

# ABSTRACT INTERPOLATION IN VECTOR-VALUED DE BRANGES-ROVNYAK SPACES

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ABSTRACT. Following ideas from the Abstract Interpolation Problem of [26] for Schur class functions, we study a general metric constrained interpolation problem for functions from a vector-valued de Branges-Rovnyak space  $\mathcal{H}(K_S)$  associated with an operator-valued Schur class function  $S$ . A description of all solutions is obtained in terms of functions from an associated de Branges-Rovnyak space satisfying only a bound on the de Branges-Rovnyak-space norm. Attention is also paid to the case that the map which provides this description is injective. The interpolation problem studied here contains as particular cases (1) the vector-valued version of the interpolation problem with operator argument considered recently in [4] (for the nondegenerate and scalar-valued case) and (2) a boundary interpolation problem in  $\mathcal{H}(K_S)$ . In addition, we discuss connections with results on kernels of Toeplitz operators and nearly invariant subspaces of the backward shift operator.

## 1. INTRODUCTION

De Branges-Rovnyak spaces play a prominent role in Hilbert space approaches to  $H^\infty$ -interpolation. However, very little work exists on interpolation for functions in de Branges-Rovnyak spaces themselves. In this paper we pursue our studies of interpolation problems for functions in de Branges-Rovnyak spaces, which started in [4]. We consider a norm constrained interpolation problem (denoted by  $\mathbf{AIP}_{\mathcal{H}(K_S)}$  in what follows), which is sufficiently fine so as to include on the one hand interpolation problems with operator argument (considered for the nondegenerate and scalar-valued case in [4]) and, on the other hand, boundary interpolation problems; it is only recent work [10, 11, 12, 22] which has led to a systematic understanding of boundary-point evaluation on de Branges-Rovnyak spaces from an operator-theoretic point of view.

In order to state the interpolation problem we first introduce some definitions and notations. As usual, for Hilbert spaces  $\mathcal{U}$  and  $\mathcal{Y}$  the symbol  $\mathcal{L}(\mathcal{U}, \mathcal{Y})$  stands for the space of bounded linear operators mapping  $\mathcal{U}$  into  $\mathcal{Y}$ , abbreviated to  $\mathcal{L}(\mathcal{U})$  in case  $\mathcal{U} = \mathcal{Y}$ . Following the standard terminology, we define the operator-valued *Schur class*  $\mathcal{S}(\mathcal{U}, \mathcal{Y})$  to be the class of analytic functions  $S$  on the open unit disk  $\mathbb{D}$  whose values  $S(z)$  are contraction operators in  $\mathcal{L}(\mathcal{U}, \mathcal{Y})$ . By  $H_{\mathcal{U}}^2$  we denote the standard Hardy space of analytic  $\mathcal{U}$ -valued functions on  $\mathbb{D}$  with square-summable sequence of Taylor coefficients. We also make use of the notation  $\text{Hol}_{\mathcal{U}}(\mathbb{D})$  for the space of all  $\mathcal{U}$ -valued holomorphic functions on the unit disk  $\mathbb{D}$ .

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Among several alternative characterizations of the Schur class there is one in terms of positive kernels and associated reproducing kernel Hilbert spaces: *A function  $S: \mathbb{D} \rightarrow \mathcal{L}(\mathcal{U}, \mathcal{Y})$  is in the Schur class  $\mathcal{S}(\mathcal{U}, \mathcal{Y})$  if and only if the associated de Branges-Rovnyak kernel*

$$(1.1) \quad K_S(z, \zeta) = \frac{I_{\mathcal{Y}} - S(z)S(\zeta)^*}{1 - z\bar{\zeta}}$$

is positive (precise definitions are recalled at the end of this Introduction). This positive kernel gives rise to a reproducing kernel Hilbert space  $\mathcal{H}(K_S)$ , the de Branges-Rovnyak space defined by  $S$  (see [15]). On the other hand, the kernel (1.1) being positive is equivalent to the operator  $M_S: f \rightarrow Sf$  of multiplication by  $S$  being a contraction in  $\mathcal{L}(H_{\mathcal{U}}^2, H_{\mathcal{Y}}^2)$ ; then the general complementation theory applied to the contractive operator  $M_S: H_{\mathcal{U}}^2 \rightarrow H_{\mathcal{Y}}^2$  provides the characterization of  $\mathcal{H}(K_S)$  as the operator range  $\mathcal{H}(K_S) = \text{Ran}(I - M_S M_S^*)^{\frac{1}{2}} \subset H_{\mathcal{Y}}^2$  with the lifted norm

$$\|(I - M_S M_S^*)^{\frac{1}{2}} f\|_{\mathcal{H}(K_S)} = \|(I - \mathbf{p})f\|_{H_{\mathcal{Y}}^2}$$

where  $\mathbf{p}$  here is the orthogonal projection onto  $\text{Ker}(I - M_S M_S^*)^{\frac{1}{2}}$ . Upon setting  $f = (I - M_S M_S^*)^{\frac{1}{2}} h$  in the last formula we get

$$(1.2) \quad \|(I - M_S M_S^*)^{\frac{1}{2}} h\|_{\mathcal{H}(K_S)} = \langle (I - M_S M_S^*)^{\frac{1}{2}} h, h \rangle_{H_{\mathcal{Y}}^2}.$$

The data set of the problem  $\mathbf{AIP}_{\mathcal{H}(K_S)}$  is a tuple

$$(1.3) \quad \mathcal{D} = \{S, T, E, N, \mathbf{x}\}$$

consisting of a Schur-class function  $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ , Hilbert space operators  $T \in \mathcal{L}(\mathcal{X})$ ,  $E \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ ,  $N \in \mathcal{L}(\mathcal{X}, \mathcal{U})$ , and a vector  $\mathbf{x} \in \mathcal{X}$ . With this data set we associate the observability operators

$$(1.4) \quad \mathcal{O}_{E,T}: x \mapsto E(I - zT)^{-1}x \quad \text{and} \quad \mathcal{O}_{N,T}: x \mapsto N(I - zT)^{-1}x,$$

which we assume map  $\mathcal{X}$  into  $\text{Hol}_{\mathcal{Y}}(\mathbb{D})$  and  $\text{Hol}_{\mathcal{U}}(\mathbb{D})$ . We also associate with the data set  $\mathcal{D}$  the  $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ -valued function

$$(1.5) \quad F^S(z) = (E - S(z)N)(I - zT)^{-1}$$

along with the multiplication operator  $M_{F^S}: x \rightarrow F^S x$ , mapping  $\mathcal{X}$  into  $\text{Hol}_{\mathcal{Y}}(\mathbb{D})$ . Using the notation (1.4), we can write  $M_{F^S}$  as

$$(1.6) \quad M_{F^S} = \mathcal{O}_{E,T} - M_S \mathcal{O}_{N,T}: \mathcal{X} \rightarrow \text{Hol}_{\mathcal{Y}}(\mathbb{D}).$$

Observe that, for an operator  $A: \mathcal{X} \rightarrow \mathcal{H}(K_S) \subset H_{\mathcal{Y}}^2$ , the adjoint operator can be taken in the metric of  $H_{\mathcal{Y}}^2$  as well as in the metric of  $\mathcal{H}(K_S)$  which are not the same unless  $S$  is inner (i.e., the multiplication operator  $M_S: H_{\mathcal{U}}^2 \rightarrow H_{\mathcal{Y}}^2$  is an isometry). To avoid confusion, in what follows we use the notation  $A^*$  for the adjoint of  $A$  in the metric of  $H_{\mathcal{Y}}^2$  and  $A^{[*]}$  for the adjoint of  $A$  in the metric of  $\mathcal{H}(K_S)$ .

**Definition 1.1.** We say that the data set (1.3) is  $\mathbf{AIP}_{\mathcal{H}(K_S)}$ -admissible if:

- (1) The operators  $\mathcal{O}_{E,T}$  and  $\mathcal{O}_{N,T}$  map  $\mathcal{X}$  into  $\text{Hol}_{\mathcal{Y}}(\mathbb{D})$  and  $\text{Hol}_{\mathcal{U}}(\mathbb{D})$ , respectively (in other words,  $(E, T)$  and  $(N, T)$  are *analytic output pairs*).
- (2) The operator  $M_{F^S}$  maps  $\mathcal{X}$  into  $\mathcal{H}(K_S)$ .
- (3) The operator  $P := M_{F^S}^{[*]} M_{F^S}$  satisfies the Stein equation

$$(1.7) \quad P - T^* P T = E^* E - N^* N.$$

We are now ready to formulate the problem  $\mathbf{AIP}_{\mathcal{H}(K_S)}$ :

Given an  $\mathbf{AIP}_{\mathcal{H}(K_S)}$ -admissible data set (1.3), find all  $f \in \mathcal{H}(K_S)$  such that

$$(1.8) \quad M_{FS}^{[*]} f = \mathbf{x} \quad \text{and} \quad \|f\|_{\mathcal{H}(K_S)} \leq 1.$$

The  $\mathbf{AIP}_{\mathcal{H}(K_S)}$ -problem as formulated here does not appear to be an interpolation problem, but in Section 6 we show that indeed the operator-argument Nevanlinna-Pick interpolation problem can be seen as a particular instance of the  $\mathbf{AIP}_{\mathcal{H}(K_S)}$ -problem.

This operator-argument problem was considered in [4] for scalar-valued functions and for the nondegenerate case where the solution  $P$  of the Stein equation (1.7) is positive definite (i.e., invertible). The eventual parametrization for the set of all solutions, which we obtain in Theorem 5.1 below, is connected with previously appearing representations for almost invariant subspaces and Toeplitz kernels in terms of an isometric multiplier between two de Branges-Rovnyak spaces. As another application of the  $\mathbf{AIP}_{\mathcal{H}(K_S)}$ -problem, we obtain an alternative characterization of Toeplitz kernels (in Corollary 7.5 below) in terms of an explicitly computable isometric multiplier on an appropriate de Branges-Rovnyak space; this is a refinement of the characterization due to Dyakonov [19].

At one level the interpolation problem  $\mathbf{AIP}_{\mathcal{H}(K_S)}$  is straightforward since de Branges-Rovnyak spaces are Hilbert spaces and consequently the set of all norm-constrained solutions splits as the orthogonal direct sum of the unique minimal-norm solution and the set of all functions satisfying the homogeneous interpolation condition and the complementary norm constraint. By viewing (1.8) as a special case of a basic linear operator equation discussed in Section 2, we get some general results on the  $\mathbf{AIP}_{\mathcal{H}(K_S)}$ -problem in Section 3. These results make no use of condition (3) (i.e., the Stein equation (1.7)) in the definition of  $\mathbf{AIP}_{\mathcal{H}(K_S)}$ -admissibility, and can be easily extended to a more general framework of contractive multipliers between any two reproducing kernel Hilbert spaces (not necessarily of de Branges-Rovnyak type). By using the full strength of  $\mathbf{AIP}_{\mathcal{H}(K_S)}$ -admissibility, in Section 5 we obtain a more explicit formula (see Theorem 5.1 below) for the parametrization of the solution set by using the connection with an associated Schur-class Abstract Interpolation Problem and its known Redheffer transform solution as worked out in [26]. The latter problem and its solution through the associated Redheffer transform is recalled in Section 4. This section also includes an analysis of the conditions under which the Redheffer transform is injective, a property which does not happen in general. The paper concludes with three sections that discuss the various applications of the  $\mathbf{AIP}_{\mathcal{H}(K_S)}$ -problem mentioned above.

The notation is mostly standard. We just mention that an operator  $X \in \mathcal{L}(\mathcal{Y})$ , for some Hilbert space  $\mathcal{Y}$ , is called *positive semidefinite* in case  $\langle Xy, y \rangle \geq 0$  for all  $y \in \mathcal{Y}$  and *positive definite* if  $X$  is positive semidefinite and invertible in  $\mathcal{L}(\mathcal{X})$ . Also, in general, given a function  $K$  defined on a Cartesian product set  $\Omega \times \Omega$  with values in  $\mathcal{L}(\mathcal{Y})$ , we say that  $K$  is a *positive kernel* if any one of the following equivalent conditions hold:

- (1)  $K$  is a positive kernel in the sense of Aronszajn: given any finite collection of points  $\omega_1, \dots, \omega_N$  in  $\Omega$  and vectors  $y_1, \dots, y_N$  in the Hilbert coefficient

space  $\mathcal{Y}$ , it holds that

$$\sum_{i,j=1}^N \langle \langle K(\omega_i, \omega_j) y_j, y_i \rangle_{\mathcal{Y}} \rangle \geq 0.$$

- (2)  $K$  is the reproducing kernel for a reproducing kernel Hilbert space  $\mathcal{H}$ : there is a Hilbert space  $\mathcal{H}(K)$  whose elements are  $\mathcal{Y}$ -valued functions on  $\Omega$  so that (i) for each  $\omega \in \Omega$  and  $y \in \mathcal{Y}$  the  $\mathcal{Y}$ -valued function  $k_{\omega}y$  given by  $k_{\omega}y(\omega') = K(\omega', \omega)y$  is an element of  $\mathcal{H}(K)$ , and (ii) the functions  $k_{\omega}y$  have the reproducing property for  $\mathcal{H}(K)$ :

$$\langle f, k_{\omega}y \rangle_{\mathcal{H}(K)} = \langle f(\omega), y \rangle_{\mathcal{Y}}$$

for all  $f \in \mathcal{H}(K)$ .

- (3)  $K$  has a Kolmogorov decomposition: there is an auxiliary Hilbert space  $\mathcal{K}$  and a function  $H: \Omega \rightarrow \mathcal{L}(\mathcal{K}, \mathcal{Y})$  so that  $K$  can be expressed as

$$K(\omega', \omega) = H(\omega')H(\omega)^*.$$

These equivalences are well-known straightforward extensions of the ideas of Aronszajn [2] to the case of operator-valued kernels in place of scalar-valued kernels.

Next we mention that on occasion we view a vector  $\mathbf{x}$  in a Hilbert space  $\mathcal{X}$  as an operator from the scalars  $\mathbb{C}$  into  $\mathcal{X}$ :  $\mathbf{x}$  maps the scalar  $c \in \mathbb{C}$  to the vector  $c\mathbf{x} \in \mathcal{X}$ . Then  $\mathbf{x}^*$  denotes the adjoint operator mapping  $\mathcal{X}$  back to  $\mathbb{C}$ :  $\mathbf{x}^*(\mathbf{y}) = \langle \mathbf{y}, \mathbf{x} \rangle \in \mathbb{C}$ . We will use the notation  $\mathbf{x}^*$  for this operator rather than the more cumbersome  $\langle \cdot, \mathbf{x} \rangle$ .

Finally we note that a crucial tool for many of the results of this paper is the manipulation of  $2 \times 2$  block matrices centering around the so-called Schur complement. Given any  $2 \times 2$  block matrix  $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$  with  $A$  invertible, we define the *Schur complement* of  $D$  (with respect to  $M$ ) to be the matrix

$$S_M(D) := D - CA^{-1}B.$$

In case  $D$  is invertible, we define the *Schur complement* of  $A$  (with respect to  $M$ ) to be the matrix

$$S_M(A) := A - BD^{-1}C.$$

Our main application is to the case where  $M = M^*$  is self-adjoint (so  $A = A^*$ ,  $D = D^*$  and  $C = B^*$ ). Assuming  $A$  is invertible, we may factor  $A$  as  $A = |A|^{1/2}J|A|^{1/2}$  where  $J := \text{sign}(A)$  and the factorization

$$\begin{bmatrix} A & B \\ B^* & D \end{bmatrix} = \begin{bmatrix} |A|^{1/2} & 0 \\ B^*|A|^{-1/2}J & I \end{bmatrix} \begin{bmatrix} J & 0 \\ 0 & D - B^*A^{-1}B \end{bmatrix} \begin{bmatrix} |A|^{1/2} & J|A|^{-1/2}B \\ 0 & I \end{bmatrix}$$

shows that  $M \geq 0$  (i.e.,  $M$  is positive-semidefinite) if and only if  $A \geq 0$  (so  $J = I$ ) and the Schur complement  $D - B^*A^{-1}B$  of  $D$  is positive semidefinite. Similarly, in case  $D$  is invertible, we see that  $M \geq 0$  if and only if  $D \geq 0$  and the Schur complement of  $A$ , namely,  $A - BD^{-1}B^*$ , is positive semidefinite. In fact, these results go through without the invertibility assumption on  $A$  or  $D$ , using Moore-Penrose inverses instead.

## 2. LINEAR OPERATOR EQUATIONS

The problem  $\mathbf{AIP}_{\mathcal{H}(K_S)}$  is a particular case of the following well-known norm constrained operator problem: *Given  $A \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_3)$  and  $B \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_3)$ , with  $\mathcal{H}_1$ ,  $\mathcal{H}_2$  and  $\mathcal{H}_3$  given Hilbert spaces, describe the operators  $X \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$  that satisfy*

$$(2.1) \quad AX = B \quad \text{and} \quad \|X\| \leq 1.$$

The solvability criterion is known as the Douglas factorization lemma [18].

**Lemma 2.1.** *There exists an  $X \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$  satisfying (2.1) if and only if  $AA^* \geq BB^*$ . In this case, there exists a unique  $X \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$  satisfying (2.1) and the additional constraints  $\text{Ran}X \subset \text{Ran}A^*$  and  $\text{Ker}X = \text{Ker}B$ .*

In case  $AA^* \geq BB^*$ , Lemma 2.1 guarantees the existence of (unique) contractions  $X_1 \in \mathcal{L}(\mathcal{H}_1, \overline{\text{Ran}A})$  and  $X_2 \in \mathcal{L}(\mathcal{H}_2, \overline{\text{Ran}A})$  so that

$$(2.2) \quad (AA^*)^{\frac{1}{2}}X_1 = B, \quad (AA^*)^{\frac{1}{2}}X_2 = A, \quad \text{Ker}X_1 = \text{Ker}B, \quad \text{Ker}X_2 = \text{Ker}A.$$

By construction,  $X_2$  is a coisometry. The next lemma gives a description of the operators  $X \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$  satisfying (2.1) in terms of the operators  $X_1$  and  $X_2$ .

**Lemma 2.2.** *Assume  $AA^* \geq BB^*$  and let  $X \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ . Then the following statements are equivalent:*

- (1)  $X$  satisfies conditions (2.1).
- (2) The operator

$$(2.3) \quad \begin{bmatrix} I_{\mathcal{H}_1} & B^* & X^* \\ B & AA^* & A \\ X & A^* & I_{\mathcal{H}_2} \end{bmatrix} : \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_3 \\ \mathcal{H}_2 \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_3 \\ \mathcal{H}_2 \end{bmatrix}$$

is positive semidefinite.

- (3)  $X$  is of the form

$$(2.4) \quad X = X_2^*X_1 + (I - X_2^*X_2)^{\frac{1}{2}}K(I - X_1^*X_1)^{\frac{1}{2}}$$

where  $X_1$  and  $X_2$  are defined as in (2.2) and where the parameter  $K$  is an arbitrary contraction from  $\overline{\text{Ran}(I - X_1^*X_1)}$  into  $\overline{\text{Ran}(I - X_2^*X_2)}$ .

Moreover, if  $X$  satisfies (2.1), then  $X$  is unique if and only if  $X_1$  is isometric on  $\mathcal{H}_1$  or  $X_2$  is isometric on  $\mathcal{H}_2$ .

*Proof.* Note that positivity of the block-matrix in (2.3) is equivalent to positivity of the Schur complement of  $I_{\mathcal{H}_2}$  in (2.3), namely

$$(2.5) \quad \begin{bmatrix} I & B^* \\ B & AA^* \end{bmatrix} - \begin{bmatrix} X^* \\ A \end{bmatrix} \begin{bmatrix} X & A^* \end{bmatrix} = \begin{bmatrix} I - X^*X & B^* - X^*A^* \\ B - AX & 0 \end{bmatrix} \geq 0.$$

Because of the zero in the (2, 2)-entry of the left hand side of (2.5), we find that the inequality (2.5) holds precisely when

$$B - AX = 0 \quad \text{and} \quad I - X^*X \geq 0,$$

which is equivalent to (2.1). On the other hand, condition (2.3) is equivalent, by taking the Schur complement of  $AA^*$  in (2.3) and making use of (2.2), to

$$\begin{bmatrix} I & X^* \\ X & I \end{bmatrix} - \begin{bmatrix} X_1^* \\ X_2^* \end{bmatrix} \begin{bmatrix} X_1 & X_2 \end{bmatrix} = \begin{bmatrix} I - X_1^*X_1 & X^* - X_1^*X_2 \\ X - X_2^*X_1 & I - X_2^*X_2 \end{bmatrix} \geq 0.$$

By Theorem XVI.1.1 from [20], the latter inequality is equivalent to the representation (2.4) for  $X$  with  $K$  some contraction in  $\mathcal{L}(\overline{\text{Ran}(I - X_1^*X_1)}, \overline{\text{Ran}(I - X_2^*X_2)})$ .

Moreover,  $X$  and  $K$  in (2.4) determine each other uniquely. The last statement in the lemma now follows from representation (2.4).  $\square$

Note that since  $X_2$  is a coisometry, it follows that  $(I - X_2^* X_2)^{\frac{1}{2}}$  is the orthogonal projection onto  $\mathcal{H}_1 \ominus \text{Ker} A = \mathcal{H}_1 \ominus \text{Ker} X_1$ . This implies that for each  $K$  in (2.4) and each  $h \in \mathcal{H}_1$ , we have

$$\|Xh\|^2 = \|X_2^* X_1 h\|^2 + \|(I - X_2^* X_2)^{\frac{1}{2}} K (I - X_1^* X_1)^{\frac{1}{2}} h\|^2,$$

so that  $X_2^* X_1$  is the minimal norm solution to the problem (2.1).

### 3. THE $\mathbf{AIP}_{\mathcal{H}(K_S)}$ -PROBLEM AS A LINEAR OPERATOR EQUATION

In this section we consider data sets  $\mathcal{D} = \{S, T, E, N, \mathbf{x}\}$  satisfying conditions (1) and (2) in the definition of an  $\mathbf{AIP}_{\mathcal{H}(K_S)}$ -admissible data set but not necessarily condition (3); condition (3) of an  $\mathbf{AIP}_{\mathcal{H}(K_S)}$ -admissible data set (i.e., the Stein equation (1.7)) comes to the fore for the derivation of the more explicit results to be presented in Section 5. We still speak of the  $\mathbf{AIP}_{\mathcal{H}(K_S)}$ -problem for this looser notion of admissible data set. Define  $F^S$  as in (1.5). If we apply Lemma 2.1 to the case where

$$(3.1) \quad A = M_{F^S}^{[*]} : \mathcal{H}(K_S) \rightarrow \mathcal{X}, \quad B = \mathbf{x} \in \mathcal{X} \cong \mathcal{L}(\mathbb{C}, \mathcal{X}),$$

then we see that solutions  $X : \mathbb{C} \rightarrow \mathcal{H}(K_S)$  to problem (2.1) necessarily have the form of a multiplication operator  $M_f$  for some function  $f \in \mathcal{H}(K_S)$ . This observation leads to the following solvability criterion.

**Theorem 3.1.** *The problem  $\mathbf{AIP}_{\mathcal{H}(K_S)}$  has a solution if and only if*

$$(3.2) \quad P \geq \mathbf{x}\mathbf{x}^*, \quad \text{where } P := M_{F^S}^{[*]} M_{F^S}.$$

**Remark 3.2.** Observe that for the unconstrained version of the problem  $\mathbf{AIP}_{\mathcal{H}(K_S)}$ , the existence criterion follows immediately from the definition (3.2) of  $P$ : *there is a function  $f \in \mathcal{H}(K_S)$  such that  $M_{F^S}^{[*]} f = \mathbf{x}$  if and only if  $\mathbf{x} \in \text{Ran} P^{\frac{1}{2}}$ .*

The next theorem characterizes solutions to the problem  $\mathbf{AIP}_{\mathcal{H}(K_S)}$  in terms of a positive kernel. We emphasize that characterizations of this type go back to the Potapov's method of fundamental matrix inequalities [31]. Here the notation  $\mathbf{x}^*$  associated with a vector  $\mathbf{x} \in \mathcal{X}$  follows the conventions explained at the end of the Introduction.

**Theorem 3.3.** *A function  $f : \mathbb{D} \rightarrow \mathcal{Y}$  is a solution of the problem  $\mathbf{AIP}_{\mathcal{H}(K_S)}$  with data set (1.3) if and only if the kernel*

$$(3.3) \quad \mathbf{K}(z, \zeta) = \begin{bmatrix} 1 & \mathbf{x}^* & f(\zeta)^* \\ \mathbf{x} & P & F^S(\zeta)^* \\ f(z) & F^S(z) & K_S(z, \zeta) \end{bmatrix} \quad (z, \zeta \in \mathbb{D}),$$

*is positive on  $\mathbb{D} \times \mathbb{D}$ . Here  $P$ ,  $F^S$  and  $K_S$  are given by (3.2), (1.5) and (1.1), respectively.*

*Proof.* By Lemma 2.2 specialized to  $A$  and  $B$  as in (3.1) and  $X = M_f$ , we conclude that  $f$  is a solution to the problem  $\mathbf{AIP}_{\mathcal{H}(K_S)}$  (that is, it meets conditions (1.8)) if

and only if the following operator is positive semidefinite:

$$\mathbf{P} := \begin{bmatrix} 1 & \mathbf{x}^* & M_f^{[*]} \\ \mathbf{x} & M_{FS}^{[*]} M_{FS} & M_{FS}^{[*]} \\ M_f & M_{FS} & I_{\mathcal{H}(K_S)} \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{x}^* & M_f^{[*]} \\ \mathbf{x} & P & M_{FS}^{[*]} \\ M_f & M_{FS} & I_{\mathcal{H}(K_S)} \end{bmatrix} \geq 0.$$

We next observe that for every vector  $g \in \mathbb{C} \oplus \mathcal{X} \oplus \mathcal{H}(K_S)$  of the form

$$(3.4) \quad g(z) = \sum_{j=1}^r \begin{bmatrix} c_j \\ x_j \\ K_S(\cdot, z_j) y_j \end{bmatrix} \quad (c_j \in \mathbb{C}, y_j \in \mathcal{Y}, x_j \in \mathcal{X}, z_j \in \mathbb{D})$$

the identity

$$(3.5) \quad \langle \mathbf{P}g, g \rangle_{\mathbb{C} \oplus \mathcal{X} \oplus \mathcal{H}(K_S)} = \sum_{j, \ell=1}^r \left\langle \mathbf{K}(z_j, z_\ell) \begin{bmatrix} c_\ell \\ x_\ell \\ y_\ell \end{bmatrix}, \begin{bmatrix} c_j \\ x_j \\ y_j \end{bmatrix} \right\rangle_{\mathbb{C} \oplus \mathcal{X} \oplus \mathcal{Y}}$$

holds. Since the set of vectors of the form (3.4) is dense in  $\mathbb{C} \oplus \mathcal{X} \oplus \mathcal{H}(K_S)$ , the identity (3.5) now implies that the operator  $\mathbf{P}$  is positive semidefinite if and only if the quadratic form on the right hand side of (3.5) is nonnegative, i.e., if and only if the kernel (3.3) is positive on  $\mathbb{D} \times \mathbb{D}$ .  $\square$

For the rest of this section we assume that the operator  $P$  in (3.2) is positive definite. Then the operator  $M_{FS} P^{-\frac{1}{2}}$  is an isometry and the space

$$(3.6) \quad \mathcal{N} = \{F^S(z)x : x \in \mathcal{X}\} \quad \text{with norm} \quad \|F^S x\|_{\mathcal{H}(S)} = \|P^{\frac{1}{2}} x\|_{\mathcal{X}}$$

is isometrically included in  $\mathcal{H}(K_S)$ . Moreover, the orthogonal complement of  $\mathcal{N}$  in  $\mathcal{H}(K_S)$  is the reproducing kernel Hilbert space  $\mathcal{H}(\tilde{K}_S)$  with reproducing kernel

$$(3.7) \quad \tilde{K}_S(z, \zeta) = K_S(z, \zeta) - F^S(z)P^{-1}F^S(\zeta)^*.$$

**Theorem 3.4.** *Assume that condition (3.2) holds and that  $P$  is positive definite. Let  $\tilde{K}_S(z, \zeta)$  be the kernel defined in (3.7). Then:*

- (1) *All solutions  $f$  to the problem  $\mathbf{AIP}_{\mathcal{H}(K_S)}$  are described by the formula*

$$(3.8) \quad f(z) = F^S(z)P^{-1}\mathbf{x} + h(z)$$

where  $h$  is a free parameter from  $\mathcal{H}(\tilde{K}_S)$  subject to

$$\|h\|_{\mathcal{H}(\tilde{K}_S)} \leq \sqrt{1 - \|P^{-\frac{1}{2}}\mathbf{x}\|^2}.$$

- (2) *The problem  $\mathbf{AIP}_{\mathcal{H}(K_S)}$  has a unique solution if and only if*

$$(3.9) \quad \|P^{-\frac{1}{2}}\mathbf{x}\| = 1 \quad \text{or} \quad \tilde{K}_S(z, \zeta) \equiv 0.$$

*Proof.* It is readily seen that

$$X_1 = P^{-\frac{1}{2}}\mathbf{x} \in \mathcal{X} \cong \mathcal{L}(\mathbb{C}, \mathcal{X}), \quad X_2 = P^{-\frac{1}{2}}M_{FS}^{[*]} \in \mathcal{L}(\mathcal{H}(K_S), \mathcal{X})$$

are the operators  $X_1$  and  $X_2$  from (2.2) after specialization to the case (3.1).

The second statement now follows from Lemma 2.2, since  $P^{-\frac{1}{2}}\mathbf{x} \in \mathcal{L}(\mathbb{C}, \mathcal{X})$  being isometric means that  $\|P^{-\frac{1}{2}}\mathbf{x}\| = 1$  and, on the other hand, the isometric property for the operator  $M_{FS} P^{-\frac{1}{2}}$  means that the space  $\mathcal{N}$  defined in (3.6) is equal to the whole space  $\mathcal{H}(K_S)$ . Thus  $\mathcal{H}(\tilde{K}_S) = \mathcal{H}(K_S) \ominus \mathcal{N} = \{0\}$  or  $\tilde{K}_S \equiv 0$ .

In the present framework, the parametrization formula (2.4) takes the form

$$(3.10) \quad M_f = M_{F^S} P^{-\frac{1}{2}} \mathbf{x} + \sqrt{1 - \|P^{-\frac{1}{2}} \mathbf{x}\|^2} \cdot (I - M_{F^S} P^{-1} M_{F^S}^{[*]})^{\frac{1}{2}} K$$

where  $K$  is equal to the operator of multiplication  $M_k : \mathbb{C} \rightarrow \mathcal{H}(\tilde{K}_S)$  by a function  $k \in \mathcal{H}(\tilde{K}_S)$  with  $\|k\| \leq 1$ . Since  $M_{F^S} P^{-\frac{1}{2}}$  is an isometry, the second term on the right hand side of (3.10) is equal to the operator  $M_h$  of multiplication by a function  $h \in \mathcal{H}(\tilde{K}_S)$  such that  $\|h\|_{\mathcal{H}(\tilde{K}_S)} = \|h\|_{\mathcal{H}(K_S)} \leq \sqrt{1 - \|P^{-\frac{1}{2}} \mathbf{x}\|^2}$ .  $\square$

**Remark 3.5.** The second term  $h$  on the right hand side of (3.8) represents in fact the general solution of the homogeneous interpolation problem (with interpolation condition  $M_{F^S}^{[*]} f = 0$ ). If  $h$  runs through the whole space  $\mathcal{H}(\tilde{K}_S)$ , then formula (3.8) produces all functions  $f \in \mathcal{H}(K_S)$  such that  $M_{F^S}^{[*]} f = \mathbf{x}$ . This unconstrained interpolation problem has a unique solution if and only if  $\tilde{K}_S(z, \zeta) \equiv 0$ . Thus, the second condition in (3.9) provides the uniqueness of an  $f$  subject to  $M_{F^S}^{[*]} f = \mathbf{x}$  in the whole space  $\mathcal{H}(K_S)$ , not just in the unit ball of  $\mathcal{H}(K_S)$ . If  $\tilde{K}_S(z, \zeta) \not\equiv 0$ , then the unconstrained problem has infinitely many solutions and, as in the general framework, the function  $F^S(z) P^{-1} \mathbf{x}$  has the minimal possible norm. Since  $M_{F^S} P^{-\frac{1}{2}}$  is an isometry, it follows from (3.6) that  $\|M_{F^S} P^{-1} \mathbf{x}\|_{\mathcal{H}(K_S)} = \|P^{-\frac{1}{2}} \mathbf{x}\|$ . Thus, if  $\|P^{-\frac{1}{2}} \mathbf{x}\| = 1$ , then uniqueness occurs since the *minimal norm solution* already has unit norm.

#### 4. REDHEFFER TRANSFORM RELATED TO THE AIP-PROBLEM ON $\mathcal{S}(\mathcal{U}, \mathcal{Y})$

To obtain a more explicit parametrization of the solution set to the **AIP** $_{\mathcal{H}(K_S)}$ -problem, we need some facts concerning the Abstract Interpolation Problem for functions in the Schur Class  $\mathcal{S}(\mathcal{U}, \mathcal{Y})$  (denoted as the **AIP** $_{\mathcal{S}(\mathcal{U}, \mathcal{Y})}$ -problem) from [26] (see also [30, 28]) which we now recall.

We consider the data set

$$(4.1) \quad \mathcal{D}' = \{P, T, E, N\}$$

consisting of operators  $P, T \in \mathcal{L}(\mathcal{X})$ ,  $E \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ ,  $N \in \mathcal{L}(\mathcal{X}, \mathcal{U})$  such that the pairs  $(E, T)$  and  $(N, T)$  are output analytic and  $P$  is a positive semidefinite solution of the Stein equation (1.7). A set with these properties is called **AIP** $_{\mathcal{S}(\mathcal{U}, \mathcal{Y})}$ -admissible.

**AIP** $_{\mathcal{S}(\mathcal{U}, \mathcal{Y})}$ : Given an **AIP** $_{\mathcal{S}(\mathcal{U}, \mathcal{Y})}$ -admissible data set (4.1), find all functions  $S : \mathbb{D} \rightarrow \mathcal{L}(\mathcal{U}, \mathcal{Y})$  such that the kernel

$$(4.2) \quad (z, \zeta) \mapsto \begin{bmatrix} P & F^S(\zeta)^* \\ F^S(z) & K_S(z, \zeta) \end{bmatrix} \quad (z, \zeta \in \mathbb{D})$$

is positive on  $\mathbb{D} \times \mathbb{D}$ , or equivalently, find all functions  $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$  so that the operator  $M_{F^S} = \mathcal{O}_{E, T} - M_S \mathcal{O}_{N, T} : x \rightarrow F^S(z)x$  maps  $\mathcal{X}$  into  $\mathcal{H}(K_S)$  and satisfies  $M_{F^S}^{[*]} M_{F^S} \leq P$ . Here  $F^S$  is the function defined in (1.5).

The equivalence of the two above formulations follows from a general result on reproducing kernel Hilbert spaces; see [8].

The parametrization of the solutions through an associated Redheffer transform is recalled in Theorem 4.1 below. To state the result we need to construct the Redheffer transform. Observe that (1.7) can equivalently be written as

$$\|P^{\frac{1}{2}} x\|^2 + \|Nx\|^2 = \|P^{\frac{1}{2}} Tx\|^2 + \|Ex\|^2 \quad \text{for all } x \in \mathcal{X}.$$



Let  $\mathcal{X}_0 = \overline{\text{Ran}} P^{\frac{1}{2}}$ . With some abuse of notation we will occasionally view  $P$  as an operator mapping  $\mathcal{X}$  into  $\mathcal{X}_0$ , or  $\mathcal{X}_0$  into  $\mathcal{X}$ , while still using  $P = P^*$ . The above identity shows that there exists a well defined isometry  $V$  with domain  $\mathcal{D}_V$  and range  $\mathcal{R}_V$  equal to

$$\mathcal{D}_V = \overline{\text{Ran}} \begin{bmatrix} P^{\frac{1}{2}} \\ N \end{bmatrix} \subseteq \begin{bmatrix} \mathcal{X}_0 \\ \mathcal{U} \end{bmatrix} \quad \text{and} \quad \mathcal{R}_V = \overline{\text{Ran}} \begin{bmatrix} P^{\frac{1}{2}}T \\ E \end{bmatrix} \subseteq \begin{bmatrix} \mathcal{X}_0 \\ \mathcal{Y} \end{bmatrix},$$

respectively, which is uniquely determined by the identity

$$(4.3) \quad V \begin{bmatrix} P^{\frac{1}{2}}x \\ Nx \end{bmatrix} = \begin{bmatrix} P^{\frac{1}{2}}Tx \\ Ex \end{bmatrix} \quad \text{for all } x \in \mathcal{X}.$$

We then define the defect spaces

$$(4.4) \quad \Delta := \begin{bmatrix} \mathcal{X}_0 \\ \mathcal{U} \end{bmatrix} \ominus \mathcal{D}_V \quad \text{and} \quad \Delta_* := \begin{bmatrix} \mathcal{X}_0 \\ \mathcal{Y} \end{bmatrix} \ominus \mathcal{R}_V,$$

and let  $\tilde{\Delta}$  and  $\tilde{\Delta}_*$  denote isomorphic copies of  $\Delta$  and  $\Delta_*$ , respectively, with unitary identification maps

$$\mathbf{i} : \Delta \rightarrow \tilde{\Delta} \quad \text{and} \quad \mathbf{i}_* : \Delta_* \rightarrow \tilde{\Delta}_*.$$

With these identification maps we define a unitary colligation matrix  $\mathbf{U}$  from  $\mathcal{D}_V \oplus \Delta \oplus \tilde{\Delta}_* = \mathcal{X} \oplus \mathcal{U} \oplus \tilde{\Delta}_*$  onto  $\mathcal{R}_V \oplus \Delta_* \oplus \tilde{\Delta} = \mathcal{X} \oplus \mathcal{Y} \oplus \tilde{\Delta}$  by

$$(4.5) \quad \mathbf{U} = \begin{bmatrix} V & 0 & 0 \\ 0 & 0 & \mathbf{i}_* \\ 0 & \mathbf{i} & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{D}_V \\ \Delta \\ \tilde{\Delta}_* \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}_V \\ \Delta_* \\ \tilde{\Delta} \end{bmatrix},$$

which we also decompose as

$$(4.6) \quad \mathbf{U} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{X}_0 \\ \mathcal{U} \\ \tilde{\Delta}_* \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{X}_0 \\ \mathcal{Y} \\ \tilde{\Delta} \end{bmatrix}.$$

Write  $\Sigma$  for the characteristic function associated with this colligation  $\mathbf{U}$ , i.e.,

$$(4.7) \quad \Sigma(z) = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & 0 \end{bmatrix} + z \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} (I - zA)^{-1} \begin{bmatrix} B_1 & B_2 \end{bmatrix} \quad (z \in \mathbb{D}),$$

and decompose  $\Sigma$  as

$$(4.8) \quad \Sigma(z) = \begin{bmatrix} \Sigma_{11}(z) & \Sigma_{12}(z) \\ \Sigma_{21}(z) & \Sigma_{22}(z) \end{bmatrix} : \begin{bmatrix} \mathcal{U} \\ \tilde{\Delta}_* \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{Y} \\ \tilde{\Delta} \end{bmatrix}.$$

A straightforward calculation based on the fact that  $\mathbf{U}$  is coisometric gives

$$(4.9) \quad \frac{I - \Sigma(z)\Sigma(\zeta)^*}{1 - z\bar{\zeta}} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} (I - zA)^{-1} (I - \bar{\zeta}A^*)^{-1} \begin{bmatrix} C_1^* & C_2^* \end{bmatrix},$$

which implies in particular that  $\Sigma$  belongs to the Schur class  $\mathcal{S}(\mathcal{U} \oplus \tilde{\Delta}_*, \mathcal{Y} \oplus \tilde{\Delta})$ . Moreover, it follows from the construction that  $\Sigma_{22}(0) = 0$ . These facts imply that the Redheffer linear fractional transform

$$(4.10) \quad S = \mathcal{R}_\Sigma[\mathcal{E}] := \Sigma_{11} + \Sigma_{12}(I - \mathcal{E}\Sigma_{22})^{-1}\mathcal{E}\Sigma_{21}$$

is well defined for every Schur-class function  $\mathcal{E} \in \mathcal{S}(\tilde{\Delta}, \tilde{\Delta}_*)$ . The next theorem (see [26] for the proof) shows that the image of the class  $\mathcal{S}(\tilde{\Delta}, \tilde{\Delta}_*)$  under the Redheffer transform  $\mathcal{R}_\Sigma$  is precisely the solution set of the problem  $\mathbf{AIP}_{\mathcal{S}(\mathcal{U}, \mathcal{Y})}$ .

**Theorem 4.1.** *Given an  $\mathbf{AIP}_{\mathcal{S}(\mathcal{U}, \mathcal{Y})}$ -admissible data set (4.1), let  $\mathcal{R}_\Sigma$  be the Redheffer transform constructed as in (4.7), (4.10). A function  $S : \mathbb{D} \rightarrow \mathcal{L}(\mathcal{U}, \mathcal{Y})$  is a solution of the problem  $\mathbf{AIP}_{\mathcal{S}(\mathcal{U}, \mathcal{Y})}$  if and only if  $S = \mathcal{R}_\Sigma[\mathcal{E}]$  for some  $\mathcal{E} \in \mathcal{S}(\tilde{\Delta}, \tilde{\Delta}_*)$ .*

The function  $\Sigma_{11}$  appears as a solution upon taking  $\mathcal{E} \equiv 0$ , and is called *the central solution* of the problem  $\mathbf{AIP}_{\mathcal{S}(\mathcal{U}, \mathcal{Y})}$ . In case the problem has only one solution, this solution must be the central solution.

**Proposition 4.2.** *Let  $\Sigma$  be the characteristic function of the unitary colligation  $\mathbf{U}$  in (4.6), decomposed as in (4.8), and let  $S = \mathcal{R}_\Sigma[\mathcal{E}] \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$  for a given  $\mathcal{E} \in \mathcal{S}(\tilde{\Delta}, \tilde{\Delta}_*)$ . Define the functions*

$$(4.11) \quad \begin{aligned} G(z) &= \Sigma_{12}(z)(I - \mathcal{E}(z)\Sigma_{22}(z))^{-1}, \\ \Gamma(z) &= (C_1 + G(z)\mathcal{E}(z)C_2)(I - zA)^{-1}. \end{aligned} \quad (z \in \mathbb{D})$$

Then  $G$  defines a contractive multiplier  $M_G : \mathcal{H}(K_\mathcal{E}) \rightarrow \mathcal{H}(K_S)$ ,  $\Gamma$  a contractive multiplier  $M_\Gamma : \mathcal{X}_0 \rightarrow \mathcal{H}(K_S)$ , and the operator

$$(4.12) \quad [M_G \quad M_\Gamma] : \begin{bmatrix} \mathcal{H}(K_\mathcal{E}) \\ \mathcal{X}_0 \end{bmatrix} \rightarrow \mathcal{H}(K_S)$$

is coisometric. Furthermore, we have

$$(4.13) \quad F^S(z) = \Gamma(z)P^{\frac{1}{2}}$$

for each  $z \in \mathbb{D}$ , where  $F^S$  is the function defined in (1.5). In particular,  $M_\Gamma$  is an isometry and  $M_G$  a partial isometry if and only if  $P = M_{F^S}^* M_{F^S}$  with  $M_{F^S} : \mathcal{X}_0 \rightarrow \mathcal{H}(K_S)$  defined by  $M_{F^S} = M_\Gamma P^{\frac{1}{2}}$ .

*Proof.* The identity  $[I \quad G(z)\mathcal{E}(z)]\Sigma(z) = [S(z) \quad G(z)]$  is an immediate consequence of (4.10) and the definition of  $G(z)$  in (4.11). Using this identity one can easily compute that

$$\begin{aligned} I - S(z)S(\zeta)^* &= G(z)(I - \mathcal{E}(z)\mathcal{E}(\zeta)^*)G(\zeta)^* + [I \quad G(z)\mathcal{E}(z)] \begin{bmatrix} I \\ \mathcal{E}(\zeta)^*G(\zeta)^* \end{bmatrix} \\ &\quad - [S(z) \quad G(z)] \begin{bmatrix} S(\zeta)^* \\ G(\zeta)^* \end{bmatrix} \\ &= G(z)(I - \mathcal{E}(z)\mathcal{E}(\zeta)^*)G(\zeta)^* \\ &\quad + [I \quad G(z)\mathcal{E}(z)](I - \Sigma(z)\Sigma(\zeta)^*) \begin{bmatrix} I \\ \mathcal{E}(\zeta)^*G(\zeta)^* \end{bmatrix}; \end{aligned}$$

(see also [9, Lemma 8.3]). Dividing both sides of the latter identity by  $1 - z\bar{\zeta}$  leads to

$$K_S(z, \zeta) = G(z)K_\mathcal{E}(z, \zeta)G(\zeta)^* + [I \quad G(z)\mathcal{E}(z)]K_\Sigma(z, \zeta) \begin{bmatrix} I \\ \mathcal{E}(\zeta)^*G(\zeta)^* \end{bmatrix}.$$

By replacing  $K_\Sigma(z, \zeta)$  by the expression on the right hand side of (4.9), we get

$$(4.14) \quad K_S(z, \zeta) = G(z)K_\mathcal{E}(z, \zeta)G(\zeta)^* + \Gamma(z)\Gamma(\zeta)^*.$$

It is easy to verify that

$$M_G^* : K_S(\cdot, \zeta)y \mapsto K_S(\cdot, \zeta)G(\zeta)^*y, \quad M_\Gamma^* : K_S(\cdot, \zeta)y \mapsto \Gamma(\zeta)^*y$$

from which one can deduce that the context of (4.14) is that  $[M_G \ M_\Gamma^*]^*$  is isometric on  $\overline{\text{span}}\{K_S(\cdot, \zeta)y : \zeta \in \mathbb{D}, y \in \mathcal{Y}\} = \mathcal{H}(K_S)$ , i.e.,  $[M_G \ M_\Gamma]$  is coisometric as asserted. In particular, we see that  $M_G$  and  $M_\Gamma$  are contractions.

To verify (4.13), recall that the very construction of the colligation  $\mathbf{U}$  implies that

$$\begin{bmatrix} A & B_1 \\ C_1 & D_{11} \\ C_2 & D_{21} \end{bmatrix} \begin{bmatrix} P^{\frac{1}{2}} \\ N \end{bmatrix} = \begin{bmatrix} P^{\frac{1}{2}}T \\ E \\ 0 \end{bmatrix},$$

so that  $AP^{\frac{1}{2}} - P^{\frac{1}{2}}T = B_1N$ ,  $C_1P^{\frac{1}{2}} = E - D_{11}N$ ,  $C_2P^{\frac{1}{2}} = -D_{21}N$ . Making use of the latter equalities and of realization formulas for  $\Sigma_{11}$  and  $\Sigma_{21}$  in (4.7) we compute for  $z \in \mathbb{D}$ ,

$$\begin{aligned} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} (I - zA)^{-1}P^{\frac{1}{2}}(I - zT) &= \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} P^{\frac{1}{2}} + z \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} (I - zA)^{-1}(AP^{\frac{1}{2}} - P^{\frac{1}{2}}T) \\ &= \begin{bmatrix} E - D_{11}N \\ -D_{21}N \end{bmatrix} - z \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} (I - zA)^{-1}B_1N \\ (4.15) \qquad \qquad \qquad &= \begin{bmatrix} E \\ 0 \end{bmatrix} - \begin{bmatrix} \Sigma_{11}(z) \\ \Sigma_{21}(z) \end{bmatrix} N. \end{aligned}$$

Upon multiplying the left-hand side expression in (4.15) by  $[I \ G(z)\mathcal{E}(z)]$  on the left and by  $(I - zT)^{-1}$  on the right, we get  $\Gamma(z)P^{\frac{1}{2}}$  by definition (4.11). Applying the same multiplications to the right hand side expression gives, on account of (4.11) and (4.10),

$$E(I - zT)^{-1} - (\Sigma_{11}(z) + G(z)\mathcal{E}(z)\Sigma_{21}(z))N(I - zT)^{-1} = (E - S(z)N)(I - zT)^{-1}$$

which is  $F^S(z)$ . Thus  $\Gamma(z)P^{\frac{1}{2}} = F^S(z)$ , and (4.13) follows.

It is now straightforward to verify that  $M_\Gamma$  is an isometry if and only if  $P = M_{F^S}^{[*]}M_{F^S}$ , while, since (4.12) is a coisometry,  $M_\Gamma$  being an isometry implies that  $M_G$  is a partial isometry.  $\square$

**4.1. Injectivity of  $R_\Sigma$  and  $M_G$ .** In this subsection we focus on two questions: (1) when is the above constructed Redheffer transform  $\mathcal{R}_\Sigma$  injective, and, (2) for a given  $\mathcal{E} \in \mathcal{S}(\tilde{\Delta}, \tilde{\Delta}_*)$ , when is the multiplication operator  $M_G : \mathcal{H}(K_\mathcal{E}) \rightarrow \mathcal{H}(K_S)$  from Proposition 4.2 injective (and thus an isometry if  $P = M_{F^S}^{[*]}M_{F^S}$ )? The next lemma provides the basis for the results to follow.

**Lemma 4.3.** *Assume  $M_{\Sigma_{12}} : \text{Hol}_{\tilde{\Delta}_*}(\mathbb{D}) \rightarrow \text{Hol}_Y(\mathbb{D})$  has a trivial kernel and  $M_{\Sigma_{21}} : \text{Hol}_U(\mathbb{D}) \rightarrow \text{Hol}_{\tilde{\Delta}}(\mathbb{D})$  has dense range. Then the Redheffer transform  $\mathcal{R}_\Sigma$  is injective, and for any  $\mathcal{E} \in \mathcal{S}(\tilde{\Delta}, \tilde{\Delta}_*)$  the multiplication operator  $M_G : \mathcal{H}(K_\mathcal{E}) \rightarrow \mathcal{H}(K_S)$  has trivial kernel.*

**Proof.** The identity

$$M_S = M_{\Sigma_{11}} + M_{\Sigma_{12}}(I - M_\mathcal{E}M_{\Sigma_{22}})^{-1}M_\mathcal{E}M_{\Sigma_{21}}$$

may not hold if we consider the multiplication operators as acting between the appropriate  $H^2$ -spaces, since  $I - M_\mathcal{E}M_{\Sigma_{11}}$  may not be boundedly invertible, but the identity does hold when the multiplication operators are viewed as operators

between the appropriate linear spaces of holomorphic functions on  $\mathbb{D}$ , i.e.,

$$M_S : \text{Hol}_{\tilde{\Delta}_*}(\mathbb{D}) \rightarrow \text{Hol}_{\mathcal{Y}}(\mathbb{D}), \quad M_{\mathcal{E}} : \text{Hol}_{\tilde{\Delta}_*}(\mathbb{D}) \rightarrow \text{Hol}_{\tilde{\Delta}}(\mathbb{D}),$$

$$\begin{bmatrix} M_{\Sigma_{11}} & M_{\Sigma_{12}} \\ M_{\Sigma_{21}} & M_{\Sigma_{22}} \end{bmatrix} : \begin{bmatrix} \text{Hol}_{\mathcal{U}}(\mathbb{D}) \\ \text{Hol}_{\tilde{\Delta}_*}(\mathbb{D}) \end{bmatrix} \rightarrow \begin{bmatrix} \text{Hol}_{\mathcal{Y}}(\mathbb{D}) \\ \text{Hol}_{\tilde{\Delta}}(\mathbb{D}) \end{bmatrix}.$$

Note that  $I - M_{\mathcal{E}}M_{\Sigma_{22}}$  is invertible as a linear map on  $\text{Hol}_{\tilde{\Delta}_*}(\mathbb{D})$  since  $\Sigma_{22}(0) = 0$  and  $\mathcal{E}(z)$  and  $\Sigma_{22}(z)$  are both contractive for  $z \in \mathbb{D}$ . Now assume  $\mathcal{E}, \mathcal{E}' \in \mathcal{S}(\tilde{\Delta}, \tilde{\Delta}_*)$  so that  $\mathcal{R}_{\Sigma}[\mathcal{E}] = \mathcal{R}_{\Sigma}[\mathcal{E}']$ . By the assumptions on  $M_{\Sigma_{12}}$  and  $M_{\Sigma_{21}}$  it follows that

$$(I - M_{\mathcal{E}}M_{\Sigma_{22}})^{-1}M_{\mathcal{E}} = (I - M_{\mathcal{E}'}M_{\Sigma_{22}})^{-1}M_{\mathcal{E}'} = M_{\mathcal{E}'}(I - M_{\Sigma_{22}}M_{\mathcal{E}'})^{-1},$$

and thus

$$\begin{aligned} M_{\mathcal{E}} - M_{\mathcal{E}}M_{\Sigma_{22}}M_{\mathcal{E}'} &= M_{\mathcal{E}}(I - M_{\Sigma_{22}}M_{\mathcal{E}'}) \\ &= (I - M_{\mathcal{E}}M_{\Sigma_{22}})M_{\mathcal{E}'} = M_{\mathcal{E}'} - M_{\mathcal{E}}M_{\Sigma_{22}}M_{\mathcal{E}'}. \end{aligned}$$

Hence  $\mathcal{E} = \mathcal{E}'$ . Since  $M_{\Sigma_{12}} : \text{Hol}_{\tilde{\Delta}_*}(\mathbb{D}) \rightarrow \text{Hol}_{\mathcal{Y}}(\mathbb{D})$  has a trivial kernel, so does  $M_{\Sigma_{12}}(I - M_{\mathcal{E}}M_{\Sigma_{22}})^{-1}$  when viewed as an operator acting on  $\text{Hol}_{\tilde{\Delta}_*}(\mathbb{D})$ , independently of the choice of  $\mathcal{E} \in \mathcal{S}(\tilde{\Delta}, \tilde{\Delta}_*)$ . In particular,  $G(z)h(z) \equiv 0$  implies  $h = 0$  for any  $h \in \mathcal{H}(K_{\mathcal{E}})$ .  $\square$

The proof of the above lemma does not take into account the particularities of the Redheffer transform associated with the problem  $\mathbf{AIP}_{\mathcal{S}(\mathcal{U}, \mathcal{Y})}$  constructed in (4.3)–(4.6), besides the fact that  $\Sigma_{22}(0) = 0$ . As we shall see, for the coefficients in the Redheffer transform we consider,  $M_{\Sigma_{21}}$  always has dense range, while  $M_{\Sigma_{12}}$  has a trivial kernel if the operator  $T^*$  is injective. As preparation for this result, we need the following lemma.

**Lemma 4.4.** *Let  $D_{21}$  and  $D_{12}$  be the operators in the unitary colligation (4.6). Then*

$$(4.16) \quad \ker D_{21}^* = \{0\} \quad \text{and} \quad \ker D_{12} = \mathbf{i}_* \overline{\begin{bmatrix} \ker T^*P^{\frac{1}{2}}|_{\mathcal{X}_0} \\ \{0\} \end{bmatrix}}.$$

*Proof.* Let  $\tilde{\delta} \in \tilde{\Delta}$  be such that  $D_{21}^*\tilde{\delta} = 0_{\mathcal{U}}$ . By construction (4.5), the vector

$$(4.17) \quad \mathbf{U}^*\tilde{\delta} = \begin{bmatrix} C_2^*\tilde{\delta} \\ D_{21}^*\tilde{\delta} \\ 0 \end{bmatrix} = \begin{bmatrix} x_0 \\ 0 \\ 0 \end{bmatrix} \in \begin{bmatrix} \mathcal{X}_0 \\ \mathcal{U} \\ \tilde{\Delta}_* \end{bmatrix}$$

belongs to  $\Delta$ , which means (by definition (4.4) of  $\Delta$ ) that

$$0 = \left\langle \begin{bmatrix} x_0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} P^{\frac{1}{2}}x \\ Nx \\ 0 \end{bmatrix} \right\rangle_{\mathcal{X}_0 \oplus \mathcal{U} \oplus \tilde{\Delta}_*} = \langle x_0, P^{\frac{1}{2}}x \rangle_{\mathcal{X}_0}$$

for every  $x \in \mathcal{X}_0$ , which is equivalent to  $P^{\frac{1}{2}}x_0 = 0$ . Since  $P^{\frac{1}{2}}|_{\mathcal{X}_0}$  is injective, we get  $x_0 = 0$ . Thus  $\mathbf{U}^*\tilde{\delta} = 0$  by (4.17) and consequently,  $\tilde{\delta} = 0$ , since  $\mathbf{U}$  is unitary. So  $\text{Ker}D_{21}^* = \{0\}$ .

To prove the second equality in (4.16) we first observe that a vector  $\begin{bmatrix} x_0 \\ y \end{bmatrix} \in \begin{bmatrix} \mathcal{X}_0 \\ \mathcal{Y} \end{bmatrix}$  belongs to  $\Delta_* := \begin{bmatrix} \mathcal{X}_0 \\ \mathcal{Y} \end{bmatrix} \ominus \mathcal{R}_V$  if and only if

$$(4.18) \quad T^*P^{\frac{1}{2}}x_0 + E^*y = 0.$$

Now let  $\tilde{\delta}_* \in \tilde{\Delta}_*$  so that  $D_{12}\tilde{\delta}_* = 0$ . Then, by (4.5) and (4.6),

$$\mathbf{U}\tilde{\delta}_* = \mathbf{i}_*^*\tilde{\delta}_* = \begin{bmatrix} B_2 \\ D_{12} \end{bmatrix} \tilde{\delta}_* = \begin{bmatrix} x_0 \\ 0 \end{bmatrix} \in \Delta_* \subset \begin{bmatrix} \mathcal{X}_0 \\ \mathcal{Y} \end{bmatrix}, \quad \text{with } x_0 = B_2\tilde{\delta}_*.$$

Since  $\begin{bmatrix} x_0 \\ 0 \end{bmatrix}$  is in  $\Delta_*$ , it follows from (4.18) that  $T^*P^{\frac{1}{2}}x_0 = 0$ , i.e.,  $x_0 \in \ker T^*P^{\frac{1}{2}}$ . Thus, since  $\mathbf{U}$  is unitary and  $\begin{bmatrix} x_0 \\ 0 \end{bmatrix} \in \Delta_*$ , we see that  $\tilde{\delta}_* = \mathbf{U}^* \begin{bmatrix} x_0 \\ 0 \end{bmatrix} = \mathbf{i}_* \begin{bmatrix} x_0 \\ 0 \end{bmatrix}$  for some  $x_0 \in \ker T^*P^{\frac{1}{2}}$ .

Conversely, for every  $x_0 \in \ker(T^*P^{\frac{1}{2}}|_{\mathcal{X}_0})$ , the vector  $\begin{bmatrix} x_0 \\ 0 \end{bmatrix}$  belongs to  $\Delta_*$  (by (4.18)) so that its image  $\tilde{\delta}_* = \mathbf{i}_* \begin{bmatrix} x_0 \\ 0 \end{bmatrix}$  belongs to  $\tilde{\Delta}_*$  and we have, on account of (4.5)-(4.6),

$$\begin{bmatrix} B_2 \\ D_{12} \end{bmatrix} \tilde{\delta}_* = \begin{bmatrix} B_2 \\ D_{12} \end{bmatrix} \mathbf{i}_* \begin{bmatrix} x_0 \\ 0 \end{bmatrix} = \mathbf{i}_*^* \mathbf{i}_* \begin{bmatrix} x_0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}.$$

Equating the bottom entries we get  $D_{12}\tilde{\delta}_* = 0$  which completes the proof of the characterization of  $\ker D_{12}$ .  $\square$

**Theorem 4.5.** *Let  $\mathcal{R}_\Sigma$  be the Redheffer transform associated with the Schur class function  $\Sigma$  defined in (4.7) from the  $\mathbf{AIP}_{\mathcal{H}(K_S)}$ -admissible data set (4.1). Assume that  $T^*$  is injective. Then  $\mathcal{R}_\Sigma : \mathcal{S}(\tilde{\Delta}, \tilde{\Delta}_*) \rightarrow \mathcal{S}(\mathcal{U}, \mathcal{Y})$  is injective, and for  $\mathcal{E} \in \mathcal{S}(\tilde{\Delta}, \tilde{\Delta}_*)$  the multiplication operator  $M_G : \mathcal{H}(K_\mathcal{E}) \rightarrow \mathcal{H}(K_S)$  has trivial kernel. If in addition  $P = M_{FS}^{[*]} M_{FS}$ , then  $M_G$  is an isometry.*

*Proof.* It is easy to see that  $M_{\Sigma_{21}}$  has dense range if and only if  $\Sigma_{21}(0) = D_{21}$  has dense range and that  $\ker \Sigma_{12}(0) = \ker D_{12} = \{0\}$  implies that  $M_{\Sigma_{12}}$  has a trivial kernel. The converse of the latter statement is not true in general. Moreover, the last statement of the theorem is immediate from Proposition 4.2. Thus Theorem 4.5 follows immediately from Lemma 4.4.  $\square$

**Remark 4.6.** In case the operator  $(I - \omega T)^{-1}$  is bounded for some  $\omega \in \mathbb{D}$ , we can define

$$\begin{aligned} \tilde{E} &= \sqrt{1 - |\omega|^2} \cdot E(I - \omega T)^{-1}, & \tilde{N} &= \sqrt{1 - |\omega|^2} \cdot N(I - \omega T)^{-1} \\ \tilde{T} &= (\bar{\omega}I - T)(I - \omega T)^{-1}, & \tilde{S}(z) &= S\left(\frac{z - \omega}{1 - z\bar{\omega}}\right). \end{aligned}$$

It is not hard to verify that if  $\mathcal{D} = \{S, T, E, N, \mathbf{x}\}$  is an  $\mathbf{AIP}_{\mathcal{H}(K_S)}$ -admissible data set, then the set  $\tilde{\mathcal{D}} = \{\tilde{S}, \tilde{T}, \tilde{E}, \tilde{N}, \mathbf{x}\}$  is also  $\mathbf{AIP}_{\mathcal{H}(K_S)}$ -admissible and moreover, a function  $f$  solves the problem  $\mathbf{AIP}_{\mathcal{H}(K_S)}$  if and only if  $\tilde{f}(z) := f\left(\frac{z - \omega}{1 - z\bar{\omega}}\right)$  solves the problem  $\mathbf{AIP}_{\mathcal{H}(K_{\tilde{S}})}$  with data set  $\tilde{\mathcal{D}}$ . Therefore, up to a suitable conformal change of variable, we get all the conclusions in Theorem 4.5 under the assumption that  $(I - \omega T)^{-1} \in \mathcal{L}(X)$  and  $(T^* - \omega I)$  is injective for some  $\omega \in \mathbb{D}$ .

In case  $T^*$  is not injective, and neither is  $(T^* - \omega I)$  for some  $\omega \in \mathbb{D}$  so that  $(I - \omega T)$  is invertible in  $\mathcal{L}(X)$ , it may still be possible to reach the conclusion of Theorem 4.5 under weaker assumptions on the operator  $T$ . We start with a preliminary result.

**Lemma 4.7.** *The operator  $M_{\Sigma_{12}} : \text{Hol}_{\tilde{\Delta}_*}(\mathbb{D}) \rightarrow \text{Hol}_{\mathcal{Y}}(\mathbb{D})$  is not injective if and only if there is a sequence  $\{g_n\}_{n \geq 1}$  of non-zero vectors in  $\mathcal{X}_0$  such that*

$$(4.19) \quad g_1 \in \ker(T^*P^{\frac{1}{2}}), \quad P^{\frac{1}{2}}g_n = T^*P^{\frac{1}{2}}g_{n+1} \quad (n \geq 1), \quad \limsup_{n \rightarrow \infty} \|B_2^*g_{n+1}\|^{\frac{1}{n}} \leq 1.$$

*Proof.* Let  $h(z) = \sum_{k=0}^{\infty} z^k h_k \in \text{Hol}_{\tilde{\Delta}_*}(\mathbb{D})$ , i.e.,  $\limsup_{k \rightarrow \infty} \|h_k\|^{\frac{1}{k}} \leq 1$  and for  $n \geq 0$ ,

$$(4.20) \quad h_n = i_* \begin{bmatrix} x_n \\ y_n \end{bmatrix} \quad \text{with} \quad \begin{bmatrix} x_n \\ y_n \end{bmatrix} \in \begin{bmatrix} \mathcal{X}_0 \\ \mathcal{Y} \end{bmatrix}, \quad \text{subject to} \quad T^* P^{\frac{1}{2}} x_n + E^* y_n = 0.$$

Note that  $x_n$  and  $y_n$  are retrieved from  $h_n$  by the identity

$$(4.21) \quad \begin{bmatrix} x_n \\ y_n \end{bmatrix} = \mathbf{i}_*^* h_n = \begin{bmatrix} B_2 \\ D_{12} \end{bmatrix} h_n.$$

Define  $g_n \in \mathcal{X}_0$  by

$$(4.22) \quad g_n = \sum_{k=0}^{n-1} A^{n-k-1} x_k \in \mathcal{X}_0 \quad \text{for} \quad n \geq 1,$$

or equivalently via the recursion

$$(4.23) \quad g_1 = x_0, \quad g_{n+1} = x_n + A g_n \quad \text{for} \quad n \geq 1.$$

Since  $\mathbf{U}$  in (4.5) and (4.6) is unitary, it follows that

$$(4.24) \quad B_2^* A = -D_{12}^* C_1.$$

Moreover, since  $\mathbf{U}$  is connected with  $V$  as in (4.5) and  $V$  is given by (4.3), we see that

$$\begin{aligned} \left\langle \begin{bmatrix} A \\ C_1 \\ C_2 \end{bmatrix} x_0, \begin{bmatrix} P^{\frac{1}{2}} T \\ E \\ 0 \end{bmatrix} x_0 \right\rangle &= \left\langle \mathbf{U} \begin{bmatrix} x_0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{U} \begin{bmatrix} P^{\frac{1}{2}} x_0 \\ N x_0 \\ 0 \end{bmatrix} \right\rangle \\ &= \left\langle \begin{bmatrix} x_0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} P^{\frac{1}{2}} x_0 \\ N x_0 \\ 0 \end{bmatrix} \right\rangle = \langle P^{\frac{1}{2}} x_0, x_0 \rangle \end{aligned}$$

from which we conclude that

$$(4.25) \quad T^* P^{\frac{1}{2}} A + E^* C_1 = P^{\frac{1}{2}}.$$

Hence for  $n \geq 1$ ,

$$\begin{aligned} T^* P^{\frac{1}{2}} g_{n+1} &= \sum_{k=0}^n T^* P^{\frac{1}{2}} A^{n-k} x_k = T^* P^{\frac{1}{2}} x_n + \sum_{k=0}^{n-1} T^* P^{\frac{1}{2}} A^{n-k} x_k \\ &= T^* P^{\frac{1}{2}} x_n + \sum_{k=0}^{n-1} (P^{\frac{1}{2}} - E^* C_1) A^{n-k-1} x_k \\ &= T^* P^{\frac{1}{2}} x_n - E^* C_1 g_n + P^{\frac{1}{2}} g_n \\ (4.26) \quad &= -E^* (y_n + C_1 g_n) + P^{\frac{1}{2}} g_n \end{aligned}$$

where we used the relation between  $x_n$  and  $y_n$  in (4.20) for the last step. Moreover, using the identity in (4.24), we get

$$\begin{aligned} B_2^* g_{n+1} &= B_2^* (x_n + A g_n) = \begin{bmatrix} B_2^* & D_{12}^* \end{bmatrix} \begin{bmatrix} x_n \\ -C_1 g_n \end{bmatrix} \\ &= \begin{bmatrix} B_2^* & D_{12}^* \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} - D_{12}^* (y_n + C_1 g_n) \\ &= \mathbf{i}_* \begin{bmatrix} x_n \\ y_n \end{bmatrix} - D_{12}^* (y_n + C_1 g_n) \\ (4.27) \quad &= h_n - D_{12}^* (y_n + C_1 g_n). \end{aligned}$$

Now assume that  $h \neq 0$  and  $\Sigma_{12}(z)h(z) \equiv 0$ , i.e.,

$$(4.28) \quad (D_{12} + zC_1(I - zA)^{-1}B_2)h(z) \equiv 0.$$

Using the power series representations

$$h(z) = \sum_{n=0}^{\infty} h_n z^n \quad \text{and} \quad \Sigma_{12}(z) = D_{12} + \sum_{k=1}^{\infty} z^k C_1 A^{k-1} B_2$$

for  $H$  and  $\Sigma_{12}$  and recalling (4.21), it follows that (4.28) is equivalent to the following system of equations:

$$(4.29) \quad y_0 = D_{12}h_0 = 0, \quad y_n + C_1g_n = D_{12}h_n + \sum_{k=0}^{n-1} C_1A^{n-k-1}B_2h_k = 0 \text{ for } n \geq 1.$$

Without loss of generality we may, and will, assume that  $h_0 \neq 0$ ; otherwise replace  $h$  by  $\tilde{h}(z) = z^{-\ell}h(z)$  for  $\ell \in \mathbb{Z}_+$  sufficiently large. Then  $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \mathbf{i}_* h_0 \neq 0$  since  $h_0 \neq 0$  by assumption. But  $y_0 = 0$  by (4.29) and hence  $x_0 \neq 0$ . From the constraint in (4.20) we see that  $0 \neq x_0 \in \text{Ker } T^*P^{\frac{1}{2}}$ . Moreover, the second identity in (4.29) combined with (4.26) and (4.27) yields

$$(4.30) \quad T^*P^{\frac{1}{2}}g_{n+1} = P^{\frac{1}{2}}g_n, \quad B_2^*g_{n+1} = h_n.$$

The second of identities (4.30) then gives us

$$(4.31) \quad \limsup_{n \rightarrow \infty} \|B_2^*g_{n+1}\|^{1/n} = \limsup_{n \rightarrow \infty} \|h_n\|^{1/n} \leq 1.$$

Finally, observe that, since  $g_1 = x_0 \neq 0$  and  $\text{Ker } P^{\frac{1}{2}}|_{\mathcal{X}_0} = \{0\}$ , the recursive relation  $T^*P^{\frac{1}{2}}g_{n+1} = P^{\frac{1}{2}}g_n$  (the first of identities (4.30)) implies that  $g_n \neq 0$  for all  $n \geq 1$ . We conclude that the sequence  $\{g_n\}_{n \geq 1}$  has all the desired properties.

Conversely, assume  $\{g_n\}_{n \geq 1}$  is a sequence in  $\mathcal{X}_0$  satisfying (4.19). Define

$$x_0 = g_1, \quad x_n = g_{n+1} - Ag_n, \quad y_0 = 0, \quad y_n = -C_1g_n \quad \text{for } n \geq 1.$$

Applying (4.26) to  $g_n$  for  $n \geq 1$  and using (4.25), we find that

$$\begin{aligned} T^*P^{\frac{1}{2}}g_{n+1} &= P^{\frac{1}{2}}g_n = T^*P^{\frac{1}{2}}Ag_n + E^*C_1g_n \\ &= T^*P^{\frac{1}{2}}g_{n+1} - T^*P^{\frac{1}{2}}x_n + E^*C_1g_n. \end{aligned}$$

We conclude that  $T^*P^{\frac{1}{2}}x_n + E^*y_n = T^*P^{\frac{1}{2}}x_n - E^*C_1g_n = 0$ . For  $n = 0$  the identity  $T^*P^{\frac{1}{2}}x_n + E^*y_n = 0$  follows from the first of conditions (4.19). Hence we obtain that  $\begin{bmatrix} x_n \\ y_n \end{bmatrix} \in \Delta_*$ .

Now define  $h_n = i_* \begin{bmatrix} x_n \\ y_n \end{bmatrix} \in \tilde{\Delta}_*$ , and  $h(z) = \sum_{k=0}^{\infty} z^k h_k$ . As before,  $x_n$  and  $y_n$

are retrieved from  $h_n$  by (4.21), and from the definition of  $x_n$  it follows that the sequence  $\{g_n\}_{n \geq 1}$  is retrieved by (4.23). Moreover, the definition of  $y_n$  shows that (4.29) holds and, in combination with the computation (4.27), that  $B_2^*g_{n+1} = h_n$ .

The latter implies that  $\limsup_{k \rightarrow \infty} \|h_k\|^{\frac{1}{k}} \leq 1$ , via (4.19) and the identities in (4.31).

In particular,  $h \in \text{Hol}_{\tilde{\Delta}_*}(\mathbb{D})$ . The fact that (4.29) holds now is equivalent to  $\Sigma_{12}(z)h(z) \equiv 0$ . Note that  $x_0 = g_1 \neq 0$  and hence  $h_0 = \mathbf{i}_* \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \neq 0$ . Thus  $h \neq 0$  and it follows that  $M_{\Sigma_{12}} : \text{Hol}_{\tilde{\Delta}_*}(\mathbb{D}) \rightarrow \text{Hol}_{\mathcal{Y}}(\mathbb{D})$  is not injective.  $\square$

Based on the previous result, we obtain the following relaxation of the the condition on  $T$  in Theorem 4.5.

**Theorem 4.8.** *Let  $\mathcal{D}' = \{P, T, E, N\}$  be an  $\mathbf{AIP}_{\mathcal{S}(U, \mathcal{Y})}$ -admissible data set, let  $\Sigma$  be constructed as in (4.7) and let us assume that  $T$  meets the condition*

$$(4.32) \quad \left( \bigcap_{k \geq 1} \text{Ran}(T^*)^k \right) \cap \text{Ker} T^* = \{0\}.$$

*Then the operator  $M_{\Sigma_{12}} : \text{Hol}_{\tilde{\Delta}_*}(\mathbb{D}) \rightarrow \text{Holo}_{\mathcal{Y}}(\mathbb{D})$  is injective.*

*Proof.* Assume  $M_{\Sigma_{12}} : \text{Hol}_{\tilde{\Delta}_*}(\mathbb{D}) \rightarrow \text{Holo}_{\mathcal{Y}}(\mathbb{D})$  is not injective. By Lemma 4.7, there exists a nonzero sequence  $\{g_n\}_{n \geq 1}$  in  $\mathcal{X}_0$  satisfying (4.19). By the first relation in (4.19),  $P^{\frac{1}{2}}g_1 \in \text{ker } T^*$ . On the other hand, iterating the second condition in (4.19) gives  $P^{\frac{1}{2}}g_1 = (T^*)^n P^{\frac{1}{2}}g_{n+1}$  for each  $n \geq 1$ . Since  $P^{\frac{1}{2}}g_1 \neq 0$ , it follows that  $\text{Ran}(T^*)^n \cap \text{ker } T^* \neq \{0\}$  for each  $n \geq 1$ . The latter is in contradiction with (4.32). Thus  $M_{\Sigma_{12}} : \text{Hol}_{\tilde{\Delta}_*}(\mathbb{D}) \rightarrow \text{Holo}_{\mathcal{Y}}(\mathbb{D})$  is injective.  $\square$

It is easy to see that not only the injectivity of  $M_{\Sigma_{12}}$ , but all the conclusions of Theorem 4.5 hold with the condition that  $T^*$  is injective replaced by the weaker condition (4.32). Although condition (4.32) is far from being necessary, it guarantees injectivity of  $M_{\Sigma_{12}}$  for important particular cases:

- (1)  $T^*$  is injective (so  $\text{Ker } T^* = \{0\}$ ),
- (2)  $T^*$  is nilpotent (so  $\bigcap_{k \geq 1} \text{Ran}(T^*)^k = \{0\}$ ), and
- (3)  $\dim \mathcal{X} < \infty$ , or, more generally e.g.,  $T = \lambda I + K$  with  $0 \neq \lambda \in \mathbb{C}$  and  $K$  compact (so  $\mathcal{X} = \text{Ran}(T^*)^p \dot{+} \text{Ker}(T^*)^p$  once  $p$  is sufficiently large).

The question of finding a condition that is both necessary and sufficient for injectivity of  $M_{\Sigma_{12}}$  remains open.

## 5. DESCRIPTION OF ALL SOLUTIONS OF THE PROBLEM $\mathbf{AIP}_{\mathcal{H}(K_S)}$

We now present the parametrization of the solution set to the problem  $\mathbf{AIP}_{\mathcal{H}(K_S)}$ . The proof relies on Theorem 3.3, Theorem 4.1 and Proposition 4.2. By Theorem 3.3, the solution set to the problem  $\mathbf{AIP}_{\mathcal{H}(K_S)}$  coincides with the set of all functions  $f : \mathbb{D} \rightarrow \mathcal{Y}$  such that the kernel  $\mathbf{K}(z, \zeta)$  defined in (3.3) is positive on  $\mathbb{D} \times \mathbb{D}$ . In particular, the function  $S$  must be such that the kernel (4.2) is positive meaning that  $S$  must be a solution to the associated problem  $\mathbf{AIP}_{\mathcal{S}(U, \mathcal{Y})}$ . By Theorem 4.1, there exists a Schur-class function  $\mathcal{E}$  such that  $S = \mathcal{R}_{\Sigma}[\mathcal{E}]$  where  $\mathcal{R}_{\Sigma}$  is the Redheffer transform constructed in (4.3)–(4.10). Define  $G$  and  $\Gamma$  as in (4.11) and let  $\tilde{\mathbf{x}}$  be the unique vector in  $\mathcal{X}_0$  so that  $\mathbf{x} = P^{\frac{1}{2}}\tilde{\mathbf{x}}$ . Making use of equalities (4.13) and (4.14) we can write  $\mathbf{K}(z, \zeta)$  in the form

$$\mathbf{K}(z, \zeta) = \begin{bmatrix} 1 & \tilde{\mathbf{x}}^* P^{\frac{1}{2}} & f(\zeta)^* \\ P^{\frac{1}{2}} \tilde{\mathbf{x}} & P & P^{\frac{1}{2}} \Gamma(\zeta)^* \\ f(z) & \Gamma(z) P^{\frac{1}{2}} & G(z) K_{\mathcal{E}}(z, \zeta) G(\zeta)^* + \Gamma(z) \Gamma(\zeta)^* \end{bmatrix}.$$

The positivity of the latter kernel is equivalent to positivity of the Schur complement of  $P$  with respect to  $\mathbf{K}(z, \zeta)$ , that is, to the condition

$$(5.1) \quad \begin{bmatrix} 1 - \|\tilde{\mathbf{x}}\|^2 & f(\zeta)^* - \tilde{\mathbf{x}}^* \Gamma(\zeta)^* \\ f(z) - \Gamma(z) \tilde{\mathbf{x}} & G(z) K_{\mathcal{E}}(z, \zeta) G(\zeta)^* \end{bmatrix} \succeq 0 \quad (z, \zeta \in \mathbb{D}).$$

We arrive at the following result.



**Theorem 5.1.** *Let  $\{S, T, E, N, \mathbf{x}\}$  be an  $\mathbf{AIP}_{\mathcal{H}(K_S)}$ -admissible data set and let us assume that  $P := M_{F^S}^{[*]} M_{F^S} \geq \mathbf{x}\mathbf{x}^*$  with  $F^S$  as in (1.5). Let  $\Sigma$  be constructed as in (4.3)–(4.8), let  $\mathcal{E}$  be a Schur-class function such that  $S = \mathcal{R}_\Sigma[\mathcal{E}]$ , let  $G$  and  $\Gamma$  be defined as in (4.11) and let  $\tilde{\mathbf{x}}$  be the unique vector in  $\mathcal{X}_0$  so that  $\mathbf{x} = P^{\frac{1}{2}}\tilde{\mathbf{x}}$ . Then:*

- (1) *The set of solutions  $f$  of the problem  $\mathbf{AIP}_{\mathcal{H}(K_S)}$  is given by the formula*

$$(5.2) \quad f(z) = \Gamma(z)\tilde{\mathbf{x}} + G(z)h(z)$$

*with parameter  $h$  in  $\mathcal{H}(K_\mathcal{E})$  subject to  $\|h\|_{\mathcal{H}(K_S)} \leq \sqrt{1 - \|\tilde{\mathbf{x}}\|^2}$ .*

- (2) *For  $f$  defined by (5.2)*

$$(5.3) \quad \|f\|_{\mathcal{H}(K_S)}^2 = \|M_\Gamma \tilde{\mathbf{x}}\|^2 + \|M_G h\|^2 = \|\tilde{\mathbf{x}}\|^2 + \|P_{\mathcal{H}(K_\mathcal{E}) \ominus \ker M_G} h\|^2$$

*and hence  $f_{\min}(z) = \Gamma(z)\tilde{\mathbf{x}}$  is the unique minimal-norm solution.*

- (3) *The problem  $\mathbf{AIP}_{\mathcal{H}(K_S)}$  admits a unique solution if and only if  $\|\tilde{\mathbf{x}}\| = 1$  or  $\overline{\text{Ran}} M_{F^S} = \mathcal{H}(K_S)$ .*

*Proof.* As we have seen, a function  $f : \mathbb{D} \rightarrow \mathcal{Y}$  solves the problem  $\mathbf{AIP}_{\mathcal{H}(K_S)}$  if and only if (5.1) holds, that is, if and only if the function  $g := f - M_\Gamma \tilde{\mathbf{x}}$  belongs to the reproducing kernel Hilbert space  $\mathcal{H}(\tilde{K})$  with reproducing kernel  $\tilde{K}(z, \zeta) = G(z)K_\mathcal{E}(z, \zeta)G(\zeta)^*$  and satisfies  $\|g\|_{\mathcal{H}(\tilde{K})} \leq \sqrt{1 - \|\tilde{\mathbf{x}}\|^2}$ . The range characterization of  $\mathcal{H}(\tilde{K})$  tells us that

$$\mathcal{H}(\tilde{K}) = \{G(z)h(z) : h \in \mathcal{H}(K_\mathcal{E})\} \text{ with norm } \|M_G h\|_{\mathcal{H}(\tilde{K})} = \|(I - \mathbf{q})h\|_{\mathcal{H}(K_\mathcal{E})}$$

where  $\mathbf{q}$  is the orthogonal projection onto the subspace  $\ker M_G \subset \mathcal{H}(K_\mathcal{E})$ . Therefore, the function  $g = f - M_\Gamma \tilde{\mathbf{x}}$  is of the form  $g = M_G h$  for some  $h \in \mathcal{H}(K_\mathcal{E})$  such that  $\|h\|_{\mathcal{H}(K_\mathcal{E})} = \|g\|_{\mathcal{H}(\tilde{K})} \leq \sqrt{1 - \|\tilde{\mathbf{x}}\|^2}$ . This proves the characterization of solutions through (5.2).

Since  $P = M_{F^S}^{[*]} M_{F^S}$ , it follows from Proposition 4.2 that the operator (4.12) is a coisometry and  $M_\Gamma$  is an isometry. From this combination the orthogonality between the minimal-norm solution  $f_{\min}(z) = \Gamma(z)\tilde{\mathbf{x}}$  and the remainder on the right hand side of (5.2), as well as the second identity in (5.3), is evident.

Since  $M_G$  is a partial isometry, it follows from (5.2) that the problem  $\mathbf{AIP}_{\mathcal{H}(K_S)}$  admits a unique solution if and only if either  $\|\tilde{\mathbf{x}}\| = 1$  (because then  $h = 0 \in \mathcal{H}(K_\mathcal{E})$  is the only admissible parameter),  $\mathcal{H}(K_\mathcal{E}) = \{0\}$  (i.e., if  $\mathcal{E}$  is an unimodular constant) or  $M_G = 0$ . Since the operator (4.12) is a coisometry and because  $M_\Gamma$  is an isometry, the last two cases are covered by the condition that  $M_\Gamma$  is unitary. Due to the relation between  $F^S$  and  $\Gamma$  (see (4.13)), this is equivalent to  $M_{F^S}$  having dense range.  $\square$

Although the correspondence  $\mathcal{E} \rightarrow S = \mathcal{R}_\Sigma[\mathcal{E}]$  established by formula (4.10) is not one-to-one in general, it follows from the proof of Theorem 5.1 that in order to find all solutions  $f$  of the problem  $\mathbf{AIP}_{\mathcal{H}(K_S)}$  it suffices to take into account just one parameter  $\mathcal{E}$  so that  $S = \mathcal{R}_\Sigma[\mathcal{E}]$ , rather than all. The further analysis in Section 4, i.e., Theorem 4.5 and Lemma 4.8, provide conditions under which the Schur class function  $\mathcal{E}$  in Theorem 5.1 is unique.

**Theorem 5.2.** *Let (1.3) be an  $\mathbf{AIP}_{\mathcal{H}(K_S)}$ -admissible data set satisfying condition (3.2) and assume that the operator  $T^*$  satisfies condition (4.32). Then:*

- (1) *There is a unique Schur-class function  $\mathcal{E}$  such that  $S = \mathcal{R}_\Sigma[\mathcal{E}]$ , where  $\mathcal{R}_\Sigma$  is the Redheffer transform constructed from the data set (1.3) via (4.3)–(4.8).*

- (2) The parametrization  $h \mapsto f$  in Theorem 5.1, via formula (5.2), of the solutions  $f$  to the problem  $\mathbf{AIP}_{\mathcal{H}(K_S)}$  is injective. That is, the operator  $M_G : \mathcal{H}(K_{\mathcal{E}}) \rightarrow \mathcal{H}(K_S)$  is isometric so that in addition

$$(5.4) \quad \|f\|_{\mathcal{H}(K_S)}^2 = \|\tilde{\mathbf{x}}\|_{\mathcal{X}_0}^2 + \|h\|_{\mathcal{H}(K_S)}^2.$$

**Remark 5.3.** Given an  $\mathbf{AIP}_{\mathcal{H}(K_S)}$ -admissible data set  $(S, E, N, T, \mathbf{x})$ , it is straightforward that  $(E, N, T, P)$  with  $P = F^{S[*]}F^S$  is an  $\mathbf{AIP}_{\mathcal{S}(\mathcal{U}, \mathcal{Y})}$ -admissible data set and that  $S$  is a solution for the associated problem  $\mathbf{AIP}_{\mathcal{S}(\mathcal{U}, \mathcal{Y})}$ . Now consider another solution  $\tilde{S} \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$  of the problem  $\mathbf{AIP}_{\mathcal{S}(\mathcal{U}, \mathcal{Y})}$ . Unlike for  $S$ , this solution  $\tilde{S}$  satisfies  $M_{F^{\tilde{S}}}^{[*]}M_{F^{\tilde{S}}} \leq P$  and equality may not hold. We may then still ask the question for which functions  $f : \mathbb{D} \rightarrow \mathcal{Y}$  the kernel in (3.3) is positive, with  $S$  is replaced by  $\tilde{S}$ . This question turns out to be equivalent to that of determining the  $f \in \mathcal{H}(K_{\tilde{S}})$  with  $\|f\|_{\mathcal{H}(K_{\tilde{S}})} \leq 1$  and such that the vector  $M_{F^{\tilde{S}}}^{[*]}f$  is close to  $\mathbf{x}$  in the sense that

$$(5.5) \quad M_{F^{\tilde{S}}}^{[*]}f = \mathbf{x} + \sqrt{1 - \|f\|_{\mathcal{H}(K_{\tilde{S}})}^2} \left( P - M_{F^{\tilde{S}}}^{[*]}M_{F^{\tilde{S}}} \right)^{\frac{1}{2}} \hat{\mathbf{x}}$$

for some  $\hat{\mathbf{x}} \in \mathcal{X}$  with  $\|\hat{\mathbf{x}}\| \leq 1$ . The solutions to this problem can still be parameterized by formula (5.2), with now  $\mathcal{E} \in \mathcal{S}(\tilde{\Delta}, \tilde{\Delta}_*)$  so that  $\tilde{S} = \mathcal{R}_{\Sigma}[\mathcal{E}]$  in the definition of  $\Gamma$  and  $G$ , with the twist that in this case, because we may not have  $M_{F^{\tilde{S}}}^{[*]}M_{F^{\tilde{S}}} = P$ , there is no guarantee that we have orthogonality as in (5.3), nor is it clear if the ‘central’ solution  $f = M_{\Gamma}\tilde{\mathbf{x}}$  is the solution with minimal norm.

To conclude this section we will briefly discuss the interplay between the uniqueness of  $S$  as a solution of the problem  $\mathbf{AIP}_{\mathcal{S}(\mathcal{U}, \mathcal{Y})}$  (with  $P = M_{F^S}^{[*]}M_{F^S}$  of the form (3.2)) and the determinacy of the related (unconstrained) problem  $\mathbf{AIP}_{\mathcal{H}(K_S)}$ . We will assume that the operator  $T$  meets the condition (4.32), leaving the general case open. Under this assumption, there are only three uniqueness and semi-uniqueness cases. Recall that  $\Delta$  and  $\Delta_*$  are the defect spaces of the isometry (4.3).

**Case 1:** Let  $\Delta_* = \{0\}$ . Then  $S = \Sigma_{11}$  is the unique solution of the problem  $\mathbf{AIP}_{\mathcal{S}(\mathcal{U}, \mathcal{Y})}$ . Furthermore, we conclude from (4.11) that

$$\Gamma(z) = C_1(I - zA)^{-1}, \quad G(z) \equiv 0 \quad \text{and} \quad \mathcal{H}(K_{\mathcal{E}}) = \{0\}.$$

By Theorem 5.1, the unconstrained problem  $\mathbf{AIP}_{\mathcal{H}(K_S)}$  has a unique solution  $f(z) = C_1(I - zA)^{-1}\tilde{\mathbf{x}}$  where  $\tilde{\mathbf{x}}$  is the unique vector in  $\mathcal{X}_0$  such that  $P^{\frac{1}{2}}\tilde{\mathbf{x}} = \mathbf{x}$ .

**Case 2:** Let  $\Delta = \{0\}$ . In this case still  $S = \Sigma_{11}$  is the unique solution of the problem  $\mathbf{AIP}_{\mathcal{S}(\mathcal{U}, \mathcal{Y})}$ . Also we have  $\Gamma(z) = C_1(I - zA)^{-1}$ . However, we now have  $G = \Sigma_{12}$  and  $\mathcal{H}(K_{\mathcal{E}}) = H_{\Delta_*}^2$ . By Theorem 5.1, all solutions  $f$  to the unconstrained problem  $\mathbf{AIP}_{\mathcal{H}(K_S)}$  are given by

$$(5.6) \quad f(z) = C_1(I - zA)^{-1}\tilde{\mathbf{x}} + \Sigma_{12}(z)h(z),$$

where  $\tilde{\mathbf{x}}$  is as above and where  $h$  varies in  $H_{\Delta_*}^2$ . One can see that the same description holds if the spaces (4.4) are nontrivial and  $S = \Sigma_{11}$  is the central (but not unique) solution to the associated problem  $\mathbf{AIP}_{\mathcal{S}(\mathcal{U}, \mathcal{Y})}$ .

**Case 3:** Let  $\Delta$  and  $\Delta_*$  be nontrivial and let us assume that  $S$  is an extremal solution to the problem  $\mathbf{AIP}_{\mathcal{S}(\mathcal{U}, \mathcal{Y})}$  (in the sense that the unique  $\mathcal{E}$  such that  $S =$

$\mathcal{R}_\Sigma[\mathcal{E}]$  is a coisometric constant). Then the unconstrained problem  $\mathbf{AIP}_{\mathcal{H}(K_S)}$  has a unique solution since in this case  $H(K_\mathcal{E}) = \{0\}$ .

## 6. INTERPOLATION WITH OPERATOR ARGUMENT

In this section we show that the interpolation problem with operator argument in the space  $\mathcal{H}(K_S)$  can be embedded into the general scheme of the problem  $\mathbf{AIP}_{\mathcal{H}(K_S)}$  considered above. Recall that a pair  $(E, T)$  with  $E \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$  and  $T \in \mathcal{L}(\mathcal{X})$  is called an analytic output pair if the observability operator  $\mathcal{O}_{E,T}$  maps  $\mathcal{X}$  into  $\text{Hol}_\mathbb{D}$ . The starting point for the operator-argument point-evaluation is a so-called *output-stable* pair  $(E, T)$  which is an analytic output pair with the additional property that  $\mathcal{O}_{E,T} \in \mathcal{L}(\mathcal{X}, H_\mathbb{D}^2)$ :

$$(6.1) \quad \mathcal{O}_{E,T} : x \mapsto E(I - zT)^{-1}x = \sum_{n=0}^{\infty} z^n ET^n x \in H_\mathbb{D}^2.$$

Given such an output-stable pair  $(E, T)$  and a function  $f \in H_\mathbb{D}^2$ , we define the *left-tangential operator-argument point-evaluation*  $(E^*f)^{\wedge L}(T^*)$  of  $f$  at  $(E, T)$  by

$$(6.2) \quad (E^*f)^{\wedge L}(T^*) = \sum_{n=0}^{\infty} T^{*n} E^* f_n \quad \text{if} \quad f(z) = \sum_{n=0}^{\infty} f_n z^n.$$

The computation

$$\left\langle \sum_{n=0}^{\infty} T^{*n} E^* f_n, x \right\rangle_{\mathcal{X}} = \sum_{n=0}^{\infty} \langle f_n, ET^n x \rangle_{\mathcal{Y}} = \langle f, \mathcal{O}_{E,T} x \rangle_{H_\mathbb{D}^2}$$

shows that the output-stability of the pair  $(E, T)$  is exactly what is needed for the infinite series in the definition of  $(E^*f)^{\wedge L}(T^*)$  in (6.2) to converge in the weak topology on  $\mathcal{X}$ . The same computation shows that tangential evaluation with operator argument amounts to the adjoint of  $\mathcal{O}_{E,T}$ :

$$(6.3) \quad (E^*f)^{\wedge L}(T^*) = \mathcal{O}_{E,T}^* f \quad \text{for} \quad f \in H_\mathbb{D}^2.$$

Evaluation (6.2) applies to functions from de Branges-Rovnyak spaces  $\mathcal{H}(K_S)$  as well, since  $\mathcal{H}(K_S) \subset H_\mathbb{D}^2$ , and suggests the following interpolation problem.

**OAP** $_{\mathcal{H}(K_S)}$ : *Given  $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ ,  $T \in \mathcal{L}(\mathcal{X})$ ,  $E \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$  and  $\mathbf{x} \in \mathcal{X}$  so that the pair  $(E, T)$  is output stable, find all functions  $f \in \mathcal{H}(K_S)$  such that*

$$(6.4) \quad \|f\|_{\mathcal{H}(K_S)} \leq 1 \quad \text{and} \quad (E^*f)^{\wedge L}(T^*) = \mathcal{O}_{E,T}^* f = \mathbf{x}.$$

In the scalar-valued case  $\mathcal{U} = \mathcal{Y} = \mathbb{C}$ , the latter problem has been considered recently in [4], with the additional assumption that  $P > 0$ . Similarly to the situation in [4], the operator-valued version contains left-tangential Nevanlinna-Pick and Carathéodory-Fejér interpolation problems as particular cases corresponding to special choices of  $E$  and  $T$ . We now show that on the other hand, the problem **OAP** $_{\mathcal{H}(K_S)}$  can be considered as a particular case of the problem **AIP** $_{\mathcal{H}(K_S)}$ .

**Lemma 6.1.** *Let  $(E, T)$  be an output stable pair with  $E \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$  and  $T \in \mathcal{L}(\mathcal{X})$ , let  $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$  be a Schur-class function and let  $N \in \mathcal{L}(\mathcal{X}, \mathcal{U})$  be defined by*

$$(6.5) \quad N := \sum_{j=0}^{\infty} S_j^* ET^j, \quad \text{where} \quad S(z) = \sum_{j=0}^{\infty} S_j z^j$$

or equivalently, via its adjoint

$$(6.6) \quad N^* = \mathcal{O}_{E,T}^* M_S |_{\mathcal{U}} : \mathcal{U} \rightarrow \mathcal{X}.$$

Then the data set  $\mathcal{D} = \{S, T, E, N, \mathbf{x}\}$  is  $\mathbf{AIP}_{\mathcal{H}(K_S)}$ -admissible for every  $\mathbf{x} \in \mathcal{X}$ . Furthermore,  $M_{FS}^{[*]} = \mathcal{O}_{E,T}^* |_{\mathcal{H}(K_S)}$ , so that the interpolation conditions (6.4) coincide with those in (1.8).

*Proof.* For  $N$  defined as in (6.5), the pair  $(N, T)$  is output stable (cf. [3, Proposition 3.1]) and the observability operator  $\mathcal{O}_{N,T} : x \mapsto N(I - zT)^{-1}x$  equals

$$(6.7) \quad \mathcal{O}_{N,T} = M_S^* \mathcal{O}_{E,T} : \mathcal{X} \rightarrow H_{\mathcal{U}}^2.$$

With  $N$  as above, we now define  $F^S$  by formula (1.5). For the multiplication operator (1.6) we have, on account of (6.7),

$$(6.8) \quad M_{FS} = \mathcal{O}_{E,T} - M_S \mathcal{O}_{N,T} = (I - M_S^* M_S) \mathcal{O}_{E,T}$$

which together with the range characterization of  $\mathcal{H}(K_S)$  implies that  $M_{FS}$  maps  $\mathcal{X}$  into  $\mathcal{H}(K_S)$ . Furthermore, it follows from (1.5), (1.2) and (6.2) that

$$\begin{aligned} \|F^S x\|_{\mathcal{H}(K_S)}^2 &= \langle (I - M_S M_S^*) \mathcal{O}_{E,T} x, \mathcal{O}_{E,T} x \rangle_{H_{\mathcal{Y}}^2} \\ &= \langle (\mathcal{O}_{E,T}^* \mathcal{O}_{E,T} - \mathcal{O}_{N,T}^* \mathcal{O}_{N,T}) x, x \rangle_{\mathcal{X}} \end{aligned}$$

for every  $x \in \mathcal{X}$ . The latter equality can be written in operator form as

$$(6.9) \quad P := M_{FS}^{[*]} M_{FS} = \mathcal{O}_{E,T}^* \mathcal{O}_{E,T} - \mathcal{O}_{N,T}^* \mathcal{O}_{N,T}.$$

It follows from the series representation (6.1) and the definition of inner product in  $H_{\mathcal{Y}}^2$  that

$$\mathcal{O}_{E,T}^* \mathcal{O}_{E,T} = \sum_{n=0}^{\infty} T^{*n} E^* E T^n$$

with convergence in the strong operator topology. Using the latter series expansion one can easily verify the identity

$$(6.10) \quad \mathcal{O}_{E,T}^* \mathcal{O}_{E,T} - T^* \mathcal{O}_{E,T}^* \mathcal{O}_{E,T} T = E^* E.$$

Since the pair  $(N, T)$  is also output stable, we have similarly

$$(6.11) \quad \mathcal{O}_{N,T}^* \mathcal{O}_{N,T} - T^* \mathcal{O}_{N,T}^* \mathcal{O}_{N,T} T = N^* N.$$

Subtracting (6.11) from (6.10) and taking into account (6.9) we conclude that  $P$  satisfies the Stein identity (1.7). Thus, the data set  $\mathcal{D}$  is  $\mathbf{AIP}_{\mathcal{H}(K_S)}$ -admissible. In view of (6.8) and (1.2), the equalities

$$\begin{aligned} \langle M_{FS}^{[*]} f, x \rangle_{\mathcal{X}} &= \langle f, M_{FS} x \rangle_{\mathcal{H}(K_S)} = \langle f, (I - M_S M_S^*) \mathcal{O}_{E,T} x \rangle_{\mathcal{H}(K_S)} \\ &= \langle f, \mathcal{O}_{E,T} x \rangle_{H_{\mathcal{Y}}^2} = \langle \mathcal{O}_{E,T}^* f, x \rangle_{\mathcal{X}} \end{aligned}$$

hold for all  $f \in \mathcal{H}(K_S)$  and  $x \in \mathcal{X}$ . Therefore,  $M_{FS}^{[*]} = \mathcal{O}_{E,T}^* |_{\mathcal{H}(K_S)}$ .  $\square$

As a consequence of Lemma 6.1, the solutions to the problem  $\mathbf{OAP}_{\mathcal{H}(K_S)}$  are obtained from Theorem 5.1, after specialization to the case under consideration. We do not state this specialization of Theorem 5.1 here because the formulas do not significantly simplify. Instead we now discuss the operator-argument interpolation problem for functions in  $H_{\mathcal{Y}}^2$ , that is, the problem  $\mathbf{OAP}_{\mathcal{H}(K_S)}$  with  $S \equiv 0$ . As we shall see, in that case the problems  $\mathbf{AIP}_{\mathcal{H}(K_S)}$  and  $\mathbf{OAP}_{\mathcal{H}(K_S)}$  coincide.

Consider an  $\mathbf{AIP}_{\mathcal{H}(K_S)}$ -admissible data set  $\{S, T, E, N, \mathbf{x}\}$  with  $S \equiv 0 \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ . Then  $\mathcal{H}(K_S) = H_{\mathcal{Y}}^2$  and  $F^S = \mathcal{O}_{E,T}$ . Thus condition (2) in Definition 1.1 just says that  $F^S = \mathcal{O}_{E,T}$  is in  $\mathcal{L}(\mathcal{X}, H_{\mathcal{Y}}^2)$ , and thus that  $(E, T)$  is output-stable. The third condition states that  $P = F^{S[*]}F^S = \mathcal{O}_{E,T}^* \mathcal{O}_{E,T}$  satisfies the Stein equation (1.7). This implies that necessarily  $N^* = 0 = (E^*S)^{\wedge L}(T^*)$ , and it follows that the problem  $\mathbf{AIP}_{\mathcal{H}(K_S)}$  reduces to the problem  $\mathbf{OAP}_{\mathcal{H}(K_S)}$  with data  $T, E$  and  $\mathbf{x}$ , and  $S \equiv 0$ . We now specify Theorem 5.1 to this case, with the additional assumption that  $P$  is positive definite.

**Theorem 6.2.** *Given an output stable pair  $(E, T)$  with  $E \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$  and  $T \in \mathcal{L}(\mathcal{X})$ , and  $\mathbf{x} \in \mathcal{X}$ . Assume that  $\mathbf{x}\mathbf{x}^* \leq P := \mathcal{O}_{E,T}^* \mathcal{O}_{E,T}$  and that  $P$  is positive definite. Then the set of all  $f \in \mathcal{H}_{\mathcal{Y}}^2$  satisfying*

$$\|f\|_{H_{\mathcal{Y}}^2} \leq 1 \quad \text{and} \quad (E^*f)^L(T^*) = \mathbf{x}$$

is given by the formula

$$(6.12) \quad f(z) = E(I - zT)^{-1}P^{-1}\mathbf{x} + B(z)h(z)$$

where  $h$  is a free parameter from the ball

$$\left\{ h \in H_{\mathcal{Y}_0}^2 : \|h\|_{H_{\mathcal{Y}_0}^2}^2 \leq 1 - \mathbf{x}^*P^{-1}\mathbf{x} \right\} \subset H_{\mathcal{Y}}^2$$

for an auxiliary Hilbert space  $\mathcal{Y}_0$ ; here  $B(z)$  is the inner function in the Schur class  $\mathcal{S}(\mathcal{Y}_0, \mathcal{Y})$  determined uniquely (up to a constant unitary factor on the right) by the identity

$$(6.13) \quad K_B(z, \zeta) := \frac{I_{\mathcal{Y}} - B(z)B(\zeta)^*}{1 - z\bar{\zeta}} = E(I - zT)^{-1}P^{-1}(I - \bar{\zeta}T^*)^{-1}E^*.$$

*Proof.* As remarked above, we are considering the problem  $\mathbf{AIP}_{\mathcal{H}(K_S)}$  with data set  $\{S, T, E, N, \mathbf{x}\}$  where  $S \equiv 0$  and  $N = 0$ . Then, for  $x \in \mathcal{X}$ , we have

$$\begin{aligned} \langle Px, x \rangle &= \lim_{N \rightarrow \infty} \left\langle \sum_{n=0}^{N-1} T^{*n} E^* E T^n x, x \right\rangle \\ &= \lim_{N \rightarrow \infty} \left\langle \sum_{n=0}^{N-1} T^{*n} (P - T^* P T) T^n x, x \right\rangle \\ &= \langle Px, x \rangle - \lim_{N \rightarrow \infty} \|P^{\frac{1}{2}} T^N x\|^2. \end{aligned}$$

and we conclude that  $\|P^{\frac{1}{2}} T^N x\|^2 \rightarrow 0$  as  $N \rightarrow \infty$ . The assumption that  $P > 0$  implies that  $P^{\frac{1}{2}}$  is invertible and we conclude that  $\|T^N x\|^2 \rightarrow 0$  as well, i.e., that  $T$  is strongly stable.

The fact that  $N = 0$  yields that in the construction of the unitary colligation  $\mathbf{U}$  in (4.3)–(4.6),  $\mathcal{D}_V = \mathcal{X}$ ,  $\Delta = \mathcal{U}$  and the isometry  $V$  is defined by the identity  $VP^{\frac{1}{2}} = \begin{bmatrix} P^{\frac{1}{2}} T \\ E \end{bmatrix}$ . Moreover, in the unitary colligation  $\mathbf{U}$  we have  $B_1 = 0$ ,  $D_{11} = 0$  and  $C_2 = 0$ , and  $A$  and  $C_1$  can be computed explicitly as

$$A = P^{\frac{1}{2}} T P^{-\frac{1}{2}}, \quad C_1 = E P^{-\frac{1}{2}}.$$

As  $T$  is strongly stable, we conclude that  $A$  is strongly stable as well. The unitary colligation  $\mathbf{U}$  then collapses to

$$(6.14) \quad \mathbf{U} = \begin{bmatrix} A & 0 & B_2 \\ C_1 & 0 & D_{12} \\ 0 & D_{21} & 0 \end{bmatrix}$$

and  $\Sigma(z)$  has the form

$$\Sigma(z) = \begin{bmatrix} \Sigma_{11}(z) & \Sigma_{12}(z) \\ \Sigma_{21}(z) & \Sigma_{22}(z) \end{bmatrix} = \begin{bmatrix} 0 & D_{12} + zC_1(I - zA)^{-1}B_2 \\ D_{21} & 0 \end{bmatrix}.$$

From the special form (6.14) of  $\mathbf{U}$ , it follows that  $D_{21}$  and  $\begin{bmatrix} A & B_2 \\ C_1 & D_{12} \end{bmatrix}$  are unitary.

As  $A$  is strongly stable, it is then well known that  $\Sigma_{12}$  is inner,  $\mathcal{O}_{C_1, A}$  maps  $\mathcal{X}$  isometrically into the de Branges-Rovnyak space  $\mathcal{H}(K_{\Sigma_{12}}) = H_{\mathcal{Y}}^2 \ominus \Sigma_{12}H_{\Delta_*}^2$ , and hence the operator

$$(6.15) \quad [M_{\Sigma_{12}} \quad \mathcal{O}_{C_1, A}] : \begin{bmatrix} H_{\Delta_*}^2 \\ \mathcal{X} \end{bmatrix} \rightarrow H_{\mathcal{Y}}^2$$

is unitary. Note that the Redheffer transform  $\mathcal{R}_{\Sigma}$  reduces to

$$S(z) = \mathcal{R}_{\Sigma}[\mathcal{E}](z) = \Sigma_{12}(z)\mathcal{E}(z)D_{21} \quad (z \in \mathbb{D}).$$

Since  $S \equiv 0$ , we have  $S = \mathcal{R}_{\Sigma}[\mathcal{E}]$  when  $\mathcal{E} \equiv 0$ . In fact, because  $\Sigma_{12}$  is inner and  $D_{21}$  unitary, the Redheffer transform  $\mathcal{R}_{\Sigma}$  is one-to-one, and thus  $\mathcal{E} \equiv 0$  is the only  $\mathcal{E} \in \mathcal{S}(\Delta, \Delta_*)$  with  $\mathcal{R}_{\Sigma}[\mathcal{E}] \equiv 0$ . Then  $\mathcal{H}(K_{\mathcal{E}}) = H_{\Delta_*}^2$  and the function  $G$  in (4.11) is equal to  $\Sigma_{12}$ , and thus is inner. To complete the proof, note that  $\Gamma = F^S P^{-\frac{1}{2}}$  and  $\tilde{\mathbf{x}} = P^{-\frac{1}{2}}\mathbf{x}$ , so that

$$\begin{aligned} \Gamma(z)\tilde{\mathbf{x}} &= F^S(z)P^{-\frac{1}{2}}P^{-\frac{1}{2}}\mathbf{x} \\ &= \mathcal{O}_{E, T}P^{-1}\mathbf{x} = E(I - zT)^{-1}P^{-1}x = C_1(I - zA)^{-1}x. \end{aligned}$$

Thus (5.2) coincides with (6.12) with  $B = \Sigma_{12}$ . The coisometric property of the unitary operator (6.15) expressed in reproducing kernel form gives us

$$\begin{aligned} \frac{I - \Sigma_{12}(z)\Sigma_{12}(\zeta)^*}{1 - z\bar{\zeta}} &= C_1(I - zA)^{-1}(I - \bar{\zeta}A^*)^{-1}C_1^* \\ &= E(I - zT)^{-1}P^{-1}(I - \bar{\zeta}T^*)^{-1}E^* \end{aligned}$$

and we see that  $B := \Sigma_{12}$  is determined from the data set as in (6.13) in Theorem 6.2.  $\square$

## 7. HOMOGENEOUS INTERPOLATION AND TOEPLITZ KERNELS

Let  $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$  be an inner function, i.e.,  $M_S \in \mathcal{L}(H_{\mathcal{U}}^2, H_{\mathcal{Y}}^2)$  is an isometry. Then  $M_S H_{\mathcal{U}}^2$  is a closed, invariant subspace of the shift operator  $M_z$  on  $H_{\mathcal{Y}}^2$ . By the Beurling-Lax-Halmos theorem, this is the general form of a closed shift-invariant subspace of  $H_{\mathcal{Y}}^2$ . Moreover, the de Branges-Rovnyak space  $\mathcal{K}_S := \mathcal{H}(K_S)$  is the orthogonal complement of  $M_S H_{\mathcal{U}}^2$ :

$$\mathcal{K}_S = H_{\mathcal{Y}}^2 \ominus M_S H_{\mathcal{U}}^2$$

and provides a general form for closed backward shift-invariant subspaces of  $H_{\mathcal{Y}}^2$ . Let, in addition,  $B \in \mathcal{S}(\mathcal{W}, \mathcal{Y})$  be inner, so that we have shift invariant subspaces  $M_S H_{\mathcal{U}}^2$  and  $M_B H_{\mathcal{W}}^2$  and backward shift invariant subspaces  $\mathcal{K}_S$  and  $\mathcal{K}_B$  of  $H_{\mathcal{Y}}^2$ .

Characterizations of the intersections  $M_S H_{\mathcal{U}}^2 \cap M_B H_{\mathcal{W}}^2$  and  $\mathcal{K}_S \cap \mathcal{K}_B$  in terms of  $S$  and  $B$  are well-known (see e.g., [32]). In this section we characterize the space

$$(7.1) \quad \mathcal{M}_{S,B} := \mathcal{K}_S \cap M_B H_{\mathcal{W}}^2.$$

Let us introduce the operators  $T \in \mathcal{L}(\mathcal{K}_B)$ ,  $E \in \mathcal{L}(\mathcal{K}_B, \mathcal{Y})$ , and  $N \in \mathcal{L}(\mathcal{K}_B, \mathcal{U})$  by

$$(7.2) \quad T: h(z) \mapsto \frac{h(z) - h(0)}{z}, \quad E: h \rightarrow h(0),$$

$$(7.3) \quad N: h(z) = \sum_{j \geq 0} h_j z^j \mapsto \sum_{j \geq 0} S_j^* h_j \quad \text{where} \quad S(z) = \sum_{j \geq 0} S_j z^j.$$

The operator  $T$  is strongly stable (i.e.,  $\lim_{n \rightarrow \infty} T^n h = 0$  for each  $h \in \mathcal{X} = \mathcal{K}_B$ ) and the pair  $(E, T)$  is output-stable. With  $N$  defined in accordance with (6.5), the data set  $\mathcal{D} = \{S, E, N, T, \mathbf{x} = 0\}$  is  $\mathbf{AIP}_{\mathcal{H}(\mathcal{K}_S)}$ -admissible, by Lemma 6.1. Furthermore the adjoint  $\mathcal{O}_{E,T}^* : H_{\mathcal{Y}}^2 \rightarrow \mathcal{K}_B$  of the observability operator  $\mathcal{O}_{E,T}$  amounts to the orthogonal projection  $P_{\mathcal{K}_B}$  onto  $\mathcal{K}_B$ . Indeed, if  $h(z) = \sum_{j \geq 0} h_j z^j \in \mathcal{K}_B$ , then  $ET^j h = h_j$  for  $j \geq 0$  and hence

$$(\mathcal{O}_{E,T} h)(z) = \sum_{j \geq 0} (ET^j h) z^j = \sum_{j \geq 0} h_j z^j = h(z).$$

Therefore, for an  $f \in H_{\mathcal{Y}}^2$  we have  $\mathcal{O}_{E,T}^* f = 0$  if and only if  $f \in H_{\mathcal{Y}}^2 \ominus \mathcal{K}_B = M_B H_{\mathcal{W}}^2$ . It is now easily checked that the space (7.1) is characterized as

$$(7.4) \quad \mathcal{M}_{S,B} = \{f \in \mathcal{K}_S : \mathcal{O}_{E,T}^* f = 0\},$$

i.e., as the solution set of the (unconstrained) homogeneous problem  $\mathbf{OAP}_{\mathcal{H}(\mathcal{K}_S)}$  with the data set  $\{S, E, T, \mathbf{x} = 0\}$ . The operator  $P$  defined by formulas (6.9) now amounts to the compression of the operator  $I_{H_{\mathcal{Y}}^2} - M_S M_S^*$  to the subspace  $\mathcal{K}_B$ :

$$(7.5) \quad P = I_{\mathcal{K}_B} - P_{\mathcal{K}_B} M_S M_S^* |_{\mathcal{K}_B}.$$

**Theorem 7.1.** *Given inner functions  $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$  and  $B \in \mathcal{S}(\mathcal{W}, \mathcal{Y})$ , let  $\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$  be the characteristic function of the unitary colligation  $\mathbf{U}$  associated via formulas (4.3)–(4.6) to the tuple  $\{P, T, E, N\}$  given in (7.2), (7.3), (7.5). Then the space  $\mathcal{M}_{S,B}$  given by (7.1) is given explicitly as*

$$(7.6) \quad \mathcal{M}_{S,B} = G \cdot \mathcal{H}(K_{\mathcal{E}})$$

where  $\mathcal{E}$  is the unique function in  $\mathcal{S}(\mathcal{U} \oplus \tilde{\Delta}_*, \tilde{\Delta}_*)$  such that  $S = \mathcal{R}_{\Sigma}[\mathcal{E}]$  and  $G(z) = \Sigma_{12}(z)(I - \mathcal{E}(z)\Sigma_{22}(z))^{-1}$ . Furthermore  $M_G$  is a unitary operator from  $\mathcal{H}(K_{\mathcal{E}})$  onto  $\mathcal{M}_{S,B}$ .

*Proof.* The parametrization formula (7.6) follows from (7.4) upon applying Theorem 5.1. The fact that in the present situation the Redheffer transform  $\mathcal{R}_{\Sigma}$  is one-to-one was established in [27] (see also [30, Theorem 5.8]). Thus the parameter  $\mathcal{E}$  such that  $S = \mathcal{R}_{\Sigma}[\mathcal{E}]$  is uniquely determined. It is also shown in [30, Proposition 5.9] that

$$(7.7) \quad \Sigma_{12}(z) = B(z)\widehat{\Sigma}_{12}(z)$$

where  $\widehat{\Sigma}_{12}$  is a  $*$ -outer function in  $\mathcal{S}(\tilde{\Delta}_*, \mathcal{Y})$ . From this identity and the definition of  $\mathcal{M}_{S,B}$  we see directly that  $\mathcal{M}_{S,B}$  is contained in  $M_B H_{\mathcal{W}}^2$ . Secondly, we see from the  $*$ -outer property of  $\widehat{\Sigma}_{12}$  and the factorization (7.7) of  $\Sigma_{12}$  that the operator of

multiplication by  $G(z) = \Sigma_{12}(z)(I - \mathcal{E}(z)\Sigma_{22}(z))^{-1}$  is injective. Since we know that  $M_G$  is a partial isometry, it now follows that  $M_G: \mathcal{H}(K_\varepsilon) \rightarrow \mathcal{M}_{S,B}$  is unitary.  $\square$

**Remark 7.2.** One gets the same parametrization of  $\mathcal{M}_{S,B}$  in case  $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$  is not inner.

As the following result indicates, spaces of the form  $\mathcal{M}_{S,B}$  come up in the description of kernels of Toeplitz operators. To formulate the result let us say that the triple  $(S, B, \Gamma)$  is an *admissible triple* if

- (1)  $S$  and  $B$  in  $\mathcal{S}(\mathcal{Y})$  with  $\mathcal{Y}$  finite-dimensional are *inner* (i.e.,  $S$  and  $B$  assume unitary values almost everywhere on the unit circle  $\mathbb{T}$ ), and
- (2)  $\Gamma \in (H_{\mathcal{L}(\mathcal{Y})}^\infty)^{\pm 1}$ , i.e., both  $\Gamma$  and  $\Gamma^{-1}$  are in  $H_{\mathcal{L}(\mathcal{Y})}^\infty$ .

We also need the following result from [7].

**Theorem 7.3.** (See [7, Theorem 4.1].) *Let  $\epsilon > 0$  and suppose that  $\mathcal{Y} \cong \mathbb{C}^n$  is a finite-dimensional coefficient Hilbert space. Suppose also that  $\Phi_u \in L_{\mathcal{L}(\mathcal{Y})}^\infty$  has unitary values almost everywhere on  $\mathbb{T}$ . Then there exists almost everywhere invertible functions  $L, K \in H_{\mathcal{L}(\mathcal{Y})}^\infty$  with  $L^{-1}, K^{-1}$  in  $L_{\mathcal{L}(\mathcal{Y})}^\infty$  such that*

$$\Phi_u = L^*K \text{ almost everywhere on } \mathbb{T}$$

and such that

$$\|L\|_\infty, \|K\|_\infty, \|L^{-1}\|_\infty, \|K^{-1}\|_\infty < 1 + \epsilon.$$

For  $\Phi$  a function in  $L_{\mathcal{L}(\mathcal{Y})}^\infty$ , the associated Toeplitz operator  $T_\Phi$  on  $H_{\mathcal{L}(\mathcal{Y})}^2$  is defined by

$$T_\Phi(f) = P_{H_{\mathcal{Y}}^2}(\Phi \cdot f).$$

We consider such operators only for the case where  $\Phi$  is invertible almost everywhere on the unit circle and in addition  $\det \Phi^* \Phi$  is log-integrable:

$$\int_{\mathbb{T}} \det(\Phi(\zeta)^* \Phi(\zeta)) |d\zeta| > -\infty.$$

We are now ready to state our result concerning Toeplitz kernels. Here we use the notation  $L_{\mathcal{L}(\mathcal{Y})}^\infty$  to denote the space of essentially uniformly bounded measurable  $\mathcal{L}(\mathcal{Y})$ -valued functions on the unit circle  $\mathbb{T}$ .

**Theorem 7.4.** *Let the coefficient Hilbert space  $\mathcal{Y}$  be finite-dimensional. A subspace  $\mathcal{M} \subset H_{\mathcal{Y}}^2$  has the form  $\mathcal{M} = \text{Ker } T_\Phi$  for some  $\Phi \in L_{\mathcal{L}(\mathcal{Y})}^\infty$  with  $\det \Phi^* \Phi$  log-integrable on  $\mathbb{T}$  if and only if there is an admissible triple  $(S, B, \Gamma)$  so that  $\mathcal{M}$  has the form*

$$\mathcal{M} = \Gamma B^{-1} \cdot \mathcal{M}_{S,B} := \Gamma B^{-1} \cdot (\mathcal{K}_S \cap M_B H_{\mathcal{Y}}^2).$$

*Proof.* Suppose that  $\Phi \in L_{\mathcal{L}(\mathcal{Y})}^\infty$  with  $\det \Phi^* \Phi$  log-integrable. Then there exists an outer function  $F \in H_{\mathcal{L}(\mathcal{Y})}^\infty$  solving the spectral factorization problem

$$\Phi(\zeta)\Phi(\zeta)^* = F(\zeta)^*F(\zeta) \text{ almost everywhere on } \mathbb{T}$$

(see e.g. [33]). If we set  $\Phi_u := F^{*-1}\Phi$ , then  $\Phi_u$  is unitary-valued on  $\mathbb{T}$  and we have the factorization  $\Phi = F^*\Phi_u$ . By Theorem 7.3, we may factor  $\Phi_u$  as

$$\Phi_u = L^*K$$



with  $L, K \in H_{\mathcal{L}(\mathcal{Y})}^\infty$  and  $L^{-1}, K^{-1} \in L_{\mathcal{L}(\mathcal{Y})}^\infty$ . Let  $L = L_i L_o$  and  $K = K_i K_o$  be the inner-outer factorizations of  $L$  and  $K$  (again we refer to [33] for details on matrix-valued Hardy space theory). Then  $\Phi$  has the representation

$$\Phi = F^* L_o^* L_i^* K_i K_o.$$

Suppose now that  $f \in H_{\mathcal{Y}}^2$  is in  $\text{Ker } T_\Phi$ . This condition can be equivalently written as

$$F^* L_o^* L_i^* K_i K_o f \in H_{\mathcal{Y}}^{2\perp},$$

or

$$L_o^* L_i^* K_i K_o f \in F^{*-1} H_{\mathcal{Y}}^{2\perp} \cap L_{\mathcal{Y}}^2.$$

Since  $F^{-1}$  is an outer Nevanlinna-class function, it follows that  $F^{*-1} H_{\mathcal{Y}}^{2\perp} \cap L_{\mathcal{Y}}^2 = H_{\mathcal{Y}}^{2\perp}$  and we are left with

$$L_o^* L_i^* K_i K_o f \in H_{\mathcal{Y}}^{2\perp}.$$

By a similar argument (even easier since  $L_o^{-1}$  is bounded), we deduce that, equivalently,  $L_i^* K_i K_o f \in H_{\mathcal{Y}}^{2\perp}$ , or

$$K_i K_o f \in L_i H_{\mathcal{Y}}^{2\perp}.$$

As  $K_i K_o f \in H_{\mathcal{Y}}^2$ , we actually have

$$(7.8) \quad K_i K_o f \in L_i H_{\mathcal{Y}}^{2\perp} \cap H_{\mathcal{Y}}^2 = \mathcal{K}_{L_i}.$$

Clearly  $K_i K_o f \in K_i H_{\mathcal{Y}}^2$  and hence (7.8) takes the sharper form

$$K_i K_o f \in \mathcal{K}_{L_i} \cap K_i H_{\mathcal{Y}}^2 =: \mathcal{M}_{L_i, K_i}.$$

Solving for  $f$  gives

$$f \in K_o^{-1} K_i^{-1} \mathcal{M}_{L_i, K_i} = \Gamma B^{-1} \cdot \mathcal{M}_{S, B}$$

where we set  $(S, B, \Gamma)$  equal to the admissible triple  $(L_i, K_i, K_o^{-1})$ . Conversely, all the steps are reversible: if  $f \in K_o^{-1} K_i^{-1} \mathcal{M}_{L_i, K_i}$ , then  $f \in \text{Ker } T_\Phi$ .

Conversely, suppose that  $(S, B, \Gamma)$  is any admissible triple. Define  $L_o \in (H_{\mathcal{L}(\mathcal{Y})}^\infty)^{\pm 1}$  as any outer solution of the spectral factorization problem

$$L_o L_o^* = S^* B \Gamma^* \Gamma B^* S$$

and set  $\Phi_u = L_o^* S^* B \Gamma^{-1}$ . Then one can check that  $\Phi_u$  is even unitary-valued on  $\mathbb{T}$  and that  $\text{Ker } T_{\Phi_u} = \Gamma B^{-1} \cdot \mathcal{M}_{S, B}$ .  $\square$

Theorem 7.4 combined with Theorem 7.1 leads to the following Corollary, where the free-parameter space  $\mathcal{M}_{S, B}$  in Theorem 7.4 is replaced by the arguably easier free-parameter space  $\mathcal{H}(K_{\mathcal{E}})$ .

**Corollary 7.5.** *Assume that the coefficient Hilbert space  $\mathcal{Y} \cong \mathbb{C}^n$  has finite dimension. A subspace  $\mathcal{M} \subset H_{\mathcal{Y}}^2$  is a Toeplitz kernel, i.e.,  $\mathcal{M} = \text{Ker } T_\Phi$ , for an  $L_{\mathcal{L}(\mathcal{Y})}^\infty$ -function  $\Phi$  pointwise-invertible on  $\mathbb{T}$  with  $\det \Phi^* \Phi$  log-integrable if and only if there is a function  $\Gamma \in (H_{\mathcal{L}(\mathcal{Y})}^\infty)^{\pm 1}$ , inner functions  $S$  and  $B$  in  $\mathcal{S}(\mathcal{Y})$ , a function  $\mathcal{E}$  in the Schur class  $\mathcal{S}(\mathcal{W}, \mathcal{V})$  for some auxiliary Hilbert spaces  $\mathcal{W}$  and  $\mathcal{V}$ , and a function  $G: \mathbb{D} \rightarrow \mathcal{L}(\mathcal{W}, \mathbb{C})$  such that  $M_G: g(z) \mapsto G(z)g(z)$  maps the de Branges-Rovnyak space  $\mathcal{H}(K_{\mathcal{E}})$  isometrically onto  $(H^2 \ominus S \cdot H^2) \cap B \cdot H^2$ , so that*

$$\mathcal{M} = \Gamma B^{-1} G \cdot \mathcal{H}(K_{\mathcal{E}}).$$

Here the function  $G$  can be constructed explicitly from the pair  $(S, B)$  by applying the construction in Theorem 7.1. In particular, there exist auxiliary coefficient

Hilbert spaces  $\mathcal{W}$  and  $\mathcal{V}$  of dimension at most equal to  $\dim(H^2 \ominus S \cdot H^2) + 1$  and a function  $\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \in \mathcal{S}(\mathcal{Y} \oplus \mathcal{W}, \mathcal{Y} \oplus \mathcal{V})$  so that

$$\begin{aligned} G(z) &= \Sigma_{12}(z)(I - \mathcal{E}(z)\Sigma_{22}(z))^{-1} \\ S(z) &= \Sigma_{11}(z) + \Sigma_{12}(z)(I - \mathcal{E}(z)\Sigma_{22}(z))^{-1}\mathcal{E}(z)\Sigma_{21}(z). \end{aligned}$$

*Proof.* Simply plug in the representation of a space  $\mathcal{M}_{S,B}$  in Theorem 7.1 into the parametrization of  $\text{Ker } T_\Phi$  in Theorem 7.4.  $\square$

**Remark 7.6.** A subspace  $\mathcal{M}$  of  $H^2$  is said to be *nearly invariant* for the backward shift  $M_z^*$  if  $f(z)/z \in \mathcal{M}$  whenever  $f \in \mathcal{M}$  and  $f(0) = 0$ . In [25], Hitt obtained the following characterization of almost invariant subspaces: *a subspace  $\mathcal{M} \subset H^2$  is nearly invariant if and only if there is an inner function  $u$  with  $u(0) = 0$  and a holomorphic function  $g$  on the disk  $\mathbb{D}$  so that*

$$(7.9) \quad \mathcal{M} = M_g \cdot (H^2 \ominus M_u H^2)$$

where  $g$  is such that the multiplication operator  $M_g: h(z) \mapsto g(z)h(z)$  acts isometrically from  $H^2 \ominus M_u H^2$  into  $H^2$ . Theorem 0.3 from [21] characterizes which functions  $g$  are such that  $M_g$  acts contractively from  $H^2 \ominus M_u H^2$  into  $H^2$  for a given inner function  $u$  with  $u(0) = 0$ : such a  $g$  must have the form

$$(7.10) \quad g(z) = a_1(z)(1 - u(z)b_1(z))^{-1}$$

for a function  $\sigma(z) = \begin{bmatrix} a_1(z) \\ b_1(z) \end{bmatrix}$  in the Schur class  $\mathcal{S}(\mathbb{C}, \mathbb{C}^2)$ . It is not hard to see that  $M_g: H^2 \ominus M_u H^2 \rightarrow H^2$  is isometric exactly when in addition

$$|a_1(\zeta)|^2 + |b_1(\zeta)|^2 = 1 \quad \text{for almost all } \zeta \in \mathbb{T}$$

from which it follows also that  $\|g\|_2 = 1$ .

If one starts with  $g \in H^2$  of unit norm for which  $M_g: H^2 \ominus M_u H^2 \rightarrow H^2$  is isometric, one can construct the representation (7.10) for  $g$  as follows. Let  $g$  have inner-outer factorization  $g = \omega \cdot f$  with  $\omega$  inner and  $f$  outer with  $f(0) > 0$ . Let  $F$  denote the Herglotz integral of  $|f|^2$ , i.e., for  $z \in \mathbb{D}$  we set

$$F(z) = \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} |f(\zeta)|^2 \frac{|d\zeta|}{2\pi}.$$

The fact that  $\|g\|_2^2 = \|f\|_2^2 = 1$  implies that  $F(0) = 1$ ; we also note that  $F(z)$  has positive real part for  $z$  in  $\mathbb{D}$ . If we then set  $b = \frac{F-1}{F+1}$ , then  $b$  is in the unit ball of  $H^\infty$  and satisfies  $b(0) = 0$ . The fact that  $M_g$  is isometric from  $H^2 \ominus M_u H^2$  into  $H^2$  forces  $b$  to be divisible by  $u$ , so we can factor  $b$  as  $b = ub_1$  with  $b_1$  in the unit ball of  $H^\infty$ . Let  $a$  be the unique outer function with  $|a(\zeta)|^2 = 1 - |b(\zeta)|^2$  for almost all  $\zeta \in \mathbb{T}$  and with  $a(0) > 0$ . Set  $a_1(z) = \omega(z)a(z)$ . Then  $g$  has the representation (7.10) with this choice of  $a_1$  and  $b_1$ . The characterization of isometric multipliers from  $H^2 \ominus M_u H^2$  into  $H^2$  in this form together with the application to Hitt's theorem is one of the main results of Sarason's paper [34]. A direct proof for the special case where  $u(z) = z$  appears in [35, Lemma 2 page 488] in connection with a different problem, namely, the characterization of Nehari pairs.

Following the terminology of [29], we say that pair of  $H^\infty$  functions  $(a, b)$  is a  $\gamma$ -generating pair if

- (i)  $a$  and  $b$  are functions in the unit ball of  $H^\infty$ ,
- (ii)  $a$  is outer and  $a(0) > 0$ ,
- (iii)  $b(0) = 0$ , and

(iv)  $|a|^2 + |b|^2 = 1$  almost everywhere on the unit circle  $\mathbb{T}$ .

Note that the pair of functions  $(a, b)$  appearing above in the representation (7.10) (with  $a_1 = \omega a$  and  $b = ub_1$ ) is  $\gamma$ -generating.

It is not hard to see that the kernel of any bounded Toeplitz operator  $\text{Ker } T_\phi \subset H^2$  with  $\phi \in L^\infty$  is always nearly invariant; hence any Toeplitz kernel  $\mathcal{M} = \text{Ker } T_\phi$  is in particular of the form (7.9) as described above. The result of Hayashi in [24] is the following characterization of which nearly invariant subspaces are Toeplitz kernels: *the subspace  $\mathcal{M} \subset H^2$  is the kernel of some bounded Toeplitz operator  $T_\phi$  if and only if  $\mathcal{M}$  has the form (7.9) with  $\omega(z) = 1$  for some  $\gamma$ -generating pair and inner function  $u$  with  $u(0) = 0$  subject to the additional condition that the function  $\left(\frac{a}{1-z\bar{u}}b\right)^2$  is an exposed point of the unit ball of  $H^1$ .* These results have now been extended to the matrix-valued case in [16] and [17].

The paper [19] of Dyakonov obtains the alternative characterization of Toeplitz kernels given in Theorem 7.4 for the scalar case; our proof is a simple adaptation of the proof in [19] to the matrix-valued case, with the matrix-valued factorization result from [7] (Theorem 7.3 above) replacing the special scalar-valued version of the result due to Bourgain [14]. The advantage of this characterization of Toeplitz kernels (as opposed to the earlier results of Hayashi [23] for the scalar case and of Chevrot [17] for the matrix-valued case) is the avoidance of mention of  $H^1$ -exposed points (as there is no useable characterization of such objects). Moreover Dyakonov formulates his results for subspaces of  $H^p$  rather than just  $H^2$ ; we expect that our Theorem 7.4 extends in the same way to the  $H^p$  setting, but we do not pursue this generalization here as Theorem 7.1 is at present formulated only for the  $H^2$  setting. Note that our characterization of Toeplitz kernels (Corollary 7.5 above) brings us back to the formulations of Hayashi and Sarason for characterizations of nearly invariant subspaces/Toeplitz kernels in two respects: (1) the characterization involves a multiplication operator which is unitary from some model space of functions to the space to be characterized, and (2) there is an explicit parametrization of which such multipliers have this unitary property.

## 8. BOUNDARY INTERPOLATION

In this section we consider a boundary interpolation problem in a de Branges-Rovnyak space  $\mathcal{H}(K_S)$ . For the sake of simplicity we focus on the scalar-valued case; it is a routine exercise to extend the results presented here to the matrix- or operator-valued case by using the notation and machinery from [10, 11, 12]. In what follows,  $f_j(z) = \frac{f^{(j)}(z)}{j!}$  stands for the  $j$ -th Taylor coefficient at  $z \in \mathbb{D}$  of an analytic function  $f$ . By  $f_j(t_0)$  we denote the boundary limit

$$(8.1) \quad f_j(t_0) := \lim_{z \rightarrow t_0} f_j(z)$$

as  $z$  tends to a boundary point  $t_0 \in \mathbb{T}$  nontangentially, provided the limit exists and is finite.

The next theorem collects from the existing literature several equivalent characterizations of the higher order Carathéodory-Julia condition for a Schur-class function  $s \in \mathcal{S}$

$$(8.2) \quad \liminf_{z \rightarrow t_0} \frac{\partial^{2n}}{\partial z^n \partial \bar{z}^n} \frac{1 - |s(z)|^2}{1 - |z|^2} < \infty,$$

where now  $z$  tends to  $t_0 \in \mathbb{T}$  unrestrictedly in  $\mathbb{D}$ .

**Theorem 8.1.** *Let  $s \in \mathcal{S}$ ,  $t_0 \in \mathbb{T}$  and  $n \in \mathbb{N}$ . The following are equivalent:*

- (1)  $s$  meets the Carathéodory-Julia condition (8.2).
- (2) The function  $\frac{\partial^n}{\partial \zeta^n} K_S(\cdot, \zeta)$  stays bounded in the norm of  $\mathcal{H}(K_S)$  as  $\zeta$  tends radially to  $t_0$ .
- (3) It holds that

$$(8.3) \quad \sum_k \frac{1 - |a_k|^2}{|t_0 - a_k|^{2n+2}} + \int_0^{2\pi} \frac{d\mu(\theta)}{|t_0 - e^{i\theta}|^{2n+2}} < \infty$$

where the numbers  $a_k$  come from the Blaschke product of the inner-outer factorization of  $s$ :

$$s(z) = \prod_k \frac{\bar{a}_k}{a_k} \cdot \frac{z - a_k}{1 - z\bar{a}_k} \cdot \exp \left\{ - \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta) \right\}.$$

- (4) The boundary limits  $s_j := s_j(t_0)$  exist for  $j = 0, \dots, n$  and the functions

$$(8.4) \quad K_{t_0, j}(z) := \frac{z^j}{(1 - z\bar{t}_0)^{j+1}} - s(z) \cdot \sum_{\ell=0}^j \frac{z^{j-\ell} \bar{s}_j}{(1 - z\bar{t}_0)^{j+1-\ell}} \quad (j = 0, \dots, n)$$

belong to  $\mathcal{H}(K_S)$ .

- (5) The boundary limits  $s_j := s_j(t_0)$  exist for  $j = 0, \dots, n$  and the function  $K_{t_0, n}(z)$  defined via formula (8.4) belongs to  $\mathcal{H}(K_S)$ .
- (6) The boundary limits  $s_j := s_j(t_0)$  exist for  $j = 0, \dots, 2n + 1$  and are such that  $|s_0| = 1$  and the matrix

$$(8.5) \quad \mathbb{P}_n^s(t_0) := \begin{bmatrix} s_1 & \cdots & s_{n+1} \\ \vdots & & \vdots \\ s_{n+1} & \cdots & s_{2n+1} \end{bmatrix} \Psi_n(t_0) \begin{bmatrix} \bar{s}_0 & \cdots & \bar{s}_n \\ & \ddots & \vdots \\ 0 & & \bar{s}_0 \end{bmatrix}$$

is Hermitian, where the first factor is a Hankel matrix, the third factor is an upper triangular Toeplitz matrix and where  $\Psi_n(t_0)$  is the upper triangular matrix given by

$$(8.6) \quad \Psi_n(t_0) = [\Psi_{j\ell}]_{j, \ell=0}^n, \quad \Psi_{j\ell} = (-1)^\ell \binom{\ell}{j} t_0^{\ell+j+1}, \quad 0 \leq j \leq \ell \leq n.$$

- (7) For every  $f \in \mathcal{H}(K_S)$ , the boundary limits  $f_j(t_0)$  exist for  $j = 0, \dots, n$ .

Moreover, if one of the conditions (1)–(7) is satisfied, and hence all, then:

- (a) The matrix (8.5) is positive semidefinite and equals

$$(8.7) \quad \mathbb{P}_n^s(t_0) = [\langle K_{t_0, i}, K_{t_0, j} \rangle_{\mathcal{H}(K_S)}]_{i, j=0}^n.$$

- (b) The functions (8.4) are boundary reproducing kernels in  $\mathcal{H}(K_S)$  in the sense that

$$(8.8) \quad \langle f, K_{t_0, j} \rangle_{\mathcal{H}(K_S)} = f_j(t_0) := \lim_{z \rightarrow t_0} \frac{f^{(j)}(z)}{j!} \quad \text{for } j = 0, \dots, n.$$

*Proof.* Equivalences (1) $\iff$ (4) $\iff$ (5), implication (5) $\implies$ (6) and statements (a) and (b) were proved in [10]; implication (6) $\implies$ (1) and equivalence (1) $\implies$ (7) appear in [13] and [12], respectively. Equivalence (1) $\iff$ (7) was established in [1] for  $s$  inner and extended in [22] to general Schur class functions. Equivalence (2) $\iff$ (7) was shown in [36, Section VII].  $\square$

Theorem 8.1 suggests a boundary interpolation problem for functions in  $\mathcal{H}(K_s)$  with data set

$$(8.9) \quad \mathfrak{D}_b = \{s, \mathbf{t}, \mathbf{k}, \{f_{ij}\}\},$$

consisting of two tuples  $\mathbf{t} = (t_1, \dots, t_k) \in \mathbb{T}^k$  and  $\mathbf{n} = (n_1, \dots, n_k) \in \mathbb{N}^k$ , a doubly indexed sequence  $\{f_{ij}\}$  (with  $0 \leq j \leq n_i$  and  $1 \leq i \leq k$ ) of complex numbers and of a Schur-class function  $s$  subject to the Carathéodory-Julia conditions

$$(8.10) \quad \liminf_{z \rightarrow t_i} \frac{\partial^{2n_i}}{\partial z^{n_i} \partial \bar{z}^{n_i}} \frac{1 - |s(z)|^2}{1 - |z|^2} < \infty \quad \text{for } i = 1, \dots, k,$$

or one of the equivalent conditions from Theorem 8.1.

We consider the problem  $\mathbf{BP}_{\mathcal{H}(K_s)}$ : *Given a data set (8.9) satisfying (8.10), find all  $f \in \mathcal{H}(K_s)$  such that  $\|f\|_{\mathcal{H}(K_s)} \leq 1$  and*

$$(8.11) \quad f_j(t_i) := \lim_{z \rightarrow t_i} \frac{f^{(j)}(z)}{j!} = f_{ij} \quad \text{for } j = 0, \dots, n_i \quad \text{and } i = 1, \dots, k.$$

According to Theorem 8.1, conditions (8.10) guarantee that all the boundary limits in (8.11) exist as well as the boundary limits

$$(8.12) \quad s_{ij} := s_j(t_i) \quad \text{for } j = 0, \dots, 2n_i + 1 \quad \text{and } i = 1, \dots, k.$$

We let  $N = \sum_{i=1}^k (n_i + 1)$  denote the total number of interpolation conditions in (8.11) and we let  $\mathcal{X} = \mathbb{C}^N$ . With the data set (8.9), we associate the matrices

$$(8.13) \quad T = \begin{bmatrix} T_1 & & 0 \\ & \ddots & \\ 0 & & T_k \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} E \\ N \\ \mathbf{x}^* \end{bmatrix} = \begin{bmatrix} E_1 & E_2 & \dots & E_k \\ N_1 & N_2 & \dots & N_k \\ \mathbf{x}_1^* & \mathbf{x}_2^* & \dots & \mathbf{x}_k^* \end{bmatrix},$$

where

$$(8.14) \quad T_i = \begin{bmatrix} \bar{t}_i & 1 & \dots & 0 \\ 0 & \bar{t}_i & & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & \bar{t}_i \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} E_i \\ N_i \\ \mathbf{x}_i^* \end{bmatrix} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ \bar{s}_{i,0} & \bar{s}_{i,1} & \dots & \bar{s}_{i,n_i} \\ \bar{f}_{i,0} & \bar{f}_{i,1} & \dots & \bar{f}_{i,n_i} \end{bmatrix}.$$

Now we define the function  $F^s$  by formula (1.5) with  $E$ ,  $T$  and  $N$  given by (8.13) and (8.14), and show that  $F^s$  can be expressed in terms of boundary kernels as

$$(8.15) \quad F^s(z) := (E - s(z)N)(I - zT)^{-1} = [\mathbf{K}_{t_1, n_1}(z) \quad \dots \quad \mathbf{K}_{t_k, n_k}(z)]$$

where

$$(8.16) \quad \mathbf{K}_{t_i, n_i}(z) = [K_{t_i, 0}(z) \quad K_{t_i, 1}(z) \quad \dots \quad K_{t_i, n_i}(z)] \quad \text{for } i = 1, \dots, k,$$

and where the functions  $K_{t_i, j}$  are the boundary kernels defined via formula (8.4). Indeed, it follows from definitions (8.14) that

$$(8.17) \quad \begin{bmatrix} E_i \\ N_i \end{bmatrix} (I - zT_i)^{-1} = \begin{bmatrix} 1 & & & z^{n_i} \\ \frac{1}{1 - z\bar{t}_i} & \dots & & \frac{1}{(1 - z\bar{t}_i)^{n_i+1}} \\ \frac{\bar{s}_{i,0}}{1 - z\bar{t}_i} & \dots & \sum_{\ell=0}^{n_i} \frac{\bar{s}_{i,\ell} z^{n_i-\ell}}{(1 - z\bar{t}_i)^{n_i+1-\ell}} & \end{bmatrix}.$$

Multiplying the latter equality by  $[1 \ -s(z)]$  on the left and taking into account (8.16) and explicit formulas (8.4) for  $K_{t_i,j}$  we obtain

$$(8.18) \quad (E_i - s(z)N_i)(I - zT_i)^{-1} = \mathbf{K}_{t_i,n_i}(z),$$

and equality (8.15) now follows from the block structure (8.13) of  $T$ ,  $E$  and  $N$ .

Now we will show that the problem  $\mathbf{AIP}_{\mathcal{H}(K_s)}$  with the  $\{s, T, E, N, \mathbf{x}\}$  taken in the form (8.13), (8.14) is equivalent to the problem  $\mathbf{BP}_{\mathcal{H}(K_s)}$ . We first check that the data is  $\mathbf{AIP}$ -admissible.

The first requirement is self-evident since all the eigenvalues of  $T$  fall onto the unit circle and therefore  $(I - zT)^{-1}$  is a rational functions with no poles inside  $\mathbb{D}$ . However, it is worth noting that the pair  $(E, T)$  is not output-stable and so  $\mathbf{BP}_{\mathcal{H}(K_s)}$  cannot be embedded into the scheme of the problem  $\mathbf{OAP}_{\mathcal{H}(K_s)}$  of Section 6. To verify that the requirements (2) concerning  $F^s$  are also fulfilled, we first observe from (8.15) and (8.16) that for a generic vector  $x = \text{Col}_{1 \leq i \leq k} \text{Col}_{0 \leq j \leq n_i} x_{ij}$  in  $\mathcal{X}$ ,

$$(8.19) \quad F^s(z)x = \sum_{i=1}^k \sum_{j=0}^{n_i} K_{t_i,j}(z)x_{ij}.$$

Now it follows from statement (3) in Theorem 8.1 that the operator  $M_{F^s}$  maps  $\mathcal{X}$  into  $\mathcal{H}(K_s)$ .

Furthermore, due to (8.19), the operator  $P = M_{F^s}^{[*]} M_{F^s}$  admits the following block matrix representation with respect to the standard basis of  $\mathcal{X} = \mathbb{C}^N$ :

$$(8.20) \quad P = [P_{ij}]_{i,j=1}^k \quad \text{where} \quad P_{ij} = [\langle K_{t_i,\ell}, K_{t_j,r} \rangle_{\mathcal{H}(K_s)}]_{\ell=0,\dots,n_i}^{r=0,\dots,n_j}$$

and the explicit formulas for  $P_{ij}$  in terms of boundary limits (8.12) are (see [11] for details):

$$(8.21) \quad P_{ij} = H_{ij} \cdot \Psi_{n_j}(t_j) \cdot \begin{bmatrix} \bar{s}_{j,0} & \dots & \bar{s}_{j,n_j} \\ & \ddots & \vdots \\ 0 & & \bar{s}_{j,0} \end{bmatrix}$$

where  $\Psi_{n_j}(t_j)$  is defined via formula (8.6), where  $H_{ii} = [s_{i,\ell+r+1}]_{\ell,r=1}^{n_i}$  is a Hankel matrix and where the matrices  $H_{ij}$  (for  $i \neq j$ ) are defined entry-wise by

$$(8.22) \quad \begin{aligned} [H_{ij}]_{r,m} &= \sum_{\ell=0}^r (-1)^{r-\ell} \binom{m+r-\ell}{m} \frac{s_{i,\ell}}{(t_i - t_j)^{m+r-\ell+1}} \\ &\quad - \sum_{\ell=0}^m (-1)^r \binom{m+r-\ell}{r} \frac{s_{j,\ell}}{(t_i - t_j)^{m+r-\ell+1}} \end{aligned}$$

for  $r = 0, \dots, n_i$  and  $m = 0, \dots, n_j$ . It was shown in [11] that the matrix  $P$  of the above structure satisfies the Stein identity (1.7), with  $T$ ,  $E$  and  $N$  given by (8.13), (8.14), whenever  $P$  is Hermitian. This works since in the present situation,  $P$  is positive semidefinite. Thus, the data set  $\{s, T, E, N, \mathbf{x}\}$  is  $\mathbf{AIP}$ -admissible.

By the reproducing property (8.8), representation (8.18) implies that for every  $f \in \mathcal{H}(K_s)$ ,

$$\langle M_{F^s}^{[*]} f, x \rangle_{\mathcal{X}} = \langle f, M_{F^s} x \rangle_{\mathcal{H}(K_s)} = \sum_{i=1}^k \sum_{j=0}^{n_i} f_j(t_i) \bar{x}_{ij}.$$

On the other hand, for  $\mathbf{x}$  defined in (8.15) and (8.16),

$$\langle \mathbf{x}, x \rangle_{\mathcal{X}} = \sum_{i=1}^k \sum_{j=0}^{n_i} f_{i,j} \bar{x}_{ij}.$$

It follows from the two last equalities that interpolation conditions (8.11) are equivalent to the equality

$$\langle M_{F^s}^{[*]} f, x \rangle_{\mathcal{X}} = \langle \mathbf{x}, x \rangle_{\mathcal{X}}$$

holding for every  $x \in \mathcal{X}$ , i.e., the equality  $M_{F^s}^{[*]} f = \mathbf{x}$  holds. We now conclude that the problem  $\mathbf{AIP}_{\mathcal{H}(K_s)}$  with the data set  $\{s, T, E, N, \mathbf{x}\}$  taken in the form (8.13), (8.14) is equivalent to the  $\mathbf{BP}_{\mathcal{H}(K_s)}$ . Thus, *the problem  $\mathbf{BP}_{\mathcal{H}(K_s)}$  has a solution if and only if  $P \geq \mathbf{x}\mathbf{x}^*$  where  $P$  is defined in terms of boundary limits (8.12) for  $s$  as in (8.21) and where  $\mathbf{x}$  is defined in (8.13), (8.14).* If this is the case, *the solution set for the problem  $\mathbf{BP}_{\mathcal{H}(K_s)}$  is parametrized as in Theorem 5.1.*

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