ON THE EXISTENCE OF A GLOBAL DIFFEOMORPHISM BETWEEN FRÉCHET SPACES

KAVEH EFTEKHARINASAB

Abstract. We provide sufficient conditions for the existence of a global diffeomorphism between tame Fréchet spaces. We prove a version of Mountain Pass theorem which plays a key ingredient in the proof of the main theorem.

1. Introduction

In this paper we consider the problem of finding sufficient conditions under which a tame map between tame Fréchet spaces becomes a global diffeomorphism. Tame maps are important because they appear not only as differential equations but also as their solutions (see [4] for examples). Although, the theory of differential equations in Fréchet spaces has a significant relation with problems in both linear and nonlinear functional analysis but not many methods for solving different type of differential equations are known. Our result would provide an approach to solve an initial value nonlinear integro-differential equation

$$x'(t) + \int_0^t \phi(t, s, x(s))ds = y(t), t \in [0, 1].$$

We follow the ideas in [5] and [6] where the analogue problem for Banach and Hilbert spaces was studied. There are two approaches to calculus on Fréchet spaces. The Gâteaux-approach (see [7]) and the so called convenient analysis (see [9]). We will apply the first one because to define the Palais-Smale condition, which plays a significant role in the calculus of variation, we need an appropriate topology on dual spaces that compatible with our notion of differentiability; only in the first approach there exists such a topology.

In [3], the author defined the Palais-Smale condition for C^k-maps between Fréchet spaces and obtained some existence results for locating critical points. In this paper by means of this condition we generalize the mountain pass theorem of Ambrosetti and Rabinowitz to Fréchet spaces. Our proof of the mountain pass theorem relies on the Ekeland's variational principle. Since, in general, we can not acquire deformation results for Fréchet spaces because of the lack of a general solvability theory for differential equations. It is worth mentioning

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that for every Fréchet space the projective limit techniques gives a way to solve a wide class of differential equations (see [1]). This technique would be a way of obtaining many results (such as deformation lemmas) for Fréchet spaces.

Roughly speaking, the main theorem states that if φ is a smooth tame map that satisfies the assumptions of the Nash-Moser inverse function theorem and if for an appropriate auxiliary functional ι, a functional $e \mapsto \iota(\varphi(e) - f)$ satisfies the Palais-Smale condition then φ is a global diffeomorphism.

2. Mountain Pass Theorem

We denote by F a Fréchet space whose topology is defined by a sequence $(\| \cdot \|_F^n)_{n \in \mathbb{N}}$ of seminorms, which we can always assume to be increasing (by considering $\max_{k \leqq n} \| \cdot \|_F^k$, if necessary). Moreover, the complete translation-invariant metric

$$d_F(x, y) := \sum_{i \in \mathbb{N}} \frac{\| x - y \|_F^n}{1 + \| x - y \|_F^n} \tag{2.1}$$

induces the same topology on F.

We recall that a family $\mathcal{B}$ of subsets of F that covers F is called a bornology on F if

- $\forall A, B \in \mathcal{B}$ there exists $C \in \mathcal{B}$ such that $A \cup B \subset C$,
- $\forall B \in \mathcal{B}$ and $\forall r \in \mathbb{R}$ there is a $C \in \mathcal{B}$ such that $r \cdot B \subset C$.

We use the Keller's definition of C_c^k-maps which is equivalent to the notion of C^k-maps in the sense of Michal and Bastiani.

Let E, F be Fréchet spaces, U an open subset of E, and $\varphi : U \to F$ a continuous map. If the directional (Gâteaux) derivatives

$$\mathrm{d}\,\varphi(x)h = \lim_{t \to 0} \frac{\phi(x + th) - \phi(x)}{t}$$

exist for all $x \in U$ and all $h \in E$, and the induced map $\mathrm{d}\,\varphi : U \times E \to F$, $(u, h) \mapsto \mathrm{d}\,\varphi(u)h$ is continuous in the product topology, then we say that φ is a C^1-map in the sense of Michal and Bastiani. Higher directional derivatives and C^k-maps, $k \geqq 2$, are defined in the obvious inductive fashion.

Let E be a Fréchet space, $\mathcal{B}$ a bornology on E and $L_{\mathcal{B}}(E, F)$ the space of all linear continuous maps from E to F. The $\mathcal{B}$-topology on $L_{\mathcal{B}}(E, F)$ is a Hausdorff locally convex topology defined by all seminorms obtained as follows:

$$\| L \|_B^n := \sup\{\| L(e) \|_F^n : e \in B\}, \tag{2.2}$$

where $B \in \mathcal{B}$. One similarly may define the space $L_{\mathcal{B}}^k(E, F)$ of k-linear jointly continuous maps from E^k to F. If $\mathcal{B}$ is generated by all compact sets, in the sense that every $B \in \mathcal{B}$ is contained in some compact set, then the $\mathcal{B}$-topology on the space $L_{\mathcal{B}}(E, \mathbb{R}) = E'_{\mathcal{B}}$ of all continuous linear functional on E, the dual of E, is the topology of compact convergence. If $\mathcal{B}$ contains all compact sets of E, then $L^k_{\mathcal{B}}(E, L^l_{\mathcal{B}}(E,F))$ is canonically isomorphic to $L^{l+k}_{\mathcal{B}}(E,F)$ as a topological vector space, see [7, Theorem 0.1.3]. In particular, $L^2_{\mathcal{B}}(E,R) = L^2_{\mathcal{B}}(E) \cong L_{\mathcal{B}}(E, E'_{\mathcal{B}})$. Under the above condition on $\mathcal{B}$, we define the differentiability of class C^k_c : Let $U \subset E$ be open, a map $\varphi : U \to F$ is called C^1_c if its directional derivatives exist and the induced map $\mathrm{d}\,\varphi : U \to L_{\mathcal{B}}(E,F)$ is continuous. Similarly we can define maps of class C^k_c, $k \in \mathbb{N} \cup \{\infty\}$, see [7, Definition 2.5.0]. A map $\varphi : U \to F$ is C^k_c, $k \geqq 1$, if and only if φ is C^k in the sense of Michal and Bastiani, see [7, Theorem 2.7.0 and Corollary 1.0.4 (2)]. In particular, φ is C^∞_c if and only in φ is C^∞.

If $\phi : E \to \mathbb{R}$ at x is C^1 and hence C^1_c, the derivative of ϕ at x, $\phi'(x)$, is an element of $E'_{\mathcal{B}}$, and the directional derivative of ϕ at x toward $h \in E$ is given by

$$\mathrm{d}\,\varphi(x)h = \langle \phi'(x), h \rangle,$$

where $\langle \cdot, \cdot \rangle$ is duality pairing.

Because of the equivalency of the notions of differentiability we shall omit the index c in denoting differentiable maps of order k. We always assume that a bornology $\mathcal{B}$ on a Fréchet space contains all its compact sets.

Definition 2.1. *Let F be a Fréchet space, $\mathcal{B}$ a bornology on F and $F'_{\mathcal{B}}$ the dual of F equipped with the $\mathcal{B}$-topology. Let $\phi : F \to \mathbb{R}$ be a C^1-functional.*

(i) *We say that ϕ satisfies the Palais-Smale condition,* PS*-condition in short, if each sequence $(x_i) \subset F$ such that $\phi(x_i)$ is bounded and*

$$\phi'(x_i) \to 0 \quad \text{in} \quad F'_{\mathcal{B}},$$

has a convergent subsequence.

(ii) *We say that ϕ satisfies the Palais-Smale condition at level $c \in \mathbb{R}$,* $(\mathrm{PS})_c$*-condition in short, if each sequence $(x_i) \subset F$ such that*

$$\phi(x_i) \to c \quad \text{and} \quad \phi'(x_i) \to 0 \quad \text{in} \quad F'_{\mathcal{B}},$$

has a convergent subsequence.

Let ϕ be a C^k-functional ($k \geqq 1$) on a Fréchet space F. As usual, a point p in the domain of ϕ is said to be a critical point of if $\phi'(p) = 0$, the corresponding value $c = \phi(p)$ will be called a critical value.

The next result is essential when we want to prove the existence of a critical point.

Theorem 2.1. [7, Corrolly 4.9] *Let F be a Fréchet space and let $\phi : F \to \mathbb{R}$ be C^1-functional bounded from below. If $(\mathrm{PS})_c$-condition holds with $c = \inf_F \phi$, then there is $x \in F$ such that $\phi(x) = c$ and $\phi'(x) = 0$.*

Consider the following weak form of the Ekeland's variational principle (cf. [2]).

Theorem 2.2. *Let (X, σ) be a complete metric space. Let a functional $\Psi : X \to (-\infty, \infty]$ be semi-continuous, bounded from below and not identical to $+\infty$. Then, for any $\epsilon > 0$ there exists $x \in X$ such that*

(1) $\Psi(x) \leqq \inf_X \Psi + \epsilon$,
(2) $\Psi(x) \leqq \Psi(y) + \epsilon\sigma(x, y), \quad \forall y \neq x \in X.$

Let $(F, \| \cdot \|_F^n)$ be a Fréchet space and let $\phi \in C^1(F, \mathbb{R})$ be a functional. Let

$$\Gamma_f := \Big\{ \gamma \in C([0,1]; F) : \gamma(0) = 0, \gamma(1) = f \in F \Big\}$$

be the set of continuous paths joining 0 and f. Consider the Fréchet space $C([0,1]; F)$ with the family of seminorms

$$\| \gamma \|_C^n = \sup_{t \in [0,1]} \| \gamma(t) \|_F^n . \tag{2.3}$$

The metric

$$d_C(\gamma, \eta) := \sum_{i \in \mathbb{N}} \frac{\| \gamma - \eta \|_C^n}{1 + \| \gamma - \eta \|_C^n} \tag{2.4}$$

is complete translation-invariant and induces the same topology on $C([0,1]; F)$. We can easily show that Γ_f is closed in $C([0,1]; F)$ and so it is a complete metric space with the metric, d_{Γ_f}, which is the restriction of d_C to Γ_f.

The proof of the following mountain pass theorem is the refinement of the proof for the Banach spaces case (see [10, Theorem 4.10]). The idea of the proof is straightforward: for a given $\phi \in C^1(F, \mathbb{R})$ that satisfies the PS-condition and a point $f \in F$ if a particular condition hold (the condition (2.5)), we define a functional Ψ on Γ_f so that it satisfies the assumptions of the Ekeland's variational principle (Theorem 2.2). Then this theorem yields that Ψ has almost minimizers points satisfying some certain conditions. We use a sequence of these points on Γ_f and associate this sequence of almost minimizers with a sequence on F, which satisfies the requirement of the PS-condition for ϕ. The limit of a subsequence of this sequence on F is a critical point of ϕ. The difficult step is to find a connection between the sequence of almost minimizers of Ψ and a sequence on F, which satisfies the requirements of the PS-condition.

Theorem 2.3 (The Mountain Pass Theorem). *Let F be a Fréchet space and $\phi : F \to \mathbb{R}$ a C^1- functional satisfying the Palais-Smale condition. Assume $f \in F$ and ϕ satisfies the condition*

$$\inf_{p \in \partial U} \phi(p) > \max\{\phi(0), \phi(f)\} = a, \tag{2.5}$$

where ∂U is the boundary of an open neighborhood U of 0 such that f does not belong to the closure $\overline{U}$, $f \notin \overline{U}$. Then ϕ has a critical value $c > a$ which can be characterized as

$$c \coloneqq \inf_{\gamma \in \Gamma_f} \max_{t \in [0,1]} \phi(\gamma(t)).$$

Proof. Let

$$\Gamma_f \coloneqq \Big\{ \gamma \in C([0,1]; F) : \gamma(0) = 0, \gamma(1) = f \in F \Big\}.$$

Suppose the metric d_{Γ_f} which is the restriction of the metric d_C (2.4) to Γ_f, defines the topology of Γ_f. With this metric Γ_f is a complete metric space.

Define the functional $\Psi : \Gamma_f \to \mathbb{R}$ by

$$\Psi(\gamma) = \max_{t \in [0,1]} \phi(\gamma(t)).$$

Since Ψ is the least upper bound of a family of lower semi-continuous functions it follows that it is lower semi-continuous too. Since U separates 0 and f for all $\gamma \in \Gamma_f$, we have

$$\gamma([0,1]) \bigcap \partial U \neq \varnothing. \tag{2.6}$$

Therefore, by (2.6)

$$c \geqq \inf_{\partial U} \phi = c_1, \tag{2.7}$$

and then it follows from (2.5) that

$$c \geqq c_1 > a.$$

Thus, Ψ is bounded from below.

Let $\widehat{\gamma} \in \Gamma_f$, we show that Ψ is continuous at $\widehat{\gamma}$. Given $\varepsilon > 0$, choose $\varrho > 0$ (here we use the continuity of ϕ) such that $\forall y \in \widehat{\gamma}([0,1])$ and $\forall x \in F$ such that $d_F(x,y) < \varrho$ we have $\mid \phi(x) - \phi(y) \mid < \varepsilon$, where $\mid \cdot \mid$ is the usual absolute modulus. Now for each $\overline{\gamma} \in \Gamma_f$ such that $d_{\Gamma_f}(\widehat{\gamma}, \overline{\gamma}) < \varrho$, we have

$$\Psi(\overline{\gamma}) - \Psi(\widehat{\gamma}) = \phi(\overline{\gamma}(t_m)) - \max_{t \in [0,1]} \phi(\widehat{\gamma}(t)) \leqq \phi(\overline{\gamma}(t_m)) - \phi(\widehat{\gamma}(t_m)),$$

where $t_m \in [0,1]$ is the point where the maximum of $\phi(\overline{\gamma}(t))$ is attained. Since

$$d_F(\widehat{\gamma}(t_m), \overline{\gamma}(t_m)) \leqq d_{\Gamma_f}(\widehat{\gamma}, \overline{\gamma}) < \varrho,$$

it follows that $\Psi(\overline{\gamma}) - \Psi(\widehat{\gamma}) < \varepsilon$. Reverting the roles of $\widehat{\gamma}$ and $\overline{\gamma}$ yields that

$$\mid \Psi(\widehat{\gamma}) - \Psi(\overline{\gamma}) \mid < \varepsilon.$$

Thus Ψ satisfies all conditions of Theorem 2.2, and hence, for every $\epsilon > 0$ there exists $\gamma_\epsilon \in \Gamma_f$ such that

$$\Psi(\gamma_\epsilon) \leqq c + \epsilon, \tag{2.8}$$

$$\Psi(\gamma_\epsilon) \leqq \Psi(\gamma) + \epsilon d_{\Gamma_f}(\gamma, \gamma_\epsilon),\ \forall \gamma \neq \gamma_\epsilon \in \Gamma_f. \tag{2.9}$$

Without loss of generality we may assume

$$0 < \epsilon < c - a. \tag{2.10}$$

Now we show that there is $s \in [0, 1]$ such that for all seminorms we have

$$\| \phi'(\gamma_\epsilon(s)) \|_B^n \leqq \epsilon, \tag{2.11}$$

We prove the inequality (2.11) by contradiction. Notice that

$$\| \phi'(\gamma_\epsilon(s)) \|_B^n = \sup_{g^n \in B} \langle \phi'(\gamma_\epsilon(s)), g^n \rangle.$$

Define the set

$$S(\epsilon) \coloneqq \{s \in [0, 1] : c - \epsilon \leqq \phi(\gamma_\epsilon(s))\}. \tag{2.12}$$

It follows from (2.10) that $a < c - \epsilon$. Since $\gamma_\epsilon(0) = 0$ and $\phi(0) \leqq a$, we obtain that $0 \notin S(\epsilon)$. Furthermore, the functional ϕ is continuous on F and the set $S(\epsilon)$ is closed so $S(\epsilon)$ is compact. Suppose for all $s \in [0, 1]$ the inequality (2.11) does not hold. Then, for all $s \in S(\epsilon)$

$$\| \phi'(\gamma_\epsilon(s)) \|_B > \epsilon, \quad \forall B \in \mathcal{B}.$$

Thus, for each $s \in S(\epsilon)$ there exist points $g_s^n \in F$ such that

$$\langle \phi'(\gamma_\epsilon(s), g_s^n \rangle < -\epsilon. \tag{2.13}$$

Since ϕ' is continuous on F, it follows from (2.13) that for each $s \in S(\epsilon)$ there exist $\alpha_s > 0$ and an open interval $B_s \subset [0, 1]$ such that

$$\langle \phi'(\gamma_\epsilon(t) + h), g_s^n \rangle < -\epsilon, \tag{2.14}$$

for all $t \in B_s$ and all $h \in F$ with $\| h \|_F^n < \alpha_s (\forall n \in \mathbb{N})$.

The family $\{B_s\}_{s \in S(\epsilon)}$ covers the compact set $S(\epsilon)$ so there exists a finite subcovering $B_{s_1}, \cdots, B_{s_k}$ of $S(\epsilon)$. Since $0 \notin S(\epsilon)$, we may assume $0 \notin B_{s_i}$. Thus, $[0, 1] \backslash B_{s_i}$ is closed and not empty for all $i = 1, \cdots k$. Therefore, if $t \in \bigcup_{i=1}^k B_{s_i}$, then

$$\sum_{i=1}^k \operatorname{dist}(t, [0, 1] \backslash B_{s_i}) > 0.$$

Now define the function $\chi_j(t):[0,1]\to[0,1]$ by

$$\chi_j(t)=\begin{cases}\sum_{i=1}^k \dfrac{\operatorname{dist}(t,[0,1]\backslash B_{s_j})}{\sum_{i=1}^k \operatorname{dist}(t,[0,1]\backslash B_{s_i})} & t\in\bigcup_{i=1}^k B_{s_i},\\ 0 & \textit{otherwise.}\end{cases}$$

It is easily seen that χ_j is continuous and

$$\sum_{j=1}^k \chi_j(t)\leqq 1 \quad \forall t\in[0,1] \tag{2.15}$$

and $\chi_j(t)=0$ if $t\notin B_j$.

Fix a continuous function $\chi:[0,1]\to[0,1]$ such that

$$\chi(t)=\begin{cases}1 & c\leqq\phi(\gamma_\epsilon(t)),\\ 0 & \phi(\gamma_\epsilon(t))\leqq c-\epsilon.\end{cases}$$

Let $\alpha=\min\{\alpha_{s_1},\cdots,\alpha_{s_k}\}$. For an arbitrary fixed $n\in\mathbb{N}$, define the continuous function $\mu_n:[0,1]\to F$ by

$$\mu_n(t)=\gamma_\epsilon(t)+\alpha\chi(t)\sum_{j=1}^k\chi_j(t)g_{s_j}^n.$$

Now we show that $\mu_n\in\Gamma_f$. By (2.10), for $t\in\{0,1\}$ we have

$$\phi(\gamma_\epsilon(t))\leqq a<c-\epsilon,$$

therefore $\chi(t)=0$ and hence $\mu_n(t)=\gamma_\epsilon(t)$. From (2.14) and the mean value theorem (cf. [8]) it follows that for each $t\in S(\epsilon)$ there is $\theta_n\in(0,1)$ such that

$$\begin{aligned}\phi(\mu_n(t))-\phi(\gamma_\epsilon(t))&=\langle\phi'\Big(\gamma_\epsilon(t)+\theta_n\alpha\chi(t)\sum_{j=1}^k\chi_j(t)g_{s_j}^n\Big),\alpha\chi(t)\sum_{j=1}^k\chi_j(t)g_{s_j}^n\rangle\\ &=\alpha\chi(t)\sum_{j=1}^k\chi_j(t)\langle\phi'\Big(\gamma_\epsilon(t)+\theta_n\alpha\chi(t)\sum_{j=1}^k\chi_j(t)g_{s_j}^n\Big),g_{s_j}^n\rangle\\ &\leqq-\epsilon\alpha\chi(t).\end{aligned} \tag{2.16}$$

The inequality (2.16) follows from (2.15) and (2.14).

Let t_1 be such that $\phi(\mu_n(t_1))=\Psi(\mu_n)$ therefore

$$\phi(\gamma_\epsilon(t_1)\geqq\phi(\mu_n(t_1))\geqq c.$$

Whence $\chi(t_1)=1$ and $t_1\in S(\epsilon)$ because if $t_1\notin S(\epsilon)$ then $\chi(t_1)=0$.

From (2.16) it follows that $\phi(\mu_n(t_1))-\phi(\gamma_\epsilon(t_1))\leqq-\epsilon\alpha$ and so

$$\Psi(\mu_n)+\epsilon\alpha\leqq\phi(\gamma_\epsilon(t_1))\leqq\Psi(\gamma_\epsilon)$$

and $\mu_n \neq \gamma_\epsilon$. But, by the definition of μ_n we have $d_{\Gamma_f}(\mu_n, \gamma_\epsilon) < \alpha$, therefore,

$$\Psi(\mu_n) + \epsilon d_{\Gamma_f}(\mu, \gamma_\epsilon) < \Psi(\gamma_\epsilon)$$

which contradicts (2.9) and complete the proof of (2.11). Therefore, for every $\epsilon > 0$ there exists $t_\varepsilon \in S(\epsilon)$ such that all seminorms satisfy

$$\| \phi'(\gamma_\epsilon(t_\epsilon)) \|_B^n \leqq \epsilon. \tag{2.17}$$

and

$$c - \epsilon \leqq \phi(\gamma_\epsilon(t_\epsilon)).$$

Therefore, by (2.8) we have

$$c - \epsilon \leqq \phi(\gamma_\epsilon(t_\epsilon)) \leqq \Psi(\gamma_\epsilon) \leqq c + \epsilon. \tag{2.18}$$

Now it suffices to consider the sequence $f_n = \gamma_{1/n}(t_{1/n})$ and use the $(PS)_c$-condition. By (2.18) we have $\phi(f_n) \to c$, and by (2.17) we have $\phi'(f_n) \to 0$, therefore, f_n has a convergent subsequence, denoted again by f_n. If the limit of f_n is h, in view of Theorem 2.1 it follows that h is a critical point and $\phi(h) = c$. □

Remark 2.1. *Note that we could have used the weaker $(PS)_c$-condition instead of PS-condition, with c being the c in the proof. But in application c is not known explicitly. This constrains us to verify $(PS)c$-condition for all possible values c.*

3. The existence of a global diffeomorphism

In this section we prove a global diffeomorphism theorem in the category of tame Fréchet spaces. With respect to the metric (2.1), d_F, we define an open ball $B_r(x)$ centered at x with radius r. Its closure and boundary is denoted by $\overline{B_r(x)}$ and $\partial B_r(x)$, respectively.

Theorem 3.1. *Let E and F be tame Fréchet spaces and $\tau : E \to F$ a smooth tame map. Let $\iota : F \to [0, \infty]$ be a C^1-functional such that $\iota(x) = 0$ if and only if $x = 0$ and $\iota'(x) = 0$ if and only if $x = 0$. If the following conditions hold*

- **C1:** *the derivative $\tau'(e)p = k$ has a unique solution $p = \nu(e)k$ for all $e \in E$ and all k, and the family of inverses $\nu : E \times F \to E$ is a smooth tame map;*
- **C2:** *for any $f \in F$ the functional ϕ_f defined on E by*

$$\phi_f(e) = \iota(\tau(e) - f)$$

satisfies the Palais-Smale condition at all levels.

Then τ is a global diffeomorphism.

Proof. The map τ satisfies the assumptions of the Nash-Moser inverse function theorem, the condition C1, which implies that τ is a local diffeomorphism. Thus, it suffices to show that τ is surjective and bijective.

To prove that τ is surjective suppose that $f \in F$ is any arbitrary point. Then, by the Palais-Smale condition the functional ϕ_f has a critical point $p \in E$, and the assumptions on ι yields that $\tau(p) = f$. The functional ϕ_f is bounded from below and is of class C^1 as it is the composition of two C^1 maps. Since ϕ_f satisfies the Palais-Smale condition it follows by Theorem 2.1 that it attains its critical point at some $p \in E$ so $\phi'_f(p) = 0$. From the chain rule [7, Corollary 1.3.2] it follows that

$$\phi'_f(p) = \iota'(\tau(p) - f) \circ \tau'(p) = 0. \tag{3.1}$$

By the assumption the map τ' is invertible and hence (3.1) implies that $\iota'(\tau(p) - f) = 0$ therefore $\tau(p) = f$. Thus, τ is surjective.

To prove that τ is injective we argue by contradiction, assume $e_1 \neq e_2 \in E$ and $\tau(e_1) = \tau(e_2) = l$. Then we will construct the functional ϕ_l on E that satisfies the assumptions of Theorem 2.3 and hence it has a critical point h which its existence violates the assumptions on ι.

Since τ is a local diffeomorphism it is an open map. Therefore, there exists $\alpha_r > 0$ such that

$$B_{\alpha_r}(l) \subset \tau(B_r(e_1)), \tag{3.2}$$

for $r > 0$. Let $\rho > 0$ be small enough such that

$$e_2 \notin \overline{B_\rho(e_1)}. \tag{3.3}$$

Consider the functional $\phi_l(e) = \iota(\tau(e) - l)$, therefore $\phi_l(e_1) = \phi_l(e_2) = 0$.

Without the loose of generality we can suppose $e_1 = 0$. For $e \in \partial B_\rho(0)$, in virtue of (3.2) we have $\tau(e) \notin B_{\alpha_\rho}(l)$ and so $\tau(e) \neq l$. Therefore, the assumption on ι yields

$$\phi_l(e) > 0 = \max\{\phi_l(0), \phi_l(e_2)\}.$$

Thus, all assumptions of Theorem 2.3 hold for the functional ϕ_l and the points 0 and e_2. Therefore, there exists a critical point $h \in E$ with $\phi_l(h) = c$ for some $c > 0$. But

$$c = \phi_l(h) = \iota(\tau(h) - l) > 0,$$

therefore, the assumption on ι implies

$$\tau(h) \neq l. \tag{3.4}$$

By the chain rule we have $\phi_l'(h) = \iota'(\tau(h) - l) \circ \tau'(h) = 0$. Thus, since τ' is invertible it follows that $\iota(\tau(h) - l) = 0$ so $\tau(h) = l$ which contradicts (3.4). □

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Topology lab., Institute of Mathematics of NAS of Ukraine, Tereshchenkivska st. 3, Kyiv, 01601 Ukraine

Email address: kaveh@imath.kiev.ua