen in $Choice^{\dagger}_{j}(h_{(\Gamma,\Delta)})$ arbitrarily, we have shown that $\mathfrak{S}, \dagger, h_{(\Gamma,\Delta)} \models [j]D$.

($\Rightarrow$). Assume that $[j]D \notin \Gamma$. By $(Ag(A), |A|)$-maxiconsistency of Γ, we must have then that $\neg[j]D \in \Gamma$, which, by Lemma 6.3, means that the pair $(\Gamma^{\Box} \cup \Gamma^{[j]} \cup \{\neg D\}, \Delta^{\Box})$ must be inseparable. By Lemma 7, we know that also the pair $(\Xi_0^{\Box} \cup \Gamma^{[j]} \cup \{\neg D\}, \Xi_1^{\Box})$ must be inseparable. We then extend the latter pair, using Lemma 3.1, to a standard pair (Γ_0, Δ_0). It is clear that $D \notin \Gamma_0$, hence, by induction hypothesis, $\mathfrak{S}, \dagger, h_{(\Gamma_0,\Delta_0)} \not\models D$. We also clearly have $\Gamma^{[j]} \subseteq \Gamma_0$, which means that $h_{(\Gamma_0,\Delta_0)} \in Choice^{\dagger}_{j}(h_{(\Gamma,\Delta)})$. Therefore, we get that $\mathfrak{S}, \dagger, h_{(\Gamma,\Delta)} \not\models [j]D$, as desired. □

We can now finish our proof of Theorem 1 by recalling the fact that we have, according to the above assumption, both $A \in \Xi_0$ and $\neg B \in \Xi_1$, so that it follows from Lemma 9, that:

$$\mathfrak{S}, \dagger, h_{(\Xi_0,\Xi_1)} \models A \wedge \neg B.$$

The latter is in contradiction with the assumption that $\vdash A \to B$, and this contradiction means that there must be an interpolant for this implication.

4 The case $n > 3$

The main result of this section looks as follows:

Theorem 2. *For every $n > 3$, stit logic does not have $(RCIP)_n$.*

Again, we start with some technicalities:

Lemma 10. *Let $j_1, j_2, j_3, j_4 \in Ag$ and propositional variables p, q, r be pairwise different. Then:*

$$\vdash \Diamond([j_1]p \wedge [j_2](p \to q)) \to \neg\Diamond([j_3]r \wedge [j_4](r \to \neg q)).$$

Proof. We reason as follows:

$\Diamond([j_1]p \wedge [j_2](p \to q)) \wedge \Diamond([j_3]r \wedge [j_4](r \to \neg q))$	(premise)	(55)
$\Diamond([j_1]p \wedge [j_2](p \to q)) \to (\Diamond[j_1]p \wedge \Diamond[j_2](p \to q))$	($\Box$ is S5)	(56)
$\Diamond([j_3]r \wedge [j_4](r \to \neg q)) \to (\Diamond[j_3]r \wedge \Diamond[j_4](r \to \neg q))$	($\Box$ is S5)	(57)
$\Diamond[j_1]p \wedge \Diamond[j_2](p \to q) \wedge \Diamond[j_3]r \wedge \Diamond[j_4](r \to \neg q)$	(from (55)–(57))	(58)
$\Diamond([j_1]p \wedge [j_2](p \to q) \wedge [j_3]r \wedge [j_4](r \to \neg q))$	(from (58), (A3))	(59)
$([j_1]p \wedge [j_2](p \to q) \wedge [j_3]r \wedge [j_4](r \to \neg q)) \to$ $\to (p \wedge (p \to q) \wedge r \wedge (r \to \neg q))$	($[j_1]$–$[j_4]$ are S5)	(60)
$([j_1]p \wedge [j_2](p \to q) \wedge [j_3]r \wedge [j_4](r \to \neg q)) \to \bot$	(from (60) by prop. logic)	(61)
$\Diamond([j_1]p \wedge [j_2](p \to q) \wedge [j_3]r \wedge [j_4](r \to \neg q)) \to \bot$	(from (61) since $\Box$ is S5)	(62)
$\bot$	(from (59) and (62))	(63)

□

Definition 2. *Let* $\mathfrak{S} = \langle Tree, \leq, Choice, V\rangle$ *and* $\mathfrak{S}' = \langle Tree', \leq', Choice', V'\rangle$ *be* (Ag, V)*-stit models, and let* $m \in Tree$ *and* $m' \in Tree'$*. Relation* $B \in H^{\mathfrak{S}}_m \times H^{\mathfrak{S}'}_{m'}$ *we will call a* bisimulation between $(\mathfrak{S}, m)$ and $(\mathfrak{S}', m')$*, iff the domain of* B *is* $H^{\mathfrak{S}}_m$*, the counter-domain of* B *is* $H^{\mathfrak{S}'}_{m'}$*, and the following holds for all* $p \in V$*, all* $j \in Ag$*, all* $h_1, h_2 \in H^{\mathfrak{S}}_m$ *and all* $h'_1, h'_2 \in H^{\mathfrak{S}'}_{m'}$*:*

$$h_1\ B\ h'_1 \Rightarrow (\mathfrak{S}, m, h_1 \models p \Leftrightarrow \mathfrak{S}', m', h'_1 \models p) \qquad \text{(atoms)}$$

$$(h_1\ B\ h'_1 \& h_2 \in Choice^m_j(h_1)) \Rightarrow (\exists h'_3 \in (Choice')^{m'}_j(h'_1))(h_2\ B\ h'_3) \qquad \text{(forth)}$$

$$(h_1\ B\ h'_1 \& h'_2 \in (Choice')^{m'}_j(h'_1)) \Rightarrow (\exists h_3 \in Choice^m_j(h_1))(h_3\ B\ h'_2) \qquad \text{(back)}$$

We show that existence of a bisimulation implies the equality of theories:

Lemma 11. *Let* $\mathfrak{S} = \langle Tree, \leq, Choice, V\rangle$ *and* $\mathfrak{S}' = \langle Tree', \leq', Choice', V'\rangle$ *be* (Ag, V)*-stit models, and let* $B \in H^{\mathfrak{S}}_m \times H^{\mathfrak{S}'}_{m'}$ *be a bisimulation between* $(\mathfrak{S}, m)$ *and* $(\mathfrak{S}', m')$*. Then, for all* $A \in \mathcal{L}^{Ag}_V$ *and all* $h_1 \in H^{\mathfrak{S}}_m$ *and* $h'_1 \in H^{\mathfrak{S}'}_{m'}$*:*

$$h_1\ B\ h'_1 \Rightarrow (\mathfrak{S}, m, h_1 \models A \Leftrightarrow \mathfrak{S}', m', h'_1 \models A).$$

Proof. By induction on the construction of A. The basis follows from (atoms), and the Boolean cases in the induction step are trivial. We consider the modal cases:

Case 1. A has the form $\Box B$. ($\Rightarrow$) Assume that $\mathfrak{S}, m, h_1 \models \Box B$ and let $h'_2 \in H^{\mathfrak{S}'}_{m'}$ be arbitrary. Then, since the counter-domain of B is $H^{\mathfrak{S}'}_{m'}$, choose any $h_2 \in H^{\mathfrak{S}}_m$ such that $h_2\ B\ h'_2$. We have $\mathfrak{S}, m, h_2 \models B$, whence, by induction hypothesis, it follows that $\mathfrak{S}', m', h'_2 \models B$. Since $h'_2 \in H^{\mathfrak{S}'}_{m'}$ was chosen arbitrarily, we infer that $\mathfrak{S}', m', h'_1 \models \Box B = A$. ($\Leftarrow$) Similarly to the ($\Rightarrow$)-part, using this time the fact that the domain of B is $H^{\mathfrak{S}}_m$.

Case 2. A has the form $[j]B$ for some $j \in Ag$. ($\Rightarrow$) Assume that $\mathfrak{S}, m, h_1 \models [j]B$ and let $h'_2 \in Choice'^{m'}_j(h'_1)$ be arbitrary. Using condition (back), choose a $h_3 \in Choice^m_j(h_1)$ such that $h_3\ B\ h'_2$. We have $\mathfrak{S}, m, h_3 \models B$, whence, by induction hypothesis, it follows that $\mathfrak{S}', m', h'_2 \models B$. Since $h'_2 \in Choice'^{m'}_j(h'_1)$ was chosen arbitrarily, we infer that $\mathfrak{S}', m', h'_1 \models [j]B = A$. ($\Leftarrow$) Similarly to the ($\Rightarrow$)-part, using this time condition (forth) instead of (back). □

Now we need to define two models: a $(\{1, 2, 3, 4\}, \{p, q\})$-stit model $\mathfrak{S} = \langle Tree, \leq, Choice, V\rangle$, and a $(\{1, 2, 3, 4\}, \{q, r\})$-stit model $\mathfrak{S}' = \langle Tree', \leq', Choice', V'\rangle$ to be used in the proof of Theorem 2. First, we define one auxiliary set:

$$4Tup := \{(a, b, c, d)^+, (a, b, c, d)^- \mid a, b, c, d \in \{0, 1\}\}.$$

Next, we start with the definitions of the models, beginning with their temporal substructures.

Definition 3. *We set:*

1. $Tree := \{\dagger\} \cup 4Tup$*.*

2. $\leq$ *is the reflexive closure of* $\{(\dagger, m) \mid m \in 4Tup\}$*.*

3. $Tree' := \{\ddagger\} \cup 4Tup$.

4. $\leq'$ *is the reflexive closure of* $\{(\ddagger, m) \mid m \in 4Tup\}$.

For an integer $1 \leq j \leq 4$, by the j-th projection of $m \in 4Tup = Tree \cap Tree'$ we will mean the j-th projection of the corresponding 4-tuple, regardless of whether m is signed by $+$ or $-$. Thus, for any appropriate $a, b, c, d \in \{0, 1\}$, the two elements $(a, b, c, d)^+$ and $(a, b, c, d)^-$ have the same j-th projection for every $1 \leq j \leq 4$. For an $m \in 4Tup$ and an integer $1 \leq j \leq 4$, the j-th projection of m will be denoted by $pr_j(m)$. The element from $\{+, -\}$ by which m is signed, we will denote $sign(m)$ so that, e.g., $sign((a, b, c, d)^+) = +$. Finally, the complete 4-tuple signed by $sign(m)$ will be called the *core* of m and will be denoted by $core(m)$ so that $core(m) = (pr_1(m), pr_2(m), pr_3(m), pr_4(m))$.

The history structure induced by these definitions is as follows. For $\mathfrak{S}$ we get that:

$$Hist(\mathfrak{S}) = \{h_m = (\dagger, m) \mid m \in 4Tup\} = H_\dagger^{\mathfrak{S}} \tag{64}$$

Similarly, for $\mathfrak{S}'$ we get that:

$$Hist(\mathfrak{S}') = \{g_m = (\ddagger, m) \mid m \in 4Tup\} = H_\ddagger^{\mathfrak{S}'} \tag{65}$$

Once we know the sets of histories induced by $\mathfrak{S}$ and $\mathfrak{S}'$, respectively, it is immediate to deduce the fans of histories passing through any given moment in these models. Namely, it follows that:

$$H_\dagger^{\mathfrak{S}} = Hist(\mathfrak{S}), \qquad H_m^{\mathfrak{S}} = \{h_m\}, \qquad \text{for all } m \in 4Tup \tag{66}$$

and:

$$H_\ddagger^{\mathfrak{S}'} = Hist(\mathfrak{S}'), \qquad H_m^{\mathfrak{S}'} = \{g_m\}, \qquad \text{for all } m \in 4Tup \tag{67}$$

This insight into the history structure allows for a handy definition of choice functions and variable evaluations for the two models:

Definition 4. *We set that:*

1. $Choice_j^\dagger = \{\{h_m \mid pr_j(m) = 0\}, \{h_m \mid pr_j(m) = 1\}\}$ *for all* $1 \leq j \leq 4$.

2. $Choice_j^m = \{H_m^{\mathfrak{S}}\} = \{\{h_m\}\}$ *for all* $m \in 4Tup$ *and* $1 \leq j \leq 4$.

3. $V(p) = \{(\dagger, h_m) \mid pr_1(m) = 0\}$,
   $V(q) = \{(\dagger, h_m) \mid (pr_1(m) = pr_2(m) = 0) \vee (pr_3(m) = pr_4(m) = 0) \vee sign(m) = +\}$.

4. $Choice_j'^\ddagger = \{\{g_m \mid pr_j(m) = 0\}, \{g_m \mid pr_j(m) = 1\}\}$ *for all* $1 \leq j \leq 4$.

5. $Choice_j'^m = \{H_m^{\mathfrak{S}'}\} = \{\{g_m\}\}$ *for all* $m \in 4Tup$ *and* $1 \leq j \leq 4$.

6. $V'(q) = \{(\ddagger, g_m) \mid (pr_3(m) = pr_4(m) = 0) \vee (sign(m) = + \& (pr_3(m) \neq 1 \vee pr_4(m) \neq 0))\}$,
   $V'(r) = \{(\ddagger, g_m) \mid pr_3(m) = 1\}$.

We now establish a number of further lemmas and corollaries.

Corollary 1. *Let* $1 \leq j \leq 4$. *Then* $Choice_j^\dagger(h_m) = \{h_{m_1} \mid pr_j(m) = pr_j(m_1)\}$ *and* $Choice_j'^\ddagger(g_m) = \{g_{m_1} \mid pr_j(m) = pr_j(m_1)\}$ *for all* $m \in 4Tup$.

Proof. The Corollary follows immediately from Definition 4.1 and 4.4, and the fact that for every $m \in 4Tup$ we have either $pr_j(m) = 0$ or $pr_j(m) = 1$. □

Lemma 12. *$\mathfrak{S}$, as given in Definitions 3 and 4, is a $(\{1,2,3,4\},\{p,q\})$-stit model, whereas $\mathfrak{S}'$, as given in the same Definitions, is a $(\{1,2,3,4\},\{q,r\})$-stit model.*

Proof. We consider $\mathfrak{S}$ first. Indeed, $\leq$ is obviously a forward-branching partial order and $\dagger$ is the $\leq$-least element in $Tree$ so that (HC) is satisfied. Also, there are no undivided histories at any moment of $Tree$ so that (NCUH) is also satisfied trivially. Next, for any $m \in 4Tup$ and $1 \leq j \leq 4$, $Choice^m_j$ is a trivial partition of $H^{\mathfrak{S}}_m$. As for $\dagger$ itself, we have, by Definition 4.1, that, for any $1 \leq j \leq 4$, $Choice^{\dagger}_j = \{\{h_m \mid pr_j(m) = 0\}, \{h_m \mid pr_j(m) = 1\}\}$, which is obviously a pair of disjoint subsets of $H^{\mathfrak{S}}_{\dagger} = Hist(\mathfrak{S})$ such that their union makes up $H^{\mathfrak{S}}_{\dagger} = Hist(\mathfrak{S})$ itself. The non-emptiness of both sets in this pair follows from the fact that $(0,0,0,0)^+$ and $(1,1,1,1)^+$ are in $4Tup$. Finally, we tackle (IA). Assume that f is defined on $\{1,2,3,4\}$ in such a way that, for a given $m \in Tree$, we have $f(j) \in Choice^m_j$ for all $1 \leq j \leq 4$. If $m \neq \dagger$, then clearly $\bigcap_{1 \leq j \leq 4} f(j) = H^{\mathfrak{S}}_m \neq \emptyset$. On the other hand, if $m = \dagger$, then, for every $1 \leq j \leq 4$, choose an $h_j \in f(j)$ so that we get $f(j) = Choice^{\dagger}_j(h_j)$ for all $1 \leq j \leq 4$. Then it follows from (64) that, for every $1 \leq j \leq 4$, there must exist an $m_j \in 4Tup$ such that $h_j = h_{m_j}$. But then, consider the 4-tuple $m_0 = (pr_1(m_1), pr_2(m_2), pr_3(m_3), pr_4(m_4))^+$. It is immediate from Definition 4.1 and Corollary 1 that for every $1 \leq j \leq 4$ we have $h_{m_0} \in Choice^{\dagger}_j(h_{m_j}) = f(j)$ whence $h_{m_0} \in \bigcap_{1 \leq j \leq 4} f(j) \neq \emptyset$.

The proof of the Lemma for $\mathfrak{S}'$ is similar. □

Lemma 13. *We have both:*

$$\mathfrak{S}, \dagger, h_m \models \Diamond([1]p \wedge [2](p \to q)),$$

and:

$$\mathfrak{S}', \ddagger, g_m \models \Diamond([3]r \wedge [4](r \to \neg q)),$$

for all $m \in 4Tup$.

Proof. As for the first part of the Lemma, let $m := (0,0,0,0)^+$ and consider h_m. If $h \in Choice^{\dagger}_1(h_m)$ is chosen arbitrarily, then, by (66), $h = h_{m_1}$ for some $m_1 \in 4Tup$ and, moreover, $pr_1(m_1) = pr_1(m) = 0$. But then, by Definition 4.3, $(\dagger, h_{m_1}) \in V(p)$ so that $\mathfrak{S}, \dagger, h_{m_1} \models p$. Since $h_{m_1} \in Choice^{\dagger}_1(h_m)$ was arbitrary, this means that $\mathfrak{S}, \dagger, h_m \models [1]p$.

Furthermore, let $h \in Choice^{\dagger}_2(h_m)$ be chosen arbitrarily. Then, again by (66), $h = h_{m_1}$ for some $m_1 \in 4Tup$ and, moreover, $pr_2(m_1) = pr_2(m) = 0$. If $\mathfrak{S}, \dagger, h_{m_1} \models p$, this means that $(\dagger, h_{m_1}) \in V(p)$ so that also $pr_1(m_1) = 0$. But in this case we will have $pr_1(m_1) = pr_2(m_1) = 0$ which means that also $\mathfrak{S}, \dagger, h_{m_1} \models q$. Thus we have shown, for an arbitrary $h_{m_1} \in Choice^{\dagger}_2(h_m)$, that whenever $\mathfrak{S}, \dagger, h_{m_1} \models p$, it is also the case that $\mathfrak{S}, \dagger, h_{m_1} \models q$ whence it follows that $\mathfrak{S}, \dagger, h_m \models [2](p \to q)$.

Summing up, we must have $\mathfrak{S}, \dagger, h_m \models [1]p \wedge [2](p \to q)$ for $m = (0,0,0,0)^+$, whence, given the semantics of $\Box$ and (66), it follows that $\mathfrak{S}, \dagger, h_m \models \Diamond([1]p \wedge [2](p \to q))$ for all $m \in 4Tup$.

Turning now to the second part of the Lemma, we set $m := (0,0,1,0)^+$ and consider g_m. If $g \in Choice'^{\ddagger}_3(g_m)$ is chosen arbitrarily, then, by (66), $g = g_{m_1}$ for some $m_1 \in 4Tup$ and, moreover, $pr_3(m_1) = pr_3(m) = 1$. But then, by Definition 4.6, $(\ddagger, g_{m_1}) \in V'(r)$ so that $\mathfrak{S}', \ddagger, g_{m_1} \models r$. Since $g_{m_1} \in Choice'^{\ddagger}_3(g_m)$ was arbitrary, this means that $\mathfrak{S}', \ddagger, g_m \models [3]r$.

Furthermore, let $g \in Choice'^{\ddagger}_4(g_m)$ be chosen arbitrarily. Then, again by (66), $g = g_{m_1}$ for some $m_1 \in 4Tup$ and, moreover, $pr_4(m_1) = pr_4(m) = 0$. If $\mathfrak{S}', \ddagger, g_{m_1} \models r$, this means that $(\ddagger, g_{m_1}) \in V'(r)$ so that also $pr_3(m_1) = 1$. But in this case we will have both $pr_3(m_1) = 1$ and $pr_4(m_1) = 0$ which means that also $\mathfrak{S}', \ddagger, g_{m_1} \models \neg q$. Thus we have shown, for an arbitrary $g_{m_1} \in Choice'^{\ddagger}_4(g_m)$, that whenever $\mathfrak{S}', \ddagger, g_{m_1} \models r$, it is also the case that $\mathfrak{S}', \ddagger, g_{m_1} \models \neg q$ whence it follows that $\mathfrak{S}', \ddagger, g_m \models [4](r \to \neg q)$.

Summing up, we must have $\mathfrak{S}', \ddagger, g_m \models [3]r \wedge [4](r \to \neg q)$ for $m = (0,0,1,0)^+$, which means, given the semantics of $\Box$ and (67), that $\mathfrak{S}', \ddagger, g_m \models \Diamond([3]r \wedge [4](r \to \neg q))$ for all $m \in 4Tup$. $\square$

In what follows we let $\mathfrak{S}_q$ and $\mathfrak{S}'_q$ stand for the reducts of $\mathfrak{S}$ and $\mathfrak{S}'$ to $(\{1,2,3,4\},\{q\})$-stit models.

Lemma 14. *The relation* $B := \{(h_m, g_{m_1}) \mid (m, m_1 \in 4Tup), \& ((\dagger, h_m) \in V(q) \Leftrightarrow (\ddagger, g_{m_1}) \in V'(q))\}$ *is a bisimulation between* $(\mathfrak{S}_q, \dagger)$ *and* $(\mathfrak{S}'_q, \ddagger)$.

Proof. We first note that it follows from Definition 4.6 that $(\ddagger, g_{(0,0,0,0)^+}) \in V'(q)$ and $(\ddagger, g_{(0,0,1,0)^+}) \notin V'(q)$. Now if $m \in 4Tup$ then either $(\dagger, h_m) \in V(q)$ or $(\dagger, h_m) \notin V(q)$. In the former case, we get $h_m \; B \; g_{(0,0,0,0)^+}$, in the latter case we get $h_m \; B \; g_{(0,0,1,0)^+}$. Therefore, by (64) and (66), the domain of B is $\{h_m \mid m \in 4Tup\} = H^{\mathfrak{S}}_{\dagger}$, as desired. As for the counterdomain, we may argue in the same fashion, noting that it follows from definition of V that $(\dagger, h_{(0,0,0,0)^+}) \in V(q)$ and $(\dagger, h_{(0,1,0,1)^-}) \notin V(q)$. Thus, we also get that the counterdomain of B is $\{g_m \mid m \in 4Tup\} = H^{\mathfrak{S}'}_{\ddagger}$.

The condition (atoms) from Definition 2 holds simply by definition of B. It remains to check the other two conditions in this definition.

Condition (forth). Assume that $m_1, m_2, m_3 \in 4Tup$ and $1 \leq j \leq 4$ are such that we have both $h_{m_1} \; B \; g_{m_2}$ and $h_{m_3} \in Choice^{\dagger}_j(h_{m_1})$. We need to consider the following cases:

Case 1. We have $(\dagger, h_{m_1}) \in V(q) \Leftrightarrow (\dagger, h_{m_3}) \in V(q)$. Then note that we have both $g_{m_2} \in Choice'^{\ddagger}_j(g_{m_2})$ and $h_{m_3} \; B \; g_{m_2}$, the latter by definition of B.

Case 2. We have $(\dagger, h_{m_1}) \in V(q)$, but $(\dagger, h_{m_3}) \notin V(q)$.

Case 2a. We have $core(m_2) \neq (a,b,0,0)$ for all $a, b \in \{0,1\}$. Then we must have $(\ddagger, g_{core(m_2)^-}) \notin V'(q)$ so that $h_{m_3} \; B \; g_{core(m_2)^-}$. On the other hand, we have, by the identity of cores and Corollary 1, that $g_{core(m_2)^-} \in Choice'^{\ddagger}_j(g_{m_2})$.

Case 2b. We have $core(m_2) = (a,b,0,0)$ for some $a, b \in \{0,1\}$. Now, if $j \in \{1,2,4\}$ we note that for $m_4 := (a,b,1,0)^+$ we have $g_{m_4} \in Choice'^{\ddagger}_j(g_{m_2})$ and also $(\ddagger, g_{m_4}) \notin V'(q)$ so that $h_{m_3} \; B \; g_{m_4}$. On the other hand, if $j = 3$, then we set $m_4 := (a,b,0,1)^-$ and, again, get $g_{m_4} \in Choice'^{\ddagger}_j(g_{m_2})$ and also $(\ddagger, g_{m_4}) \notin V'(q)$ so that $h_{m_3} \; B \; g_{m_4}$.

Case 3. We have $(\dagger, h_{m_1}) \notin V(q)$, but $(\dagger, h_{m_3}) \in V(q)$. Then, by $h_{m_1} \; B \; g_{m_2}$, also $(\ddagger, g_{m_2}) \notin V'(q)$ which means that $core(m_2) \neq (a,b,0,0)$ for all $a, b \in \{0,1\}$.

Case 3a. We have, moreover, that $core(m_2) \neq (a,b,1,0)$ for all $a, b \in \{0,1\}$. Then we must have $(\ddagger, g_{core(m_2)^+}) \in V'(q)$ so that $h_{m_3} \; B \; g_{core(m_2)^+}$. On the other hand, we have, by the identity of cores and Corollary 1, that $g_{core(m_2)^+} \in Choice'^{\ddagger}_j(g_{m_2})$.

Case 3b. We have $core(m_2) = (a, b, 1, 0)$ for some $a, b \in \{0, 1\}$. Now, if $j \in \{1, 2, 4\}$ we note that for $m_4 := (a, b, 0, 0)^+$ we have $g_{m_4} \in Choice'^{\ddagger}_j(g_{m_2})$ and also $(\ddagger, g_{m_4}) \in V'(q)$ so that $h_{m_3}\ B\ g_{m_4}$. On the other hand, if $j = 3$, then we set $m_4 := (a, b, 1, 1)^+$ and, again, get $g_{m_4} \in Choice'^{\ddagger}_j(g_{m_2})$ and also $(\ddagger, g_{m_4}) \in V'(q)$ so that $h_{m_3}\ B\ g_{m_4}$.

Condition (back). Assume that $m_1, m_2, m_3 \in 4Tup$ and $1 \leq j \leq 4$ are such that we have both $h_{m_1}\ B\ g_{m_2}$ and $g_{m_3} \in Choice'^{\ddagger}_j(g_{m_2})$. We need to consider the following cases:

Case 1. We have $(\ddagger, g_{m_2}) \in V'(q) \Leftrightarrow (\ddagger, g_{m_3}) \in V'(q)$. Then note that we have both $h_{m_1} \in Choice^{\dagger}_j(h_{m_1})$ and $h_{m_1}\ B\ g_{m_3}$, the latter by definition of B.

Case 2. We have $(\ddagger, g_{m_2}) \in V'(q)$, but $(\ddagger, g_{m_3}) \notin V'(q)$.

Case 2a. For all $a, b \in \{0, 1\}$, we have both $m_1 \neq (a, b, 0, 0)$ and $m_1 \neq (0, 0, a, b)$. Then we must have $(\dagger, h_{core(m_1)^-}) \notin V(q)$ so that $h_{core(m_1)^-}\ B\ g_{m_3}$. On the other hand, we have, by the identity of cores, that $h_{core(m_1)^-} \in Choice^{\dagger}_j(h_{m_1})$.

Case 2b. We have $core(m_1) = (0, 0, 0, 0)$. Now, if $j \in \{1, 3\}$, we note that for $m_4 := (0, 1, 0, 1)^-$ we have $g_{m_4} \in Choice^{\dagger}_j(h_{m_1})$ and also $(\dagger, h_{m_4}) \notin V(q)$ so that $h_{(0,1,0,1)^-}\ B\ g_{m_3}$. On the other hand, if $j \in \{2, 4\}$, then we set $m_4 := (1, 0, 1, 0)^-$ and, again, get $h_{(1,0,1,0)^-} \in Choice^{\dagger}_j(h_{m_1})$ and also $(\dagger, h_{(1,0,1,0)^-}) \notin V(q)$ so that $h_{(1,0,1,0)^-}\ B\ g_{m_3}$.

Case 2c. We have $core(m_1) = (0, 0, a, b)$ for some $a, b \in \{0, 1\}$ such that $(a, b) \neq (0, 0)$. Then we have to instantiate j:

For $j = 1$, we set $m_4 := (0, 1, a, b)^-$.

For $j \in \{2, 3, 4\}$, we set $m_4 := (1, 0, a, b)^-$.

Under these settings, we always get both $h_{m_4} \in Choice^{\dagger}_j(h_{m_1})$ for the respective j, and $(\dagger, h_{m_4}) \notin V(q)$ so that $h_{m_4}\ B\ g_{m_3}$.

Case 2d. We have $core(m_1) = (a, b, 0, 0)$ for some $a, b \in \{0, 1\}$ such that $(a, b) \neq (0, 0)$. Then we have to instantiate j:

For $j \in \{1, 2, 3\}$, we set $m_4 := (a, b, 0, 1)^-$.

For $j = 4$, we set $m_4 := (a, b, 1, 0)^-$.

Under these settings, we always get both $h_{m_4} \in Choice^{\dagger}_j(h_{m_1})$ for the respective j, and $(\dagger, h_{m_4}) \notin V(q)$ so that $h_{m_4}\ B\ g_{m_3}$.

Case 3. We have $(\ddagger, g_{m_2}) \notin V'(q)$, but $(\ddagger, g_{m_3}) \in V'(q)$. Then we must have $(\dagger, h_{core(m_1)^+}) \in V(q)$ so that $h_{core(m_1)^+}\ B\ g_{m_3}$. On the other hand, we have, by the identity of cores and Corollary 1, that $h_{core(m_1)^+} \in Choice^{\dagger}_j(h_{m_1})$. □

We are now in a position to prove Theorem 2.

Proof of Theorem 2. Assume for *reductio*, that stit logic has $(RCIP)_n$ for some $n > 3$. Then $n \geq 4$ and both $A := \Diamond([1]p \wedge [2](p \to q))$ and $B := \neg\Diamond([3]r \wedge [4](r \to \neg q))$ are in $\mathcal{L}^{\{1,\ldots,n\}}_{\{p,q,r\}}$. By Lemma 10, we have $\vdash A \to B$, therefore, by Definition 1, there must be a $C \in \mathcal{L}^{\{1,2,3,4\}}_{\{q\}}$ such that both $\vdash A \to C$ and $\vdash C \to B$. We choose such a C and note that, by Lemma 13, we have $\mathfrak{S}, \dagger, h_{(0,0,0,0)^+} \models A$, therefore, by $\vdash A \to C$ and the strong completeness of $\mathbb{S}$ w.r.t. stit logic, we must also have $\mathfrak{S}, \dagger, h_{(0,0,0,0)^+} \models C$. The latter means that, moreover, $\mathfrak{S}_q, \dagger, h_{(0,0,0,0)^+} \models C$, since $C \in \mathcal{L}^{\{1,2,3,4\}}_{\{q\}}$. Note that it follows from the definition of B as given in Lemma 14 that $h_{(0,0,0,0)^+}\ B\ g_{(0,0,0,0)^+}$, therefore, it follows from Lemmas 14 and 11 that also $\mathfrak{S}'_q, \ddagger, g_{(0,0,0,0)^+} \models C$. Again, by the fact that $C \in \mathcal{L}^{\{1,2,3,4\}}_{\{q\}}$, we infer that $\mathfrak{S}', \ddagger, g_{(0,0,0,0)^+} \models C$, whence it follows by $\vdash C \to B$, that we must also have $\mathfrak{S}', \ddagger, g_{(0,0,0,0)^+} \models B$. But the latter is in contradiction with Lemma 13 which says that, on the contrary, $\mathfrak{S}', \ddagger, g_{(0,0,0,0)^+} \not\models B$. So we have got our contradiction in place. □

5 Further developments and ramifications

The main topic of this paper is the Restricted Interpolation Property as given by Definition 1. This property is much weaker than the simple Craig Interpolation Property which has attracted much more attention in the existing literature, and for a good reason. In the context of stit logic, we may formulate the Craig Interpolation Property as follows:

Definition 5. *Stit logic has the* n-Craig Interpolation Property *(abbreviated by* $(CIP)_n$*) iff for any set of propositional variables* V*, and all* $A, B \in \mathcal{L}^{\{1,\ldots,n\}}_V$*, whenever* $\vdash A \to B$*, then there exists a* $C \in \mathcal{L}^{Ag(A)\cup Ag(B)}_{|A|\cap|B|}$ *such that both* $\vdash A \to C$ *and* $\vdash C \to B$*.*

Then the relevance of the above results to this latter much more important version of interpolation can be summed up in two following corollaries:

Corollary 2. *For all positive integers* n*, if stit logic does not have* $(RCIP)_n$*, then stit logic does not have* $(CIP)_n$*.*

Proof. Immediately from Definition 1 and Definition 5. □

Corollary 3. *For all* $n > 3$*, stit logic does not have* $(CIP)_n$*.*

Proof. Immediately from Corollary 2 and Theorem 2. □

Thus we may infer from the results of the above sections that stit logic fails $(CIP)_n$ for *almost* all positive integers n. The failure of $(CIP)_n$ further entails, by the standard argument, the failure of the Robinson Consistency Property for the respective values of n. Furthermore, Theorem 1 allows us to considerably limit our search for counterexamples to $(CIP)_n$ for the remaining few values of n. Namely, it follows from Theorem 1 that whenever $\vdash A \to B$ does not have an interpolant in the sense of Definition 5, then we must have $Ag(A) \cap Ag(B) \neq \emptyset$.

Turning again to the Robinson Consistency Property and its variants, Definition 1 raises a natural question whether $(RCIP)_n$ has its accompanying restricted version of the Robinson Consistency Property. The answer is yes, and the respective version of the Robinson Consistency Property can be formulated as follows:

Definition 6. *Stit logic has the* Restricted n-Robinson Consistency Property *(abbreviated by* $(RRCP)_n$*) iff for any set of propositional variables* V*, and all* $\Gamma, \Delta \subseteq \mathcal{L}^{\{1,\ldots,n\}}_V$*, if* (Γ, Δ) *is inseparable, then* $\Gamma \cup \Delta$ *is consistent.*

On the basis of this definition and the proofs given in Sections 3 and 4, the following theorem can be established:

Theorem 3. *For every positive integer* n*, stit logic has* $(RRCP)_n$ *iff it has* $(RCIP)_n$*.*

Proof (a sketch). By a standard argument, one can show that whenever stit logic fails $(RCIP)_n$, it also fails $(RRCP)_n$. In the other direction, an obvious modification of the proof of Theorem 1 given above shows that stit logic has $(RRCP)_n$ for all $n \leq 3$. □

Finally, we tackle the question of the logical status of action modalities. Definition 1 treats action modalities of the form $[j]$ for a $j \in Ag$ as logical symbols, and this is in accordance with the standard view of modalities. But it is easy to see that one can also argue in favor of non-logical status of these modalities, since the agent indices are often treated as proper names of respective agents, and proper names are non-logical. If this attitude is carried out systematically, then we get the following strengthening of Definition 1:

Definition 7. *Stit logic has the* Strong Restricted n-Craig Interpolation Property *(abbreviated by $(SRCIP)_n$) iff for any set of propositional variables V, and all $A, B \in \mathcal{L}^{\{1,\ldots,n\}}_V$, whenever $\vdash A \to B$ and $Ag(A) \cap Ag(B) = \emptyset$, then there exists a $C \in \mathcal{L}^{\emptyset}_{|A| \cap |B|}$ such that both $\vdash A \to C$ and $\vdash C \to B$.*

One immediately sees that $(SRCIP)_n$ only differs from $(RCIP)_n$ in placing stricter requirements on the interpolant. Therefore, for any given positive integer n, the failure of $(RCIP)_n$ for stit logic entails the failure of $(SRCIP)_n$ so that it follows from Theorem 2 that stit logic fails $(SRCIP)_n$ for all positive integers $n > 3$. This result, however, can be improved as follows:

Theorem 4. *For every $n > 1$, stit logic does not have $(SRCIP)_n$.*

In order to prove this theorem, we again need to establish a number of technical claims:

Lemma 15. *Let $j_1, j_2 \in Ag$ be different and let p be a propositional variable. Then:*

$$\vdash \Diamond[j_1]p \to \neg\Diamond[j_2]\neg p.$$

Proof. We reason as follows:

$$(\Diamond[j_1]p \wedge \Diamond[j_2]\neg p) \to \Diamond([j_1]p \wedge [j_2]\neg p) \qquad \text{(by (A3))} \qquad (68)$$

$$([j_1]p \wedge [j_2]\neg p) \to \bot \qquad ([j_1],\ [j_2] \text{ are S5}) \qquad (69)$$

$$\Diamond([j_1]p \wedge [j_2]\neg p) \to \bot \qquad \text{(from (69) since } \Box \text{ is S5)} \qquad (70)$$

$$(\Diamond[j_1]p \wedge \Diamond[j_2]\neg p) \to \bot \qquad \text{(from (68) and (70))} \qquad (71)$$

□

Lemma 16. *Let $\mathfrak{S} = \langle Tree, \leq, Choice, V\rangle$ and $\mathfrak{S}' = \langle Tree', \leq', Choice', V'\rangle$ be an (Ag, V)-stit model and an (Ag', V)-stit model, respectively, and let $m \in Tree$ and $m' \in Tree'$. Let relation $B \subseteq H^{\mathfrak{S}}_m \times H^{\mathfrak{S}'}_{m'}$ be such that the domain of B is $H^{\mathfrak{S}}_m$, the counter-domain of B is $H^{\mathfrak{S}'}_{m'}$, and assume that B satisfies condition* (atoms)*. Then, whenever $A \in \mathcal{L}^{\emptyset}_V$, we will have, for all $h_1 \in H^{\mathfrak{S}}_m$ and $h'_1 \in H^{\mathfrak{S}'}_{m'}$:*

$$h_1\ B\ h'_1 \Rightarrow (\mathfrak{S}, m, h_1 \models A \Leftrightarrow \mathfrak{S}', m', h'_1 \models A).$$

Proof. We reason in the same way as in the proof of Lemma 11, the only difference being that Case 2 in the induction step can be omitted. □

We are now in a position to prove Theorem 4.

Proof of Theorem 4. Consider the following sets and structures:

- $Tr = \{m, m_0, m_1\}$.
- $\trianglelefteq$ is the reflexive closure of the relation $\{(m, m_0), (m, m_1)\}$.

The two histories induced by $(Tr, \trianglelefteq)$ are $h_0 = \{m, m_0\}$ and $h_1 = \{m, m_1\}$. We now define two further sets:

- $U = \{(m, h_0)\}$.
- $F = \{(m, \{\{h_0\}, \{h_1\}\}), (m_0, \{\{h_0\}\}), (m_1, \{\{h_1\}\})\}$.

It is immediate to establish that the structure $\mathfrak{M}_{j,p} = (Tr, \trianglelefteq, F_j, U_p)$, in which if F_j interprets F as the choice function for a given single agent j and U_p interprets U as the evaluation for a given single propositional variable p, is a $(\{j\}, \{p\})$-stit structure.

We now consider two stit models, $\mathfrak{M}_{1,p}$ and $\mathfrak{M}_{2,p}$, and we set B as the diagonal of $Hist(Tr, \trianglelefteq)$, in other words, we set $B := \{(h_0, h_0), (h_1, h_1)\}$. It is clear that B satisfies the conditions of Lemma 16 so that for every $C \in \mathcal{L}^{\emptyset}_{\{p\}}$ which contains no action modalities, we will have:

$$\mathfrak{M}_{1,p}, m, h_0 \models C \Leftrightarrow \mathfrak{M}_{2,p}, m, h_0 \models C. \tag{72}$$

Now assume that $(SRCIP)_n$ holds for any n greater than one. We will show that this assumption leads to a contradiction. Indeed, it follows then from Lemma 15 that there must be a formula $C \in \mathcal{L}^{\emptyset}_{\{p\}}$ such that the following holds:

$$\vdash \Diamond[1]p \to C \tag{73}$$

$$\vdash C \to \neg\Diamond[2]\neg p \tag{74}$$

Choose any such C. We obviously have $\mathfrak{M}_{1,p}, m, h_0 \models \Diamond[1]p$ so that it follows from (73) and the soundness of $\mathbb{S}$ that $\mathfrak{M}_{1,p}, m, h_0 \models C$, whence, by (72), also $\mathfrak{M}_{2,p}, m, h_0 \models C$. From the latter, together with (74), it follows that we should have $\mathfrak{M}_{2,p}, m, h_0 \models \neg\Diamond[2]\neg p$, whereas the direct check shows that we in fact have $\mathfrak{M}_{2,p}, m, h_0 \models \Diamond[2]\neg p$. Thus we have got our contradiction in place. □

The Strong Restricted Craig Interpolation Property admits of the following unrestricted companion:

Definition 8. *Stit logic has the* Strong n-Craig Interpolation Property *(abbreviated by $(SCIP)_n$) iff for any set of propositional variables V, and all $A, B \in \mathcal{L}^{\{1,\ldots,n\}}_V$, there exists a $C \in \mathcal{L}^{Ag(A)\cap Ag(B)}_{|A|\cap|B|}$ such that both $\vdash A \to C$ and $\vdash C \to B$.*

Of course, for a given positive integer n, $(SCIP)_n$ is at least as strong as $(SRCIP)_n$, whence we get the following corollary to Theorem 4:

Theorem 5. *For every $n > 1$, stit logic does not have $(SCIP)_n$.*

6 Conclusion

In the preceding text, we have looked into the question of whether stit logic has the Restricted n-Craig Interpolation Property, showing that the answer is in the affirmative iff $n \leq 3$. We have also briefly looked into some related properties, showing that the Restricted Craig Interpolation for stit logic has its natural accompanying version of the Robinson Consistency Property which turns out to be equivalent to the Restricted Craig Interpolation for every positive integer n. From these results, we have drawn the corollary that the unrestricted n-Craig Interpolation fails for stit logic under every instantiation of $n > 3$, that is to say, for almost all positive integers n. We have also shown that if one treats action modalities as non-logical symbols, the scope of interpolation failures extends to include the case when $n \in \{2, 3\}$, and this extension occurs for the strengthened versions of both unrestricted and restricted n-Craig Interpolation Property.

The import of this almost universal failure of Craig Interpolation for stit logic can be seen sharper if one takes into an account that the axiomatic system $\mathbb{S}$ for this logic, as given in Section 2 above, suggests that stit logic is an extension of propositional multi-S5. It is a well-known fact, see e.g. [12], that multi-S5 has the Craig Interpolation Property.[5] Thus the fact that this property fails for stit logic highlights the fact that the difference between multi-S5 and stit logic is quite substantial. Another conclusion is that, in extending multi-S5, stit logic upsets the delicate balance between deductive power and expressivity which is present in multi-S5.

As the main problem for the future research remains the question whether unrestricted n-Craig Interpolation Property holds for all or at least some $n \leq 3$ and whether the natural Robinson Consistency companions of the n-Craig Interpolation Property can be distinguished from this property on this, rather limited, set of values.

7 Acknowledgements

To be inserted.

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