

# Remarks on Banach spaces determined by their finite dimensional subspaces

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## Abstract

A separable Banach space  $X$  is said to be *finitely determined* if for each separable space  $Y$  such that  $X$  is finitely representable (f.r.) in  $Y$  and  $Y$  is f.r. in  $X$  then  $Y$  is isometric to  $X$ . We provide a direct proof (without model theory) of the fact that every finitely determined space  $X$  (isometrically) contains every (separable) space  $Y$  which is finitely representable in  $X$ . We also point out how a similar argument proves the Krivine-Maurey theorem on stable Banach spaces, and give the model theoretic interpretations of some results.

## 1 Introduction

This paper is a kind of companion-piece to [4], although here we are mainly concerned with direct proofs (without any use of model theory) of some results of [4] and some new results. The main results are Theorem 1.5 on connections of isometry groups and existence of classical sequence spaces, Theorem 1.8 on isometry group of ‘finitely determined spaces’, Corollary 1.9 showing that the finitely determined spaces contain ‘good’ subspaces, and Remark 1.10 pointing out that the Krivine-Maurey theorem has an equivalent formulation in the language of general topology.

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We study Banach spaces which have ‘large’ isometry groups. These spaces are very ‘symmetrical’ and have ‘good’ substructures. Recall that a separable Banach space  $X$  is said to be *determined by its first order theory* (or  $\aleph_0$ -categorical) provided that  $X$  is isometric to every space  $Y$  such that ultrapowers  $X_{\mathcal{U}}$  and  $Y_{\mathcal{U}}$  are isometric for some ultrafilter  $\mathcal{U}$ . In fact, a space is determined by its first order theory if it is the only separable model of its first order theory in the sense of Continuous Logic (see [1]). In [4], it is shown that every  $\aleph_0$ -categorical space contains some  $\ell_p$  or  $c_0$ . In this paper we show that every space which is determined by its finite dimensional subspaces (see Definition 1.7 below) contains  $\ell_p$  for each  $p$  in its spectrum. In fact the latter is a consequence of the prior, and in this paper we give direct proofs of them and some new results.

One reason for restricting our attention to direct proofs (without model theory) is to make the paper more accessible to Banach-theorists and other interested readers.

## 1.1 Isomery groups and classical sequences

Let  $(X, d)$  be a complete metric space and  $G$  a closed subgroup of isometry group  $\text{Iso}(X)$  with the natural topology (i.e. the topology pointwise convergence). For  $x \in X^{\mathbb{N}}$ , we let  $[x]$  (or  $[x]_G$ , if there is a risk of ambiguity) denote the closure of the orbit  $x$  with respect to the product topology on  $X^{\mathbb{N}}$ . We fix a metric inducing the product topology and it is denoted by  $d$  again.

**Definition 1.1** ([2]). Let  $(X, d)$  be a complete metric space and  $G$  a closed subgroup of isometry group  $\text{Iso}(X)$ . We equip the set of orbit closures

$$X//G = \{[x]_G : x \in X\}$$

with the metric induced from  $X$

$$d([x], [y]) = \inf\{d(u, v) : u \in [x], v \in [y]\}.$$

We say that the action  $G$  on  $X$  is *approximately oligomorphic* (or short,  $G$  is *approximately oligomorphic*) if  $X^n//G$  is compact for all  $n$ .

**Remark 1.2.** (i) Since elements of  $G$  are isometry, then  $[x] \cap [y] \neq \emptyset$  if and only if  $[x] = [y]$ .

(ii) Since  $G$  is a subgroup of isometries,  $d$  is a metric on  $X//G$ , i.e. the triangle inequality holds and  $d([x], [y]) = 0$  implies  $[x] = [y]$ .

- (iii)  $G$  is approximately oligomorphic iff  $X^{\mathbb{N}}//G$  is compact (see the discussion after Definition 2.1 in [2]).
- (iv) If  $(X, d)$  is complete then  $X^{\mathbb{N}}//G$  is complete. So the latter space is compact iff it is totally bounded.

Now we can prove some results, after introducing some notations:

**Notation 1.3.** Let  $X$  be a Banach spaces and  $x_1, \dots, x_n, y_1, \dots, y_n \in X$ . For  $\epsilon > 0$ , we write  $(x_1, \dots, x_n) \sim_{\epsilon} (y_1, \dots, y_n)$  if for each  $r_1, \dots, r_n \in \mathbb{R}$ ,

$$(1 - \frac{1}{\epsilon}) \left\| \sum_1^n r_i y_i \right\| \leq \left\| \sum_1^n r_i x_i \right\| \leq (1 + \frac{1}{\epsilon}) \left\| \sum_1^n r_i y_i \right\|.$$

We write  $(x_1, \dots, x_n) \sim_{\epsilon} (e_1, \dots, e_n)$  (where  $(e_n)$  is the standard basis of  $\ell_p$ ) if for each  $r_1, \dots, r_n \in \mathbb{R}$ ,

$$(1 - \frac{1}{\epsilon}) \left( \sum_1^n |r_i|^p \right)^{\frac{1}{p}} \leq \left\| \sum_1^n r_i x_{n,i} \right\| \leq (1 + \frac{1}{\epsilon}) \left( \sum_1^n |r_i|^p \right)^{\frac{1}{p}}.$$

We give an easy lemma.

**Lemma 1.4.** *Let  $X$  be a Banach space and  $x_1, \dots, x_n, y_1, \dots, y_n \in X$  such that  $(x_1, \dots, x_n) \sim_{\epsilon} (y_1, \dots, y_n)$  for some  $\epsilon > 0$ . Then for every linear isometry  $g$  on  $X$ ,  $(g(x_1), \dots, g(x_n)) \sim_{\epsilon} (y_1, \dots, y_n)$ .*

*Proof.* Immediate, since  $g$  is linear and isometry.  $\square$

**Theorem 1.5.** *Suppose that  $X$  is a separable Banach space and  $G$  is a closed subgroup of linear isometries on  $X$ . If  $X^{\mathbb{N}}//G$  is compact then  $X$  contains  $c_0$  or  $\ell_p$  (for some  $1 \leq p < \infty$ ).*

*Proof.* By Krivine's theorem, there is some  $p \in [1, \infty]$  such that  $\ell_p$  is finitely representable in  $X$ . So, for each  $n \in \mathbb{N}$ , let  $x_{n,1}, x_{n,2}, \dots, x_{n,n}$  be in  $X$  such that  $(x_{n,1}, x_{n,2}, \dots, x_{n,n}) \sim_{\frac{1}{n}}$   $(e_1, \dots, e_n)$ . Set  $x_n = (x_{n,1}, \dots, x_{n,n}, z_{n+1}, z_{n+2}, \dots)$  where  $z_i$  are arbitrary in  $\overset{\mathbb{N}}{X}$ . By compactness, the sequence  $([x_n])_{n \in \mathbb{N}}$  has a cluster point  $[x]$  in  $X^{\mathbb{N}}//G$ . Now, it is easy to verify that  $[x] = [(x_1, x_2, \dots)]$  contains  $\ell_p$ . Indeed, for each  $\epsilon > 0$ , there is a  $[x_n]$  such that  $d([x_n], [x]) \leq \epsilon$ . Since elements of  $G$  are linear isometry, as  $(x_{n,1}, \dots, x_{n,n}) \sim_{\epsilon} (x_1, \dots, x_n)$  we have  $(x_1, \dots, x_n) \sim_{2\epsilon} (e_1, \dots, e_n)$ . As  $\epsilon$  is arbitrary, the proof is completed.  $\square$

For non-specialists in Banach space theory, we mention that a Banach space  $X$  is said to be *finitely representable* (f.r.) in another Banach space  $Y$  if for each finite dimensional subspace  $X_n$  of  $X$  and each number  $\lambda > 1$ , there is an isomorphism  $T_n$  of  $X_n$  into  $Y$  for which  $\lambda^{-1}\|x\| \leq \|T_n(x)\| \leq \lambda\|x\|$  if  $x \in X_n$ .

**Remark 1.6.** In the above (1.5), by a similar argument, one can show that if a Banach space  $Y$  which has a basis is f.r. in  $X$ , and  $X^{\mathbb{N}}//G$  is compact, then  $X$  (isometrically) contains  $Y$ .

**Definition 1.7.** A separable Banach space  $X$  is said to be *determined by its finite dimensional subspaces* (or short *finitely determined*) if for each separable space  $Y$  such that  $X$  is finitely representable (f.r.) in  $Y$  and  $Y$  is f.r. in  $X$  then  $Y$  is isometric to  $X$ .

Now we recall some facts from general topology. Let  $X$  be a topological space,  $(x_i)_{i \in I}$  an indexed family in  $X$ , and  $\mathcal{U}$  an ultrafilter on  $I$ . For  $x \in X$ , we write  $\lim_{\mathcal{U}, i} x_i = x$  and say that  $x$  is the  $\mathcal{U}$ -limit of  $(x_i)_{i \in I}$  if for every neighborhood  $U$  of  $x$ , the set  $\{i \in I : x_i \in U\}$  is in  $\mathcal{U}$ . A basic fact in general topology is that  $X$  is compact and Hausdorff iff for every indexed family  $(x_i)_{i \in I} \in X$  and every ultrafilter  $\mathcal{U}$  on  $I$ , the  $\mathcal{U}$ -limit exists and is unique. We use this fact in the following.

**Theorem 1.8.** *Suppose that  $X$  is determined by its finite dimensional subspaces, and  $G_L$  the group of all linear isometries on  $X$ . Then  $X^{\mathbb{N}}//G_L$  is compact.*

*Proof.* Suppose, if possible, that  $X^{\mathbb{N}}//G_L$  is not compact. So, there is an ultrafilter  $\mathcal{U}$  on  $\mathbb{N}$  and sequence  $\{x_n\} \in X^{\mathbb{N}}//G$  such that the  $\mathcal{U}$ -limit does not exist (in  $X^{\mathbb{N}}//G$ ), this means that for every  $u_n \in x_n$ , the  $\mathcal{U}$ -limit of the sequence  $\{u_n\}$  does not exist in  $X^{\mathbb{N}}$ . Let  $X_{\mathcal{U}}$  be the  $\mathcal{U}$ -ultrapower of  $X$  (see [3], for the definition). By Theorem 6.3 of [3],  $X_{\mathcal{U}}$  is f.r. in  $X$ . For each  $n$ , let  $(x_{n,1}, x_{n,2}, \dots) \in x_n$ . Then, for each  $m$ , the sequence  $\{x_{n,m}\}_{n=1}^{\infty}$  determines an element  $y_m = (x_{n,m})_{\mathcal{U}}$  in  $X_{\mathcal{U}}$  (see again [3] for definition of elements of  $X_{\mathcal{U}}$ ). So,  $\{y_1, y_2, \dots\}$  is f.r. in  $X$ . Let  $\hat{X}$  be the Banach space generated by  $X \cup \{y_1, y_2, \dots\}$ . Clearly,  $\hat{X}$  and  $X$  are not isometrically isomorphic. Indeed, suppose, if possible, that  $f : \hat{X} \rightarrow X$  is an isometrically isomorphism and  $f(y_n) = z_n$  for all  $n$ . Since  $\lim_{\mathcal{U}, n} x_{n,m} = y_m$ , so  $(z_1, z_2, \dots)$  is the  $\mathcal{U}$ -limit of the sequence  $\{(f(x_{n,1}), f(x_{n,2}), \dots)\}_{n=1}^{\infty}$ , but  $(f(x_{n,1}), f(x_{n,2}), \dots) \in x_n$  for all  $n$  (note that  $f$  is a *linear isometry*), and the sequence  $\{x_n\}$  was not

convergent. To summarize, the space  $\hat{X}$  is different from  $X$ , but  $\hat{X}$  is f.r. in  $X$  and  $X$  is f.r. in  $\hat{X}$ . This is a contradiction.  $\square$

**Corollary 1.9.** *Suppose that  $X$  is determined by its finite dimensional subspaces. Then  $X$  contains  $c_0$  or  $\ell_p$ . Moreover,  $X$  contains every Banach space which has a basis and is f.r. in  $X$ .*

*Proof.* Putting 1.5 and 1.8 together we get the first part. For the general case, an argument similar to 1.5 for any such space  $Y$  works well.  $\square$

**Remark 1.10.** (i) Recall that two spaces  $X$  and  $Y$  are called *almost isometric* if for each  $\lambda > 1$ , they are  $\lambda$ -isomorphic, that is, there is a linear isomorphism  $T : X \rightarrow Y$  such that for all  $x \in X$ ,  $\lambda^{-1}\|x\| \leq \|T(x)\| \leq \lambda\|x\|$ . In the above, one can work with *almost* linear isometries instead of linear isometries, and get similar results. Indeed, we say that  $\bar{x}, \bar{y} \in X^{\mathbb{N}}$  are in the same class, and write  $\bar{x} \sim \bar{y}$ , if for each  $\lambda > 1$  there is a linear  $\lambda$ -isomorphism such that  $T(\bar{x}) = \bar{y}$ . Now, if  $X^{\mathbb{N}}/\sim$  is compact (with respect to the metric which is defined similar to Definition 1.1), then  $X$  almost isometrically contains any space  $Y$  which has a basis and is f.r. in  $X$ . To summarize, we say that a Banach space  $X$  is *almost determined by its finite dimensional subspaces* if for any space  $Y$  such that  $X$  is f.r. in  $Y$  and  $Y$  is f.r. in  $X$ , we have  $X$  and  $Y$  are almost isometric. So, every almost determined space (almost isometrically) contains any space which is f.r. in the space.

(ii) By a similar argument, one can show that  $\aleph_0$ -categoricity implies compactness of  $X^{\mathbb{N}}/\text{Aut}(X)$ , where  $\text{Aut}(X)$  is the *group of automorphisms of structure  $X$*  (see [1] for the definition). Note that  $\text{Aut}(X)$  is a closed subgroup of  $G_L$ . So,

$$\text{finitely determined} \implies \aleph_0\text{-categorical} \implies X^{\mathbb{N}}/\text{Aut}(X) \text{ compact} \implies X^{\mathbb{N}}/G_L \text{ compact}$$

(iii) **(stability):** By a similar argument one can show that a stable Banach space (in the sense of Krivine and Maurey) almost isometrically contains some  $\ell_p$ . Indeed, for a space  $X$ , let  $X_{\mathcal{U}}$  be an ultrapower of  $X$ , and  $G_L$  the isometry group of  $X_{\mathcal{U}}$ . Then we consider the space  $X_{\mathcal{U}}/G_L$ . Let  $A = \{[x] : x \in X^{\mathbb{N}}\}$ . One can prove that a space  $X$  is stable iff the closure of  $A$ , denoted by  $\bar{A}$ , is compact in  $X_{\mathcal{U}}/G_L$ . Clearly, if  $\bar{A}$  is compact, then some  $\ell_p$  is a limit of a *sequence* in  $A$ , by Krivine's theorem.

## 2 Model theoretic interpretations

Recall that a separable Banach space  $X$  is said to be *determined by its first order theory* if it is the only separable model of its first order theory in the sense of Continuous Logic (see [1]). In this case,  $X$  is called  $\aleph_0$ -categorical or *separably categorical*.

**Remark 2.1.** (i)  $X$  is f.r. in  $Y$  if and only if there is an ultrafilter  $\mathcal{U}$  such that  $X$  is isometric to a subspace of the ultrapower  $Y_{\mathcal{U}}$ . (See [3], Theorem 6.3.)  
(ii) If  $X$  and  $Y$  have the same first order theory, denoted by  $Th(X) = Th(Y)$  or  $X \equiv Y$ , then there is an ultrafilter  $\mathcal{U}$  such that the ultrapowers  $X_{\mathcal{U}}$  and  $Y_{\mathcal{U}}$  are isometric. (See [1], Theorem 5.7.)

**Corollary 2.2.** *Every separable space which is determined by its finite dimensional subspaces is also determined by its first order theory, equivalently, it is  $\aleph_0$ -categorical.*

*Proof.* Putting (i) and (ii) of Remark 2.1 together the proof is completed.  $\square$

The converse of the above does not hold in general. For example,  $L_p[0, 1]$  is determined by its theory but  $\ell_p$  is f.r. in  $L_p[0, 1]$  and vice versa. In fact,  $X$  is f.r. in  $Y$  if and only if  $Th\exists(X) \subseteq Th(Y)$ , where  $Th\exists(X)$  is the existential theory of  $X$ .

**Fact 2.3** (Corollary 1.9 revisited). *Suppose that  $X$  is determined by its finite dimensional subspaces. Then  $X$  (isometrically) contains every (separable) space that is f.r. in  $X$ .*

*Proof.* This is a consequence of Downward Löwenheim–Skolem Theorem for continuous logic. Indeed, suppose that  $Y$  is f.r. in  $X$ . Then, there is an ultrafilter  $\mathcal{U}$  such that  $Y$  is isometric to a subspace of  $X_{\mathcal{U}}$ . By Proposition 7.3 in [1], there is a separable subspace  $Z$  of  $X_{\mathcal{U}}$  such that  $Z$  contains  $Y$ , and  $X$  and  $Z$  have the same first order theory. Now, since  $X$  is determined by its finite dimensional subspaces, by Corollary 2.2,  $X$  and  $Z$  are isometric, and so  $X$  contains  $Y$ .  $\square$

**Remark 2.4.** (i) By Dvoretzky’s theorem, the Hilbert space  $\ell_2$  is finitely representable in every space. So every space which is (almost) determined by its finite dimensional subspaces (almost) isometrically contains  $\ell_2$ .  
(ii) The Hilbert space  $\ell_2$  is determined by its finite dimensional subspaces.

(iii) The argument of the proof of Fact 2.3 is similar to the argument due to Ward Henson for existence of  $\ell_2$ . We thank professor Henson for communicating to us his argument. On the other hand, Fact 2.3 can be considered as a consequence of the main theorem of [4].

(iv) The above observations (i.e. Theorems 1.5, 1.8) seem to be new for the Banach spaces theorists. We thank William Johnson for useful comments.

**Question 2.5.** *Is every (separable, infinite-dimensional) Banach space which is (almost) determined by its finite dimensional subspaces (almost) isometric to  $\ell_2$ ? (We thank William Johnson and Timothy Gowers for guiding us to this question.)*

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