

OBSTRUCTIONS TO DEFORMING CURVES ON A PRIME FANO 3-FOLD

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ABSTRACT. We prove that for every smooth prime Fano 3-fold V , the Hilbert scheme $\text{Hilb}^{sc} V$ of smooth connected curves on V contains a generically non-reduced irreducible component of Mumford type. We also study the deformations of degenerate curves C in V , i.e., curves C contained in a smooth anticanonical member $S \in |-K_V|$ of V . We give a sufficient condition for C to be stably degenerate, i.e., every small (and global) deformation of C in V is contained in a deformation of S in V . As a result, by using the Hilbert-flag scheme of V , we determine the dimension and the smoothness of $\text{Hilb}^{sc} V$ at the point $[C]$, assuming that the class of C in $\text{Pic } S$ is generated by $\mathbf{h} := -K_V|_S$ together with the class of a line, or a conic on V .

1. INTRODUCTION

We work over an algebraically closed field k of characteristic 0. Let X be a smooth Fano 3-fold of index r with Picard group $\text{Pic } X \simeq \mathbb{Z}$. Then by [9, 10], all such X are classified into 17 classes up to deformation equivalence and we have $1 \leq r \leq 4$. Let $\text{Hilb}^{sc} X$ denote the Hilbert scheme of smooth connected curves in X . Mumford [20] first proved that if $r = 4$ (i.e. $X \simeq \mathbb{P}^3$), then $\text{Hilb}^{sc} X$ contains a generically non-reduced (irreducible) component. This example was generalized in [5, 12, 3], etc., for $X = \mathbb{P}^3$, and in [18, 21] for many uniruled 3-folds X . It is known that if $r = 3$ (i.e. $X \simeq Q^3 \subset \mathbb{P}^4$) or $r = 2$ (i.e. X is a *del Pezzo* 3-fold), then $\text{Hilb}^{sc} X$ contains (infinitely many) generically non-reduced components.

In this paper, we discuss the existence of a generically non-reduced component of $\text{Hilb}^{sc} X$ for every X with $r = 1$, i.e., a *prime* Fano 3-fold X , in the view point of a further generalization of Mumford's example. Let V be a prime Fano 3-fold of genus g ($:= (-K_V)^3/2 + 1$), and let $\text{Hilb}_{d,p}^{sc} V$ denote the subscheme of $\text{Hilb}^{sc} V$ parametrising curves of degree d and genus p .

Theorem 1.1. *The Hilbert scheme $\text{Hilb}_{4g,4g+1}^{sc} V$ contains a generically non-reduced irreducible component (of Mumford type) of dimension $5g + 1$, whose general member C satisfies:*

- (1) C is contained in a smooth anticanonical member $S \in |-K_V|$,

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- (2) C belongs to the class $-2K_V|_S + 2E$ in $\text{Pic } S$ for a good conic $E \simeq \mathbb{P}^1$ on V , i.e., a conic with trivial normal bundle $N_{E/V} \simeq \mathcal{O}_{\mathbb{P}^1}^2$, and
- (3) $h^0(C, N_{C/V}) = 5g + 2$.

This is a generalization of a result in [22] (for $g = 3$). We will sketch its proof. As is well known, V contains good conics E , which are parametrised by an open subset Γ' of the Fano surface $\Gamma := \text{Hilb}_{2,0} V$ of V (cf. Lemma 2.1). The pairs (E, S) of E and a smooth member $S \in |-K_V|$ containing E are parametrised by an open subset U of a \mathbb{P}^{g-2} -bundle over Γ' . (Thus $\dim U = g$.) We consider the maximal family W of curves C contained in a smooth $S \in |-K_V|$, and belonging to the linear equivalence class in (2) for some E . Then W is isomorphic to an open subset of a \mathbb{P}^{4g+1} -bundle over U , and thus $\dim W = 5g + 1$. (See §3.1 for the construction of W .) On the other hand, we compute that $h^0(C, N_{C/V}) = 5g + 2$, using the fact that the Hilbert-flag scheme $\text{HF}^{sc} V$ is nonsingular at (C, S) of expected dimension $5g + 1$ (cf. Lemma 2.12). By using a result in [22] (cf. Lemma 2.14), we show that every C is obstructed in V . Then as in Mumford's example, there exists an inequality $\dim W \leq \dim_{[C]} \text{Hilb}^{sc} V < h^0(N_{C/V}) = \dim W + 1$, and this inequality immediately implies that the closure \overline{W} of W in $\text{Hilb}^{sc} V$ is an irreducible component of $(\text{Hilb}^{sc} V)_{\text{red}}$, and $\text{Hilb}^{sc} V$ is generically non-reduced along \overline{W} . In Table 1, for every integer $1 \leq r \leq 4$ and every smooth Fano 3-fold X of index r with $\text{Pic } X \simeq \mathbb{Z}$, we list a series of generically non-reduced components $W \subset \text{Hilb}^{sc} X$ "of Mumford type":

TABLE 1. Generically non-reduced components of Mumford type

r	class of S	class of C	E	$\dim W$	
4	$-\frac{3}{4}K_X$	$-K_X _S + 2E$	line	56	Mumford [20] (cf. Ex. 3.4)
3	$-\frac{2}{3}K_X$			42	[18] (cf. Ex. 3.5)
2	$-\frac{1}{2}K_X$			$4n + 4^a$	[18, 21]
1	$-K_X$	$-2K_X _S + 2E$	conic	$5g + 1$	[22] ($g = 3$)

^a Here n denotes the degree of a del Pezzo 3-fold X (i.e. $n = (-K_X)^3/8$).

As the second topic of this paper, we study the deformations of degenerate curves C in V , i.e., curves contained in some member $S \in |-K_V|$. We are interested in (i) the stability of the degeneration of C , and also (ii) the (un)obstructedness of C in V . Here we say C is *stably degenerate* if every small and global deformation C' of C in V is contained in some deformation S' of S in V (cf. Definition 2.6).

Theorem 1.2. *Let C be a smooth connected curve of genus $g(C)$ on V contained in a smooth anticanonical member $S \in |-K_V|$ of V , and let \mathbf{h} denote the class of $-K_V|_S$ in $\text{Pic } S$. Suppose that we have either*

- [i] $C \sim n\mathbf{h}$ for some integer n ,

- [ii] S contains a good line E (cf. Example 2.9) on V and $C \sim a\mathbf{h} + bE$ for some integers $a, b \geq 0$ with $(a, b) \neq (0, 1)$, or
- [iii] S contains a good conic E on V and $C \sim a\mathbf{h} + bE$ for some integers $a, b \geq 0$ with $(a, b) \neq (0, 1)$.

Then

- (1) C is stably degenerate in V .
- (2) $\dim_{[C]} \text{Hilb}^{sc} V = \begin{cases} g + g(C) + 1 & \text{if [i] with } n \geq 2 \text{ holds, and} \\ g + g(C) & \text{otherwise.} \end{cases}$
- (3) C is obstructed in V if and only if [iii] with $a = b \geq 2$ holds.

The same subject was studied in [21] for degenerate curves on a del Pezzo 3-fold. By [14], if S is general in $|-K_V|$, then every smooth curve C on S is of type [i]. We call a curve C on S of this type a *complete intersection in S* . On the other hand, in the studies of Fano 3-folds V (cf. [9, 10]), lines and conics on V play an important role. Thus, the curves on S of type [ii] or [iii] seem to be of the next natural class to consider their deformations in V , other than the complete intersections in S . In Propositions 4.1, 4.3 and 4.4, we give the dimension of $\text{Hilb}^{sc} V$ at $[C]$ more explicitly, in terms of (a, b) and n .

The organization of this paper is as follows. In §2.1, we recall some basic results on prime Fano 3-folds. In §2.2, we discuss the Hilbert-flag schemes. We consider the image of the first projection $pr_1 : \text{HF}^{sc} V \rightarrow \text{Hilb}^{sc} V$ sending (C, S) to $[C]$, and prove Theorem 2.4, which is a generalization of a result due to Kleppe [12] for $V = \mathbb{P}^3$, although its proof is not new. In §2.3, we apply this theorem to our case: $C \subset S \subset V$, where C is a curve, S is a $K3$ surface, and V is a Fano 3-fold (cf. Lemma 2.13), and prepare the two key lemmas (cf. Lemmas 2.12 and 2.14). We prove Theorems 1.1 in §3 and 1.2 in §4, respectively.

2. PRELIMINARIES

2.1. Prime Fano 3-folds. In this section, we recall some basic facts on prime Fano 3-folds. We refer to Iskovskih [10], or a survey in [24] for the details. A smooth projective 3-fold V is called a *Fano 3-fold* if the anticanonical divisor $-K_V$ of V is ample. The maximal integer r such that $-K_V \sim rH$ for some Cartier divisor H is called the (*Fano*) *index* of V , and we have $1 \leq r \leq 4$ for every V . We consider a *prime* Fano 3-fold V , i.e., a Fano 3-fold V with $r = 1$, and such that $\text{Pic } V$ is generated by $H := -K_V$. Then the genus $g := H^3/2 + 1$ of V can be any integer between 2 and 12, except 11. The linear system $|H|$ on V defines a morphism $\Phi_{|H|} : V \rightarrow \mathbb{P}^{g+1}$, which is an embedding, or a finite morphism of degree 2 onto its image in \mathbb{P}^{g+1} . In the latter case, V is said to be *hyperelliptic*. By virtue of the classification due to Iskovskih, together with Mukai's work ([15, 16]), in which the anticanonical models of V (i.e. $\Phi_{|H|}(V) \subset \mathbb{P}^{g+1}$) were described as linear or quadratic sections of homogeneous spaces Σ ($g = 7, 9, 10$), every prime Fano 3-fold V of genus g is isomorphic to V_{2g-2} ($2 \leq g \leq 12$) or V'_4 in Table 2 (cf. [17]). Every

TABLE 2. Prime Fano 3-folds

g	anticanonical model (or morphism)
2	$V_2 \xrightarrow{2:1} \mathbb{P}^3$: a double cover branched along $(6) \subset \mathbb{P}^3$
3	$V_4 = (4) \subset \mathbb{P}^4$: a quartic hypersurface
	$V'_4 \xrightarrow{2:1} (2) \subset \mathbb{P}^4$: a double cover branched along $(2) \cap (4) \subset \mathbb{P}^4$
4	$V_6 = (2) \cap (3) \subset \mathbb{P}^5$: a complete intersection of a quadric and a cubic
5	$V_8 = (2) \cap (2) \cap (2) \subset \mathbb{P}^6$: a complete intersection of three quadrics
6	$V_{10} = [V_5^a \subset \mathbb{P}^7] \cap (2)$: a quadratic hypersurface section of a del Pezzo 4-fold
7	$V_{12} = [\Sigma_{12}^{10} = SO(10)/U(5) \subset \mathbb{P}^{15}] \cap \mathbb{P}^8$: a linear section of a orthogonal Grassmannian
8	$V_{14} = [G(2, 6) \subset \mathbb{P}^{14}] \cap \mathbb{P}^9$: a linear section of a Grassmannian
9	$V_{16} = [\Sigma_{16}^6 = Sp_6(6)/U(3) \subset \mathbb{P}^{13}] \cap \mathbb{P}^{10}$: a linear section of a symplectic Grassmannian
10	$V_{18} = [\Sigma_{18}^5 \subset \mathbb{P}^{13}] \cap \mathbb{P}^{11}$: a linear section of a G_2 -variety
12	$V_{22}^b \subset \mathbb{P}^{13}$: a Mukai-Umemura 3-fold (cf. [19])

^a V_5 is a del Pezzo 4-fold $V_5 = [G(2, 5) \subset \mathbb{P}^9] \cap \mathbb{P}^7$, or a cone over a quintic del Pezzo 3-fold.

^b V_{22} is isomorphic to the variety $G(3, 7, N) \subset \mathbb{P}^{13}$ associated with a non-degenerate 3-dimensional subspace $N \subset \wedge^2 k^7$.

general member of $| -K_V |$ is a smooth $K3$ surface (cf. [26]), and a smooth projective 3-fold $X \subset \mathbb{P}^{g+1}$ (for $g \geq 3$) is a prime Fano 3-fold (of genus g) if every general linear section $[X \subset \mathbb{P}^{g+1}] \cap \mathbb{P}^{g-1}$ of codimension 2 is a canonical curve (of genus g in \mathbb{P}^{g-1}).

We next recall the geometry of lines and conics on prime Fano 3-folds. By a *line* (resp. a *conic*) on V , we mean a reduced irreducible rational curve E on V with $(E.H)_V = 1$ (resp. $(E.H)_V = 2$). There exists a line (cf. [25]) and a conic on every V , and V is not covered by the family of lines (because its dimension is at most 1), but that of conics. Let E be a conic on V . Then since V does not contain a plane, by e.g., [10, Lemma 4.2 and Proposition 4.3], E is one of the following type:

$$(2.1) \quad (k, -k) : \quad N_{E/V} \simeq \mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(-k), \quad k = 0, 1, 2.$$

In this paper, if $N_{E/V}$ is trivial, E is called a *good conic*, and a *bad conic* otherwise. For every V , the Hilbert scheme $\Gamma (= \text{Hilb}_{2,0}^{sc} V)$ of conics on V is a smooth surface (but possibly reducible, e.g., for $g = 6$), and called the *Fano surface* of conics on V . Every general point of Γ corresponds to a good conic on V (of type $(0, 0)$). (Here we use the assumption that $\text{char } k = 0$.) For the proof, we refer to [1] for $g = 2$, [2] for $g = 3$, and [24, Proposition 4.2.5] for $g \geq 4$ (whose proof works also for $g = 4$). Consequently, we have

Lemma 2.1. *Every prime Fano 3-fold V contains a good conic E , i.e., a conic of type $(0, 0)$. There exists an open dense subset of the Fano surface of V , which parametrises good conics $E \subset V$.*

2.2. Hilbert-flag schemes. In this subsection, we recall some basic results on Hilbert-flag schemes. See [12, 23] for the proofs. Given a projective scheme Z over k , we denote by $\text{HF } Z$ the *Hilbert-flag schemes* (or the *nested Hilbert scheme*) of Z , which parametrises all pairs (X, Y) of closed subschemes X and Y of Z satisfying $X \subset Y$. There are two natural projections $pr_i : \text{HF } Z \rightarrow \text{Hilb } Z$ ($i = 1, 2$) to the Hilbert scheme $\text{Hilb } Z$ of Z , sending (X, Y) to $[X]$ for $i = 1$, and to $[Y]$ for $i = 2$. We denote by $N_{(X,Y)/Z}$ the normal sheaf of (X, Y) in Z (see [23, 4.5.2] for its definition), which is a sheaf on Z with support contained in Y . By definition, as \mathcal{O}_Z -modules, $N_{(X,Y)/Z}$ is isomorphic to a subsheaf of the direct sum $N_{X/Z} \oplus N_{Y/Z}$ of the normal sheaves of X and Y . Moreover, there exists a natural cartesian square

$$(2.2) \quad \begin{array}{ccc} N_{(X,Y)/Z} & \xrightarrow{\pi_2} & N_{Y/Z} \\ \pi_1 \downarrow & \square & \downarrow |_X \\ N_{X/Z} & \xrightarrow{\pi_{X/Y}} & N_{Y/Z}|_X \end{array}$$

of homomorphisms of sheaves on Z , by which $N_{(X,Y)/Z}$ is characterized, where $|_X$ is the restriction of sheaves, $\pi_{X/Y} : N_{X/Z} \rightarrow N_{Y/Z}|_X$ is the natural projection of normal sheaves, and π_i ($i = 1, 2$) are induced by the projections to direct summands.

Suppose now that the two closed embeddings $X \hookrightarrow Y$ and $Y \hookrightarrow Z$ are *regular embeddings*, whose definition can be found in [23, §D.1]. Then by [23, Proposition 4.5.3], $H^0(Z, N_{(X,Y)/Z})$ and $H^1(Z, N_{(X,Y)/Z})$ respectively represent the tangent space and the obstruction space of $\text{HF } Z$ at (X, Y) . Moreover, it follows from a general theory that

$$(2.3) \quad h^0(Z, N_{(X,Y)/Z}) - h^1(Z, N_{(X,Y)/Z}) \leq \dim_{(X,Y)} \text{HF } Z \leq h^0(Z, N_{(X,Y)/Z}),$$

and $\text{HF } Z$ is nonsingular at (X, Y) if and only if $\dim_{(X,Y)} \text{HF } Z = h^0(Z, N_{(X,Y)/Z})$ (cf. [12, Lemma 7]). The induced map $p_i = H^0(Z, \pi_i)$ by π_i on the space of global sections is the tangent map of pr_i for each $i = 1, 2$. Then the diagram (2.2) extends to a commutative diagram

$$(2.4) \quad \begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & \mathcal{I}_{X/Y} \otimes_Y N_{Y/Z} & \xlongequal{\quad} & \mathcal{I}_{X/Y} \otimes_Y N_{Y/Z} & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N_{X/Y} & \longrightarrow & N_{(X,Y)/Z} & \xrightarrow{\pi_2} & N_{Y/Z} & \longrightarrow & 0 \\ & & \parallel & & \pi_1 \downarrow & & \downarrow |_X & & \\ 0 & \longrightarrow & N_{X/Y} & \longrightarrow & N_{X/Z} & \xrightarrow{\pi_{X/Y}} & N_{Y/Z}|_X & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

of exact sequences of sheaves of \mathcal{O}_Z -modules. Taking a long exact sequence, we deduce from the first column of (2.4) the fundamental exact sequence

$$(2.5) \quad \begin{array}{ccccccc} 0 & \longrightarrow & H^0(Y, \mathcal{I}_{X/Y} \otimes_Y N_{Y/Z}) & \longrightarrow & H^0(Z, N_{(X,Y)/Z}) & \xrightarrow{p_1} & H^0(X, N_{X/Z}) \\ & & \longrightarrow & & H^1(Z, N_{(X,Y)/Z}) & \xrightarrow{o_1} & H^1(X, N_{X/Z}) \\ & & \longrightarrow & & H^2(Y, \mathcal{I}_{X/Y} \otimes_Y N_{Y/Z}) & \longrightarrow & \dots \end{array}$$

of cohomology groups, and similarly from the first row of (2.4) the exact sequence

$$(2.6) \quad \begin{array}{ccccccc} 0 & \longrightarrow & H^0(X, N_{X/Y}) & \longrightarrow & H^0(Z, N_{(X,Y)/Z}) & \xrightarrow{p_2} & H^0(Y, N_{Y/Z}) \\ & & \longrightarrow & & H^1(Z, N_{(X,Y)/Z}) & \xrightarrow{o_2} & H^1(Y, N_{Y/Z}) \\ & & \longrightarrow & & H^2(X, N_{X/Y}) & \longrightarrow & \dots \end{array}$$

(cf. [12, (2.7)]). Here o_i represents the maps on the obstruction spaces induced by pr_i for $i = 1, 2$. By [12, Lemma A10], if $H^1(Y, \mathcal{I}_{X/Y} \otimes_Y N_{Y/Z}) = 0$, then the morphism pr_1 is smooth at (X, Y) (see also [11, Theorem 1.3.4]). Similarly one can deduce from (2.6) the fact that if $H^1(X, N_{X/Y}) = 0$, then pr_2 is smooth at (X, Y) (cf. [11, Proposition 1.3.7]).

Lemma 2.2 (cf. [12]). (1) *If $H^1(Y, N_{Y/Z}) = 0$, then*

$$H^1(Z, N_{(X,Y)/Z}) \simeq \text{coker } \alpha_{X/Y},$$

where $\alpha_{X/Y}$ is the composition $\alpha_{X/Y} : H^0(Y, N_{Y/Z}) \xrightarrow{|_X} H^0(X, N_{Y/Z}|_X) \xrightarrow{\partial_{X/Y}} H^1(X, N_{X/Y})$ of the restriction map $|_X$ and the coboundary map $\partial_{X/Y}$ of the exact sequence in the second row of (2.4).

(2) *If $H^1(X, N_{X/Z}) = 0$, then*

$$H^1(Z, N_{(X,Y)/Z}) \simeq \text{coker } \beta_{X/Y},$$

where $\beta_{X/Y}$ is the composition $\beta_{X/Y} : H^0(X, N_{X/Z}) \xrightarrow{\pi_{X/Y}} H^0(Y, N_{Y/Z}|_X) \xrightarrow{\cup \mathbf{k}_X} H^1(Y, \mathcal{I}_{X/Y} \otimes_Y N_{Y/Z})$ of $\pi_{X/Y}$ and the coboundary map $\cup \mathbf{k}_X$ of the exact sequence in the second column of (2.4).

Proof. We show (1). By assumption, $H^1(Z, N_{(X,Y)/Z})$ is isomorphic to the cokernel of the coboundary map $\partial'_{X/Y} : H^0(Y, N_{Y/Z}) \rightarrow H^1(X, N_{X/Y})$ of the exact sequence in the first row of (2.4). By commutativity, we see that $\partial'_{X/Y}$ factors through $\partial_{X/Y}$. Similarly, (2) follows from the diagram. \square

Remark 2.3. The number in the left hand side of the inequality (2.3) is called the *expected dimension* of HF Z at (X, Y) . The Euler characteristic $\chi(Z, N_{(X,Y)/Z})$ of $N_{(X,Y)/Z}$ can be computed by the equation

$$\chi(Z, N_{(X,Y)/Z}) = \chi(X, N_{X/Y}) + \chi(Y, N_{Y/Z}) = \chi(Y, \mathcal{I}_{X/Y} \otimes_Y N_{Y/Z}) + \chi(X, N_{X/Z}),$$

which follows from the additivity on Euler characteristics and (2.4). By definition, the support of $N_{(X,Y)/Z}$ is a closed subset of Y . Hence we have $H^i(Z, N_{(X,Y)/Z}) = 0$ for all integers $i > \dim Y$.

If $H^0(Y, \mathcal{I}_{X/Y} \otimes_Y N_{Y/Z}) = 0$, then pr_1 induces an embedding of a neighborhood $U \subset \text{HF } Z$ of (X, Y) into $\text{Hilb } Z$. The following theorem is useful for estimating the (local) codimension of the image of $\text{HF } Z$ in $\text{Hilb } Z$ at $[X]$, and will be applied to Lemma 2.13.

Theorem 2.4 (cf. [12, 13]). *Let $\mathcal{O}_{\text{Hilb } Z, [X]}$ and $\mathcal{O}_{\text{HF } Z, (X,Y)}$ denote the local rings of $\text{Hilb } Z$ and $\text{HF } Z$ at $[X]$ and (X, Y) , respectively. Let $\mathcal{I}_{X/Y} N_Y$ denote the sheaf $\mathcal{I}_{X/Y} \otimes_Y N_{Y/Z}$ on Y , and suppose that $H^0(Y, \mathcal{I}_{X/Y} N_Y) = 0$. Suppose furthermore that $H^i(Z, N_{(X,Y)/Z}) = 0$ for $i = 1, 2$. Then we have*

(2.7)

$$h^1(Y, \mathcal{I}_{X/Y} N_Y) - h^2(Y, \mathcal{I}_{X/Y} N_Y) \leq \dim \mathcal{O}_{\text{Hilb } Z, [X]} - \dim \mathcal{O}_{\text{HF } Z, (X,Y)} \leq h^1(Y, \mathcal{I}_{X/Y} N_Y).$$

Here the inequality to the right is strict if and only if $\text{Hilb } Z$ is singular at $[X]$. Let $\mathcal{W}_{X,Y}$ denote the unique irreducible component of $\text{HF } Z$ passing through (X, Y) . Then,

- (1) If $h^1(Y, \mathcal{I}_{X/Y} N_Y) = 0$ or $h^2(Y, \mathcal{I}_{X/Y} N_Y) = 0$, then X is unobstructed in Z . Moreover, $\text{Hilb } Z$ is generically smooth along $pr_1(\mathcal{W}_{X,Y})$.
- (2) If $h^1(Y, \mathcal{I}_{X/Y} N_Y) = 0$, then $pr_1(\mathcal{W}_{X,Y})$ is an irreducible component of $(\text{Hilb } Z)_{\text{red}}$.
- (3) If $h^2(Y, \mathcal{I}_{X/Y} N_Y) = 0$, then $pr_1(\mathcal{W}_{X,Y})$ is of codimension $h^1(Y, \mathcal{I}_{X/Y} N_Y)$ in $\text{Hilb } Z$ at $[X]$.

Proof. We see that the proof of [12, Theorem 10] works in our general setting, although it is assumed there that $V = \mathbb{P}^3$, Y is a smooth surfaces (of degree s) and X is a smooth curve (of degree $d > s^2$). In fact, it follows from a general theory that

$$(2.8) \quad h^0(X, N_{X/Z}) - h^1(X, N_{X/Z}) \leq \dim \mathcal{O}_{\text{Hilb } Z, [X]} \leq h^0(X, N_{X/Z}),$$

and moreover, we have $\dim \mathcal{O}_{\text{Hilb } Z, X} = h^0(X, N_{X/Z})$ if and only if $\text{Hilb } Z$ is nonsingular at $[X]$. Since $H^1(Z, N_{(X,Y)/Z}) = 0$, we have $\dim \mathcal{O}_{\text{HF } Z, (X,Y)} = h^0(Z, N_{(X,Y)/Z})$. By subtracting this number from (2.8), we have

$$\begin{aligned} h^0(X, N_{X/Z}) - h^0(Z, N_{(X,Y)/Z}) - h^1(X, N_{X/Z}) &\leq \dim \mathcal{O}_{\text{Hilb } Z, X} - \dim \mathcal{O}_{\text{HF } Z, (X,Y)} \\ &\leq h^0(X, N_{X/Z}) - h^0(Z, N_{(X,Y)/Z}). \end{aligned}$$

It follows from the exact sequence (2.5) and $H^1(Z, N_{(X,Y)/Z}) = H^0(Y, \mathcal{I}_{X/Y} N_Y) = 0$ that $h^0(X, N_{X/Z}) - h^0(Z, N_{(X,Y)/Z}) = h^1(Y, \mathcal{I}_{X/Y} N_Y)$. Since we have $H^2(Z, N_{(X,Y)/Z}) = 0$ also, the same sequence shows that $h^1(X, N_{X/Z}) = h^2(Y, \mathcal{I}_{X/Y} N_Y)$. Thus we have obtained (2.7). The first part of (1) is clear, because we have $\dim \mathcal{O}_{\text{Hilb } Z, X} - \dim \mathcal{O}_{\text{HF } Z, (X,Y)} \geq 0$ by assumption, while the last part of (1), (2) and (3) follow from the upper semicontinuity on cohomology groups. \square

Remark 2.5. (1) As is mentioned in [12, Remark 11], by replacing the middle term

$$\dim \mathcal{O}_{\text{Hilb } Z, [X]} - \dim \mathcal{O}_{\text{HF } Z, (X, Y)}$$

in (2.7) with

$$(2.9) \quad \dim \mathcal{O}_{\text{Hilb } Z, [X]} - \dim \mathcal{O}_{\text{HF } Z, (X, Y)} + h^0(Y, \mathcal{I}_{X/Y} N_Y),$$

we can still prove Theorem 2.4 without assuming that $H^0(Y, \mathcal{I}_{X/Y} N_Y) = 0$. In fact, since $h^0(Y, \mathcal{I}_{X/Y} N_Y)$ is greater than or equal to the dimension of the fiber at $[X]$ of the restriction $pr'_1 : \mathcal{W}_{X, Y} \rightarrow \text{Hilb } Z$ of pr_1 to $\mathcal{W}_{X, Y}$, the number (2.9) is non-negative, and similarly we obtain all of (1), (2) and (3) in Theorem 2.4.

(2) Theorem 2.4 (2) more directly follows from the smoothness (or more precisely, the flatness) of the morphism pr_1 at (X, Y) (cf. [11, Thm. 1.3.4 and Cor. 1.3.5]).

2.2.1. Stably degenerate curves, Y -maximal family. From now on, we assume that $\text{Hilb } Z$ is nonsingular at $[Y]$, and X is a smooth connected curve. Let W_Y denote the irreducible component of $\text{Hilb } Z$ passing through $[Y]$, and let $Z \times W_Y \supset \mathcal{Y} \xrightarrow{pr_2} W_Y$ be the universal subscheme over W_Y . We consider the Hilbert scheme $\text{Hilb}^{sc} \mathcal{Y}$ of smooth connected curves in \mathcal{Y} , i.e., the relative Hilbert scheme of \mathcal{Y}/W_Y . Then $\text{Hilb}^{sc} \mathcal{Y}$ is isomorphic to an open subscheme of the Hilbert-flag scheme $\text{HF}^{sc} Z$ ($:= pr_1^{-1}(\text{Hilb}^{sc} Z)$), where $pr_1 : \text{HF } Z \rightarrow \text{Hilb } Z$ is the first projection. Let $pr'_1 : \text{Hilb}^{sc} \mathcal{Y} \rightarrow \text{Hilb}^{sc} Z$ denote the restriction of pr_1 to $\text{Hilb}^{sc} \mathcal{Y} \subset \text{HF } Z$.

Definition 2.6. X is *stably (Y -)degenerate* if pr'_1 is surjective in a (Zariski) open neighborhood of $[X] \in \text{Hilb}^{sc} Z$.

By definition, X is stably degenerate if and only if there exists an open neighborhood $U_X \subset \text{Hilb}^{sc} Z$ of $[X]$ such that for any member X' of U_X , there exists a deformation of Y' of Y in Z such that $X' \subset Y'$ and $[Y'] \in W_Y$. The following is one of the most fundamental results on the stability of degenerate curves.

Lemma 2.7. *If $\text{HF}^{sc} Z$ is nonsingular at (X, Y) and pr_1 is smooth at (X, Y) , then X is stably degenerate and unobstructed in Z .*

Proof. By a property of smooth morphisms, $\text{Hilb}^{sc} Z$ is nonsingular at $[X]$. Since the smoothness is a local property, pr'_1 is smooth at (X, Y) and $\text{Hilb}^{sc} \mathcal{Y}$ is nonsingular at (X, Y) . Let $\mathcal{W}_{X, Y}$ denote the unique irreducible component of $\text{Hilb}^{sc} \mathcal{Y}$ passing through (X, Y) . Then by the smoothness, its image $W_{X, Y} := pr'_1(\mathcal{W}_{X, Y})$ is an irreducible component of $\text{Hilb}^{sc} Z$, and this is the only one passing through $[X]$. Hence pr'_1 is dominant near $[X]$. \square

In general, the image $W_{X, Y}$ of an irreducible component $\mathcal{W}_{X, Y} \subset \text{Hilb}^{sc} \mathcal{Y}$ is just an irreducible closed subset of $\text{Hilb}^{sc} Z$, and called the *Y -maximal family of curves* (containing X) (cf. [18, 22]).

2.3. Curves and $K3$ surfaces in a Fano 3-fold. In this section, we recall some results from [22], concerned with the deformations of curves and $K3$ surfaces in a smooth Fano 3-fold. Lemmas 2.12 and 2.14 are two key lemmas to prove Theorems 1.1 and 1.2.

Let V be a smooth Fano 3-fold, S a smooth member of $|-K_V|$, i.e., a smooth $K3$ surface, C a smooth connected curve on S . Since $K_S \sim 0$, we have by adjunction that $N_{C/S} \simeq K_C$ and $N_{S/V} \simeq -K_V|_S$, and then $H^1(C, N_{C/S}) \simeq k$. By the ampleness of $-K_V$, we see that $H^i(S, N_{S/V}) = 0$ for all integers $i > 0$, and hence S is unobstructed in V . Then by Lemma 2.8 below, the Hilbert-flag scheme $\text{HF}^{\text{sc}} V$ of V is nonsingular at (C, S) of expected dimension $\chi(V, N_{(C,S)/V})$, i.e., we have $H^1(V, N_{(C,S)/V}) = 0$, if and only if there exists a first order deformation \tilde{S} of S in V to which C does not lift. By Remark 2.3, or more directly from [22, Lemma 2.10], the Euler characteristic of $N_{(C,S)/V}$ is computed as

$$(2.10) \quad \chi(V, N_{(C,S)/V}) = (-K_V)^3/2 + g(C) + 1,$$

where $g(C)$ denotes the arithmetic genus of C .

Lemma 2.8 (cf. [22]). *Let V be a smooth projective scheme, $S \subset V$ a smooth surface, $C \subset S$ a smooth curve.*

- (1) *Suppose that $H^1(S, N_{S/V}) = 0$ and $H^1(C, N_{C/S}) \simeq k$. Then $H^1(V, N_{(C,S)/V}) = 0$ if and only if there exists a first order deformation \tilde{S} of S in V , to which C does not lift.*
- (2) *If $H^2(S, N_{S/V}) = 0$, then $H^i(V, N_{(C,S)/V}) = 0$ for $i > 1$.*

Proof. (1) By (2.6) and assumption, there exists an exact sequence

$$H^0(V, N_{(C,S)/V}) \xrightarrow{p_2} H^0(S, N_{S/V}) \longrightarrow H^1(C, N_{C/S}) \longrightarrow H^1(V, N_{(C,S)/V}) \longrightarrow 0.$$

Then since $H^1(C, N_{C/S})$ is of dimension 1, we have $H^1(V, N_{(C,S)/V}) = 0$ if and only if p_2 is not surjective. Thus we have proved (1). By Remark 2.3, it suffices to show $H^2(V, N_{(C,S)/V}) = 0$ for (2), which follows from (2.6) and that $\dim C = 1$. \square

The following example shows that even if the second projection $pr_2 : \text{HF}^{\text{sc}} V \rightarrow \text{Hilb}^{\text{sc}} V$ is not surjective in any neighborhood of $[S]$, its tangent map $p_2 : H^0(V, N_{(C,S)/V}) \rightarrow H^0(S, N_{S/V})$ at (C, S) can be surjective (and hence $H^1(V, N_{(C,S)/V}) \neq 0$).

Example 2.9. Suppose that V is a prime Fano 3-fold of genus g , and let E be a line on V (i.e. $E \simeq \mathbb{P}^1$ and $(-K_V \cdot E)_V = 1$). Then E is called a *good line* on V , if $N_{E/V}$ is of type $(0, -1)$, and a *bad line*, otherwise, i.e., $N_{E/V}$ is of type $(1, -2)$. Suppose that S contains E . Then we have

$$H^1(V, N_{(E,S)/V}) \simeq \begin{cases} 0 & (E: \text{good}) \\ k & (E: \text{bad}) \end{cases}.$$

Moreover, $\text{HF}^{\text{sc}} V$ is nonsingular at (E, S) if and only if E is good. In fact, the exact sequence $0 \rightarrow N_{E/S} \rightarrow N_{E/V} \rightarrow N_{S/V}|_E \rightarrow 0$ on E splits if and only if E is bad. Suppose

that E is good. Then by $H^1(E, N_{E/V}) = 0$, the coboundary map $\partial_{E/S} : H^0(E, N_{S/V}|_E) \rightarrow H^1(E, N_{E/S}) (\simeq k)$ is surjective. Note that the restriction map $H^0(S, N_{S/V}) \xrightarrow{|_E} H^0(E, N_{S/V}|_E)$ is also surjective, because E is a line. Then by Lemma 2.2, we have $H^1(V, N_{(E,S)/V}) \simeq \text{coker } \alpha_{E/S} = 0$. Conversely, we suppose that E is bad. Then by splitting, we have $\partial_{E/S} = 0$ and $H^1(V, N_{(E,S)/V}) \simeq H^1(E, N_{E/S}) \simeq k$. We also note that pr_1 is smooth at (E, S) by $H^1(S, N_{S/V}(-E)) = 0$. Since $\text{Hilb}^{sc} V$ is singular at $[E]$, so is $\text{HF}^{sc} V$ at (E, S) . Since E is not a complete intersection in S , i.e., $E \not\sim n(-K_V|_S)$ for any $n \in \mathbb{Z}$, we see that pr_2 is not surjective in any neighborhood of $[S]$ by [14], although p_2 is surjective for (E, S) .

Repeating the same argument in Example 2.9 for a conic E on V , we obtain the following lemma.

Lemma 2.10. *Suppose that V is prime, and E is a line or conic on V contained in S . If E is good, or a conic of type $(1, -1)$ (cf. (2.1)), then we have $H^1(V, N_{(E,S)/V}) = 0$.*

For the proof of Lemma 2.12 below, we recall a criterion for the lifting of invertible sheaves. Given a smooth projective scheme X and an invertible sheaf \mathcal{L} on X , we denote by $c(\mathcal{L})$ the Atiyah extension class of \mathcal{L} in $H^1(X, \Omega_X) \simeq \text{Ext}^1(T_X, \mathcal{O}_X)$. Here $c(\mathcal{L})$ is the image of the class of \mathcal{L} in $\text{Pic } X \simeq H^1(X, \mathcal{O}_X^\times)$, under the map induced by the map $d \log : \mathcal{O}_X^\times \rightarrow \Omega_X^1$ taking logarithmic derivatives (cf. [6, V, Ex. 1.8]). Given an element $\tau \in H^1(X, T_X)$, i.e., an abstract first order deformation \tilde{X} of X , \mathcal{L} lifts to an invertible sheaf on \tilde{X} if and only if the cup product $\tau \cup c(\mathcal{L})$ via the pairing $H^1(X, T_X) \times H^1(X, \Omega_X) \xrightarrow{\cup} H^2(X, \mathcal{O}_X)$ is zero (cf. [7, Ex. 10.6]).

Lemma 2.11. *Let V be a smooth projective scheme, S a smooth closed subscheme of V , \mathcal{L} an invertible sheaf on V , and $\mathcal{M} := \mathcal{L}|_S$ its restriction to S . If $\tau \in H^1(S, T_S)$ is contained in the image of the coboundary map $\delta : H^0(S, N_{S/V}) \rightarrow H^1(S, T_S)$ of the exact sequence*

$$(2.11) \quad 0 \longrightarrow T_S \xrightarrow{\iota} T_V|_S \longrightarrow N_{S/V} \longrightarrow 0,$$

then we have $\tau \cup c(\mathcal{M}) = 0$ in $H^2(S, \mathcal{O}_S)$.

Proof. Let ι be the inclusion of sheaves in (2.11). Then its dual ι^\vee induces a map $H^1(\iota^\vee) : H^1(S, \Omega_V|_S) \rightarrow H^1(S, \Omega_S)$ on the cohomology groups. There exists a commutative diagram

$$\begin{array}{ccc} H^1(V, \mathcal{O}_V^\times) & \xrightarrow{c} & H^1(V, \Omega_V) \\ \text{is} \downarrow & & \text{is} \downarrow \\ H^1(S, \mathcal{O}_S^\times) & \xrightarrow{c} & H^1(S, \Omega_S) \xleftarrow{H^1(\iota^\vee)} H^1(S, \Omega_V|_S). \end{array}$$

Therefore, we have $c(\mathcal{M}) = H^1(\iota^\vee)(c(\mathcal{L})|_S)$ in $H^1(S, \Omega_S)$. Then

$$\tau \cup c(\mathcal{M}) = \tau \cup H^1(\iota^\vee)(c(\mathcal{L})|_S) = H^1(\iota)(\tau) \cup c(\mathcal{L})|_S,$$

where the map $H^1(\iota) : H^1(S, T_S) \rightarrow H^1(S, T_V|_S)$ is induced by ι . By assumption, we have $H^1(\iota)(\tau) = 0$, and hence we have finished the proof. \square

Lemma 2.12 (char $k \neq 0$). *Let $i : S \hookrightarrow V$ be the closed embedding of S into V , and E an effective Cartier divisor on S satisfying $H^1(S, \mathcal{O}_S(E)) = H^1(V, N_{(E,S)/V}) = 0$. If the class of $C - bE$ in $\text{Pic } S$ is contained in the image of the pullback map $i^* : \text{Pic } V \rightarrow \text{Pic } S$ for some integer $b \neq 0$, then we have $H^1(V, N_{(C,S)/V}) = 0$.*

Proof. By Lemma 2.8, there exists a first order deformation \tilde{S} of S in V to which E does not lift. Let $\alpha \in H^0(S, N_{S/V})$ be the global section corresponding to \tilde{S} , and $\tau := \delta(\alpha)$ its image in $H^1(S, T_S)$ by the coboundary map δ of (2.11). Then by Lemma 2.11, we have $\tau \cup c(\mathcal{O}_S(C - bE)) = 0$ in $H^2(S, \mathcal{O}_S)$. On the other hand, since $c : H^1(S, \mathcal{O}_S^\times) \rightarrow H^1(S, \Omega_S)$ is a group homomorphism, we have $\tau \cap c(\mathcal{O}_S(C)) = b\tau \cap c(\mathcal{O}_S(E))$. If the invertible sheaf $\mathcal{O}_S(E)$ on S lifts to \tilde{S} , then so does E as a closed subscheme of S by $H^1(S, \mathcal{O}_S(E)) = 0$ (cf. [4, Remark 4.5]). Therefore, we have $\tau \cap c(\mathcal{O}_S(E)) \neq 0$ and hence $\tau \cap c(\mathcal{O}_S(C)) \neq 0$ by $b \neq 0$. Then $\mathcal{O}_S(C)$ does not lift to \tilde{S} , hence neither does C (cf. [7, Ex. 6.7]). Thus we have finished the proof by Lemma 2.8 again. \square

We define a Cartier divisor D on S by

$$(2.12) \quad D := C + K_V|_S.$$

Then since $\mathcal{I}_{C/S} \otimes_S N_{S/V} \simeq \mathcal{O}_S(-D)$, the Serre duality shows that

$$(2.13) \quad H^i(S, \mathcal{I}_{C/S} \otimes_S N_{S/V}) \simeq H^{2-i}(S, D)^\vee$$

for all integers i . Applying Theorem 2.4, we have the following lemma.

Lemma 2.13. *Let $W_{C,S} \subset \text{Hilb}^{\text{sc}} V$ be the S -maximal family of curves containing C . (See §2.2.1 for its definition.) Suppose that $H^0(S, -D) = H^1(V, N_{(C,S)/V}) = 0$. Then $\dim W_{C,S} = (-K_V)^3/2 + g(C) + 1$, and we have*

$$h^1(S, D) - h^0(S, D) \leq \dim \mathcal{O}_{\text{Hilb}^{\text{sc}} V, [C]} - \dim W_{C,S} \leq h^1(S, D).$$

Moreover, $\text{Hilb}^{\text{sc}} V$ is singular along $W_{C,S}$ if and only if the inequality to the right is strict. In particular,

- (1) *If $h^1(S, D) = 0$ or $h^0(S, D) = 0$, then C is unobstructed in V , and moreover $\text{Hilb}^{\text{sc}} V$ is generically smooth along $W_{C,S}$.*
- (2) *If $h^1(S, D) = 0$, then $W_{C,S}$ is an irreducible component of $(\text{Hilb}^{\text{sc}} V)_{\text{red}}$. In particular, C is stably degenerate.*
- (3) *If $h^0(S, D) = 0$, then $W_{C,S}$ is of codimension $h^1(S, D)$ in $\text{Hilb}^{\text{sc}} V$.*

Proof. Since $H^0(S, \mathcal{I}_{C/S} \otimes_S N_{S/V}) \simeq H^0(S, -D) = 0$, $\text{HF}^{\text{sc}} V$ and $W_{C,S}$ are locally isomorphic in a neighborhood of (C, S) , and hence $\dim W_{C,S} = \dim \mathcal{O}_{\text{HF}^{\text{sc}} V, (C,S)} = (-K_V)^3/2 + g(C) + 1$. \square

Finally we recall a result from [22]. Given a curve E on S , we define the π -map

$$(2.14) \quad \pi_{E/S}(E) : H^0(E, N_{E/V}(E)) \longrightarrow H^0(E, N_{S/V}(E)|_E)$$

for E (or for the pair (E, S)) as the map on cohomology groups induced by the sheaf homomorphism $[N_{E/V} \xrightarrow{\pi_{E/S}} N_{S/V}|_E] \otimes_E \mathcal{O}_E(E)$. The following is another key lemma to prove Theorems 1.1 and 1.2.

Lemma 2.14 ([22, Theorem 1.2 and Corollary 1.3]). *Suppose that $H^1(V, N_{(C,S)/V}) = 0$. Let D be the divisor on S defined by (2.12), and suppose that $D \geq 0$ and $D^2 \geq 0$. If there exists a (-2) -curve E on S such that*

- (a) $E.D = -2$,
- (b) $H^1(S, D - 3E) = 0$, and
- (c) the π -map $\pi_{E/S}(E)$ is not surjective,

then C is stably degenerate, and obstructed in V .

Remark 2.15. If an effective divisor D on a $K3$ surface S with $D^2 \geq 0$ satisfy the above two conditions (a) and (b) for some (-2) -curve E , then we have $h^1(S, D) = 1$. (See [22, Claim 4.1] for the proof.) Thus in applying Lemma 2.14, the tangent map p_1 of the first projection pr_1 is not surjective and $\text{coker } p_1$ is of dimension 1 (cf. (3.3)).

3. NON-REDUCED COMPONENTS OF HILBERT SCHEMES

In this section, we prove Theorem 1.1. As a result, we obtain an example of a generically non-reduced component of the Hilbert scheme $\text{Hilb}^{sc} X$ (of Mumford type) for every smooth Fano 3-fold X with $\text{Pic } X \simeq \mathbb{Z}$ of any index r (cf. Corollary 3.6, Table 1).

3.1. Construction. Let V be a prime Fano 3-fold of genus g . By Lemma 2.1, there exists a good conic $E \simeq \mathbb{P}^1$ on V . Then we have $(-K_V.E)_V = 2$ and $N_{E/V}$ is trivial. As in the proof of [24, Lemma 4.2.1], there exists a smooth member $S \in |-K_V|$ containing E . Then S is a $K3$ surface, and we have the self-intersection number $E^2 = -2$ on S . Put $\mathbf{h} := -K_V|_S$, an ample divisor on S . Then $\mathbf{h}^2 = (-K_V|_S)^2 = (-K_V)^3 = 2g - 2$ and $\mathbf{h}.E = 2$. We consider a complete linear system

$$\Lambda := |2\mathbf{h} + 2E|$$

of divisors on S . By using the intersection numbers on S , we can show that $\mathbf{h} + E$ is nef and big. It is known that for every nef and big line bundle L on a $K3$ surface, L^k is globally generated if $k \geq 2$ (cf. [8, Chap. 3, Remark 3.4]). Therefore, by Bertini's theorem, Λ contains a smooth connected curve C . The degree and the genus of C are computed as $d(C) = C.\mathbf{h} = 4g$ and $g(C) = C^2/2 + 1 = 4g + 1$, respectively.

Let $W \subset \text{Hilb}_{4g, 4g+1}^{sc} V$ be the family of such curves $C \subset V$, i.e., smooth connected curves C contained in a smooth $S \in |-K_V|$, and such that C is a member of Λ for some good conic E on V . For each $C \in W$, the surface S and the conic E are uniquely

determined by C , because we deduce $h^0(V, \mathcal{I}_{C/V} \otimes_V \mathcal{O}_V(-K_V)) \simeq k$ from the exact sequence

$$(3.1) \quad [0 \longrightarrow \mathcal{O}_V(K_V) \longrightarrow \mathcal{I}_{C/V} \xrightarrow{|\mathcal{S}|} \mathcal{O}_S(-C) \longrightarrow 0] \otimes_V \mathcal{O}_V(-K_V)$$

and moreover, E is recovered from C and S as the unique base component of the linear system $|C + \mathbf{h}|$ on S . Thus there exists a morphism $W \rightarrow \mathrm{HF}^{sc} V$ to the Hilbert-flag scheme $\mathrm{HF}^{sc} V$ of V , sending $[C]$ to (E, S) , and the fiber at (E, S) is isomorphic to an open subset of $\Lambda \simeq \mathbb{P}^{4g+1} = \mathbb{P}^{g(C)}$ of smooth curves. As in §1, the pairs (E, S) such that $E \subset S \subset V$ are parametrised by an open subset U of \mathbb{P}^{g-2} -bundle over an open surface $\Gamma' \subset \Gamma$. Thus there exists a diagram

$$(3.2) \quad \begin{array}{ccccc} C & \in & W^{(5g+1)} & \subset & \mathrm{Hilb}_{4g, 4g+1}^{sc} V \\ \downarrow & & \downarrow \mathbb{P}^{4g+1}\text{-bundle} & & \\ (E, S) & \in & U^{(g)} & \subset & \mathrm{HF}^{sc} V \\ \downarrow & & \downarrow \mathbb{P}^{g-2}\text{-bundle} & & \downarrow \\ E & \in & \Gamma'^{(2)} & \subset & \mathrm{Hilb}_{2,0}^{sc} V \end{array}$$

of fiber bundles, where the upper script (d) of $X^{(d)}$ denotes $d = \dim X$. Since E is a good conic on V , by Lemmas 2.10 and 2.12, we have $H^1(V, N_{(C,S)/V}) = 0$. Then the Hilbert-flag scheme $\mathrm{HF}^{sc} V$ is nonsingular at (C, S) of expected dimension

$$h^0(V, N_{(C,S)/V}) = \chi(V, N_{(C,S)/V}) = (g-1) + (4g+1) + 1 = 5g+1$$

by (2.10). Let D be the divisor on S defined by (2.12). Then it follows from (2.5) and (2.13) that there exists an exact sequence

$$(3.3) \quad 0 \longrightarrow H^0(V, N_{(C,S)/V}) \xrightarrow{p_1} H^0(C, N_{C/V}) \longrightarrow H^1(S, D)^\vee \longrightarrow 0.$$

We will show that $H^1(S, D) \simeq k$, which implies that $h^0(C, N_{C/V}) = h^0(V, N_{(C,S)/V}) + 1 = 5g+2$. In fact, since $D \sim \mathbf{h} + 2E$, we have $H^i(S, D - E) = 0$ for $i = 1, 2$. Then it follows from the exact sequence

$$(3.4) \quad 0 \longrightarrow \mathcal{O}_S(D - E) \longrightarrow \mathcal{O}_S(D) \longrightarrow \mathcal{O}_E(D) \longrightarrow 0$$

on S that $H^1(S, D) \simeq H^1(E, \mathcal{O}_E(D))$. Since $E \cdot D = -2$ and $E \simeq \mathbb{P}^1$, we have $H^1(S, D) \simeq k$. Thus we have a dichotomy between

- (A) The closure \overline{W} of W in $\mathrm{Hilb}^{sc} V$ is an irreducible component of $(\mathrm{Hilb}^{sc} V)_{\mathrm{red}}$, and moreover $\mathrm{Hilb}^{sc} V$ is singular along W , and
- (B) There exists an irreducible component Z such that $\dim Z > \dim \overline{W}$ and $\mathrm{Hilb}^{sc} V$ is generically nonsingular along W .

3.2. Proof of non-reducedness. We prove that the case (B) of the dichotomy in the previous subsection does not occur.

Proof of Theorem 1.1. Let $W \subset \text{Hilb}^{sc} V$ be the family of curves C in V as above. Then by the dichotomy, it suffices to show that C is obstructed in V . We apply Lemma 2.14 to C . It is easy to see that $D^2 = 2g - 2 > 0$. Moreover, since $N_{E/V}$ is globally generated, the π -map $\pi_{E/S}(E)$ (cf. (2.14)) is not surjective by [22, Lemma 2.14]. Thus we have only to check that $H^1(S, D - 3E) = H^1(S, \mathbf{h} - E) = 0$. Since E is a conic on $V \subset \mathbb{P}^{g+1}$, E is linearly normal, i.e., $H^1(\mathbb{P}^{g+1}, \mathcal{I}_{E/\mathbb{P}}(1)) = 0$. Therefore the restriction map $H^0(V, -K_V) \rightarrow H^0(E, -K_V|_E)$ to E is surjective, and so is the map $H^0(S, \mathbf{h}) \rightarrow H^0(E, \mathbf{h}|_E)$. Thus we deduce $H^1(S, \mathbf{h} - E) = 0$ from (3.4) with $D = \mathbf{h}$ and that $H^1(S, \mathbf{h}) = 0$. Thus we have finished the proof. \square

Remark 3.1. By construction, the closure \overline{W} of W in $\text{Hilb}^{sc} V$ is nothing but the S -maximal family $W_{C,S}$ of curves in V containing C (cf. §2.2.1). In fact, we have $W \subset W_{C,S}$ and $\dim W = W_{C,S} = 5g + 1$ ($= \dim \mathcal{O}_{\text{HF}^{sc} V, (C,S)}$).

Remark 3.2. By using the same construction and the same proof, we can show that $\text{Hilb}^{sc} V$ contains *infinitely many* generically non-reduced components (cf. [22, Example 5.8]). In fact, for every integer $n \geq 2$, we define a complete linear system Λ_n on S by

$$\Lambda_n := |n\mathbf{h} + nE|.$$

Then every general member C_n is a smooth connected curve on S of degree $2ng$ and genus $n^2g + 1$. Moreover, C_n is parametrised by an irreducible locally closed subset $W_n \subset \text{Hilb}_{2ng, n^2g+1}^{sc} V$ of dimension $(n^2 + 1)g + 1$ and we have $h^0(C_n, N_{C_n/V}) = \dim W_n + 1$. The above argument (for $n = 2$) works for W_n in general. Indeed, the generic member C_n of W_n is obstructed in V (cf. Proposition 4.4). Thereby, the closure \overline{W}_n of W_n is an irreducible component of $(\text{Hilb}_{2ng, n^2g+1}^{sc} V)_{\text{red}}$ and $\text{Hilb}_{2ng, n^2g+1}^{sc} V$ is generically non-reduced along \overline{W}_n . In particular, $\text{Hilb}^{sc} V$ contains infinitely many generically non-reduced components.

Remark 3.3. For some prime Fano 3-folds V , the Hilbert scheme $\text{Hilb}_{1,0}^{sc} V$ of lines on V is known to contain a generically non-reduced component (cf. [24, Proposition 4.2.2]). In the following two cases, $\text{Hilb}_{1,0}^{sc} V$ contains a generically non-reduced component Γ , whose general point corresponds to a bad line on V , i.e., a line on V of type $(1, -2)$ (cf. Example 2.9):

- (1) V is a smooth quartic hypersurface $V_4 \subset \mathbb{P}^4$ (i.e. $g = 3$), and there exists a ruled surface R swept out by lines from Γ and R is a cone over a smooth plane curve of degree 4, e.g., V_4 is the Fermat quartic $V_{\text{Fer}} = (\sum_{i=0}^4 x_i^2 = 0) \subset \mathbb{P}_{x_0, \dots, x_4}^4$.
- (2) R is swept out by projective tangent lines to some curve $B \subset V$, e.g., V is a Mukai-Umemura 3-fold $V_{22} \subset \mathbb{P}^{13}$ (cf. [19]).

By the following examples, for every smooth Fano 3-fold X of index 4 and 3, there exists a generically non-reduced component of $\text{Hilb}^{sc} X$ (of Mumford type).

Example 3.4 (Mumford [20], $r = 4$). Let $S \subset \mathbb{P}^3$ be a smooth cubic surface, E a (-1) - \mathbb{P}^1 on S and $C \subset S$ a smooth member of the linear system $|-K_{\mathbb{P}^3}|_S + 2E| \simeq \mathbb{P}^{37}$ on S . Then C is of degree 14 and genus 24. Such C 's are parametrized by $W^{(56)} \subset \text{Hilb}^{sc} \mathbb{P}^3$, which is an open subset of a \mathbb{P}^{37} -bundle over $|\mathcal{O}_{\mathbb{P}^3}(3)| \simeq \mathbb{P}^{19}$. Then the closure $\overline{W}^{(56)}$ is an irreducible component of $(\text{Hilb}^{sc} \mathbb{P}^3)_{\text{red}}$ and $\text{Hilb}^{sc} \mathbb{P}^3$ is everywhere non-reduced along W^{56} .

Example 3.5 ($r = 3$). Let Q be a smooth hyperquadric in \mathbb{P}^4 , and S a smooth complete intersection of Q with some other hyperquadric, i.e., $S \sim (-2/3)K_Q$. Let $\mathbf{h} \sim \mathcal{O}_S(1) \in \text{Pic } S$ be the class of hyperplane sections of S . Since S is a del Pezzo surface of degree 4 (namely, $-K_S \simeq \mathbf{h}$ and $\mathbf{h}^2 = 4$), S is isomorphic to a blown up of \mathbb{P}^2 (at 5 points). Thus there exists a (-1) -curve $E \simeq \mathbb{P}^1$ on S such that $E \cdot \mathbf{h} = 1$ (i.e. a line E). We consider a complete linear system $|-K_Q|_S + 2E| = |3\mathbf{h} + 2E|$ on S . Then its general member C is a smooth connected curve on S of degree 14 and genus 16. Since $N_{C/S} \simeq K_C(1)$ and $N_{S/Q} \simeq K_S(3)$, we have for all $i > 0$ that $H^i(C, N_{C/S}) = H^i(S, N_{S/Q}) = 0$ and hence $H^i(Q, N_{(C,S)/Q}) = 0$ by (2.6), which implies that $\text{HF } Q$ is nonsingular at (C, S) of expected dimension $\chi(C, N_{C/S}) + \chi(S, N_{S/Q}) = \chi(C, K_C(1)) + \chi(S, -2K_S) = (d(C) + g(C) - 1) + 13 = 42$. Since $H^1(S, N_{S/Q}(-C)) \simeq H^1(S, -\mathbf{h} - 2E) \simeq k$, it follows from (2.5) that $h^0(C, N_{C/Q}) = 43$. Then the S -maximal family $W_{C,S} \subset \text{Hilb}^{sc} Q$ of curves containing C is a closed subset of $\text{Hilb}^{sc} Q$ of codimension 1 in $H^0(C, N_{C/Q})$. Since for the generic member C' of $W_{C,S}$, the line E' on S determined by $E' = \text{Bs } |C' - 2\mathbf{h}|$ is a good line on Q (i.e. $N_{E'/Q}$ is of type $(1, 0)$), by using the same technique in [18], we can show that C' is obstructed in Q . Then by the same argument shown in §3.1 using a dichotomy, we see that $W_{C,S}$ is an irreducible component of $(\text{Hilb}^{sc} Q)_{\text{red}}$, and hence $\text{Hilb}^{sc} Q$ is generically non-reduced along $W_{C,S}$. Consequently, $\text{Hilb}^{sc} Q$ contains a generically non-reduced component of Mumford type.

For the index 2 case, we refer to [18, 21]. As a result, we have proved the following.

Corollary 3.6. *For every smooth Fano 3-fold X with $\text{Pic } X \simeq \mathbb{Z}$, the Hilbert scheme $\text{Hilb}^{sc} X$ of smooth connected curves on X contains a generically non-reduced component of Mumford type (cf. Table1).*

4. DEFORMATIONS OF DEGENERATE CURVES

In this section, we discuss the deformations of curves C on a prime Fano 3-fold V and prove Theorem 1.2. We focus on the problem on determining whether C is stably degenerate or not (cf. Definition 2.6) for curves C lying on a smooth anticanonical member S of V . In [21], the same problem was discussed for a smooth del Pezzo 3-fold V , with curves lying on a smooth half-anticanonical member $S \in |-\frac{1}{2}K_V|$.

Let V be a prime Fano 3-fold of genus g , and C a smooth connected curve on V . We suppose that there exists a smooth member $S \in |-K_V|$ containing C . We recall that a

curve C on S is called a *complete intersection in S* if $C \sim n\mathbf{h}$ in $\text{Pic } S$ for some integer n , where \mathbf{h} is the class $-K_V|_S$ (of hyperplane sections of S).

4.1. Complete intersection case. We first consider the complete intersection case. Given a positive integer n , let $\mathcal{W}_n \subset \text{HF}^{sc} V$ denote the (maximal) family of pairs (C, S) of a smooth member $S \in |-K_V|$ and a smooth curve $C \subset S$ satisfying $C \sim n\mathbf{h}$ in $\text{Pic } S$. Then the second projection pr_2 induces a morphism $pr'_2 : \mathcal{W}_n \rightarrow |-K_V| \simeq \mathbb{P}^{g+1}$ to the Hilbert scheme of hyperplane sections of V . It is clear that pr'_2 is dominant, and its fiber at $[S]$ is isomorphic to an open subset of the projective space $|\mathcal{O}_S(C)| \simeq |\mathcal{O}_S(n\mathbf{h})|$. Therefore, \mathcal{W}_n is an irreducible locally closed subset of $\text{HF}^{sc} V$, and hence so is its image $W_n := pr_1(\mathcal{W}_n)$ in $\text{Hilb}^{sc} V$ by the first projection pr_1 .

Proposition 4.1. *Let $n \geq 1$ be an integer, and let C and $W_n \subset \text{Hilb}^{sc} V$ be as above. Then C is unobstructed and stably degenerate in V . Moreover, the closure \overline{W}_n of W_n is an irreducible component of $\text{Hilb}^{sc} V$ of dimension $(n^2 + 1)(g - 1) + 3$ for $n \geq 2$, and $2g$ for $n = 1$.*

Proof. By using the Riemann-Roch theorem on S , we compute that $\dim |\mathcal{O}_S(C)| = g(C) = (n\mathbf{h})^2/2 + 1 = n^2(g-1) + 1$. Hence $\dim \mathcal{W}_n = \dim |-K_V| + \dim |\mathcal{O}_S(C)| = (n^2 + 1)(g - 1) + 3$. Let pr'_1 denote the restriction of pr_1 to \mathcal{W}_n . Then its fiber $pr'^{-1}_1([C])$ at $[C]$ is isomorphic to the linear system $\Lambda := |\mathcal{I}_{C/V} \otimes_V \mathcal{O}_V(-K_V)|$ on V . It follows from (3.1) and $H^1(V, \mathcal{O}_V) = 0$ that

$$\dim \Lambda = h^0(S, -K_V|_S - C) = h^0(S, (1 - n)\mathbf{h}),$$

which is equal to 1 if $n = 1$, and 0 otherwise. Thus we obtain the dimension of W_n as stated in the proposition. It is also easy to see that $h^0(V, N_{C/V}) = \dim W_n$. In fact, there exists an exact sequence $0 \rightarrow K_C \rightarrow N_{C/V} \rightarrow \mathcal{O}_C(-K_V) \rightarrow 0$ on C , because $N_{S/V} \simeq -K_V|_S$. Since C is a complete intersection in S , this sequence splits and the restriction map $H^0(S, -K_V|_S) \rightarrow H^0(C, -K_V|_C)$ to C is surjective. Therefore, we compute that $h^0(C, N_{C/V}) = h^0(C, K_C) + h^0(C, -K_V|_C) = \dim |\mathcal{O}_S(C)| + \dim |-K_V| - \dim \Lambda = \dim W_n$. Since $\dim W_n \leq \dim \mathcal{O}_{\text{Hilb}^{sc} V, [C]}$, we see by (2.8) that C is unobstructed in V . Then it follows from $H^1(S, N_{S/V}(-C)) = 0$ that pr'_1 is smooth at (C, S) . This implies that $\text{HF}^{sc} V$ is nonsingular at (C, S) and hence C is stably degenerate by Lemma 2.7. \square

Remark 4.2. We note that in Proposition 4.1, $\text{HF}^{sc} V$ is nonsingular, but not of expected dimension at (C, S) , because $H^1(V, N_{(C,S)/V}) \simeq k$. Moreover, W_n coincides with the S -maximal family $W_{C,S}$ containing C for every member C of W_n and a member $S \in \Lambda$ (cf. §2.2.1).

4.2. Non-complete intersection case. We next consider the case where C is not a complete intersection in S . For a technical reason (cf. Remark 4.6), we assume that the class of C in $\text{Pic } S$ is generated by \mathbf{h} together with a line E (cf. Proposition 4.3) or a conic E (cf. Proposition 4.4) with non-negative coefficients, i.e., $C \sim a\mathbf{h} + bE$ with two

integers $a \geq 0$ and $b \geq 0$. Since every line and every conic on V is stably degenerate, and there are many references, e.g., [10, 24], for their (un)obstructedness in V , in what follows, we also assume that C is not a complete intersection in S (i.e. $b \neq 0$), and $C \neq E$ (i.e. $(a, b) \neq (0, 1)$). Let $W_{C,S} \subset \text{Hilb}^{sc} V$ denote the S -maximal family of curves containing C .

Proposition 4.3. *Suppose that E is a good line on V . Then*

- (1) C is stably degenerate, and unobstructed in V , and
- (2) $W_{C,S}$ is an irreducible component of $(\text{Hilb}^{sc} V)_{\text{red}}$ of dimension $(a^2 + 1)(g - 1) + b(a - b) + 2$.

Proof. We have $E \cdot \mathbf{h} = 1$ and $E^2 = -2$. Since $C \neq E$, we have $C \cdot E \geq 0$, which implies that $a \geq 2b$. By assumption, we have $b \geq 1$, and hence $H^1(V, N_{(C,S)/V}) = 0$ by Lemmas 2.10 and 2.12. Let $D := C + K_V|_S = (a - 1)\mathbf{h} + bE$. Since D is effective and $D \not\sim 0$, we have $H^0(S, -D) = 0$. We note that $H^1(E, D|_E) = 0$ by $D \cdot E \geq -1$. Since $D - E = (a - 2b + 1)\mathbf{h} + (b - 1)(2\mathbf{h} + E)$ and $2\mathbf{h} + E$ is nef and big, we see that $D - E$ is ample and hence $H^1(S, D - E) = 0$. Thus it follows from the exact sequence (3.4) that we have $H^1(S, D) = 0$. Then by Lemma 2.13, we obtain (1). We compute the genus $g(C)$ of C as $g(C) = C^2/2 + 1 = a^2(g - 1) + ab - b^2 + 1$. Thus by the same lemma, we have proved (2). \square

Proposition 4.4. *Suppose that E is a good conic on V . Then*

- (1) C is stably degenerate,
- (2) C is obstructed in V if and only if $a = b \geq 2$, and
- (3) $W_{C,S}$ is an irreducible component of $(\text{Hilb}^{sc} V)_{\text{red}}$ of (the maximal) dimension $(a^2 + 1)g - (a - b)^2 + 1$ (passing through $[C]$).

Proof. The proof is similar to that of Proposition 4.3. We have $E \cdot \mathbf{h} = 2$ and $E^2 = -2$. Since $C \neq E$ and $b \geq 1$, we have $a \geq b$ and $H^1(V, N_{(C,S)/V}) = 0$, respectively. Put $D := C + K_V|_S = (a - 1)\mathbf{h} + bE$, a divisor on S , as before. Then $H^0(S, -D) = 0$ by $D \geq 0$ and $D \neq 0$. We consider the exact sequence (3.4) for the computation of $H^1(S, D)$. First we note that $D - E = (a - b)\mathbf{h} + (b - 1)(\mathbf{h} + E)$. Since $\mathbf{h} + E$ is nef and big, so is $D - E$, and hence we have $H^1(S, D - E) = 0$ by the Kodaira-Ramanujam vanishing theorem. Since $D \cdot E = 2(a - b) - 2$, we see that if $a > b$, then $H^1(E, D|_E) = 0$ and hence $H^1(S, D) = 0$ by (3.4). Then in this case, by Lemma 2.13, we obtain the conclusions (1), (2) and (3), together with $\dim W_{C,S} = (-K_V)^3/2 + g(C) + 1$, where $g(C) = a^2g - (a - b)^2 + 1$. If $a = b = 1$, then we have $H^1(S, D) = H^1(S, E) = 0$, and hence the proof is done.

Suppose now that $a = b = n \geq 2$. Then we have $D^2 = (2g - 2)(n - 1)^2 > 0$, $D \geq 0$, $D \cdot E = -2$ and $D - 3E = (n - 1)\mathbf{h} + (n - 3)E$. Since we have $D - 3E = \mathbf{h} - E$ for $n = 2$, and $D - 3E = 2\mathbf{h} + (n - 3)(\mathbf{h} + E)$ for $n \geq 3$, we conclude that $H^1(S, D - 3E) = 0$. Since E is a good conic on V , the π -map $\pi_{E/S}(E)$ is not surjective, as in the proof of Theorem 1.1.

Thus by Lemma 2.14, C is stably degenerate and obstructed in V . Moreover, we have $h^1(S, D) = 1$ by Remark 2.15, and hence $h^0(C, N_{C/V}) = h^0(V, N_{(C,S)/V}) + 1 = \dim W_{C,S} + 1$ by (3.3). Then we have the dichotomy between (A) and (B) in §3.1 for $W = W_{C,S}$ again. Then the obstructedness immediately shows that the case (B) does not occur. Hence $W_{C,S}$ is an irreducible component of $(\text{Hilb}_{sc} V)_{\text{red}}$ of the maximal dimension passing through $[C]$. \square

Consequently, Theorem 1.2 has been proved as a combination of Propositions 4.1, 4.3 and 4.4.

Remark 4.5. We have assumed that $(a, b) \neq (0, 1)$, i.e., C is not E itself, in Propositions 4.3, 4.4, and hence in Theorem 1.2. If the condition [ii] or [iii] of this theorem holds with $(a, b) = (0, 1)$ (i.e. $C = E$), then all the conclusions of the theorem are true, except for the dimension of $\text{Hilb}^{sc} V$ at $[C]$, which is computed as follows. Since $C = E$ is a good line or a good conic on V , we have $H^1(E, N_{E/V}) = 0$, and it follows from (2.8) that $\dim_{[E]} \text{Hilb}^{sc} V = \chi(E, N_{E/V}) = (-K_V \cdot E)_V$, that is equal to 1 for [ii], and 2 for [iii].

Remark 4.6. Here we restrict ourselves to the case where the class $[C]$ of C in $\text{Pic } S$ is generated by \mathbf{h} together with a line or a conic on S . Of course, this is not enough for determining the stability of every S -degenerate non-complete intersection curves C in V . It follows from a general principle (cf. Lemmas 2.7 and 2.13) that if $H^1(V, N_{(C,S)/V}) = H^1(S, D) = 0$, then C is stably degenerate and unobstructed in V . On the other hand, for some curves C with $H^1(S, D) \neq 0$, it is difficult to prove that C is stably degenerate. For example, if $[C]$ is generated by \mathbf{h} and a (-2) -curve E on S of degree $E \cdot \mathbf{h} > 2$, then the intersection number $D \cdot E = (C - \mathbf{h}) \cdot E$ can be less than -2 (and then $h^1(S, D) > 1$), and thereby Lemma 2.14 does not apply to C . For a detailed study of the stability of S -degenerate curves on a smooth quartic 3-fold V ($g = 3$), we refer to [22], in which S is assumed to be of Picard rank 2.

Remark 4.7. Let $d(C)$ denotes the degree $(-K_V \cdot C)_V$ of C . Then by counting dimensions, we see that if $H^1(V, N_{(C,S)/V}) = 0$ and $g(C) < d(C) - g$, then C is not stably degenerate. In fact, by assumption we have

$$\dim_{(C,S)} \text{HF}^{sc} V = (-K_V)^3/2 + g(C) + 1 = g(C) + g,$$

while every component of the Hilbert scheme $\text{Hilb}^{sc} V$ is of dimension at least $\chi(C, N_{C/V}) = d(C)$ by (2.8). Thus if $pr'_1 : \text{Hilb}^{sc} \mathcal{S}(\subset \text{HF}^{sc} V) \rightarrow \text{Hilb } V$ is surjective in a neighborhood of $[C]$, then we have $g(C) + g \geq d(C)$.

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