

Convexity of Balls in Gromov–Hausdorff Space

Daria P. Klibus

Abstract

In this paper we study the space \mathcal{M} of all nonempty compact metric spaces considered up to isometry, equipped with the Gromov–Hausdorff distance. We show that each ball in \mathcal{M} with center at the one-point space is convex in the weak sense, i.e., every two points of such a ball can be joined by a shortest curve that belongs to this ball; however, such a ball is not convex in the strong sense: it is not true that every shortest curve joining the points of the ball belongs to this ball. We also show that a ball of sufficiently small radius with center at a space of general position is convex in the weak sense.

Introduction

The Gromov–Hausdorff distance was defined in 1975 by D. Edwards in article ”The Structure of Superspace” [1], then in 1981 it was rediscovered by M. Gromov [2].

We will investigate the geometry of the space \mathcal{M} of all nonempty compact metric spaces (considered up to isometry) with the Gromov–Hausdorff distance. It is well-known that the Gromov–Hausdorff distance is a metric on \mathcal{M} [3]. The Gromov–Hausdorff space is Polish (complete separable) and path-connected. Also A.O. Ivanov, N.K. Nikolaeva and A.A. Tuzhilin showed that the Gromov–Hausdorff metric is strictly intrinsic [4].

The present paper is devoted to the following question: are balls in the Gromov–Hausdorff space convex. There are two concepts: convexity in the weak sense (every two points of such a ball can be joined by a shortest curve that belongs to this ball) and strong sense (each shortest curve connecting any pair of points of the set, belongs to this set). We show that a ball of nonzero radius with center at the one-point space is convex in the weak sense, but not convex in the strong one. We also show that a ball of sufficiently small radius with center at a generic position space is convex in the weak sense.

I am grateful to my scientific adviser professor Alexey A. Tuzhilin and to professor Alexander O. Ivanov for stating the problem and regular attention to my work.

The work was supported by the Russian Foundation for Basic Research (grant No. 16-01-00378-a) and the program ”Leading Scientific Schools” (grant no. NSh-6399.2018.1).

1 Preliminaries

Let X be an arbitrary metric space. By $|xy|$ we denote the distance between it is two points x and y . For any point $x \in X$ and a real number $r > 0$ we denote by $U_\varepsilon(x) = \{y \in X : |xy| < \varepsilon\}$ the *open ball of radius ε centered at x* ; for each nonempty $A \subset X$ and a real number $r > 0$ we put $U_\varepsilon(A) = \cup_{a \in A} U_\varepsilon(a)$ an call it *open r -neighborhood of the set A* . The *closed ball of radius*

ε centered at x is $B_\varepsilon(x) = \{y \in X : |xy| \leq \varepsilon\}$. For $x \in X$ and nonempty $A \subset X$ we put $|xA| = \inf\{|xa| : a \in A\}$. For nonempty $A \subset X$ and non-negative r (possibly equal ∞), the *closed r -neighborhood of the set A* is $B_r(A) = \{x \in X : |xA| \leq r\}$.

Definition 1.1. Let X and Y be two nonempty subsets of a metric space. The *Hausdorff distance between X and Y* is $d_H(X, Y) = \inf\{\varepsilon > 0 \mid (U_\varepsilon(X) \supset Y) \ \& \ (U_\varepsilon(Y) \supset X)\}$.

By $\mathcal{H}(X)$ we denote the family of all nonempty closed bounded subsets of a metric space X .

Proposition 1.2 ([3]). *The function d_H is a metric on $\mathcal{H}(X)$.*

Proposition 1.3 ([5]). *For any metric space X , any $A \in \mathcal{H}(X)$, and any nonnegative r we have $B_r(A) \in \mathcal{H}(X)$.*

Definition 1.4 ([5]). Let W be any metric space, $a, b \in W$, $|ab| = r$, $s \in [0, r]$. A point $c \in W$ is *in s -position between a and b* , if $|ac| = s$ and $|cb| = r - s$.

Proposition 1.5 ([5]). *Let X be any metric space and $A, B \in \mathcal{H}(X)$, $r = d_H(A, B)$, $s \in [0, r]$. If a set $C \in \mathcal{H}(X)$ is in s -position between A and B , then $C \subset B_s(A) \cap B_{r-s}(B)$.*

Designation 1.6 ([5]). In what follows, we denote the set $B_s(A) \cap B_{r-s}(B)$ by $C_s(A, B)$.

Definition 1.7. Let X and Y be metric spaces. A triple (X', Y', Z) that consists of a metric space Z and its subsets X' and Y' isometric to X and Y , respectively, is called a *realization of the pair (X, Y)* . The *Gromov–Hausdorff distance $d_{GH}(X, Y)$ between X and Y* is the infimum of real numbers ρ such that there exists a realization (X', Y', Z) of the pair (X, Y) with $d_H(X', Y') \leq \rho$.

By \mathcal{M} we denote the set of all compact metric spaces, considered up to an isometry, with the Gromov–Hausdorff distance. The restriction of $d_{GH}(X, Y)$ onto \mathcal{M} is a metric [3].

A metric on a set X is *strictly intrinsic*, if any two points $x, y \in X$ are joined by a curve whose length is equal to the distance between x and y (this curve is called *shortest*).

Theorem 1 ([5]). *Let X be a complete locally compact space with intrinsic metric. Then for any $A, B \in \mathcal{H}(X)$, $r = d_H(A, B)$, $s \in [0, r]$, the set $C_s(A, B)$ belongs to $\mathcal{H}(X)$ and is in s -position between A and B .*

Corollary 1.8 ([3]). *Let X be a complete locally compact space with intrinsic metric (X is boundedly compact with strictly intrinsic metric [1]). Then $\mathcal{H}(X)$ is boundedly compact, and Hausdorff metric is strictly intrinsic.*

Corollary 1.9 ([5]). *Let X be a complete locally compact space with intrinsic metric, $A, B \in \mathcal{H}(X)$, and $r = d_H(A, B)$. Then $\gamma(s) = C_s(A, B)$, $s \in [a, b]$ is a shortest curve connecting A and B , where the length of curve γ is equal to $d_H(A, B)$, and the parameter s is natural.*

Definition 1.10. A nonempty subset M of a metric space X with strictly intrinsic metric is *convex in the weak sense*, if for any two points from M , some shortest curve connecting them belongs to M .

Definition 1.11. A nonempty subset M of a metric space X with strictly intrinsic metric is *convex in the strong sense*, if for any two points from M , any shortest curve connecting them belongs to M .

Let X and Y be arbitrary nonempty sets. Recall that a *relation* between the sets X and Y is a subset of the Cartesian product $X \times Y$. By $\mathcal{P}(X, Y)$ we denote the set of all nonempty relations between X and Y . Let $\pi_X: (X, Y) \rightarrow X$ and $\pi_Y: (X, Y) \rightarrow Y$ be the canonical projections, i.e., $\pi_X(x, y) = x$ and $\pi_Y(x, y) = y$. In the same way we denote the restrictions of the canonical projections to each relation $\sigma \in \mathcal{P}(X, Y)$.

Let us consider each relation $\sigma \in \mathcal{P}(X, Y)$ as a multivalued mapping whose domain may be less than X . Then, similarly with the case of mappings, for any $x \in X$ and any $A \subset X$ their images $\sigma(x)$ and $\sigma(A)$ are defined, and for any $y \in Y$ and any $B \subset Y$ their preimages $\sigma^{-1}(y)$ and $\sigma^{-1}(B)$ are also defined.

Definition 1.12. A relation $R \subset X \times Y$ between X and Y is called a *correspondence*, if the restrictions of the canonical projections π_X and π_Y onto R are surjective. By $\mathcal{R}(X, Y)$ we denote the set of all correspondences between X and Y .

Definition 1.13. Let X and Y be arbitrary metric spaces. The *distortion* $\text{dis } \sigma$ of a *relation* $\sigma \in \mathcal{P}(X, Y)$ is the value

$$\text{dis } \sigma = \sup \left\{ \left| |xx'| - |yy'| \right| : (x, y), (x', y') \in \sigma \right\}.$$

Proposition 1.14 ([3]). *For any metric spaces X and Y we have*

$$d_{GH}(X, Y) = \frac{1}{2} \inf \{ \text{dis } R : R \in \mathcal{R}(X, Y) \}.$$

Definition 1.15. A relation $R \in \mathcal{R}(X, Y)$ is called *optimal*, if $d_{GH}(X, Y) = \frac{1}{2} \text{dis } R$. The set of all optimal correspondences between X and Y is denoted by $\mathcal{R}_{\text{opt}}(X, Y)$.

Proposition 1.16 ([6]). *For any $X, Y \in \mathcal{M}$ we have $\mathcal{R}_{\text{opt}}(X, Y) \neq \emptyset$.*

Proposition 1.17 ([6]). *For any $X, Y \in \mathcal{M}$ and each $R \in \mathcal{R}_{\text{opt}}(X, Y)$ the family R_t , $t \in [0, 1]$, of compact metric spaces such that $R_0 = X$, $R_1 = Y$, and for $t \in (0, 1)$ the space R_t is equal to (R, ρ_t) , where $\rho_t((x, y), (x', y')) = (1-t)|xx'| + t|yy'|$, is a shortest curve in \mathcal{M} connecting X and Y .*

For a metric space X by $\text{diam}(X)$ we denote its *diameter*:

$$\text{diam}(X) = \sup_{x, x' \in X} |xx'|.$$

Let Δ_1 be a single-point space.

Assertion 1.18 ([3]). *For any metric space X we have $d_{GH}(X, \Delta_1) = \text{diam}(X)/2$.*

Definition 1.19. We say that a finite metric space M is *in general position*, or is a *space of general position*, if all its nonzero distances are distinct, and all triangle inequalities are strict.

For a metric space X we define the following values:

$$s(X) = \inf \{ |xy| : x \neq y \}, \quad e(X) = \inf \left\{ \left| |xy| - |zw| \right| : x \neq y, z \neq w, \{x, y\} \neq \{z, w\} \right\}.$$

Proposition 1.20 ([7]). *Let $M = \{1, \dots, n\}$ be a metric space. Then for any $0 < \varepsilon \leq s(M)/2$ and each $X \in \mathcal{M}$ such that $2d_{GH}(M, X) < \varepsilon$, there exists a partition $X = \sqcup_{i=1}^n X_i$ unique up to numeration by points of M , possessing the following properties:*

- (1) $\text{diam } X_i < \varepsilon$;
- (2) for any $i, j \in M$ and any $x \in X_i$ and $x' \in X_j$ (here the indices i and j may be equal to each other) it holds $\left| |xx'| - |ij| \right| < \varepsilon$.

Proposition 1.21 ([7]). *Let $M = \{1, \dots, n\}$ be a metric space. Then for any $0 < \varepsilon \leq s(M)/2$, any $X \in \mathcal{M}$, $2d_{GH}(M, X) < \varepsilon$, and each $R \in \mathcal{R}_{\text{opt}}(M, X)$ the family $\{R(i)\}_{i=1}^n$ is a partition of the set X , satisfying the following properties:*

- (1) $\text{diam } X_i < \varepsilon$;
- (2) for any $i, j \in M$, $x \in R(i)$, $x' \in R(j)$ it holds $\left| |xx'| - |ij| \right| < \varepsilon$.

Moreover, if R' is another optimal correspondence between M and X , then the partitions $\{R(i)\}_{i=1}^n$ and $\{R'(i)\}_{i=1}^n$ may differ from each other only by numerations generated by the correspondences $i \mapsto R(i)$ and $i \mapsto R'(i)$.

Definition 1.22 ([7]). The family $\{X_i\}$ from Proposition 1.20 we call the *canonical partition* of the space X with respect to M .

Proposition 1.23 ([7]). *Let $M = \{1, \dots, n\}$ be a metric space, $n \geq 3$, $e(M) > 0$. Choose an arbitrary $0 < \varepsilon \leq \frac{1}{4} \min\{s(M), e(M)\}$, any $X, Y \in \mathcal{M}$, $2d_{GH}(M, X) < \varepsilon$, $2d_{GH}(M, Y) < \varepsilon$, and let $\{X_i\}$ and $\{Y_i\}$ denote the canonical partitions of X and Y , respectively, w.r.t. M . Then for each $R \in \mathcal{R}_{\text{opt}}(X, Y)$ there exist $R_i \in R(X_i, Y_i)$ such that $R = \sqcup_{i=1}^n R_i$.*

2 The main results

Theorem 2. *A ball with center at the one-point metric space is convex in the weak sense.*

Proof. Let $B = B_r(\Delta_1)$ be a closed ball with center at Δ_1 , where $r > 0$, and $X, Y \in B$. By Proposition 1.18,

$$d_{GH}(\Delta_1, X) = \text{diam}(X)/2 \leq r, \quad d_{GH}(\Delta_1, Y) = \text{diam}(Y)/2 \leq r.$$

We choose some correspondence $R \in \mathcal{R}_{\text{opt}}(X, Y)$ which exists by Proposition 1.16. We construct the space $R_t = (R, \rho_t)$ with metric $\rho_t((x, y), (x', y')) = (1-t)|xx'| + t|yy'|$, where $t \in (0, 1)$, and put $R_0 = X$, $R_1 = Y$. Then, by Proposition 1.17, the curve R_t , $t \in [0, 1]$, connecting X and Y is shortest.

We show that the curve R_t lies in the ball B . To do that, we estimate the Gromov–Hausdorff distance between the center Δ_1 and the space R_t . We have

$$d_{GH}(\Delta_1, R_t) = \text{diam}(R_t)/2.$$

For any $x, x' \in X$ and $y, y' \in Y$ it holds

$$|xx'| \leq \text{diam } X \leq \max(\text{diam } X, \text{diam } Y); \quad |yy'| \leq \text{diam } Y \leq \max(\text{diam } X, \text{diam } Y).$$

Therefore,

$$\begin{aligned} \text{diam}(R_t) &= \max |(x, y)(x', y')|_{\rho_t} = \max((1-t)|xx'| + t|yy'|) \leq \\ &\leq (1-t) \max(\text{diam } X, \text{diam } Y) + t \max(\text{diam } X, \text{diam } Y) = \max(\text{diam } X, \text{diam } Y), \end{aligned}$$

thus, $d_{GH}(\Delta_1, R_t) \leq \frac{1}{2} \max(\text{diam } X, \text{diam } Y) = \max(d_{GH}(\Delta_1, X), d_{GH}(\Delta_1, Y)) \leq r$. \square

Theorem 3. *A ball with center at the one-point metric space is not convex in the strong sense.*

Proof. To prove that, we construct a shortest curve connecting some spaces $A, B \in B_r(\Delta_1) \subset \mathcal{M}$, but not containig in $B_r(\Delta_1)$. Let $A = [0, 2r] \subset \mathbb{R}$ and $B = \{0, 2r\} \subset \mathbb{R}$. We choose some correspondence $R \in \mathcal{R}_{\text{opt}}(B, A)$ (it exists by Proposition 1.16) and estimate it:

$$\begin{aligned} \text{dis } R &= \sup \left\{ |aa'| : a, a' \in R(0); |aa'| : a, a' \in R(2r); |2r - |aa' || : a \in R(0), a' \in R(2r) \right\} = \\ &= \sup \left\{ \text{diam } R(0), \text{diam } R(2r), |2r - |aa' || : a \in R(0), a' \in R(2r) \right\} \leq 2r. \end{aligned}$$

1) If $R(0) \cap R(2r) \neq \emptyset$, then choosing $a = a' \in R(0) \cap R(2r)$, we have $\text{dis } R = 2r$.

2) If $R(0) \cap R(2r) = \emptyset$, then for any $\varepsilon > 0$ there exist $a \in R(0)$, $a' \in R(2r)$, such that $|aa'| < \varepsilon$, thus $\text{dis } R = 2r$.

Then, by Proposition 1.14, we have $d_{GH}(A, B) = r$. Since $d_H(A, B) = r$, it holds $d_{GH}(A, B) = d_H(A, B)$. For $t \in [0, r]$ we put $\gamma(t) = C_t(A, B) = B_t(A) \cap B_{r-t}(B)$. Applying the corollary 1.9, we see, that $\gamma(t)$ is a shortest curve in $\mathcal{H}(\mathbb{R})$.

For any partition $t_0 = 0 < t_1 < \dots < t_n = r$ of the segment $[0, r]$ we have

$$d_{GH}(A, B) \leq \sum_{i=1}^n d_{GH}(\gamma(t_{i-1}), \gamma(t_i)) \leq \sum_{i=1}^n d_H(\gamma(t_{i-1}), \gamma(t_i)) = d_H(A, B) = d_{GH}(A, B),$$

so

$$d_{GH}(A, B) = \sum_{i=1}^n d_{GH}(\gamma(t_{i-1}), \gamma(t_i)).$$

Since the length of the curve γ is equal to the supremum of the sums $\sum_{i=1}^n d_{GH}(\gamma(t_{i-1}), \gamma(t_i))$ over all possible partitions of the segment $[0, r]$, and all these sums are the same and equal $d_{GH}(A, B)$, then the length of the curve γ is equal to $d_{GH}(A, B)$, therefore γ is a shortest curve.

We show that this curve does not lie entirely in the ball $B_r(\Delta_1)$. To do that we calculate $d_{GH}(C_t(A, B), \Delta_1)$. By Assertion 1.18, $d_{GH}(C_t(A, B), \Delta_1) = \frac{\text{diam}(C_t(A, B))}{2}$. Notice that for $t = \frac{r}{2}$ we have $\text{diam}(C_{\frac{r}{2}}(A, B)) = 3r$, so $d_{GH}(C_{r/2}(A, B), \Delta_1) = 3r/2 > r$, thus $\gamma(r/2) \notin B_r(\Delta_1)$. \square

Theorem 4. *For any space $M \in \mathcal{M}$ in general position and any $0 < r \leq \frac{1}{4} \min\{s(M), e(M)\}$ the ball with center at M and radius r is convex in the weak sense.*

Proof. Let $\varepsilon = 2r$, $M = \{1, \dots, n\}$, and $X, Y \in B_{\varepsilon/2}(M)$. By Proposition 1.20, there exist unique (up to a numeration of points of M) partitions $X = \sqcup_{i=1}^n X_i$ and $Y = \sqcup_{i=1}^n Y_i$, possessing the following properties: for any $x_i \in X_i, x_j \in X_j, y_i \in Y_i, y_j \in Y_j$ it holds $||x_i x_j| - |ij|| < \varepsilon$ and

$\left| |y_i y_j| - |ij| \right| < \varepsilon$. By Proposition 1.23, for any $R \in \mathcal{R}_{\text{opt}}(X, Y)$ there are $R_i \in R(X_i, Y_i)$, where $R = \sqcup_{i=1}^n R_i$. We choose some correspondence $R \in \mathcal{R}_{\text{opt}}(X, Y)$ (it exists by Proposition 1.16, it exists).

We construct a shortest curve R_t , as in Proposition 1.17. To prove convexity in the weak sense, we show that $d_{GH}(M, R_t) \leq \varepsilon/2$. Let us introduce a the correspondence $R' \in \mathcal{R}(M, R_t)$ us $R' = \sqcup_{i=1}^n \{i\} \times R_i$. We have

$$\begin{aligned} d_{GH}(M, R_t) &\leq \frac{1}{2} \text{dis } R' = \frac{1}{2} \sup \left\{ \left| |ij| - |p_i p_j|_t \right| : i, j \in M, (i, p_i), (j, p_j) \in R' \right\} = \\ &= \frac{1}{2} \sup \left\{ \left| |ij| - (1-t)|x_i x_j| - t|y_i y_j| \right| : i, j \in M, (x_i, y_i) = p_i \in R_i, (x_j, y_j) = p_j \in R_j \right\} = \\ &= \frac{1}{2} \sup \left\{ \left| (1-t)|ij| + t|ij| - (1-t)|x_i x_j| - t|y_i y_j| \right| \right\} = \\ &= \frac{1}{2} \sup \left\{ \left| (1-t)(|ij| - |x_i x_j|) + t(|ij| - |y_i y_j|) \right| \right\} \leq \\ &\leq \frac{1}{2}(1-t) \sup \left\{ \left| |ij| - |x_i x_j| \right| \right\} + \frac{1}{2}t \sup \left\{ \left| |ij| - |y_i y_j| \right| \right\} \leq \frac{1}{2}(1-t)\varepsilon + \frac{1}{2}t\varepsilon = \frac{\varepsilon}{2}. \end{aligned}$$

□

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