SHARP OFF-DIAGONAL WEIGHTED WEAK TYPE ESTIMATES FOR SPARSE OPERATORS

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Abstract. We prove sharp weak type weighted estimates for a class of sparse operators that includes majorants of standard singular integrals, fractional integral operators, and square functions. These bounds are knows to be sharp in many cases, and our main new result is the optimal bound

$$[w]_{A_{p,q}}^{\frac{1}{q}}[w^q]_{A_\infty}^{\frac{1}{2}-\frac{1}{p}} \lesssim [w]_{A_{p,q}}^{\frac{1}{2}-\frac{\alpha}{d}}$$

for proper conditions which satisfy that three index p, q and α ensure weak type norm of fractional square functions on $L^q(w^q)$ with $p > 2$.

1. Introduction

We study weighted inequalities for sparse operators, which can be defined by

$$\mathcal{A}_{\alpha,\nu}^{\mathcal{S}}(f) := \Big(\sum_{Q\in\mathcal{S}} \langle f\rangle_{\alpha,Q}^{\nu}\mathbf{1}_Q\Big)^{\frac{1}{\nu}}, \quad \langle f\rangle_{\alpha,Q} = \frac{1}{|Q|^{1-\frac{\alpha}{d}}}\int_Q f, \tag{1.1}$$

where $\nu > 0$, $0 \le \alpha < d$ and $\mathcal{S}$ is a sparse collection of dyadic cubes, i.e. all (dyadic) cubes $Q \in \mathcal{S}$, there exists $E_Q \subset Q$ which are pairwise disjoint and $|E_Q| \ge \gamma|Q|$ with $0 < \gamma < 1$. Note that $\langle f\rangle_Q$ denote $\langle f\rangle_{\alpha,Q}$ with $\alpha = 0$. And so far it it know that the operator $\mathcal{A}_{\alpha,\nu}^{\mathcal{S}}$ dominate large classes of classical operators T, relying upon the sparse domination formula

$$|Tf(x)| \lesssim \sum_{i=1}^{N} \mathcal{A}_{\alpha,\nu}^{\mathcal{S}_i}(|f|)(x), \tag{1.2}$$

where the collections $\mathcal{S}_i$ depend on the function f. For $\nu = 1$ and $\nu = 2$ with $\alpha = 0$, T becomes the Calderón-Zygmund singular integrals [13, 20] and Littlewood-Paley square functions [17, 19], respectively. Thus, the various norm inequalities that we prove for $\mathcal{A}_{\alpha,\nu}^{\mathcal{S}}$ immediately translate to corresponding estimates for these classses of classical operators.

A weight w on $\mathbb{R}^d$ is a locally integrable function w: $\mathbb{R}^d \to (0, +\infty)$. The class of all A_∞ weights consists of all weights w for which their A_∞ characteristic

$$[w]_{A_\infty} := \sup_Q \frac{1}{w(Q)}\int_Q M(\mathbf{1}_Q w) < \infty,$$

where M is the Hardy-Littlewood maximal function and the suprema take over cubes of sides are parallel to the coordinate axes.

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More precisely, we are concerned with quantifying the dependence of various weighted operator norms on a mixture of the two weight $A^{\alpha}_{p,q}$ characteristic

$$[w,\sigma]_{A^{\alpha}_{p,q}} := \sup_{Q\in\mathcal{S}} |Q|^{q(\frac{\alpha}{d}-1)} w(Q)\sigma(Q)^{\frac{q}{p'}} < \infty.$$

The study of such maixed bounds was initiated in [12]. All our estimates will be stated in a dual-weight formulation, in which the classical one-weight off-diagonal case $A_{p,q}$ as defined below.

Since we dealing with sparse operators, we also consider the sparse versions of the weight characteristics, where the supremums above are over dyadic cubes only. This is a standing convention throughout this paper without further notice.

Throughout this paper, $1 < p, p', q < \infty$, p and p' are conjugate indices, i.e. $1/p + 1/p' = 1$. Formally, we will also define $p = 1$ as conjugate to $p' = \infty$ and vice versa.

Now, we formulate our main results as follows.

Theorem 1.1. *Let $0 < \nu < \infty$, $0 \le \alpha < d$ and $1 < p \le q < \infty$. Let w, σ be a pair of weights. Then*

$$\|\mathcal{A}^{\mathcal{S}}_{\alpha,\nu}(\cdot\sigma)\|_{L^p(\sigma)\to L^{q,\infty}(w)} \lesssim [w,\sigma]^{\frac{1}{q}}_{A^{\alpha}_{p,q}} \begin{cases} [w]^{\frac{1}{\nu}(1-(\frac{\nu}{p})^2)}_{A_\infty}[\sigma]^{\frac{1}{\nu}(\frac{\nu}{p})^2}_{A_\infty}, & p = q > \nu \quad \text{and} \quad \alpha > 0, \\ [\sigma]^{\frac{1}{q}}_{A_\infty}, & p \le \nu \le q, \\ [w]^{(\frac{1}{\nu}-\frac{1}{p})_+}_{A_\infty}, & \text{other case.} \end{cases} \tag{1.3}$$

where $x_+ := \max(x, 0)$ in the exponent. Here and below, we simplify case analysis by interpreting $[w]^0_{A_\infty} = 1$, whether or not $[w]_{A_\infty}$ is finite.

Lacey and Scurry [16] provided a method to proof of the case $q < \nu$ of Theorem 1.1, and we merely repeat their one-weight proof in the two-weight off-diagonal case. For $p > \nu$, we bound

$$[w,\sigma]^{\frac{1}{q}}_{A^{\alpha}_{p,q}}[w^q]^{\frac{1}{\nu}-\frac{1}{p}}_{A_\infty} \lesssim [w]^{\frac{1}{q}}_{A_{p,q}}[w]^{\frac{1}{\nu}-\frac{1}{p}}_{A_{p,q}} = [w]^{\frac{1}{\nu}-\frac{\alpha}{d}}_{A_{p,q}} \tag{1.4}$$

is new even in the one weight case for $\frac{1}{q} + \frac{\alpha}{d} = \frac{1}{p}$. For $\nu \le q \le \frac{\nu}{1-\frac{\nu\alpha}{d}}$, we also obtain the bounds $[w]^{\frac{1}{q}}_{A_{p,q}}$ and it has an additional logarithmic factor, taking the form $(1 + \log[w^q]_{A_\infty})^{\frac{1}{\nu}}$. This form bouds which will be proved in Section 4.

Theorem 1.1 include several known cases, the Sobolev type case $\frac{1}{q} + \frac{\alpha}{d} = \frac{1}{p}$ of these results, together with strong type estimate and multilinear extensions, can also be recovered from Fackler and Hytönen [5], Zorin-Kranich [24] the recent general framework, respectively.

For $\nu = 1$ and $\alpha = 0$, (1.2) holds for all Calderón-Zygmund operators. Lerner [20] first prove the result, and Lacey [13] give the most general version, with a simplified proof in the paper [21]. The bound (1.3) in this case was obtained in [12] for $p = q = 1$. In [11], Hänninen and Lorist consider the sparse domination for the lattice Hardy-Littlewood maximal operator, and their obtained sharp weighted weak L^p estimates.

For $\nu = 2$ and $\alpha = 0$, (1.2) holds for several square function operators of Littlewood-Paley type [6, 16, 17]. For $p = q$, the mixed bound (1.3), even for general $\nu > 0$, is from [11, 14]. This improves the pure A_p bound of [6, 16, 17].

For $\nu = 1$ and $0 < \alpha < d$, (1.2) holds for the fractional integral operator [15]

$$I_\alpha f(x) := \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{n-\alpha}} dy. \tag{1.5}$$

In the case for $p < q$, (1.3) are due to [3]. The Sobolev type case with $\frac{1}{q} + \frac{\alpha}{d} = \frac{1}{p}$ was obtained by the same authors in [4]. Additional complications with $p = q$, which lead to the weaker version of our bound (1.3), have been observed and addressed in different ways in [3, 4].

For $\nu > 0$ and $\alpha = 0$, the bound (1.3) in the case was obtained by Hytönen and Li [11] for $p = q \in (1, \infty)$.

Theorem 1.1 with $\nu = 2$ completes the picure of sharp weighted inequalities for fractional square functions, aside from the remaining case of $2 \le q \le \frac{2}{1-\frac{2\alpha}{d}}$. Namely, $[w]_{A_{p,q}}^{\max(\frac{1}{q}, \frac{1}{2}-\frac{\alpha}{d})}$ is the optimal bound among all possible bounds of form $\Phi([w]_{A_{p,q}})$ with an incrasing function Φ. This was shown by Hytönen and Li [11], Lacey and Scurry [16] in the category of power type function $\Phi(t) = t^\beta$; a variant of their argument proves the general claim, as we show in the last section.

To prove the above results, we need the following characterization, which is essentially due to Lai [18]; we supply the necessary details to cover the cases that were not explicitly treated in [18].

Theorem 1.2. *Let $1 < p \le q < \infty$, $\nu > 0$, $p > \nu$ and $0 \le \alpha < d$. Let w, σ be a pair of weights. Then*

$$\|\mathcal{A}_{\alpha,\nu}^{\mathcal{S}}(\cdot\sigma)\|_{L^p(\sigma)\to L^{q,\infty}(w)}^\nu \simeq \mathscr{T}^*,$$

where the testing constants defined by

$$\mathscr{T}^* := \sup_{R\in\mathcal{S}} w(R)^{-\frac{1}{(\frac{q}{\nu})'}} \Big\| \sum_{\substack{Q\in\mathcal{S}\\ Q\subset R}} \langle\sigma\rangle_{\alpha,Q}^{\nu-1} \langle w\rangle_{\alpha,Q} \mathbf{1}_Q \Big\|_{L^{(\frac{p}{\nu})'}(\sigma)}.$$

The case $p > \nu$ of Theorem 1.1 is a consequence of Theorem 1.2. The estimation of the testing $\mathscr{T}^*$ given by Fackler and Hytönen [5] and their obtained following result.

Proposition 1.3. *Let $\nu > 0$, $0 \le \alpha < d$, $p > \nu$ and $1 < p \le q < \infty$. For $\mathscr{T}^*$ as in Theorem 1.2, we have*

$$\mathscr{T}^* \lesssim [w,\sigma]_{A_{p,q}^\alpha}^{\frac{\nu}{q}} \begin{cases} [w]_{A_\infty}^{1-(\frac{\nu}{p})^2} [\sigma]_{A_\infty}^{(\frac{\nu}{p})^2}, & p = q \quad \textit{and} \quad \alpha > 0, \\ [w]_{A_\infty}^{1-\frac{\nu}{p}}, & \textit{other case.} \end{cases}$$

The plan of the paper is as follows: We come with the proof of Theorem 1.2, this completes the proof of Theorem 1.1 in the case of $p > \nu$. The remaining case of Theorem 1.1 for $p \le \nu$ is then handled in Section 3. In the final scetion, we discuss the sharpness of our weak type estimates by modifying the example given by Lacey and Scurry [16].

2. Proof of Theorem 1.2

As mentioned, Theorem 1.2 is essentially duo to Hytönen and Li [11].

First, we give the following lemma.

Lemma 2.1. *Let w, σ be a pair of weights and $p > \nu > 0$.*

$$\|\mathcal{A}_{\alpha,\nu}^{\mathcal{S}}(\cdot\sigma)\|_{L^p(\sigma)\to L^{q,\infty}(w)} \simeq \sup_{\|f\|_{L^p(\sigma)}=1} \Big\| \sum_{Q\in\mathcal{S}} \langle\sigma\rangle_{\alpha,Q}^{\nu} \langle f^\nu\rangle_Q^\sigma \mathbf{1}_Q \Big\|_{L^{\frac{q}{\nu},\infty}(w)}$$

Proof. By the definition of $\mathcal{A}_{\alpha,\nu}^{\mathcal{S}}$, we have

$$\begin{aligned}
\|\mathcal{A}_{\alpha,\nu}^{\mathcal{S}}(\cdot\sigma)\|_{L^p(\sigma)\to L^{q,\infty}(w)} &= \sup_{\|f\|_{L^p(\sigma)}=1} \Big\| \sum_{Q\in\mathcal{S}} \langle f\sigma\rangle_{\alpha,Q}^{\nu} \mathbf{1}_Q \Big\|_{L^{\frac{q}{\nu},\infty}(w)} \\
&= \sup_{\|f\|_{L^p(\sigma)}=1} \Big\| \sum_{Q\in\mathcal{S}} \langle\sigma\rangle_{\alpha,Q}^{\nu} (\langle f\rangle_Q^\sigma)^\nu \mathbf{1}_Q \Big\|_{L^{\frac{q}{\nu},\infty}(w)} \\
&\le \sup_{\|f\|_{L^p(\sigma)}=1} \Big\| \sum_{Q\in\mathcal{S}} \langle\sigma\rangle_{\alpha,Q}^{\nu} \langle (M_\sigma(f))^\nu\rangle_Q^\sigma \mathbf{1}_Q \Big\|_{L^{\frac{q}{\nu},\infty}(w)} \\
&= \sup_{\|f\|_{L^p(\sigma)}=1} \Big\| \sum_{Q\in\mathcal{S}} \langle\sigma\rangle_{\alpha,Q}^{\nu} \left\langle \left(\frac{M_\sigma(f)}{\|M_\sigma(f)\|_{L^p(\sigma)}} \right)^\nu \right\rangle_Q^\sigma \mathbf{1}_Q \Big\|_{L^{\frac{q}{\nu},\infty}(w)} \|M_\sigma(f)\|_{L^p(\sigma)}^\nu \\
&\lesssim \sup_{\|g\|_{L^p(\sigma)}=1} \Big\| \sum_{Q\in\mathcal{S}} \langle\sigma\rangle_{\alpha,Q}^{\nu} \langle g^\nu\rangle_Q^\sigma \mathbf{1}_Q \Big\|_{L^{\frac{q}{\nu},\infty}(w)},
\end{aligned}$$

where in the last step, we used the boundedness of M_σ on $L^p(\sigma)$, and the bound is independent of σ. For the other direction, notice that

$$\langle f^\nu\rangle_Q^\sigma \le \inf_{x\in Q} M_\sigma(f^\nu)(x) = (\inf_{x\in Q} M_{\sigma,\nu}(f)(x))^\nu \le (\langle M_{\sigma,\nu}(f)\rangle_Q^\sigma)^\nu,$$

where $M_{\sigma,\nu}(f) := (M_\sigma(f^\nu))^{1/\nu}$, with this observation, we have

$$\begin{aligned}
&\sup_{\|f\|_{L^p(\sigma)}=1} \Big\| \sum_{Q\in\mathcal{S}} \langle\sigma\rangle_{\alpha,Q}^{\nu} \langle f^\nu\rangle_Q^\sigma \mathbf{1}_Q \Big\|_{L^{\frac{q}{\nu},\infty}(w)} \\
&\le \sup_{\|f\|_{L^p(\sigma)}=1} \Big\| \sum_{Q\in\mathcal{S}} \langle\sigma\rangle_{\alpha,Q}^{\nu} (\langle M_{\sigma,\nu}(f)\rangle_Q^\sigma)^\nu \mathbf{1}_Q \Big\|_{L^{\frac{q}{\nu},\infty}(w)} \\
&\le \sup_{\|f\|_{L^p(\sigma)}=1} \big\|\mathcal{A}_{\alpha,\mathcal{S}}^{\nu}(\cdot\sigma)\big\|_{L^p(\sigma)\to L^{q,\infty}(w)} \|M_{\sigma,\nu}(f)\|_{L^p(\sigma)}^\nu \\
&\lesssim \big\|\mathcal{A}_{\alpha,\mathcal{S}}^{\nu}(\cdot\sigma)\big\|_{L^p(\sigma)\to L^{q,\infty}(w)},
\end{aligned}$$

where in the last step, we use the boundedness of $M_{\sigma,\nu}$ on $L^p(\sigma)$ since $p > \nu$, and the bound is independent of σ. This completes the proof of Lemma 2.1. □

Now suppose that B is the sharp constant such that

$$\Big\| \sum_{Q\in\mathcal{S}} \langle\sigma\rangle_{\alpha,Q}^{\nu} \langle f^\nu\rangle_Q^\sigma \mathbf{1}_Q \Big\|_{L^{\frac{q}{\nu},\infty}(w)} \le B\|f\|_{L^p(\sigma)}^\nu,$$

that is,

$$\Big\| \sum_{Q\in\mathcal{S}} \langle\sigma\rangle_{\alpha,Q}^{\nu} \langle f\rangle_Q^\sigma \mathbf{1}_Q \Big\|_{X^{\frac{q}{\nu}}(w)} \le B\|f\|_{L^{\frac{p}{\nu}}(\sigma)}, \tag{2.1}$$

Then

$$\|\mathcal{A}_{\alpha,\mathcal{S}}(\cdot\sigma)\|_{L^p(\sigma)\to X^q(w)} \simeq B^{\frac{1}{\nu}}.$$

Hence, we have reduced the problem to study (2.1). We need the following result given by Lacey, Sawyer and Uriarte-Tuero [22].

Proposition 2.2. *Let $\tau = \{\tau : Q \in \mathcal{Q}\}$ be nonnegative constants, w, σ be weights and define linear operators by*

$$T_\tau := \sum_{Q\in\mathcal{Q}} \tau_Q \langle f\rangle_Q \mathbf{1}_Q.$$

Then for $1 < p \le q < \infty$, there holds

$$\|T_\tau(\cdot\sigma)\|_{L^p(\sigma)\to L^{q,\infty}(w)} \simeq \sup_{R\in\mathcal{Q}} w(R)^{-\frac{1}{q'}} \Big\| \sum_{\substack{Q\in\mathcal{Q}\\ Q\subset R}} \tau_Q \langle w\rangle_Q \mathbf{1}_Q \Big\|_{L^{p'}(\sigma)}$$

Observing that for (2.1), we have

$$\Big\| \sum_{Q\in\mathcal{S}} \langle \sigma\rangle_{\alpha,Q}^{\nu} \langle f\rangle_Q^{\sigma} \mathbf{1}_Q \Big\|_{L^{\frac{q}{\nu},\infty}(w)} = \|T_\tau(f\sigma)\|_{L^{\frac{q}{\nu},\infty}(w)}$$

with $\tau_Q = \langle\sigma\rangle_{\alpha,Q}^{\nu-1}|Q|^{\frac{\alpha}{d}}$. Theorem 1.2 follows immediately from Proposition 2.2.

The following proposition is weighted weak estimate for fractional maximal operator, which can found in the paper[8].

Proposition 2.3. *Given $1 < p \le q < \infty$, $0 \le \alpha < d$ and a pair of wights (w, σ). Then for all measurable functions f,*

$$\|M_\alpha(f\sigma)\|_{L^{q,\infty}(w)} \lesssim [w,\sigma]_{A^\alpha_{p,q}} \|f\|_{L^p(\sigma)}.$$

3. Proof of the weak type bound for $1 < p \le \nu$

We are left to prove Theorem 1.1 in the case that $1 < p \le \nu$. Actually, the method stem from Hytönen and Li [11], they have investigated the two-weight case. Following their method, it is easy to give the off-diagonal two-weight estimate as well. For completeness, we give the deails.

4.1. **The case for** $1 < p \le q < \nu$. We want to bound the following inequality,

$$\sup_{\lambda>0} \lambda w(\{x\in\mathbb{R}^n : \mathcal{A}^{\mathcal{S}}_{\alpha,\nu}(f\sigma) > \lambda\})^{\frac{1}{q}} \lesssim [w,\sigma]^{\frac{1}{q}}_{A^\alpha_{p.q}} \|f\|_{L^p(\sigma)}.$$

By scaling it suffices to give an uniform estimate for

$$\lambda_0 w(\{x\in\mathbb{R}^n : \mathcal{A}^{\mathcal{S}}_{\alpha,\nu}(f\sigma) > \lambda_0\})^{\frac{1}{q}},$$

where λ_0 is some constant to be determined later. It is also free to further sparsify $\mathcal{S}$ such that

$$\Big| \bigcup_{\substack{Q'\subsetneq Q\\ Q',Q\in\mathcal{S}}} Q' \Big| \le \frac{1}{4^{1-\frac{\alpha}{d}}}|Q|.$$

Now set

$$\mathcal{S}_m := \{Q\in\mathcal{S} : 2^{-m-1} < \langle f\sigma\rangle_{\alpha,Q} \le 2^{-m}\}, \qquad m \ge 0, \tag{3.1}$$

and

$$\mathcal{S}' := \{Q\in\mathcal{S} : \langle f\sigma\rangle_{\alpha,Q} > 1\}. \tag{3.2}$$

Then for $Q\in\mathcal{S}_m$, $m\ge 0$, denote by $\mathrm{ch}_{\mathcal{S}_m}(Q)$ the maximal subcubes of Q in $\mathcal{S}_m$ and define

$$E_Q := Q\setminus \bigcup_{Q'\in \mathrm{ch}_{\mathcal{S}_m}(Q)} Q'. \tag{3.3}$$

Then

$$\begin{aligned}\langle f\sigma\mathbf{1}_{E_Q}\rangle_{\alpha,Q} &= \frac{1}{|Q|^{1-\frac{\alpha}{d}}}f\sigma dx - \frac{1}{|Q|^{1-\frac{\alpha}{d}}}\sum_{Q'\in \mathrm{ch}_{\mathcal{S}_m}(Q)}\int_{Q'} f\sigma dx \\ &= \frac{1}{|Q|^{1-\frac{\alpha}{d}}}f\sigma dx - \sum_{Q'\in \mathrm{ch}_{\mathcal{S}_m}(Q)}\left(\frac{Q'}{|Q|}\right)^{1-\frac{\alpha}{d}}\frac{1}{|Q'|}\int_{Q'} f\sigma dx \\ &\geq \frac{1}{|Q|^{1-\frac{\alpha}{d}}}f\sigma dx - \frac{1}{4}2^{-l} \geq \frac{1}{2}\langle f\sigma\rangle_{\alpha,Q}.\end{aligned} \tag{3.4}$$

Also, we set $\mathcal{A}^{\mathcal{S}_m}_{\alpha,\nu}$ and $\mathcal{A}^{\mathcal{S}'}_{\alpha,\nu}$ to be the sparse operators associated with $\mathcal{S}_m$ and $\mathcal{S}'$, respectively

$$(\mathcal{A}^{\mathcal{S}_m}_{\alpha,\nu}(f))^\nu := \sum_{Q\in\mathcal{S}_m}\langle f\rangle^\nu_{\alpha,Q}\mathbf{1}_Q \qquad \text{and} \qquad (\mathcal{A}^{\mathcal{S}'}_{\alpha,\nu}(f))^\nu := \sum_{Q\in\mathcal{S}'}\langle f\rangle^\nu_{\alpha,Q}\mathbf{1}_Q. \tag{3.5}$$

Thus, it is easy to know that

$$\mathcal{A}^{\mathcal{S}}_{\alpha,\nu} := \sum_{Q\in\mathcal{S}}\langle f\rangle^\nu_{\alpha,Q}\mathbf{1}_Q = \sum_{m\in\mathbb{N}}(\mathcal{A}^{\mathcal{S}_m}_{\alpha,\nu}(f))^\nu + (\mathcal{A}^{\mathcal{S}'}_{\alpha,\nu}(f))^\nu. \tag{3.6}$$

By (3.5) and (3.6), we conclude that

$$\begin{aligned}&w(\{x\in\mathbb{R}^n : \mathcal{A}^{\mathcal{S}}_{\alpha,\nu}(f\sigma) > \lambda_0\}) \\ &\leq w(\{x\in\mathbb{R}^n : \sum_{m\geq 0}(\mathcal{A}^{\mathcal{S}_m}_{\alpha,\nu}(f))^\nu > \frac{\lambda_0^\nu}{2}\}) + w(\{x\in\mathbb{R}^n : (\mathcal{A}^{\mathcal{S}'}_{\alpha,\nu}(f))^\nu > \frac{\lambda_0^\nu}{2}\}) \\ &= w(\{x\in\mathbb{R}^n : \sum_{m\geq 0}\sum_{Q\in\mathcal{S}_m}\langle f\sigma\rangle^\nu_{\alpha,Q}\mathbf{1}_Q > \frac{\lambda_0^\nu}{2}\}) + w(\{x\in\mathbb{R}^n : \sum_{Q\in\mathcal{S}'}\langle f\sigma\rangle^\nu_{\alpha,Q}\mathbf{1}_Q > \frac{\lambda_0^\nu}{2}\}) := II_1 + II_2.\end{aligned}$$

The second term estimation is trival. In fact, it follows immediately from Proposition 2.3,

$$II_2 \leq w(\bigcup_{Q\in\mathcal{S}'} Q) \leq w(\{x\in\mathbb{R}^n : M_\alpha(f\sigma) > 1\}) \lesssim [w,\sigma]_{A^\alpha_{p,q}}\|f\|^q_{L^p(\sigma)}.$$

Now let $\frac{\lambda_0^\nu}{2} = \sum_{m\geq 0} 2^{-\varepsilon m}$, where $\varepsilon := (\nu - q)/2$. By (3.4), we can estimate

$$\begin{aligned}II_1 &\leq \sum_{m\geq 0} w(\{x\in\mathbb{R}^n : \sum_{Q\in\mathcal{S}_m}\langle f\sigma\rangle^\nu_{\alpha,Q}\mathbf{1}_Q > 2^{-\varepsilon m}\}) \\ &\leq \sum_{m\geq 0} w(\{x\in\mathbb{R}^n : \sum_{Q\in\mathcal{S}_m}\langle f\sigma\mathbf{1}_Q\rangle^q_{\alpha,Q}\mathbf{1}_Q > 2^{(\nu-q)m}2^{-\varepsilon m}\}) \\ &\leq \sum_{m\geq 0} w(\{x\in\mathbb{R}^n : \sum_{Q\in\mathcal{S}_m}\langle f\sigma\mathbf{1}_{E_Q}\rangle^q_{\alpha,Q}\mathbf{1}_Q > 2^{-q}2^{(\nu-q)m}2^{-\varepsilon m}\}) \\ &\leq \sum_{m\geq 0} 2^{(q-\nu+\varepsilon)m+q}\int_{\mathbb{R}^n}\sum_{Q\in\mathcal{S}_m}\langle f\sigma\mathbf{1}_{E_Q}\rangle^q_{\alpha,Q}\mathbf{1}_Q dw \lesssim [w,\sigma]_{A^\alpha_{p,q}}\|f\|^q_{L^p(\sigma)}\end{aligned}$$

where in the last inequality we have use the following the fact

$$\begin{aligned}\int_{\mathbb{R}^n}\sum_{Q\in\mathcal{S}_m}\langle f\sigma\mathbf{1}_{E_Q}\rangle_{\alpha,Q}^q\mathbf{1}_Q dw &= \sum_{Q\in\mathcal{S}_m}\langle f\sigma\mathbf{1}_{E_Q}\rangle_{\alpha,Q}^q w(Q)\\ &\le \sum_{Q\in\mathcal{S}_m}\left(\frac{1}{\sigma(E_Q)^{1-\frac{1}{p}+\frac{1}{q}}}\int_{E_Q} f\sigma\right)^q |Q|^{q(\frac{\alpha}{d}-1)}w(Q)\sigma(Q)^{\frac{q}{p'}}\sigma(E_Q)\\ &\le [w,\sigma]_{A^\alpha_{p,q}}\|f\|_{L^p(\sigma)}.\end{aligned}$$

Combining the above II_1 and II_2, we get

$$\|\mathcal{A}^{\mathcal{S}}_{\alpha,\nu}(f\sigma)\|_{L^{q,\infty}(w)} \lesssim [w,\sigma]^{\frac{1}{q}}_{A^\alpha_{p,q}}\|f\|_{L^p(\sigma)}.$$

4.2. **The cases for** $p\le q=\nu$ or $p\le\nu<q$. We can estimate for the case by [5, Theorem 1.1]

$$\|\mathcal{A}^{\mathcal{S}}_{\alpha,\nu}(f\sigma)\|_{L^{q,\infty}(w)} \le \|\mathcal{A}^{\mathcal{S}}_{\alpha,\nu}(f\sigma)\|_{L^{q}(w)} \lesssim [w,\sigma]^{\frac{1}{q}}_{A^\alpha_{p,q}}[\sigma]^{\frac{1}{q}}_{A_\infty}\|f\|_{L^p(\sigma)}.$$

□

4. Sharpness of the weak type bounds for fractional square function

In this section, we will show that the case for $\nu=2$, which called fractional square function, i.e.

$$\mathcal{A}^{\mathcal{S}}_{\alpha,2}(f) = \Big(\sum_{Q\in\mathcal{S}}\langle f\rangle^2_{\alpha,Q}\mathbf{1}_Q\Big)^{\frac{1}{2}}, \tag{4.1}$$

and p,q,α satisfy condition $\frac{1}{q}+\frac{\alpha}{d}=\frac{1}{p}$. We only consider one weight theory estimate for $L^p(w^p)\to L^{q,\infty}(w^q)$ in here. The governing weight class is a generalization of Muckenhoupt A_p weights, and was introduced by Muckenhoupt and Wheeden [23].

$$[w]_{A_{p,q}} := \sup_Q\left(\frac{1}{|Q|}\int_Q w^q\right)\left(\frac{1}{|Q|}\int_Q w^{-p'}\right)^{\frac{q}{p'}} < \infty.$$

Its relation to two weight characteristic is $[w^q,w^{-p'}]_{A^\alpha_{p,q}} = [w]_{A_{p,q}}$ with $\frac{1}{q}+\frac{\alpha}{d}=\frac{1}{p}$. Moreover, it is straightforward to show that the following are equivalent:

$$\text{(a)}\quad w\in A_{p,q}; \qquad\qquad \text{(b)}\quad w^q\in A_{1+\frac{q}{p'}} \quad\text{and}\quad w^{-p'}\in A_{1+\frac{p'}{q}}. \tag{4.2}$$

We will show that the norm bound

$$\|\mathcal{A}^{\mathcal{S}}_{\alpha,2}\|_{L^p(w^p)\to L^{q,\infty}(w^q)} \le [w]^{\max(\frac{1}{q},\frac{1}{2}-\frac{\alpha}{d})}_{A_{p,q}}$$

is unimprovable. Actually, a lower bound with the exponent $\frac{1}{q}$ holds uniformly over all weights, which is the content of the next Theorem. The optimality of the exponent $\frac{1}{2}-\frac{\alpha}{d}$ is slightly more tricky, and is based on a example of a specific weight $A_{p,q}$. Also, by Theorem 1.1 we give the following mixed $A_{p,q}-A_\infty$ estimate.

Corollary 4.1. *Let $0<\alpha<d$ and $1<p\le q<\infty$ with $\frac{1}{q}+\frac{\alpha}{d}=\frac{1}{p}$. Then*

$$\|\mathcal{A}^{\mathcal{S}}_{\alpha,2}\|_{L^p(w^p)\to L^{q,\infty}(w^q)} \lesssim [w,\sigma]^{\frac{1}{q}}_{A^\alpha_{p,q}}\begin{cases}[w^{-p'}]^{\frac{1}{q}}_{A_\infty}, & 2\le q\le \dfrac{2}{1-\frac{2\alpha}{d}},\\ [w^q]^{(\frac{1}{2}-\frac{1}{p})_+}_{A_\infty}, & \text{other case.}\end{cases}$$

Notice that (4.2), we easilly know that

$$[w^q]_{A_{1+\frac{q}{p'}}} = [w]_{A_{p,q}} \qquad \text{and} \qquad [w^{-p'}]_{A_{1+\frac{p'}{q}}} = [w]^{\frac{p'}{q}}_{A_{p,q}}. \tag{4.3}$$

And Lerner [19] show that $[w]_{A_\infty} \lesssim [w]_{A_p}$. Hence, using this relation to Corollary 4.1 we obtain the following pure $A_{p,q}$ estimate.

Corollary 4.2. *Let $0 < \alpha < d$ and $1 < p \le q < \infty$ with $\frac{1}{q} + \frac{\alpha}{d} = \frac{1}{p}$. Then*

$$\|\mathcal{A}^{\mathcal{S}}_{\alpha,2}\|_{L^p(w^p)\to L^{q,\infty}(w^q)} \lesssim \begin{cases} [w]^{\frac{p'}{q}(1-\frac{\alpha}{d})}_{A_{p,q}}, & 2 \le q \le \dfrac{2}{1-\frac{2\alpha}{d}}, \\ [w]^{(\frac{1}{2}-\frac{\alpha}{d})}_{A_{p,q}}, & \dfrac{2}{1-\frac{2\alpha}{d}} < q < \infty, \\ [w]^{\frac{1}{q}}_{A_{p,q}}, & 1 \le q < 2. \end{cases}$$

However, the exponent $\frac{p'}{q}(1-\frac{\alpha}{d})$ is not optimal of the case for $2 \le q \le \frac{2}{1-\frac{2\alpha}{d}}$. and the best exponent $\frac{1}{q}$ will appear following estimate. For generally, we consider case for $\nu \ge 1$, and we are concerned with the weak-type bounds, which have cases of $\nu \le q \le \frac{\nu}{1-\frac{\nu\alpha}{d}}$, which a $(\log_1[w^q]_{A_\infty})^{\frac{1}{\nu}}$ appears in the sharp estimate.

Theorem 4.3. *Let $\nu \ge 1$, $0 \le \alpha < d$ and $1 \le p \le q < \infty$ with $\frac{1}{p} + \frac{\alpha}{d} = \frac{1}{q}$, there holds for any weight $w \in A_{p,q}$*

$$\|\mathcal{A}^{\mathcal{S}}_{\alpha,\nu}(f)\|_{L^{q,\infty}(w^q)} \lesssim [w]^{\max(\frac{1}{q},\frac{1}{\nu}-\frac{\alpha}{d})}_{A_{p,q}} \phi([w^q]_{A_\infty})\|wf\|_{L^p},$$

where

$$\phi([w^q]_{A_\infty}) = \begin{cases} (\log_1[w^q]_{A_\infty})^{\frac{1}{\nu}}, & \nu \le q \le \dfrac{\nu}{1-\frac{\nu\alpha}{d}}; \\ 1, & \text{other case.} \end{cases}$$

and $\log_1(x) = 1 + \log_+(x)$.

As a Corollary of Theorem 4.3, the following result of fractional square function is sharp.

Corollary 4.4. *Let $0 \le \alpha < d$ and $1 \le p \le q < \infty$ with $\frac{1}{p} + \frac{\alpha}{d} = \frac{1}{q}$, there holds for any weight $w \in A_{p,q}$*

$$\|\mathcal{A}^{\mathcal{S}}_{\alpha,2}(f)\|_{L^{q,\infty}(w^q)} \lesssim [w]^{\max(\frac{1}{q},\frac{1}{2}-\frac{\alpha}{d})}_{A_{p,q}} \phi_1([w^q]_{A_\infty})\|wf\|_{L^p},$$

where

$$\phi_1([w^q]_{A_\infty}) = \begin{cases} (\log_1[w^q]_{A_\infty})^{\frac{1}{2}}, & 2 \le q \le \dfrac{2}{1-\frac{2\alpha}{d}}; \\ 1, & \text{other case.} \end{cases}$$

A basic tool for us is the following classical reverse Hölder's inequality with optimal bound, which can be found in [12].

Proposition 4.5. *There is a dimensional constant $c > 0$ such that for $w \in A_\infty$, and $r(w) = 1 + c[w]_{A_\infty}$, there holds*

$$\langle w^{r(w)}\rangle_Q^{\frac{1}{r(w)}} \le 2\langle w\rangle_Q, \qquad Q \text{ a cube.} \tag{4.4}$$

We also need the following off-diagonal extrapolation given by Duoandikoetxra [7].

Proposition 4.6. *Let $1 \le p_0 < \infty$ and $0 < q_0 < \infty$. Assume that that for some family of nonnegative couples (f, g) and for all $w \in A_{p_0,q_0}$ we have*

$$\|wg\|_{L^{q_0}} \le CN([w]_{A_{p_0,q_0}})\|wf\|_{L^{p_0}},$$

where N is an increasing function and the constant C does not depend on w. Set $\gamma = \frac{1}{q_0} + \frac{1}{p_0'}$. Then for $1 < p < \infty$ and $0 < q < \infty$, such that

$$\frac{1}{q} - \frac{1}{p} = \frac{1}{q_0} - \frac{1}{q_0},$$

and all $w \in A_{p,q}$ we have

$$\|wg\|_{L^p} \le CK(w)\|wf\|_{L^p},$$

where

$$K(w) = \begin{cases} N([w]_{A_{p,q}}(2\|M\|_{L^{\gamma q}(w^q)})^{\gamma(q-q_0)}), & q < q_0; \\ N([w]_{A_{p,q}}^{\frac{\gamma q_0 - 1}{\gamma q - 1}}(2\|M\|_{L^{\gamma p'}(w^{-p'})})^{\frac{\gamma(q-q_0)}{\gamma q - 1}}), & q > q_0. \end{cases}$$

In particular, $K(w) \le C_1 N(C_2[w]_{A_{p,q}}^{\max(1, \frac{q_0 p'}{q p_0'})})$ for $w \in A_{p,q}$.

The following estimate based on Domingo-Salazar, Lacey, and Rey [6].

Theorem 4.7. *Let $\nu \ge 1$, $0 \le \alpha < d$ and $1 \le p \le q < \infty$ with $\frac{1}{p} + \frac{\alpha}{d} = \frac{1}{q}$, there holds for any weight $w \in A_{p,q}$*

$$\|\mathcal{A}_{\alpha,\nu}^{\mathcal{S}}(f)\|_{L^{q,\infty}(w^q)} \lesssim [w]_{A_{p,q}}^{\max(\frac{1}{q}, \frac{1}{\nu} - \frac{\alpha}{d})} \phi_2([w^q]_{A_\infty})\|wf\|_{L^p}, \tag{4.5}$$

where

$$\phi_2([w^q]_{A_\infty}) = \begin{cases} 1, & 1 \le q < \nu; \\ (\log_1 [w^q]_{A_\infty})^{\frac{1}{\nu}}, & \nu \le q < \infty. \end{cases}$$

Theorem 4.3 follows immediately from Theorems 1.1 and 4.7.

In order to prove Theorem 4.7, we need following estimate.

Lemma 4.8. *Let $\nu \ge 1$, $q \ge \nu$, $0 \le \alpha < d$ and $1 \le p \le q < \infty$ with $\frac{1}{q} + \frac{\alpha}{d} = \frac{1}{p}$, then*

$$\|\mathcal{A}_{\alpha,\nu}^{\mathcal{S}_m}\|_{L^{q,\infty}(w^q)} \lesssim [w]_{A_{p,q}}^{\frac{1}{\nu} - \frac{\alpha}{d}} \|wf\|_{L^p}.$$

where and given $0 < m < \log_1 [w^q]_{A_\infty}$.

Proof. We only need to prove the case for $q = \frac{\nu}{1 - \frac{\nu\alpha}{d}}$, by off-diagonal extrapolation in Proposition 4.6, it yields the cases for $\nu \le q < \frac{\nu}{1 - \frac{\nu\alpha}{d}}$ and $\frac{\nu}{1 - \frac{\nu\alpha}{d}} < q < \infty$. By Minkowski's inequality and (3.4), we can estimate

$$\Big(\int_{\mathbb{R}^n}\big(\sum_{Q\in\mathcal{S}_m}\langle f\rangle_{\alpha,Q}^{\nu}\mathbf{1}_Q\big)^{\frac{q}{\nu}}\upsilon\Big)^{\frac{1}{q}}\le\Big(\sum_{Q\in\mathcal{S}_m}\Big(\int_{\mathbb{R}^n}\langle f\rangle_{\alpha,Q}^{q}\mathbf{1}_Q\upsilon\Big)^{\frac{\nu}{q}}\Big)^{\frac{1}{\nu}}=\Big(\sum_{Q\in\mathcal{S}_m}\langle f\rangle_{\alpha,Q}^{\nu}\upsilon^{\frac{\nu}{q}}(Q)\Big)^{\frac{1}{\nu}}$$
$$\lesssim\Big(\sum_{Q\in\mathcal{S}_m}\langle f\mathbf{1}_{E_m(Q)}\rangle_{\alpha,Q}^{\nu}\upsilon^{\frac{\nu}{q}}(Q)\Big)^{\frac{1}{\nu}}$$
$$\le\Big(\sum_{Q\in\mathcal{S}_m}\langle f^p\mathbf{1}_{E_m(Q)}w^p\rangle_{\alpha,Q}\langle\sigma^{p'}\rangle_{\alpha,Q}^{\frac{p}{p'}}\upsilon^{\frac{p}{q}}(Q)\Big)^{\frac{1}{p}}$$
$$\le[w]_{A_{q,\nu}}^{\frac{1}{q}}\Big(\sum_{Q\in\mathcal{S}_m}\int_{E_m(Q)}f^p\mathbf{1}_{E_m(Q)}w^p\Big)^{\frac{1}{p}}\le[w]_{A_{q,\nu}}^{\frac{1}{\nu}-\frac{\alpha}{d}}\|wf\|_{L^p},$$

where $p=\nu$ in the above estimate. □

The good property of Lebesgue measure appear in the paper [6].

Proposition 4.9. *Let any $\lambda>0$, $\mathcal{S}_m$ defined as (3.1) and $b=\sum_{Q'\in\mathcal{S}_m}\mathbf{1}_{Q'}$, then we have that for any dyadic cube $Q\in\mathcal{S}_m$*

$$|\{x\in Q:\ b(x)>\lambda\}|\lesssim\exp(-c\lambda)|Q|.$$

For $\log_1[w^q]_{A_\infty}\le m<\infty$, we also have following estimate.

Lemma 4.10. *Let υ denote the weight w^q, for all integers $m_0>0$, then*

$$\upsilon\big(\sum_{m=m_0}^{\infty}(\mathcal{A}_{\alpha,\nu}^{\mathcal{S}_m}(f))^{\nu}>1\big)\lesssim[w]_{A_{p,q}}\Big(\frac{[w]_{A_\infty}}{2^{m_0}}\Big)^{q}\|wf\|_{L^p}^{q}. \tag{4.6}$$

Proof. Define

$$\mathcal{S}_m^*:=\{Q\text{ maximal s.t. }Q\in\mathcal{S}_m\}\quad\text{and}\quad B_m:=\bigcup\{Q:\ Q\in\mathcal{S}_m^*\}.$$

By the definitions of $\mathcal{S}_m$ and $(\mathcal{A}_{\alpha,\nu}^{\mathcal{S}_m}(f))^{\nu}$, we can write $(\mathcal{A}_{\alpha,\nu}^{\mathcal{S}_m}(f))^{\nu}$ as $2^{-\nu m}b_m$, where

$$b_m\le\sum_{Q\in\mathcal{S}_m}\mathbf{1}_Q\quad\text{and}\quad\text{supp}(b_m)\subset B_m.$$

For any dyadic cube $Q\in\mathcal{S}_m$, by Proposition 4.9, we know that the function b_m is locally exponentially integrable. By the sharp weak-type estimate for the fractional maximal function [15], we know that

$$\upsilon(B_m)\lesssim 2^{qm}[w]_{A_{p,q}}\|wf\|_{L^p}^{q}.$$

The left hand side of (4.6) can be estimated as

$$\upsilon\big(\sum_{m=m_0}^{\infty}(\mathcal{A}_{\alpha,\nu}^{\mathcal{S}_m}(f))^{\nu}>1\big)=\upsilon\big(\sum_{m=m_0}^{\infty}2^{-\nu m}b_m>\sum_{m=m_0}^{\infty}2^{m_0-m-1}\big)$$
$$\le\sum_{m=m_0}^{\infty}\upsilon(b_m>2^{m_0+(\nu-1)m-1}).$$

Taking

$$\beta(Q) := \{x \in Q : b_m(x) > 2^{m_0+(\nu-1)m-1}\}$$

for any dyadic cube $Q \in \mathcal{S}_m^*$, by the definition of $\mathcal{S}_m^*$ and Proposition 4.9, we show that

$$|\beta(Q)| \lesssim \exp(-c2^{m_0+(\nu-1)m})|Q|.$$

Using the A_∞ property for $A_{1+\frac{q}{p'}}$ weights with υ-measure and Proposition 4.4, there holds

$$\begin{aligned}\upsilon(\beta(Q)) &= \langle \upsilon \mathbf{1}_{\beta(Q)}\rangle_Q |Q| \le \langle \mathbf{1}_{\beta(Q)}\rangle_Q^{(\frac{1}{r(\upsilon)})'} \langle \upsilon^{r(\upsilon)}\rangle_Q^{\frac{1}{r(\upsilon)}} |Q| \\ &\lesssim \left[\frac{|\beta(Q)|}{|Q|}\right]^{(c[\upsilon]_{A_\infty})^{-1}} \upsilon(Q) \lesssim \upsilon(Q) \exp\left(-c\frac{2^{m_0+(\nu-1)m}}{[\upsilon]_{A_\infty}}\right),\end{aligned}$$

where $r(\upsilon)$ as in (4.4).

Summing over the disjoint cubes in $\mathcal{S}_m^*$, we obtain

$$\upsilon\Big(\sum_{m=m_0}^{\infty} (\mathcal{A}_{\alpha,\nu}^{\mathcal{S}_m}(f))^\nu > 1\Big) \lesssim [w]_{A_{p,q}} \|wf\|_{L^p}^q \sum_{m=m_0}^{\infty} 2^{mq} \exp\left(-c\frac{2^{m_0+(\nu-1)m}}{[\upsilon]_{A_\infty}}\right). \tag{4.7}$$

The sum in the right hand side of (4.7), we can be controlled by

$$\begin{aligned}\sum_{m=m_0}^{\infty} 2^{mq} \exp\left(-c\frac{2^{m_0+(\nu-1)m}}{[\upsilon]_{A_\infty}}\right) &\le \int_{m_0}^{\infty} 2^{qx} \exp\left(-c\frac{2^{m_0+(\nu-1)x}}{[\upsilon]_{A_\infty}}\right) dx \\ &\approx \int_{2^{(\nu-1)m_0}}^{\infty} y^q \exp\left(-c\frac{2^{m_0}}{[\upsilon]_\infty} y\right) \frac{dy}{y} \\ &= \left(\frac{[\upsilon]_\infty}{2^{m_0}}\right)^q \int_{\frac{2^{\nu m_0}}{[\upsilon]_\infty}}^{\infty} y^q e^{-cy} \frac{dy}{y} \lesssim \left(\frac{[\upsilon]_\infty}{2^{m_0}}\right)^q.\end{aligned} \tag{4.8}$$

Combining (4.7) and (4.8), we obtain the desired result. This completes the proof Lemma 4.10. □

Proof of Theorem 4.7. The case for $1 \le q < \nu$ is easy and contained in Theorem 1.1,as so only our attention on the case for $q \ge \nu$. By scaling the left hand side of (4.5) suffices to estimate

$$\lambda^q \upsilon(\{x \in \mathbb{R}^n : \mathcal{A}_{\alpha,\nu}^{\mathcal{S}} > \lambda\}). \tag{4.9}$$

Now, we assume that $\lambda = 3^{\frac{1}{\nu}}$, $\|f\|_{L^p(w^p)} = 1$ and notice that (3.6), the (4.9) can be estimated as

$$\upsilon((\mathcal{A}_{\alpha,\nu}^{\mathcal{S}}(f))^\nu > 3) \le \upsilon((\mathcal{A}_{\alpha,\nu}^{\mathcal{S}'}(f))^\nu > 1) + \upsilon\bigg(\sum_{m=0}^{m_0-1} (\mathcal{A}_{\alpha,\nu}^{\mathcal{S}_m}(f))^\nu > 1\bigg) + \upsilon\bigg(\sum_{m=m_0}^{\infty} (\mathcal{A}_{\alpha,\nu}^{\mathcal{S}_m}(f))^\nu > 1\bigg).$$

By the sharp weak-type estimate for the fractional maximal function [15], the first term to arrive at the bound

$$\upsilon((\mathcal{A}_{\alpha,\nu}^{\mathcal{S}'}(f))^\nu > 1) \lesssim [w]_{A_{p,q}}^{\frac{1}{q}}. \tag{4.10}$$

By Chebysheff's inequality and Minkowski's inequality for $q \geq \nu$, the second term from Lemma 4.8

$$\begin{aligned}
\upsilon\bigg(\sum_{m=0}^{m_0-1}(\mathcal{A}_{\alpha,\nu}^{\mathcal{S}_m}(f))^\nu > 1\bigg) &\leq \bigg\|\sum_{m=0}^{m_0-1}(\mathcal{A}_{\alpha,\nu}^{\mathcal{S}_m}(f))^\nu\bigg\|_{L^{\frac{q}{\nu}}(\upsilon)}^{\frac{q}{\nu}} \\
&\leq \bigg(\sum_{m=0}^{m_0-1}\|(\mathcal{A}_{\alpha,\nu}^{\mathcal{S}_m}(f))^\nu\|_{L^{\frac{q}{\nu}}(\upsilon)}\bigg)^{\frac{q}{\nu}} \\
&= \bigg(\sum_{m=0}^{m_0-1}\|\mathcal{A}_{\alpha,\nu}^{\mathcal{S}_m}(f)\|_{L^q(w^q)}^\nu\bigg)^{\frac{q}{\nu}} \lesssim (m_0[w]_{A_{p,q}}^{\frac{1}{\nu}-\frac{\alpha}{d}})^{\frac{q}{\nu}}. \qquad (4.11)
\end{aligned}$$

By Lemma 4.10, the third term can be estimate as

$$\upsilon\bigg(\sum_{m=m_0}^{\infty}(\mathcal{A}_{\alpha,\nu}^{\mathcal{S}_m}(f))^\nu > 1\bigg) \lesssim [w]_{A_{p,q}}\left(\frac{[w^q]_\infty}{2^{m_0}}\right)^q. \qquad (4.12)$$

Combining (4.10), (4.11) and (4.12), we get

$$\|\mathcal{A}_{\alpha,\nu}^{\mathcal{S}}\|_{L^{q,\infty}(w^q)} \lesssim [w]_{A_{p,q}}^{\frac{1}{q}} + m_0^{\frac{1}{\nu}}[w]_{A_{p,q}}^{\frac{1}{\nu}-\frac{\alpha}{d}} + [w]_{A_{p,q}}^{\frac{1}{q}}[w^q]_{A_\infty}2^{-m_0} \approx [w]_{A_{p,q}}^{\max(\frac{1}{q},\frac{1}{\nu}-\frac{\alpha}{d})}(\log_1[w^q]_{A_\infty})^{\frac{1}{\nu}},$$

due to $m_0 \approx \log_1[w^q]_{A_\infty}$. This finishes the proof Theorem 4.7. □

However, this is not the end of the story; we can prove even more. Here we present our full statement of the main theorem. This estimate is sharp in the following sense.

Theorem 4.11. *For any weight w, we have*

$$\|\mathcal{A}_{\alpha,\nu}^{\mathcal{S}}\|_{L^p(w^p)\to L^{q,\infty}(w^q)} \geq [w]_{A_{p,q}}^{\frac{1}{q}}.$$

Proof. Let υ denote the weight w^q and consider $f = |f|\chi_Q$, then we obtain for $Q \in \mathcal{S}$

$$\mathcal{A}_{\alpha,2}^{\mathcal{S}}(f) \geq \langle |f|\rangle_{\alpha,Q}.$$

Taking $N := \|\mathcal{A}_{\alpha,\nu}^{\mathcal{S}}(f)\|_{L^p(w^p)\to L^{q,\infty}(\upsilon)}$, by the inequality of norm $\mathcal{A}_{\alpha,\nu}^{\mathcal{S}}(f)$, we have

$$N\|f\|_{L^p(w^p)} \geq \|\mathcal{A}_{\alpha,\nu}^{\mathcal{S}}(f)\|_{L^{q,\infty}(\upsilon)} \geq \|\langle |f|\rangle_{\alpha,Q}\|_{L^{q,\infty}(\upsilon)} = \frac{\upsilon(Q)^{\frac{1}{q}}}{|Q|^{1-\frac{\alpha}{d}}}\int_Q |f| = \frac{\upsilon(Q)^{\frac{1}{q}}}{|Q|^{1-\frac{\alpha}{d}}}\int_Q |f|w^{-p}w^p$$

for all positive functions $|f|$ on Q. By the converse to Hölder's inequality, this shows that

$$N \geq \frac{\upsilon(Q)^{\frac{1}{q}}}{|Q|^{1-\frac{\alpha}{d}}}\|w^{-p}\|_{L^{p'}(w^p)} = \frac{\upsilon(Q)^{\frac{1}{q}}\sigma(Q)^{\frac{1}{p'}}}{|Q|^{1-\frac{\alpha}{d}}},$$

and taking the supremuum over all Q proves this theorem. □

Theorem 4.12. *Let $\nu \geq 1$, $0 \leq \alpha < d$ and $1 \leq p \leq q < \infty$ with $\frac{1}{q}+\frac{\alpha}{d}=\frac{1}{p}$. If Φ be an increasing function such that*

$$\|\mathcal{A}_{\alpha,\nu}^{\mathcal{S}}\|_{L^p(w^p)\to L^{q,\infty}(w^q)} \leq \Phi([w]_{A_{p,q}})$$

for all $w \in A_{p,q}$, then $\Phi(t) \gtrsim ct^{\frac{1}{\nu}-\frac{\alpha}{d}}$.

Lacey and Scurry [16] show that this in class of power functions, namely, they proved that there cannot be a bound the form $\Phi(t) = t^{\frac{1}{2}-\eta}$ for $\eta > 0$. We will extend their methods to general case.

Proof. We will consider two cases to prove this theorem: $\nu > 1$ and $\nu = 1$.

Case 1: $\nu > 1$. Following the same arguments as that in [11, 16], the assumption implies

$$\big\|\big(\sum_Q \langle a_Q \cdot w^q\rangle_{\alpha,Q}^{\nu}\mathbf{1}_Q\big)^{\frac{1}{\nu}}\big\|_{L^{p'}(w^{-p'})} \lesssim \Phi([w]_{A_{p,q}})\big\|\big(\sum_Q a_Q^{\nu}\big)^{\frac{1}{\nu}}\big\|_{L^{q',1}(w^q)} \tag{4.13}$$

for all sequences of measurable functions a_Q. For $\vartheta \in (0,1)$, we consider $w(x) = |x|^{\frac{\vartheta-1}{q}}$ and a sequence of functions

$$a_{[0,2^{-k})}(x) := a_k(x) := \vartheta^{\frac{1}{\nu-1}-\frac{1}{\nu}}\sum_{j=k+1}^{\infty} 2^{-\vartheta(j-k)}\mathbf{1}_{[2^{-j},2^{-j+1})}(x), \qquad k\in\mathbb{N}.$$

Then it is easy to check that

$$[w]_{A_{p,q}} = [w^q]_{A_{1+\frac{q}{p'}}} \simeq \vartheta^{-1} \qquad \text{and} \qquad \sum_k a_k^{\nu}(x) \lesssim \vartheta^{\frac{\nu}{\nu-1}-2}\mathbf{1}_{[0,1]}.$$

In fact, we choose $I_k = [0,2^{-k}]$ and $x\in(2^{-(l+1)},2^{-l}]$ with $l\in\mathbb{N}_0$ such that

$$a_k(x) \simeq \vartheta^{\frac{1}{\nu-1}-\frac{1}{\nu}}|I_k|^{-\vartheta}|x|^{\vartheta}\mathbf{1}_{I_k}(x).$$

A simple calculation shows that

$$\begin{aligned}\sum_{k=0}^{\infty} a_k^{\nu}(x) &= \vartheta^{\frac{\nu}{\nu-1}-1}|x|^{\nu\vartheta}\sum_{k=0}^{\infty}|I_k|^{-\nu\vartheta}\mathbf{1}_{I_k}(x) = \vartheta^{\frac{\nu}{\nu-1}-1}|x|^{\nu\vartheta}\sum_{k=0}^{l}(2^{\nu\vartheta})^k\\ &= \vartheta^{\frac{\nu}{\nu-1}-1}|x|^{\nu\vartheta}\frac{2^{\nu(l+1)\vartheta}-1}{2^{\nu\vartheta}-1} \lesssim \vartheta^{\frac{\nu}{\nu-1}-2}|x|^{\nu\vartheta}2^{\nu l\vartheta} \lesssim \vartheta^{\frac{\nu}{\nu-1}-2}\mathbf{1}_{[0,1]}.\end{aligned}$$

This directly for the right hand side of (4.13)

$$\begin{aligned}\big\|\big(\sum_{k=1}^{\infty} a_k(x)^{\nu}\big)^{\frac{1}{\nu}}\big\|_{L^{q',1}(w^q)} &\lesssim q'\int_0^{\infty}\left(\int_{\{x\in[0,1]:\, c\vartheta^{\frac{\nu}{\nu-1}-2}>s\}}|x|^{\vartheta-1}dx\right)^{\frac{1}{q'}}ds\\ &\le \int_0^{c\vartheta^{\frac{\nu}{\nu-1}-2}}\left(\int_0^1|x|^{\vartheta-1}dx\right)^{\frac{1}{q'}}ds \simeq \vartheta^{\frac{\nu}{\nu-1}-2}\vartheta^{-\frac{1}{q'}}.\end{aligned}$$

On the other hand, the left hand side of (4.13) can be estimated as

$$\langle a_k\cdot w^q\rangle_{\alpha,[0,2^{-k})} \simeq \vartheta^{\frac{1}{\nu-1}-\frac{1}{\nu}}2^{k(1-\frac{\alpha}{d})}\sum_{j=k+1}^{\infty}2^{-\vartheta(j-k)}2^{-\vartheta j} \simeq \vartheta^{\frac{1}{\nu-1}-\frac{1}{\nu}-1}2^{k(1-\frac{\alpha}{d}-\vartheta)},$$

It follows that

$$\begin{aligned}\int_{[0,1]}\big(\sum_{k=1}^{\infty}\langle a_k\cdot w^q\rangle_{\alpha,[0,2^{-k})}^{\nu}\mathbf{1}_{[0,2^{-k})}\big)^{\frac{p'}{\nu}}w^{-p'} &\simeq \vartheta^{\frac{p'}{\nu-1}-\frac{p'}{\nu}-p'}\int_0^1|x|^{(\vartheta-(1-\frac{\alpha}{d})p'}|x|^{-\frac{(\vartheta-1)p'}{q}}dx\\ &= \vartheta^{\frac{p'}{\nu-1}-\frac{p'}{\nu}-p'}\int_0^1|x|^{\frac{\vartheta p'}{q'}-1}dx = \frac{q'}{p'}\vartheta^{\frac{p'}{\nu-1}-\frac{p'}{\nu}-p'-1}.\end{aligned}$$

By assumption, this implies

$$\vartheta^{\frac{1}{\nu-1}-\frac{1}{p'}-\frac{1}{\nu}-1} \lesssim \Phi([w]_{A_{p,q}})\vartheta^{\frac{\nu}{\nu-1}-2}\vartheta^{-\frac{1}{q'}} \lesssim \Phi(c\vartheta^{-1})\vartheta^{\frac{\nu}{\nu-1}-2}\vartheta^{-\frac{1}{q'}}.$$

Hence, we show that $\Phi(t)\gtrsim t^{\frac{1}{\nu}-\frac{\alpha}{d}}$, this finishes the Case 1 of the estimate.

Case 2: $\nu=1$. This case upper bound follows from [15], and we show that

$$\|\mathcal{A}_{\alpha,1}^{\mathcal{S}}(f)\|_{L^{q,\infty}(w^q)} \lesssim \Phi([w]_{A_{p,q}})\|wf\|_{L^p} \tag{4.14}$$

holds for $\Phi(t) \geq ct^{1-\frac{\alpha}{d}}$.

By (4.3), we show that

$$\|\mathcal{A}^{\mathcal{S}}_{\alpha,1}(f)\|_{L^{q,\infty}(w^q)} \lesssim \Phi([w^q]_{A_{1+q/p'}})\|wf\|_{L^p}, \tag{4.15}$$

and if we let $u = w^q$, then

$$\|\mathcal{A}^{\mathcal{S}}_{\alpha,1}(f)\|_{L^{q,\infty}(u)} \lesssim \Phi([u]_{A_{1+q/p'}})\|f\|_{L^p(u^{p/q})}. \tag{4.16}$$

Assume now that $u \in A_1$, then (4.16) it yields that

$$\|\mathcal{A}^{\mathcal{S}}_{\alpha,1}(f)\|_{L^{q,\infty}(u)} \lesssim \Phi([u]_{A_1})\|f\|_{L^p(u^{p/q})}. \tag{4.17}$$

Since $\frac{p}{q} = 1 - \frac{p\alpha}{d}$, this is equivalent to

$$\|\mathcal{A}^{\mathcal{S}}_{\alpha,1}(u^{\frac{\alpha}{d}}f)\|_{L^{q,\infty}(u)} \lesssim \Phi([u]_{A_1})\|f\|_{L^p(u)}. \tag{4.18}$$

We now prove that (4.18) holds for $\Phi(t) \geq ct^{1-\frac{\alpha}{d}}$. Let

$$u(x) = |x|^{\vartheta - n}$$

with $0 < \vartheta < 1$. Then standard computations shows that

$$[u]_{A_1} \simeq \vartheta^{-1}. \tag{4.19}$$

Consider the function $f = \chi_B$ where B is the unit ball, we can compute its norm to be

$$\|f\|_{L^p(u)} = u(B)^{\frac{1}{p}} \simeq \vartheta^{-\frac{1}{p}}. \tag{4.20}$$

By sparse domination formula, we know there exists a sparse family $\mathcal{S}$ such that

$$\mathcal{A}^{\mathcal{S}}_{\alpha,1}(|f|)(x) \gtrsim |I_\alpha f(x)|, \tag{4.21}$$

where I_α is defined by (1.5). Let $0 < x_\vartheta < 1$ be a parameter whose value will be chosen soon. By (4.21), we have that

$$\begin{aligned}
\|\mathcal{A}^{\mathcal{S}}_{\alpha,1}(u^{\frac{\alpha}{d}}f)\|_{L^{q,\infty}(u)} &\gtrsim \|I_\alpha u^{\frac{\alpha}{d}} f\|_{L^{q,\infty}(u)} \\
&\geq \sup_{\lambda>0}\left(u\{|x| < x_\vartheta : \int_B \frac{|y|^{(\vartheta-1)\alpha/d}}{|x-y|^{1-\alpha/d}}dx > \lambda\}\right)^{\frac{1}{q}} \\
&\geq \sup_{\lambda>0}\left(u\{|x| < x_\vartheta : \int_{B\setminus B(0,|x|)} \frac{|y|^{(\vartheta-1)\alpha/d}}{|x-y|^{1-\alpha/d}}dx > \lambda\}\right)^{\frac{1}{q}} \\
&\geq \sup_{\lambda>0}\left(u\{|x| < x_\vartheta : \int_{B\setminus B(0,|x|)} \frac{|y|^{(\vartheta-1)\alpha/d}}{(2|y|)^{1-\alpha/d}}dx > \lambda\}\right)^{\frac{1}{q}} \\
&= \sup_{\lambda>0}\left(u\{|x| < x_\vartheta : \frac{c_{\alpha,d}}{\vartheta}(1-|x|^{\vartheta\alpha/d}) > \lambda\}\right)^{\frac{1}{q}} \\
&\geq \frac{c_{\alpha,d}}{2\vartheta}\left(u\{|x| < x_\vartheta : \frac{c_{\alpha,d}}{\vartheta}(1-|x|^{\vartheta\alpha/d}) > \frac{c_{\alpha,d}}{2\vartheta}\}\right)^{\frac{1}{q}} \\
&= \frac{c_{\alpha,d}}{2\vartheta}u(B(0,x_\vartheta))^{\frac{1}{q}},
\end{aligned}$$

where taking $x_\vartheta = (\frac{1}{2})^{d/\alpha\vartheta}$ in the last step. It now follows that for $0 < \vartheta < 1$,

$$\|\mathcal{A}^{\mathcal{S}}_{\alpha,1}(u^{\frac{\alpha}{d}} f)\|_{L^{q,\infty}(u)} \gtrsim \frac{1}{\vartheta}\left(\frac{x_\vartheta}{\vartheta}\right)^{\frac{1}{q}} \simeq \vartheta^{-1-\frac{1}{q}}. \tag{4.22}$$

Finally, combining (4.19), (4.20), (4.22), and using that $\frac{1}{q} + \frac{\alpha}{d} = \frac{1}{p}$, we have that (4.18) holds for $\Phi(t) \geq ct^{1-\frac{\alpha}{d}}$, which gives the desired bound by the monotonicity of Φ. □

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