# EXISTENCE OF EQUIVARIANT MODELS OF $G$-VARIETIES

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Abstract. Let $k_0$ be a field of characteristic 0, and let $k$ be a fixed algebraic closure of $k_0$. Let $G$ be an algebraic $k$-group, and let $Y$ be a $G$-variety over $k$. Let $G_0$ be a $k_0$-model ($k_0$-form) of $G$. We ask whether $Y$ admits a $G_0$-equivariant $k_0$-model $Y_0$.

We assume that $Y$ admits a $G_\diamondsuit$-equivariant $k_0$-model $Y_\diamondsuit$, where $G_\diamondsuit$ is an inner form of $G_0$. We give a Galois-cohomological criterion for the existence of a $G_0$-equivariant $k_0$-model $Y_0$ of $Y$. We apply this criterion to certain spherical homogeneous varieties $Y = G/H$.

## 1. Introduction

**1.1.** Let $k_0$ be a field of characteristic 0, and let $k$ be a fixed algebraic closure of $k_0$. Set $\Gamma = \mathrm{Gal}(k/k_0)$.

Let $G$ be a connected algebraic group over $k$ (not necessarily linear). Let $Y$ be a $G$-variety, that is, an irreducible algebraic variety over $k$ together with a morphism

$$\theta\colon G \times_k Y \to Y$$

defining an action of $G$ on $Y$. We say that $(Y, \theta)$ is a *$G$-$k$-variety* or just that $Y$ is a $G$-$k$-variety.

Let $G_0$ be a *$k_0$-model* ($k_0$-form) of $G$, that is, an algebraic group over $k_0$ together with an isomorphism of algebraic $k$-groups

$$\nu_G\colon G_0 \times_{k_0} k \xrightarrow{\sim} G.$$

By a *$G_0$-equivariant $k_0$-model of the $G$-$k$-variety* $(Y, \theta)$ we mean a $G_0$-$k_0$-variety $(Y_0, \theta_0)$ together with an isomorphism $\nu_Y\colon Y_0 \times_{k_0} k \xrightarrow{\sim} Y$ such that the following diagram commutes:

$$\begin{array}{ccc} G_{0,k} \times_k Y_{0,k} & \xrightarrow{\theta_{0,k}} & Y_{0,k} \\ \downarrow{\scriptstyle \nu_G \times \nu_Y} & & \downarrow{\scriptstyle \nu_Y} \\ G \times_k Y & \xrightarrow{\theta} & Y \end{array}$$

Inspired by the works of Akhiezer and Cupit-Foutou [2], [1], for a given $k_0$-model $G_0$ of $G$ we ask whether there exists a $G_0$-equivariant $k_0$-model $Y_0$ of $Y$.

**1.2.** With the above notation, we consider the group $\mathrm{Aut}(G)$ of automorphisms of $G$. We regard $\mathrm{Aut}(G)$ as an abstract group. Any $g \in G(k)$ defines an *inner automorphism*

$$i_g\colon G \to G, \quad x \mapsto gxg^{-1} \text{ for } x \in G(k).$$

We obtain a homomorphism

$$i\colon G(k) \to \mathrm{Aut}(G).$$

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We denote by $\mathrm{Inn}(G) \subset \mathrm{Aut}(G)$ the image of the homomorphism $i$ and we say that $\mathrm{Inn}(G)$ is the group of inner automorphisms of $G$. We may identify $\mathrm{Inn}(G)$ with $\overline{G}(k)$, where

$$\overline{G} = G/Z(G)$$

and $Z(G)$ is the center of $G$.

Let $G_\diamondsuit$ be a $k_0$-model of $G$. We write $Z_\diamondsuit$ for the center $Z(G_\diamondsuit)$, then $\overline{G}_\diamondsuit := G_\diamondsuit/Z_\diamondsuit$ is a $k_0$-model of $\overline{G}$. Let $c\colon \Gamma \to \overline{G}_\diamondsuit(k)$ be a *1-cocycle*, that is, a locally constant map such that the following cocycle condition is satisfied:

$$c_{st} = c_s \cdot {}^s c_t \quad \text{for all } s,t \in \Gamma. \tag{1.3}$$

We denote the set of such 1-cocycles by $Z^1(\Gamma, \overline{G}_\diamondsuit(k))$ or by $Z^1(k_0, \overline{G}_\diamondsuit)$. For $c \in Z^1(k_0, \overline{G})$ one can define the *c-twisted inner form* ${}_c(G_\diamondsuit)$ of $G_\diamondsuit$; see Subsection 2.3 below. For simplicity we write ${}_cG_\diamondsuit$ for ${}_c(G_\diamondsuit)$.

**1.4.** It is well known that if $G$ is a connected reductive $k$-group, then any $k_0$-model $G_0$ of $G$ is an inner form of a quasi-split model; see e.g., Springer [14, Proposition 16.4.9]. In other words, there exist a quasi-split model $G_{\mathrm{qs}}$ of $G$ and a 1-cocycle $c \in Z^1(k_0, \overline{G}_{\mathrm{qs}})$ such that $G_0 = {}_cG_{\mathrm{qs}}$. In some cases it is clear that $Y$ admits a $G_{\mathrm{qs}}$-equivariant $k_0$-model. For example, assume that $Y = G/U$, where $U = R_u(B)$, the unipotent radical of a Borel subgroup $B$ of $G$. Since $G_{\mathrm{qs}}$ is a *quasi-split* model, there exists a Borel subgroup $B_{\mathrm{qs}} \subset G_{\mathrm{qs}}$ (defined over $k_0$). Set $U_{\mathrm{qs}} = R_u(B_{\mathrm{qs}})$, then $G_{\mathrm{qs}}/U_{\mathrm{qs}}$ is a $G_{\mathrm{qs}}$-equivariant $k_0$-model of $Y = G/U$.

**1.5.** In the setting of 1.1 and 1.2, let $G_\diamondsuit$ be a $k_0$-model of $G$ and let $G_0 = {}_cG_\diamondsuit$, where $c \in Z^1(k_0, \overline{G}_\diamondsuit)$. Motivated by 1.4, *we assume* that $Y$ admits a $G_\diamondsuit$-equivariant $k_0$-model $Y_\diamondsuit$, and we ask whether $Y$ admits a $G_0$-equivariant $k_0$-model $Y_0$.

We consider the short exact sequence

$$1 \to Z_\diamondsuit \to G_\diamondsuit \to \overline{G}_\diamondsuit \to 1$$

and the connecting map

$$\delta\colon H^1(k_0, \overline{G}_\diamondsuit) \to H^2(k_0, Z_\diamondsuit);$$

see Serre [11, I.5.7, Proposition 43]. If $c \in Z^1(k_0, \overline{G}_\diamondsuit)$, we write $[c]$ for the corresponding cohomology class in $H^1(k_0, \overline{G}_\diamondsuit)$. By abuse of notation we write $\delta[c]$ for $\delta([c])$.

We consider the group $\mathcal{A} := \mathrm{Aut}^G(Y)$ of $G$-equivariant automorphisms of $Y$, which we regard as an abstract group. The $G_\diamondsuit$-equivariant $k_0$-model $Y_\diamondsuit$ of $Y$ defines a $\Gamma$-action on $\mathcal{A}$, see Subsection 3.2 below, and we denote the obtained $\Gamma$-group by $\mathcal{A}_\diamondsuit$. One can define the second Galois cohomology set $H^2(\Gamma, \mathcal{A}_\diamondsuit)$. See Springer [13, 1.14] for a definition of $H^2(\Gamma, \mathcal{A}_\diamondsuit)$ in the case when the $\Gamma$-group $\mathcal{A}_\diamondsuit$ is nonabelian.

For $z \in Z_\diamondsuit(k)$ we consider the $G$-equivariant automorphism

$$y \mapsto z \cdot y\colon\ Y \to Y.$$

We obtain a $\Gamma$-equivariant homomorphism

$$\varkappa\colon Z_\diamondsuit(k) \to \mathcal{A}_\diamondsuit,$$

which induces a map

$$\varkappa_*\colon H^2(k_0, Z_\diamondsuit) \to H^2(\Gamma, \mathcal{A}_\diamondsuit).$$

**Theorem 1.6** (Theorem 3.5). *Let $k$, $G$, $Y$, $k_0$, $G_\diamondsuit$, $Y_\diamondsuit$, $\mathcal{A}_\diamondsuit$, $\delta$, $\varkappa_*$ be as in Subsections 1.1 and 1.5. In particular, we assume that $Y$ admits a $G_\diamondsuit$-equivariant $k_0$-model $Y_\diamondsuit$. We assume also that $Y$ is quasi-projective. Let $c \in Z^1(k_0, \overline{G}_\diamondsuit)$ be a 1-cocycle, and consider its class $[c] \in H^1(k_0, \overline{G}_\diamondsuit)$. Set $G_0 = {}_cG_\diamondsuit$ (the inner twisted form of $G_\diamondsuit$ defined by the 1-cocycle $c$). Then the $G$-variety $Y$ admits a $G_0$-equivariant $k_0$-model if and only if the cohomology class*

$$\varkappa_*(\delta[c]) \in H^2(\Gamma, \mathcal{A}_\diamondsuit)$$

*is neutral.*

**Remark 1.7.** In the case when $\mathcal{A}_\diamondsuit$ is abelian, the condition "$\varkappa_*(\delta[c])$ is neutral" means that $\varkappa_*(\delta[c]) = 1$.

Theorem 1.6 is the main result of this paper. Theorems 1.8, 1.12, and 1.14 below are applications of Theorem 1.6 to the case when $Y = G/H$ is a homogeneous space of $G$. In this case $\mathcal{A} = A(k)$, where $A = \mathcal{N}_G(H)/H$ and $\mathcal{N}_G(H)$ denotes the normalizer of $H$ in $G$; see e.g. [4, Lemma 5.1].

In the following theorem, $G$ is a connected reductive group.

**Theorem 1.8** (Theorem 4.3)**.** *Let $G$ be a reductive group over an algebraically closed field $k$ of characteristic 0. Let $H \subset G$ be a $k$-subgroup. Let $k_0 \subset k$ be a subfield such that $k$ is an algebraic closure of $k$. Let $G_0$ be a $k_0$-model of $G$. Write $G_0 = {}_cG_{\rm qs}$, where $G_{\rm qs}$ is a quasi-split $k_0$-model of $G$ and $c \in Z^1(k_0, G_{\rm qs}/Z(G_{\rm qs}))$. Assume that*

$$(*)\qquad G/H \text{ admits a } G_{\rm qs}\text{-equivariant } k_0\text{-model } Y_{\rm qs}.$$

*The $k_0$-model $Y_{\rm qs}$ defines a $k_0$-model $A_{\rm qs} = \mathrm{Aut}^{G_{\rm qs}}(Y_{\rm qs})$ of $A = \mathrm{Aut}^G(Y)$. Then $G/H$ admits a $G_0$-equivariant $k_0$-model $Y_0$ if and only if the image in $H^2(k_0, A_{\rm qs})$ of the Tits class $t(\widetilde{G}_0) \in H^2(k_0, Z(\widetilde{G}_{\rm qs}))$ is neutral (see Section 4 below for the definition of the Tits class).*

**Remark 1.9.** In Theorem 1.8, if there exists a $G_0$-equivariant $k_0$-model $Y_0$ of $G/H$, then the set of isomorphism classes of such models is in a canonical bijection with the set $H^1(k_0, \mathrm{Aut}^{G_0}(Y_0))$.

**1.10.** Let

$$\tilde{c}\colon \Gamma \to G_\diamondsuit(k)$$

be a 1-cocycle *with values in* $G_\diamondsuit$, that is, $\tilde{c} \in Z^1(k_0, G_\diamondsuit)$. Consider $i \circ \tilde{c} \in Z^1(k_0, \overline{G}_\diamondsuit)$, then by abuse of notation we write ${}_{\tilde{c}}G_\diamondsuit$ for ${}_{i\circ\tilde{c}}G_\diamondsuit$. We say that ${}_{\tilde{c}}G_\diamondsuit$ is a *pure inner form* of $G_\diamondsuit$. For a pure inner form $G_0 = {}_{\tilde{c}}G_\diamondsuit$, the $G$-variety $Y$ clearly admits a $G_0$-equivariant $k_0$-model: we may take $Y_0 = {}_{\tilde{c}}Y_\diamondsuit$; see Lemma 2.4 below. It follows from the cohomology exact sequence (3.3) below that for a cocycle $c \in Z^1(k_0, \overline{G}_\diamondsuit)$, the twisted form ${}_cG_\diamondsuit$ is a pure inner form of $G_\diamondsuit$ if and only if $\delta[c] = 1$.

**1.11.** Let $H$ be a connected linear $k$-group, and set $G = H \times_k H$. Let $Y = H$, where $G$ acts on $Y$ by

$$(h_1, h_2) * y = h_1 y h_2^{-1}.$$

Note that $Y = G/\Delta$, where $\Delta \subset H \times_k H$ is the diagonal, that is, $\Delta$ is $H$ embedded in $G$ diagonally. Let $H_0^{(1)}$ and $H_0^{(2)}$ be two $k_0$-models of $H$. We set $G_0 = H_0^{(1)} \times_{k_0} H_0^{(2)}$ and ask whether $Y$ admits a $G_0$-equivariant $k_0$-model.

**Theorem 1.12** (Theorem 5.2)**.** *With the notation and assumptions of 1.11, $Y = (H \times H)/\Delta$ admits an $H_0^{(1)} \times_{k_0} H_0^{(2)}$-equivariant $k_0$-model if and only if $H_0^{(2)}$ is a pure inner form of $H_0^{(1)}$.*

**Example 1.13.** Let $k = \mathbb{C}$, $k_0 = \mathbb{R}$, then $\Gamma = \{1, s\}$, where $s$ is the complex conjugation. Let $H = \mathrm{SL}(4, \mathbb{C})$. Consider the diagonal matrices

$$I_4 = \mathrm{diag}(1,1,1,1) \quad \text{and} \quad I_{2,2} = \mathrm{diag}(1,1,-1,-1).$$

Consider the real models $\mathrm{SU}(2,2)$ and $\mathrm{SU}(4)$ of $G$:

$$H_0^{(1)} = \mathrm{SU}(2,2), \text{ where } \mathrm{SU}(2,2)(\mathbb{R}) = \{g \in \mathrm{SL}(4,\mathbb{C}) \mid g \cdot I_{2,2} \cdot {}^s g^{\rm tr} = I_{2,2}\},$$

$$H_0^{(2)} = \mathrm{SU}(4), \text{ where } \mathrm{SU}(4)(\mathbb{R}) = \{g \in \mathrm{SL}(4,\mathbb{C}) \mid g \cdot I_4 \cdot {}^s g^{\rm tr} = I_4\},$$

where $g^{\mathrm{tr}}$ denotes the transpose of $g$. Consider the 1-cocycle

$$c\colon \Gamma \to \mathrm{SU}(2,2)(\mathbb{R}), \quad 1 \mapsto I_4,\ s \mapsto I_{2,2}.$$

A calculation shows that ${}_c\mathrm{SU}(2,2) \simeq \mathrm{SU}(4)$. Thus $\mathrm{SU}(4)$ is a pure inner form of $\mathrm{SU}(2,2)$. By Theorem 1.12, there exists an $\mathrm{SU}(2,2) \times_{\mathbb{R}} \mathrm{SU}(4)$-equivariant real model $Y_0$ of $Y = (H \times H)/\Delta$. We describe this model explicitly. We may take for $Y_0$ the *transporter*

$$Y_0 = \{g \in \mathrm{SL}(4,\mathbb{C}) \mid g \cdot I_4 \cdot {}^s g^{\mathrm{tr}} = I_{2,2}\}.$$

Clearly $Y_0$ is defined over $\mathbb{R}$. It is well known that $Y_0$ is nonempty but it has no $\mathbb{R}$-points. The group $G_0 := H_0^{(1)} \times_{\mathbb{R}} H_0^{(2)}$ acts on $Y_0$ by

$$(h_1, h_2) * g = h_1 g h_2^{-1}.$$

It is clear that $Y_0$ is a principal homogeneous space of both $H_0^{(1)}$ and $H_0^{(2)}$. Thus $Y_0$ is a $G_0$-equivariant $k_0$-model of $Y$. Compare [4, Example 10.11].

In the following theorem, $G$ is a connected reductive group and $Y = G/U$.

**Theorem 1.14** (Theorem 6.1)**.** *Let $k$ and $k_0$ be as in 1.1, and let $G$ be a connected reductive group over $k$. Let $B \subset G$ be a Borel subgroup, and write $U$ for the unipotent radical of $B$. Consider the homogeneous space $Y = G/U$. Let $G_0$ be a $k_0$-model of $G$. Then $Y$ admits a $G_0$-equivariant $k_0$-model if and only if $G_0$ is a pure inner form of a quasi-split model of $G$.*

**Example 1.15.** Let $k = \mathbb{C}$, $k_0 = \mathbb{R}$, $G = \mathrm{SL}(4,\mathbb{C})$, $Y = G/U$, where $U$ is as in Theorem 1.14. Let $G_0 = \mathrm{SU}(4)$. Since $G_0$ is a pure inner form of the quasi-split group $\mathrm{SU}(2,2)$, by Theorem 1.14 the variety $G/U$ admits an $\mathrm{SU}(4)$-equivariant $\mathbb{R}$-model $Y_0$. This model has no $\mathbb{R}$-points (because the stabilizer of an $\mathbb{R}$-point would be a unipotent subgroup of $G_0$ defined over $\mathbb{R}$).

The plan for the rest of the paper is as follows. In Section 2 we recall basic definitions and results. In Section 3 we prove Theorem 1.6. In Section 4 we prove Theorem 1.8. In Section 5 we prove Theorem 1.11. In Section 6 we prove Theorem 1.14.

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## 2. Preliminaries

**2.1.** Let $k_0$, $k$, and $\Gamma$ be as in Subsection 1.1. By a $k_0$-model of a $k$-scheme $Y$ we mean a $k_0$-scheme $Y_0$ together with an isomorphism of $k$-schemes

$$\nu_Y\colon Y_0 \times_{k_0} k \xrightarrow{\sim} Y.$$

We write $\Gamma = \mathrm{Gal}(k/k_0)$. For $s \in \Gamma$, denote by $s^*\colon \operatorname{Spec} k \to \operatorname{Spec} k$ the morphism of schemes induced by $s$. Notice that $(st)^* = t^* \circ s^*$.

Let $(Y, p_Y\colon Y \to \operatorname{Spec} k)$ be a $k$-scheme. A *$k/k_0$-semilinear automorphism of $Y$* is a pair $(s, \mu)$ where $s \in \Gamma$ and $\mu\colon Y \to Y$ is an isomorphism of schemes such that the diagram below commutes:

$$\begin{CD} Y @>{\mu}>> Y \\ @V{p_Y}VV @VV{p_Y}V \\ \operatorname{Spec} k @>{(s^*)^{-1}}>> \operatorname{Spec} k \end{CD}$$

In this case we say also that $\mu$ is an $s$-semilinear automorphism of $Y$. We shorten "$s$-semilinear automorphism" to "$s$-semi-automorphism". Note that if $(s, \mu)$ is a semi-automorphism of $Y$, then $\mu$ uniquely determines $s$; see [4, Lemma 1.2].

We denote $\mathrm{SAut}(Y)$ the group of all $s$-semilinear automorphisms $\mu$ of $Y$, where $s$ runs over $\Gamma = \mathrm{Gal}(k/k_0)$. By a *semilinear action* of $\Gamma$ on $Y$ we mean a homomorphism of groups

$$\mu\colon \Gamma \to \mathrm{SAut}(Y), \quad s \mapsto \mu_s$$

such that for each $s \in \Gamma$, $\mu_s$ is $s$-semilinear.

If we have a $k_0$-scheme $Y_0$, then the formula

$$s \mapsto \mathrm{id}_{Y_0} \times (s^*)^{-1} \tag{2.2}$$

defines a semilinear action of $\Gamma$ on

$$Y := Y_0 \times_{k_0} k = Y_0 \times_{\mathrm{Spec}\, k_0} \mathrm{Spec}\, k.$$

Thus a $k_0$-model of $Y$ induces a semilinear action of $\Gamma$ on $Y$.

Let $(G, p_G\colon G \to \mathrm{Spec}\, k)$ be a $k$-group-scheme. A $k/k_0$-semi-linear automorphism of $G$ is a pair $(s, \tau)$ where $s \in \Gamma$ and $\tau\colon G \to G$ is a morphism of schemes such that the following diagram commutes

$$\begin{array}{ccc} G & \xrightarrow{\ \tau\ } & G \\ {\scriptstyle p_G}\downarrow & & \downarrow{\scriptstyle p_G} \\ \mathrm{Spec}\, k & \xrightarrow{(s^*)^{-1}} & \mathrm{Spec}\, k \end{array}$$

and the $k$-morphism

$$\tau_\natural\colon s_*G \to G$$

is an isomorphism of algebraic groups over $k$; see [4, Definition 2.2] for the notations $\tau_\natural$ and $s_*G$.

We denote by $\mathrm{SAut}_{k/k_0}(G)$, or just by $\mathrm{SAut}(G)$, the group of all $s$-semilinear automorphisms $\tau$ of $G$, where $s$ runs over $\Gamma = \mathrm{Gal}(k/k_0)$. By a semilinear action of $\Gamma$ on $G$ we mean a homomorphism

$$\sigma\colon \Gamma \to \mathrm{SAut}(G), \quad s \mapsto \sigma_s$$

such that for all $s \in \Gamma$, $\sigma_s$ is $s$-semilinear. As above, a $k_0$-model $G_0$ of $G$ induces a semilinear action of $\Gamma$ on $G$.

Let $G$ be an algebraic group over $k$ and let $Y$ be a $G$-$k$-variety. Let $G_0$ be a $k_0$-model of $G$. It gives rise to a semilinear action $\sigma\colon \Gamma \to \mathrm{SAut}(G), s \mapsto \sigma_s$. Let $Y_0$ be a $G_0$-equivariant $k_0$-model of $Y$. It gives rise to a semilinear action $\mu\colon \Gamma \to \mathrm{SAut}(Y)$ such that for all $s$ in $\Gamma$ we have

$$\mu_s(g \cdot y) = \sigma_s(g) \cdot \mu_s(y) \quad \text{for all } y \in Y(k),\ g \in G(k).$$

We say then that $\mu_s$ is *$\sigma_s$-equivariant.*

**2.3.** Let $k_0$, $k$, and $\Gamma$ be as in Subsection 1.1. Let $G_0$ be a $k_0$-model of $G$; it defines a semilinear action

$$\sigma\colon \Gamma \to \mathrm{SAut}(G).$$

This action induces an action of $\Gamma$ on the abstract group $\mathrm{Aut}(G)$. Recall that a map

$$c\colon \Gamma \to \mathrm{Aut}(G)$$

is called a *1-cocycle* if the map $c$ is locally constant and satisfies the cocycle condition (1.3). The set of such 1-cocycles is denoted by $Z^1(\Gamma, \mathrm{Aut}(G))$ or $Z^1(k_0, \mathrm{Aut}(G))$. For $c \in Z^1(k_0, \mathrm{Aut}(G))$, we consider the $c$-twisted semilinear action

$$\sigma'\colon \Gamma \to \mathrm{SAut}(G), \quad s \mapsto c_s \circ \sigma_s.$$

Then, clearly, $\sigma'_s$ is an $s$-semi-automorphism of $G$ for any $s \in \Gamma$. It follows from the cocycle condition (1.3) that

$$\sigma'_{st} = \sigma'_s \circ \sigma'_t \ \text{ for all } s, t \in \Gamma.$$

Since $G$ is an algebraic group, the semilinear action $\sigma'$ comes from some $k_0$-model $G_0'$ of $G$; see Serre [10, Section V.4.20, Corollary 2(ii) of Proposition 12] and Serre [11, III.1.3, Proposition 5]. We write $G_0' = {}_cG_0$ and say that $G_0'$ is the *twisted form of* $G_0$ *defined by the 1-cocycle* $c$.

**Lemma 2.4.** *Let $G$ be a linear algebraic group over $k$, and let $Y$ be a* quasi-projective *$G$-$k$-variety. Let $G_\diamondsuit$ be a $k_0$-model of $G$, and assume that $Y$ admits a $G_\diamondsuit$-equivariant $k_0$-model $Y_\diamondsuit$. Let $\tilde{c} \in Z^1(k_0, G_\diamondsuit)$ be a 1-cocycle. Consider the pure inner form ${}_{\tilde{c}}G_\diamondsuit$. Then $Y$ admits a ${}_{\tilde{c}}G_\diamondsuit$-equivariant $k_0$-model.*

*Proof.* Write $G_0 = {}_{\tilde{c}}G_\diamondsuit$. We take $Y_0 =_{\tilde{c}} Y_\diamondsuit$, then $Y_0$ is a $G_0$-equivariant $k_0$-model.

We give details. The $k_0$-models $G_\diamondsuit$ and $Y_\diamondsuit$ define semilinear actions

$$\sigma\colon \Gamma \to \mathrm{SAut}(G) \quad \text{and} \quad \mu\colon \Gamma \to \mathrm{SAut}(Y)$$

such that for any $s \in \Gamma$ the semi-automorphism $\mu_s$ is $\sigma_s$-equivariant, that is,

$$\mu(g \cdot y) = \sigma_s(g) \cdot \mu_s(y) \quad \text{for all } g \in G(k),\ y \in Y(k).$$

Let $\tilde{c}\colon \Gamma \to G(k)$ be a 1-cocycle, that is, $\tilde{c} \in Z^1(k_0, G_\diamondsuit)$. Consider the pure inner form $G_0 = {}_{\tilde{c}}G_\diamondsuit$, then

$$\sigma_s^0(g) = \tilde{c}_s \cdot \sigma_s(g) \cdot \tilde{c}_s^{-1} \text{ for } s \in \Gamma,\ g \in G(k).$$

where $\sigma^0$ is the semilinear action defined by $G_0$. Now we define the twisted form ${}_{\tilde{c}}Y_\diamondsuit$ as follows. We set

$$\mu_s^0(y) = \tilde{c}_s \cdot \mu_s^0(y).$$

Since $\tilde{c}$ is a 1-cocycle, we have

$$\mu_{st}^0 = \mu_s^0 \circ \mu_t^0 \quad \text{for all } s, t \in \Gamma.$$

Since $Y$ is quasi-projective, by Borel and Serre [3, Lemme 2.12] the semilinear action $\mu^0\colon \Gamma \to \mathrm{SAut}(Y)$ defines a $k_0$-model $Y_0$ of $Y$. An easy calculation shows that

$$\mu^0(g \cdot y) = \sigma_s^0(g) \cdot \mu_s^0(y) \quad \text{for all } g \in G(k),\ y \in Y(k),$$

hence by Galois descent we obtain an action of $G_0$ on $Y_0$ (defined over $k_0$); see Jahnel [6, Theorem 2.2(b)]. Thus $Y$ admits a $G_0$-equivariant $k_0$-model $Y_0 = {}_{\tilde{c}}Y_\diamondsuit$. □

## 3. Model for an inner twist of the group

**3.1.** Let $k$ be an algebraically closed field of characteristic 0. Let $G$ be an algebraic group over $k$. Let $Y$ be a $G$-$k$-variety. Let $Z(G)$ denote the center of $G$. We consider the algebraic group $\overline{G} := G/Z(G)$. The group $\overline{G}(k)$ naturally acts on $G$:

$$gZ(G)\colon\ x \mapsto gxg^{-1} \quad \text{for } gZ(G) \in \overline{G}(k),\ x \in G(k).$$

Let $k_0$ be a subfield of $k$ such that $k/k_0$ is an algebraic extension. We write $\Gamma = \mathrm{Gal}(k/k_0)$, which is a profinite group.

Let $G_\diamondsuit$ be a $k_0$-model of $G$. We write $\overline{G}_\diamondsuit = G_\diamondsuit/Z(G_\diamondsuit)$, where $Z(G_\diamondsuit)$ is the center of $G_\diamondsuit$. The $k_0$-model $\overline{G}_\diamondsuit$ of $\overline{G}$ defines a semilinear action:

$$\sigma\colon \Gamma \to \mathrm{SAut}(\overline{G});$$

cf. (2.2). We write $\overline{G}_\diamondsuit(k)$ for the group of $k$-points of the algebraic $k_0$-group $\overline{G}_\diamondsuit$, then we have an action of $\Gamma$ on $\overline{G}_\diamondsuit(k)$:

$$(s, g) \mapsto {}^s g = \sigma_s(g) \quad \text{for } s \in \Gamma,\ g \in \overline{G}_\diamondsuit(k) = \overline{G}(k).$$

Let $c \in Z^1(k_0, \overline{G}_\diamondsuit)$ be a 1-cocycle, that is, a locally constant map

$$c\colon \Gamma \to \overline{G}_\diamondsuit(k) \quad \text{such that} \quad c_{st} = c_s \cdot {}^s c_t \quad \text{for all } s, t \in \Gamma.$$

We denote by $G_0 = {}_cG_\diamondsuit$ the corresponding inner twisted form of $G_\diamondsuit$, see Subsection 2.3. This means that $G_0(k) = G_\diamondsuit(k)$, but the Galois action is twisted by $c$:

$$\sigma_s^0 = c_s \circ \sigma_s \quad \text{for } s \in \Gamma,$$

where we embed $\overline{G}_\diamondsuit(k)$ into $\mathrm{Aut}(G)$.

In this section we *assume* that there exists a $G_\diamondsuit$-equivariant $k_0$-model $Y_\diamondsuit$ of $Y$. We give a criterion for the existence of a $G_0$-equivariant $k_0$-model $Y_0$ of $Y$, where $G_0 = {}_cG_\diamondsuit$.

**3.2.** We write $[c] \in H^1(k_0, \overline{G}_\diamondsuit)$ for the cohomology class of $c$. We consider the short exact sequence

$$1 \to Z(G_\diamondsuit) \to G_\diamondsuit \to \overline{G}_\diamondsuit \to 1$$

and the corresponding connecting map

$$\delta\colon H^1(k_0, \overline{G}_\diamondsuit) \to H^2(k_0, Z(G_\diamondsuit))$$

from the cohomology exact sequence

$$H^1(k_0, Z(G_\diamondsuit)) \to H^1(k_0, G_\diamondsuit) \to H^1(k_0, \overline{G}_\diamondsuit) \xrightarrow{\ \delta\ } H^2(k_0, Z(G_\diamondsuit)); \tag{3.3}$$

see Serre [11, I.5.7, Proposition 43]. We obtain $\delta[c] \in H^2(k_0, Z(G_\diamondsuit)$.

The $G_\diamondsuit$-equivariant $k_0$-model $Y_\diamondsuit$ of $Y$ defines an action of $\Gamma$ on $\mathcal{A} := \mathrm{Aut}^G(Y)$ by

$$({}^s a)({}^s y) = {}^s(a(y)) \quad \text{for } s \in \Gamma,\ a \in \mathcal{A},\ y \in Y(k).$$

We denote by $\mathcal{A}_\diamondsuit$ the corresponding $\Gamma$-group. We obtain homomorphisms

$$\mu\colon \Gamma \to \mathrm{SAut}(Y), \quad s \mapsto \mu_s, \quad \text{where } \mu_s(y) = {}^s y \ \text{ for } s \in \Gamma,\ y \in Y(k) = Y_\diamondsuit(k),$$

and

$$\tau\colon \Gamma \to \mathrm{Aut}(\mathcal{A}), \quad s \mapsto \tau_s, \quad \text{where } \tau_s(a) = {}^s a \ \text{ for } s \in \Gamma,\ a \in \mathcal{A}.$$

The center $Z_\diamondsuit \subset G_\diamondsuit$ acts on $Y_\diamondsuit$, and this action clearly commutes with the action of $G_\diamondsuit$. Thus we obtain a canonical $\Gamma$-equivariant homomorphism

$$\varkappa\colon Z_\diamondsuit(k) \to \mathcal{A}_\diamondsuit.$$

**3.4.** We need the nonabelian cohomology set $H^2(\Gamma, \mathcal{A}_\diamondsuit)$; see Springer [13, 1.14]. Recall that an (abelian) 2-cocycle $z \in Z^2(k_0, Z_\diamondsuit)$ is a locally constant map

$$a\colon \Gamma \times \Gamma \to Z_\diamondsuit(k), \quad (s,t) \mapsto z_{s,t}$$

such that

$${}^s d_{t,u} \cdot d_{s,tu} = d_{s,t} \cdot d_{st,u} \quad \text{for all } s,t,u \in \Gamma.$$

Then $\varkappa_*([z]) \in H^2(\Gamma, \mathcal{A}_\diamondsuit)$ is by definition the class of the 2-cocycle $(\tau, \varkappa \circ z)$. This class is called *neutral* if there exists a locally constant map $a\colon \Gamma \to \mathcal{A}_\diamondsuit$ such that

$$a_s \cdot {}^s a_t \cdot \varkappa(z_{s,t}) \cdot a_{st}^{-1} = 1 \quad \text{for all } s,t \in \Gamma.$$

**Theorem 3.5.** *Let $G$, $H$, $Y$, $k_0$, $G_\diamondsuit$, $Y_\diamondsuit$, $A_\diamondsuit$, $\delta$ be as in Subsections 3.1 and 3.2. In particular we assume that $Y$ admits a $G_\diamondsuit$-equivariant $k_0$-model $Y_\diamondsuit$. We assume also that $Y$ is quasi-projective. Let $c \in Z^1(k_0, \overline{G}_\diamondsuit)$ be a 1-cocycle, and consider it class $[c] \in H^1(k_0, \overline{G}_\diamondsuit)$. Set $G_0 = {}_cG_\diamondsuit$ (the inner twisted form of $G_\diamondsuit$ defined by the 1-cocycle $c$). The $G$-variety $Y$ admits a $G_0$-equivariant $k_0$-model if and only if the cohomology class*

$$\varkappa_*(\delta[c]) \in H^2(\Gamma, \mathcal{A}_\diamondsuit)$$

*is neutral.*

*Proof.* The $k_0$-model $G_\diamondsuit$ of $G$ defines a homomorphism

$$\sigma\colon \Gamma \to \mathrm{SAut}(G), \quad s \mapsto \sigma_s,$$

where each $\sigma_s$ is an $s$-semi-automorphism of $G$. The $G_\diamondsuit$-equivariant $k_0$-model $Y_\diamondsuit$ of $Y$ defines a homomorphism

$$\mu\colon \Gamma \to \mathrm{SAut}(Y), \quad s \mapsto \mu_s$$

such that each $\mu_s$ is an $s$-semi-automorphism of $Y$ and is $\sigma_s$-equivariant, that is,

$$\mu_s(g\cdot y) = \sigma_s(g)\cdot \mu_s(y) \quad \text{for all } g \in G(k),\ y\in Y(k). \tag{3.6}$$

Since the map $s \mapsto \mu_s$ is a homomorphism, we have

$$\mu_{st} = \mu_s \circ \mu_t \quad \text{for all } s,t\in\Gamma. \tag{3.7}$$

We lift the 1-cocycle

$$c\colon \Gamma \to \overline{G}(k)$$

to a locally constant map

$$\tilde c\colon \Gamma \to G(k),$$

which does not have to be a 1-cocycle. Let $\sigma^0\colon \Gamma \to \mathrm{SAut}(G)$ denote the homomorphism corresponding to the twisted form $G_0 = {}_cG_\diamondsuit$, then by definition

$$\sigma^0_s(g) = \tilde c_s \cdot \sigma_s(g)\cdot \tilde c_s^{-1}.$$

For $g \in G(k)$, we write $l(g)$ for the automorphism $y \mapsto g\cdot y$ of $Y$. We have

$$l(g)\circ a = a\circ l(g) \quad \text{for all } g\in G(k),\ a\in \mathcal{A}_\diamondsuit, \tag{3.8}$$

because $a$ is a $G$-equivariant automorphism of $Y$. By (3.6) we have $\mu_s(g\cdot y) = \sigma_s(g)\cdot\mu_s(y)$, hence

$$\mu_s\circ l(g) = l(\sigma_s(g))\circ \mu_s \quad \text{for all } s\in\Gamma,\ g\in G_\diamondsuit(k). \tag{3.9}$$

Similarly, $\tau_s(a)(\mu_s(y)) = \mu_s(a(y))$, hence,

$$\mu_s\circ a = \tau_s(a)\circ\mu_s \quad \text{for all } s\in\Gamma,\ a\in\mathcal{A}_\diamondsuit. \tag{3.10}$$

By definition (Serre [11, I.5.6])

$$\delta[c] \in H^2(k_0, Z(G_\diamondsuit))$$

is the class of the 2-cocycle given by

$$(s,t)\mapsto \tilde c_s\cdot {}^s\tilde c_t \cdot \tilde c_{st}^{-1} \ \in Z(G_\diamondsuit)(k) \quad (s,t\in\Gamma).$$

Then $\varkappa_*(\delta[c])$ is the class of the 2-cocycle

$$(s,t)\mapsto \varkappa(\tilde c_s\cdot {}^s\tilde c_t\cdot \tilde c_{st}^{-1}) \in \mathcal{A}_\diamondsuit.$$

Let

$$a\colon \Gamma\to\mathcal{A}_\diamondsuit$$

be a locally constant map. We define

$$\mu^0_s = a_s\circ l(\tilde c_s)\circ\mu_s = l(\tilde c_s)\circ a_s\circ \mu_s.$$

**Lemma 3.11.** *For any $s\in\Gamma$, the $s$-semi-automorphism $\mu^0_s$ is $\sigma^0_s$-equivariant.*

*Proof.* Using (3.8) and (3.9), we compute:

$$\begin{aligned}\mu^0_s(g\cdot y) &= (a_s\circ l(\tilde c_s))(\mu_s(g\cdot y))\\ &= a_s(\tilde c_s\cdot\sigma_s(g)\cdot\mu_s(y))\\ &= \tilde c_s\sigma_s(g)\tilde c_s^{-1}\cdot a_s(\tilde c_s\cdot\mu_s(y)) = \sigma^0_s(g)\cdot\mu^0_\sigma(y).\end{aligned}$$

□

**Lemma 3.12.** *The map $s\to\mu^0_s$ is a homomorphism if and only if*

$$a_s\cdot {}^s a_t\cdot\varkappa(\tilde c_s\, {}^s\tilde c_t\,\tilde c_{st}^{-1})\cdot a_{st}^{-1} = 1 \quad \textit{for all } s,t\in\Gamma. \tag{3.13}$$

*Proof.* Let $s, t \in \Gamma$. Using (3.8), (3.9), and (3.10), we compute:

$$\begin{aligned}\mu_s^0 \circ \mu_t^0 \circ (\mu_{st}^0)^{-1} &= a_s \circ l(\tilde{c}_s) \circ \mu_s \circ a_t \circ l(\tilde{c}_t) \circ \mu_t \circ \mu_{st}^{-1} \circ l(\tilde{c}_{st})^{-1} \circ a_{st}^{-1} \\ &= a_s \circ \tau_s(a_t) \circ l(\tilde{c}_s) \circ l(\sigma_s(\tilde{c}_t)) \circ \mu_s \circ \mu_t \circ \mu_{st}^{-1} \circ l(\tilde{c}_{st})^{-1} \circ a_{st}^{-1}.\end{aligned}$$

By (3.7) we obtain that

$$\begin{aligned}\mu_s^0 \circ \mu_t^0 \circ (\mu_{st}^0)^{-1} &= a_s \circ \tau_s(a_t) \circ l(\tilde{c}_s) \circ l(\sigma_s(\tilde{c}_t)) \circ l(\tilde{c}_{st})^{-1} \circ a_{st}^{-1} \\ &= a_s \cdot {}^s a_t \cdot \varkappa(\tilde{c}_s \, {}^s\tilde{c}_t \, \tilde{c}_{st}^{-1}) \cdot a_{st}^{-1}.\end{aligned}$$

We see that

$$\mu_s^0 \circ \mu_t^0 \circ (\mu_{st}^0)^{-1} = 1$$

if and only if (3.13) holds. Thus the map $s \to \mu_s^0$ is a homomorphism if and only if (3.13) holds, which completes the proof of Lemma 3.12. □

Now assume that $\varkappa_*(\delta[c]) \in H^2(\Gamma, \mathcal{A}_\diamond)$ is neutral. This means that there exists a locally constant map

$$a\colon \Gamma \to \mathcal{A}_\diamond$$

such that (3.13) holds. Then by Lemma 3.12 the map

$$\mu^0\colon \Gamma \to \mathrm{SAut}(Y), \quad s \mapsto \mu_s^0$$

is a homomorphism, hence it satisfies hypothesis (i) of [4], Lemma 6.3. By Lemma 3.11 $\mu_s^0$ is $\sigma_s^0$-equivariant, hence $\mu^0$ satisfies hypothesis (iv) of [4], Lemma 6.3. The variety $Y = G/H$ is quasi-projective, hence hypothesis (iii) of this lemma is satisfied. It is easy to see that the restriction of the homomorphism $\mu_0$ to $\mathrm{Gal}(k/k_1)$ for some finite Galois extension $k_1/k_0$ in $k$ comes from a $G_1$-equivariant $k_1$-model $Y_1$ of $Y$, where $G_1 = G_0 \times_{k_0} k_1$. Thus the homomorphism $\mu^0$ satisfies hypothesis (ii) of [4], Lemma 6.3. By this lemma the variety $Y$ admits a $G_0$-equivariant $k_0$-model $Y_0$ inducing the homomorphism $s \mapsto \mu_s^0$, as required.

Conversely, assume that $Y$ admits a $G_0$-equivariant $k_0$-model $Y_0$ inducing a homomorphism

$$\mu^0\colon \Gamma \to \mathrm{SAut}(G), \quad s \mapsto \mu_s^0.$$

Then by Lemma 3.11 (in the case $a_s = 1$) the $s$-semi-automorphism $l(c_s) \circ \mu_s$ of $Y$ is $\sigma_s^0$-equivariant for any $s \in \Gamma$. Since $\mu_s^0$ is $\sigma_s^0$-equivariant as well, we have

$$\mu_s^0 = a_s \circ l(c_s) \circ \mu_s$$

for some locally constant map

$$a\colon \Gamma \to \mathcal{A}_\diamond, \quad s \mapsto a_s.$$

Since the map $s \mapsto \mu_s^0$ is a homomorphism, by Lemma 3.12 the equality (3.13) holds and hence, $\varkappa_*(\delta[c])$ is neutral in $H^2(\Gamma, \mathcal{A}_\diamond)$. This completes the proof of Theorem 3.5. □

## 4. Model of a homogeneous space of a reductive group

Let $k$, $k_0$, and $\Gamma$ be as in Subsection 1.1. In this section $G$ is a connected reductive group over $k$. Let $H \subset G$ be a $k$-subgroup (not necessarily spherical). We consider the homogeneous $G$-variety $Y = G/H$. Consider the abstract group $\mathcal{A} = \mathrm{Aut}^G(G/H)$ and the algebraic group $A = \mathcal{N}_G(H)/H$, then there is a canonical isomorphism $A(k) \xrightarrow{\sim} \mathcal{A}$; see e.g. [4, Lemma 5.1]. Let $G_{\mathrm{qs}}$ be a quasi-split $k_0$-model of $G$ and let $Y_{\mathrm{qs}}$ be a $G_{\mathrm{qs}}$-equivariant model of $G/H$, then we obtain a $\Gamma$-action on $A(k) = \mathcal{A}$ and hence, a $k_0$-model $A_{\mathrm{qs}}$ of $A$. We need the following result:

**Proposition 4.1.** *Let $k_0 \subset k$ be a subfield such that $k$ is a Galois extension of $k$. Let $G$ be a connected reductive group over $k$. Let $G_0$ be any $k_0$-model of $G$. Then there exist a quasi-split inner $k_0$-model $G_{\rm qs}$ of $G$ and a cocycle $c \in Z^1(k_0, {\rm Inn}(G_{\rm qs}))$ such that $G_0 \simeq {}_cG_{\rm qs}$ (we say that $G_{\rm qs}$ is a quasi-split inner $k_0$-form of $G_0$). Moreover, if $G_{\rm qs}$ and $G'_{\rm qs}$ are two quasi-split inner $k_0$-forms of $G_0$, then they are isomorphic.*

*Proof.* See "The Book of Involutions" [7, Proposition(31.5)], or Conrad [5, Proposition 7.2.12], the existence only in Springer [14, Proposition 16.4.9]. □

**4.2.** Let $G$ be a connected reductive group over $k$, and let $G_0$ be a $k_0$-model of $G$. Write $\overline{G} = G/Z(G)$ for the corresponding adjoint group, and $\widetilde{G}$ for the universal cover of the connected semisimple group $[G, G]$. By Proposition 4.1 we may write $G_0 = {}_cG_{\rm qs}$, where $G_{\rm qs}$ is a quasi-split $k_0$-model of $G$ and $c \in Z^1(k_0, G_{\rm qs}/Z(G_{\rm qs}))$. We fix $G_{\rm qs}$ and $c$. We write $\overline{G}_{\rm qs} = G_{\rm qs}/Z(G_{\rm qs})$.

We write $\widetilde{Z}_{\rm qs}$ for the center $Z(\widetilde{G}_{\rm qs})$ of the universal cover $\widetilde{G}_{\rm qs}$ of the connected semisimple group $[G_{\rm qs}, G_{\rm qs}]$. Similarly, we write $\widetilde{Z}_0$ for the center $Z(\widetilde{G}_0)$ of the universal cover $\widetilde{G}_0$ of the connected semisimple group $[G_0, G_0]$. The short exact sequence

$$1 \to \widetilde{Z}_{\rm qs} \to \widetilde{G}_{\rm qs} \to \overline{G}_{\rm qs} \to 1$$

induces a cohomology exact sequence

$$H^1(k_0, \widetilde{Z}_{\rm qs}) \to H^1(k_0, \widetilde{G}_{\rm qs}) \to H^1(k_0, \overline{G}_{\rm qs}) \xrightarrow{\tilde{\delta}} H^2(k_0, \widetilde{Z}_{\rm qs}).$$

By definition, the *Tits class* $t(\widetilde{G}_0)$ is the image of $[c] \in Z^1(k_0, \overline{G}_{\rm qs})$ in $H^2(k, \widetilde{Z}_{\rm qs})$ under the connecting map $\tilde{\delta}\colon H^1(k_0, \overline{G}_{\rm qs}) \to H^2(k_0, \widetilde{Z}_{\rm qs})$; compare [7], Section 31, before Proposition (31.7).

**Theorem 4.3.** *Let $G$ be a reductive group over an algebraically closed field $k$ of characteristic 0. Let $H \subset G$ be an algebraic subgroup. Let $k_0 \subset k$ be a subfield such that $k$ is an algebraic closure of $k$. Let $G_0$ be a $k_0$-model of $G$. Write $G_0 = {}_cG_{\rm qs}$, where $G_{\rm qs}$ is a quasi-split inner form of $G_0$ and where $c \in Z^1(k_0, \overline{G}_{\rm qs})$. Assume that $G/H$ admits a $G_{\rm qs}$-equivariant $k_0$-model. Then $G/H$ admits a $G_0$-equivariant $k_0$-model if and only if the image in $H^2(k_0, A_{\rm qs})$ of the Tits class $t(\widetilde{G}_0) \in H^2(k_0, Z(\widetilde{G}_{\rm qs}))$ is neutral.*

*Proof.* By Theorem 3.5 the homogeneous variety $G/H$ admits a $G_0$-equivariant $k_0$-form if and only if the image

$$\varkappa(\delta[c]) \in H^2(k_0, A_{\rm qs})$$

is neutral. We write $Z_{\rm qs}$ for $Z(G_{\rm qs})$ and $\widetilde{Z}_{\rm qs}$ for $Z(\widetilde{G}_{\rm qs})$. From the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \widetilde{Z}_{\rm qs} & \longrightarrow & \widetilde{G}_{\rm qs} & \longrightarrow & \overline{G}_{\rm qs} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \rm id} & & \\ 1 & \longrightarrow & Z_{\rm qs} & \longrightarrow & G_{\rm qs} & \longrightarrow & \overline{G}_{\rm qs} & \longrightarrow & 1 \end{array}$$

we obtain a commutative diagram

$$\begin{array}{ccc} H^1(k_0, \overline{G}_{\rm qs}) & \xrightarrow{\tilde{\delta}} & H^2(k_0, \widetilde{Z}_{\rm qs}) \\ {\scriptstyle \rm id}\downarrow & & \downarrow{\scriptstyle \lambda} \\ H^1(k_0, \overline{G}_{\rm qs}) & \xrightarrow{\delta} & H^2(k_0, Z_{\rm qs}), \end{array}$$

which shows that

$$\delta[c] = \lambda(\tilde{\delta}[c]).$$

By definition

$$t(\widetilde{G}_0) = \tilde{\delta}[c] \in H^2(k_0, \widetilde{Z}_{\rm qs}).$$

Thus $\varkappa(\delta[c])$ is the image in $H^2(k_0, A_{\rm qs})$ of $t(\widetilde{G}_0)$ under the map

$$H^2(k_0, \widetilde{Z}_{\rm qs}) \to H^2(k_0, Z_{\rm qs}) \to H^2(k_0, A_{\rm qs})$$

induced by the homomorphism $\widetilde{Z}_{\rm qs} \to Z_{\rm qs} \to A_{\rm qs}$. We conclude that the homogeneous variety $G/H$ admits a $G_0$-equivariant $k_0$-form if and only if the image of $t(\widetilde{G}_0)$ in $H^2(k_0, A_{\rm qs})$ is neutral, as required. □

## 5. Models of $(H \times H)/\Delta$

**5.1.** Let $H$ be a connected algebraic $k$-group, and set $G = H \times_k H$. Let $Y = H$, where $G$ acts on $Y$ by

$$(h_1, h_2) * y = h_1 y h_2^{-1}.$$

Note that $Y = G/\Delta$, where $\Delta \in H \times_k H$ is the diagonal, that is, $\Delta$ is $H$ embedded in $G$ diagonally. Let $H_0^{(1)}$ and $H_0^{(2)}$ be two $k_0$-models of $H$. We set $G_0 = H_0^{(1)} \times_{k_0} H_0^{(2)}$ and ask whether $Y$ admits a $G_0$-equivariant $k_0$-model.

**Theorem 5.2.** *With the notation and assumptions of 5.1, $Y = (H \times H)/\Delta$ admits an $H_0^{(1)} \times_{k_0} H_0^{(2)}$-equivariant $k_0$-model if and only if $H_0^{(2)}$ is a pure inner form of $H_0^{(1)}$.*

*Proof of Theorem 5.2.* Set $G_1 = H_0^{(1)} \times_{k_0} H_0^{(1)}$, then $Y$ admits a $k_0$-model $Y_1 = H_0^{(1)}$ (with the natural action of $G_1$). Assume that $H^{(2)}$ is a pure inner form of $H^{(1)}$, then $G_0 := H_0^{(1)} \times_{k_0} H_0^{(2)}$ is a pure inner form of $G_1$, and by Lemma 2.4 the $G$-variety $Y$ admits a $G_0$-equivariant $k_0$-model. Explicitly, let $P_{\tilde{c}}$ denote the *torsor* (principal homogeneous space) of $H^{(1)}$ corresponding to the 1-cocycle $\tilde{c}$; see Serre [11, Section I.5.2]. Then $H^{(1)}$ acts on $P_{\tilde{c}}$, and $H(k) = H^{(1)}(k)$ acts on $P$ simply transitively. Moreover, $H^{(2)} = {}_{\tilde{c}}(H^{(1)})$ acts on $P_{\tilde{c}}$ as well, and these two actions commute; see Serre [11, Section I.5.3, Corollary of Proposition 34]. Thus $P_{\tilde{c}}$ is an $H_0^{(1)} \times_{k_0} H_0^{(2)}$-equivariant $k_0$-model of $Y$.

Conversely, assume that $Y$ admits an $H_0^{(1)} \times_{k_0} H_0^{(2)}$-equivariant $k_0$-model. First we show that then $H^{(2)}$ is an *inner form* of $H^{(1)}$. Indeed, let

$$\sigma^{(i)} \colon \Gamma \to \mathrm{SAut}(H)$$

denote the semilinear actions corresponding to the model $H_0^{(i)}$ of $H$ for $i = 1, 2$. Recall that $\Delta(k) = \{(h, h) \mid h \in H(k)\}$. Then for any $s \in \Gamma$ we have

$$(\sigma_s^{(1)} \times \sigma_s^{(2)})(\Delta(k)) = \{(\sigma_s^{(1)}(h), \sigma_s^{(2)}(h)) \mid h \in H(k)\}.$$

Since $Y$ admits an $H_0^{(1)} \times_{k_0} H_0^{(2)}$-equivariant $k_0$-model, the subgroup $(\sigma_s^{(1)} \times \sigma_s^{(2)})(\Delta)$ is conjugate to $\Delta$ in $G = H \times H$; see, e.g., [12, Lemma 4.1]. This means that there exists a pair $(h_1, h_2) \in H(k) \times H(k)$ such that

$$(\sigma_s^{(1)}(h), \sigma_s^{(2)}(h)) = (h_1 h h_1^{-1}, h_2 h h_2^{-1}) \quad \text{for all} \quad h \in H(k).$$

It follows that

$$\sigma_s^{(1)}(h) = (h_1 h_2^{-1}) \cdot \sigma_s^{(2)}(h) \cdot (h_1 h_2^{-1})^{-1}.$$

We see that for any $s \in \Gamma$, the $s$-semi-automorphism $\sigma_s^{(2)}$ of $H$ differs from $\sigma_s^{(2)}$ by an inner automorphism of $H$. This means that $H_0^{(2)}$ is an inner form of $H_0^{(1)}$.

Now we know that $H_0^{(2)} = {}_c(H_0^{(1)})$ for some 1-cocycle $c \in Z^1(k_0, \overline{H}^{(1)})$. Set $G_1 = H_0^{(1)} \times_{k_0} H_0^{(1)}$, then $Y_1 := H_0^{(1)}$ with the natural action of $G_1$ is a $G_1$-equivariant $k_0$-model of $Y$. Moreover, $G_0 = H_0^{(1)} \times_{k_0} H_0^{(2)}$ is the inner twisted form of $G_1$ given by the 1-cocycle $(1,c) \in Z^1(k_0, G_1)$. Then

$$\delta[(1,c)] \ \in\ H^2(k_0, Z(G_1)) = H^2(k_0, Z(H_0^{(1)})) \times H^2(k_0, Z(H_0^{(1)}))$$

is $(1, \delta_H[c])$, where

$$\delta_H \colon H^1(k_0, \overline{H}_0^{(1)}) \to H^2(k_0, Z(H_0^{(1)}))$$

is the connecting map. By Theorem 3.5, $Y$ admits an $H_0^{(1)} \times_{k_0} H_0^{(2)}$-equivariant $k_0$-model if and only if $\varkappa_*(\delta[1,c]) = 0$, that is, if and only if $\varkappa_*(1, \delta_H[c]) = 1$. An easy calculation shows that

$$\mathcal{N}_G(\Delta) = Z(G) \cdot \Delta \quad \text{and} \quad A := \mathcal{N}_G(\Delta)/\Delta = Z(G)/Z(\Delta) = (Z(H) \times Z(H))/Z(\Delta).$$

Similarly, over $k_0$ we obtain

$$\mathcal{N}_{G_1}(\Delta_1) = Z(G_1)\cdot\Delta_1 \text{ and } A_1 := \mathcal{N}_{G_1}(\Delta_1)/\Delta_1 = Z(G_1)/Z(\Delta_1) = (Z(H_0^{(1)}) \times Z(H_0^{(1)}))/Z(H_0^{(1)}).$$

It is easy to see that the morphism of abelian $k_0$-groups

$$Z(H_0^{(1)}) \to A_1, \quad z \mapsto (1, z) \cdot Z(\Delta)$$

is an isomorphism. It follows that the induced map on cohomology

$$H^2(k_0, Z(H_0^{(1)})) \to H^2(k_0, A_1)$$

is an isomorphism of abelian groups. Therefore, $Y$ admits an $H_0^{(1)} \times_{k_0} H_0^{(2)}$-equivariant $k_0$-model if and only if $\delta_H[c] = 1$, that is, if and only if $H_0^{(2)}$ is a pure inner form of $H_0^{(1)}$, as required. □

**Lemma 5.3.** *Let $H_0$ be a simply connected semisimple group over a $p$-adic field $k_0$. Then any pure inner form of $H_0$ is isomorphic to $H_0$.*

*Proof.* Indeed, by Kneser's theorem we have $H^1(k_0, H_0) = 1$; see Platonov and Rapinchuk [8, Theorem 6.4]. □

**Corollary 5.4.** *In Theorem 5.2, if $k_0$ is a $p$-**adic** field and $H$ is a **simply connected** semisimple group over $k$, then $Y$ admits an $H_0^{(1)} \times_{k_0} H_0^{(2)}$-equivariant $k_0$-model if and only if $H_0^{(2)}$ is isomorphic to $H_0^{(1)}$.*

*Proof.* Indeed, by Theorem 5.2 the variety $Y$ admits an $H_0^{(1)} \times_{k_0} H_0^{(2)}$-equivariant $k_0$-model if and only if $H_0^{(2)}$ is a pure inner form of $H_0^{(1)}$, and by Lemma 5.3 any pure inner form of $H_0^{(1)}$ is isomorphic to $H_0^{(1)}$. □

## 6. Models of $G/U$

**Theorem 6.1.** *Let $k$ be a fixed algebraic closure of a field $k$ of characteristic 0, and let $G$ be a connected reductive group over $k$. Let $B \subset G$ be a Borel subgroup, and write $U$ for the unipotent radical of $B$. Consider the homogeneous space $Y = G/U$. Let $G_0$ be a $k_0$-model of $G$. Then $Y$ admits a $G_0$-equivariant $k_0$-model if and only if $G_0$ is a pure inner form of a quasi-split model of $G$.*

*Proof.* It is well known that $G_0$ is an inner form of a quasi-split model $G_{\rm qs}$ of $G$; see Springer [14, Proposition 16.4.9]) or "The Book of Involutions" [7, Proposition (31.5)], or Conrad [5, Proposition 7.2.12]. This means that $G_0 = {}_cG_{\rm qs}$, where $c \in Z^1(k_0, \overline{G}_{\rm qs})$. Since $G_{\rm qs}$ is quasi-split, there exists a Borel subgroup $B_{\rm qs} \subset G_{\rm qs}$ (defined over $k_0$). Set $U_{\rm qs} = R_u(B_{\rm qs})$, then $G_{\rm qs}/U_{\rm qs}$ is a $G_{\rm qs}$-equivariant $k_0$-model of $Y = G/U$. By Theorem 3.5, $Y$ admits a $G_0$-equivariant $k_0$-model if and only if $\varkappa_*(\delta[c]) \subset H^2(k_0, A_{\rm qs})$ vanishes, where $A_{\rm qs} = \mathcal{N}_G(U_{\rm qs})/U_{\rm qs} \cong T_{\rm qs}$ and $T_{\rm qs} \subset B_{\rm qs}$ is a maximal torus. Note that $\varkappa\colon Z_{\rm qs} \to A_{\rm qs} = T_{\rm qs}$ is the canonical embedding, where $Z_{\rm qs} = Z(G_{\rm qs})$.

We show that the homomorphism

$$\varkappa_*\colon H^2(k_0, Z_{\rm qs}) \to H^2(k_0, T_{\rm qs})$$

is injective. Indeed, we have a short exact sequence

$$1 \to Z_{\rm qs} \xrightarrow{\varkappa} T_{\rm qs} \to \overline{T}_{\rm qs} \to 1,$$

which induces a cohomology exact sequence

$$\dots \to H^1(k_0, \overline{T}_{\rm qs}) \to H^2(k_0, Z_{\rm qs}) \xrightarrow{\varkappa_*} H^2(k_0, T_{\rm qs}) \to \dots$$

Since $G_{\rm qs}$ is quasi-split, by Lemma 6.2 below we have $H^1(k_0, \overline{T}_{\rm qs}) = 1$, hence the homomorphism $\varkappa_*$ is injective, as required.

We see that $\varkappa_*(\delta[c]) = 1$ if and only if $\delta[c] = 1$. Now consider the cohomology exact sequence

$$\dots \to H^1(k_0, G_{\rm qs}) \to H^1(k_0, \overline{G}_{\rm qs}) \xrightarrow{\delta} H^2(k_0, Z_{\rm qs}) \to \dots$$

It follows from the construction of the map $\delta$ (see Serre [11, Section I.5.6]) that $\delta[c] = 1$ if and only if $c$ can be lifted to a 1-cocycle $\tilde{c} \in Z^1(k_0, G)$, that is, if and only if $G_0 = {}_cG_{\rm qs}$ is a pure inner form of $G_{\rm qs}$, as required.

We conclude that $Y$ admits a $G_0$-equivariant $k_0$-model if and only if $G_0$ is a pure inner form of $G_{\rm qs}$. □

**Lemma 6.2.** *Let $G_0$ be a quasi-split semisimple group of adjoint type, $B_0 \subset G_0$ be a Borel subgroup defined over $k_0$, and $T_0 \subset B_0$ be a maximal torus. Then $H^1(k_0, B_0) = H^1(k_0, T_0) = 1$.*

*Proof.* Note that $T_0 \simeq B_0/R_u(B_0)$, which gives a canonical bijection $H^1(k_0, B_0) \xrightarrow{\sim} H^1(k_0, T_0)$; see Sansuc [9, Lemme 1.13]. Since $G_0$ is a group of adjoint type, the set of simple roots $S = S(G_{0,k}, T_{0,k}, B_{0,k})$ is a basis of the character group $\mathsf{X}^*(T_{0,k})$; see Springer [14, 8.1.11]. Since $B_{0,k}$ is defined over $k_0$, the action of $\Gamma$ on $\mathsf{X}^*(T_{0,k})$ preserves the basis $S$. In other words, $\mathsf{X}^*(T_{0,k})$ is a *permutation* $\Gamma$-module, hence $T_0$ is a *quasi-trivial* $k_0$-torus, and therefore, $H^1(k_0, T_0) = 1$; see Sansuc [9, Lemme 1.9]. □

**Remark 6.3.** In Theorem 6.1 assume that $G$ is a semisimple group of adjoint type. Then a $k_0$-model of $G/U$, if exists, is unique. Indeed, then $A_{\rm qs} \cong T_{\rm qs}$, and by Lemma 6.2 we have

$$H^1(k_0, A_{\rm qs}) = H^1(k_0, T_{\rm qs}) = 1.$$

**Corollary 6.4.** *In Theorem 6.1 assume that $k_0$ is a $p$-adic field and that $G$ is semisimple and simply connected. Then $G/U$ admits a $G_0$-equivariant $k_0$-model if and only if $G_0$ is quasi-split.*

*Proof.* Indeed, by Theorem 6.1 the $G$-variety $G/U$ admits a $G_0$-equivariant $k_0$-model if and only if $G_0$ is a pure inner form of a quasi-split group $G_{\rm qs}$. Since $k_0$ is a $p$-adic field, by Lemma 5.3 then $G_0$ is isomorphic to $G_{\rm qs}$. □

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