

ALL SOLUTIONS TO AN OPERATOR NEVANLINNA-PICK INTERPOLATION PROBLEM

A.E. FRAZHO, S. TER HORST, AND M.A. KAASHOEK

ABSTRACT. The main results presented in this paper provide a complete and explicit description of all solutions to the left tangential operator Nevanlinna-Pick interpolation problem assuming the associated Pick operator is strictly positive. The complexity of the solutions is similar to that found in descriptions of the sub-optimal Nehari problem and variation on the Nevanlinna-Pick interpolation problem in the Wiener class that have been obtained through the band method. The main techniques used to derive the formulas are based on the theory of co-isometric realizations, and use the Douglas factorization lemma and state space calculations. A new feature is that we do not assume an additional stability assumption on our data, which allows us to view the Leech problem and a large class of commutant lifting problems as special cases. Although the paper has partly the character of a survey article, all results are proved in detail and some background material has been added to make the paper accessible to a large audience including engineers.

1. INTRODUCTION

Nevanlinna-Pick interpolation problems have a long and interesting history which goes back to the papers of G. Pick [42] and R. Nevanlinna [41] for scalar functions. Since then interpolation problems with metric constraints involving matrix or operator-valued functions, in one or several variables, has been a topic of intense study with rich applications to system and control theory, prediction theory and geophysics. See, for example, the introductions of the books [18, 19], Chapter 7 in the book [5], the papers [36] and [37], several variable papers [2, 3], and references therein.

In the present paper we deal with the left tangential Hilbert space operator Nevanlinna-Pick interpolation problem in one variable with the unknowns being operators. Our aim is to give a self-contained presentation combining the best techniques from commutant lifting [18, 19], the band method [27, 28, 29], state space analysis [2, 3, 13], and other interpolation methods [1, 16, 36, 37, 17, 45]. In particular, the technique of extending a partial isometry used in the present paper goes back to work of Sz.-Nagy-Koranyi [45] and also appears in the so-called “lurking Isometry” method of Ball and co-authors [11] and Arov-Grossman [6], to name only a few. In [8] this problem was considered in the more general setting of the Drury-Arveson space and solved via a modification of the Potapov methodology.

2010 *Mathematics Subject Classification.* Primary 47A57; Secondary 47A48, 47A56, 47A62, 28D20.

Key words and phrases. Nevanlinna-Pick interpolation, linear fractional transformations, co-isometric systems, operator optimisation problems, entropy.

This work is based on the research supported in part by the National Research Foundation of South Africa (Grant Number 90670 and 93406).

Our proofs are not based on the commutant lifting method, and the approach taken here avoids the complications that arise in describing the solutions when the isometric lifting is not minimal, as is typically the case in the commutant lifting reformulation of the operator interpolation problem. As main tools we use the theory of co-isometric realizations, the Douglas factorization lemma and state space calculations, which are common in mathematical system theory.

As a by-product of our method we present in Subsection A.4 an alternative way to construct co-isometric realizations of Schur class functions, which seems to be new and could be of interest in the multi-variable case. In the appendix we also give an alternative proof of the Beurling-Lax-Halmos theorem and present a new approach to the maximum entropy principle. We made an effort for the paper to be readable by someone whose has an elementary knowledge of Hilbert space operator theory with state space techniques from systems and control theory. On the other hand in order to achieve self-containedness, the appendix provides background material that is used throughout the paper.

Let us now introduce the Hilbert space operator Nevanlinna-Pick interpolation problem we shall be dealing with and review some of our main new results. The data for the problem is a triplet of bounded linear Hilbert space operators $\{W, \widetilde{W}, Z\}$, where, for given Hilbert spaces \mathcal{Z} , \mathcal{Y} and \mathcal{U} , we have

$$Z : \mathcal{Z} \rightarrow \mathcal{Z}, \quad W : \ell_+^2(\mathcal{Y}) \rightarrow \mathcal{Z}, \quad \widetilde{W} : \ell_+^2(\mathcal{U}) \rightarrow \mathcal{Z},$$

with $\ell_+^2(\mathcal{Y})$ (respectively $\ell_+^2(\mathcal{U})$) the Hilbert space of square summable unilateral sequences of vectors from \mathcal{Y} (respectively \mathcal{U}), and where the following intertwining relations are satisfied

$$(1.1) \quad ZW = WS_{\mathcal{Y}} \quad \text{and} \quad Z\widetilde{W} = \widetilde{W}S_{\mathcal{U}}.$$

Here $S_{\mathcal{U}}$ and $S_{\mathcal{Y}}$ are the unilateral forward shift operators on $\ell_+^2(\mathcal{U})$ and $\ell_+^2(\mathcal{Y})$, respectively.

We say that F is a *solution to the operator Nevanlinna-Pick (LTONP for short) interpolation problem with data set $\{W, \widetilde{W}, Z\}$* if

$$(1.2) \quad F \in \mathcal{S}(\mathcal{U}, \mathcal{Y}) \quad \text{and} \quad WT_F = \widetilde{W}.$$

Here T_F is the Toeplitz operator with defining function F mapping $\ell_+^2(\mathcal{U})$ into $\ell_+^2(\mathcal{Y})$. Moreover, $\mathcal{S}(\mathcal{U}, \mathcal{Y})$ is the Schur class of operator-valued functions whose values map \mathcal{U} into \mathcal{Y} , that is, the set of all operator-valued analytic functions F in the open unit disc \mathbb{D} whose values map \mathcal{U} into \mathcal{Y} such that $\|F\|_{\infty} = \sup\{\|F(\lambda)\| : \lambda \in \mathbb{D}\} \leq 1$.

Note that this class of Nevanlinna-Pick interpolation problems has the same point evaluation interpolation condition as the one considered in Section 1.4 of [19], but is larger in the sense that, unlike in [19], we do not assume the spectral radius of Z to be strictly less than one. To see that the point evaluation condition coincides with that of [19], note that the fact that W and \widetilde{W} satisfy (1.1) implies that they are the controllability operators (cf., [19, page 20]) of the pairs $\{Z, B\}$ and $\{Z, \widetilde{B}\}$, respectively, where B and \widetilde{B} are the operators given by

$$(1.3) \quad B = WE_{\mathcal{Y}} : \mathcal{Y} \rightarrow \mathcal{Z} \quad \text{and} \quad \widetilde{B} = \widetilde{W}E_{\mathcal{U}} : \mathcal{U} \rightarrow \mathcal{Z}.$$

Here $E_{\mathcal{Y}}$ and $E_{\mathcal{U}}$ are the operators embedding \mathcal{Y} and \mathcal{U} , respectively, into the first component of $\ell_+^2(\mathcal{Y})$ and $\ell_+^2(\mathcal{U})$, respectively; see the final paragraph of this section

for more details. Then for $F \in S(\mathcal{U}, \mathcal{Y})$, the operator WT_F is also a controllability operator, namely for the pair $\{Z, (BF)(Z)_{\text{left}}\}$, where

$$(BF)(Z)_{\text{left}} = \sum_{k=0}^{\infty} Z^k BF_k,$$

with F_0, F_1, F_2, \dots being the Taylor coefficients of F at zero. Then $WT_F = \widetilde{W}$ is equivalent to the left tangential operator argument condition $(BF)(Z)_{\text{left}} = \widetilde{B}$.

Although the LTONP interpolation problem has a simple formulation, it covers two relevant special cases that will be discussed in Sections 8 and 9 below. In both cases it is essential that we do not demand that the spectral radius is strictly less than one. In Section 8 we discuss a large class of commutant lifting problems that can be written in the form of a LTONP interpolation problem. Conversely, any LTONP interpolation problem can be rewritten as a commutant lifting problem from this specific class. Hence the problems are equivalent in this sense. In this case, the operator Z will be a compression of a unilateral forward shift operator and will typically not have spectral radius less than one. The connection with commutant lifting is already observed in [19, Section II.2] and also appears in the more general setting of the Drury-Arveson space in [8].

The second special case, discussed in Section 9, is the Leech problem. This problem, and its solution, originates from a paper by R.B. Leech, which was written in 1971-1972, but published only recently [40]; see [35] for an account of the history behind this paper. The Leech problem is another nontrivial example of a LTONP interpolation problem for which the operator Z need not have spectral radius less than one, in fact, in this case, the operator Z is equal to a unilateral forward shift operator and hence its spectral radius is equal to one. Our analysis of the rational Leech problem [22, 23, 24] inspired us to study in detail the class of LTONP interpolation problems. It led to new results and improvements on our earlier results on the Leech problem.

Next we will present our main results. This requires some preparation. Let $\{W, \widetilde{W}, Z\}$ be a LTONP data set. Set $P = WW^*$ and $\widetilde{P} = \widetilde{W}\widetilde{W}^*$. The intertwining relations in (1.1) imply that

$$(1.4) \quad P - ZPZ^* = BB^*, \text{ where } B = WE_{\mathcal{Y}} : \mathcal{Y} \rightarrow \mathcal{Z},$$

$$(1.5) \quad \widetilde{P} - Z\widetilde{P}Z^* = \widetilde{B}\widetilde{B}^*, \text{ where } \widetilde{B} = \widetilde{W}E_{\mathcal{U}} : \mathcal{U} \rightarrow \mathcal{Z}.$$

Here, as before (see (1.3)), the maps $E_{\mathcal{Y}}$ and $E_{\mathcal{U}}$ are the operators embedding \mathcal{Y} and \mathcal{U} , respectively, into the first component of $\ell_+^2(\mathcal{Y})$ and $\ell_+^2(\mathcal{U})$, respectively; see the final paragraph of this section for more details. The operator $\Lambda = P - \widetilde{P}$ is called the *Pick operator* associated with the data set $\{W, \widetilde{W}, Z\}$.

If the LTONP interpolation problem is solvable, then necessarily the Pick operator is non-negative. Indeed, assume there exists a function F in $S(\mathcal{U}, \mathcal{Y})$ satisfying $WT_F = \widetilde{W}$. Then T_F is a contraction so that

$$\langle \widetilde{P}x, x \rangle = \|\widetilde{W}^*x\|^2 = \|T_F^*W^*x\|^2 \leq \|W^*x\|^2 = \langle Px, x \rangle, \quad x \in \mathcal{Z}.$$

Hence $\Lambda = P - \widetilde{P} \geq 0$.

The converse is also true. If the Pick operator is non-negative, then the LTONP interpolation problem is solvable (see Theorem 2.1 in the next section). In this

paper our aim is to describe all solutions, in particular for the case when Λ is strictly positive.

To state our first main theorem we need two auxiliary operators. Assume $P = WW^*$ is strictly positive, which is the case if Λ is strictly positive. Then there exist a Hilbert space \mathcal{E} and a pair of operators $C : \mathcal{Z} \rightarrow \mathcal{E}$ and $D : \mathcal{Y} \rightarrow \mathcal{E}$ such that

$$(1.6) \quad \begin{bmatrix} D & C \\ B & Z \end{bmatrix} \begin{bmatrix} I_{\mathcal{Y}} & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} D^* & B^* \\ C^* & Z^* \end{bmatrix} = \begin{bmatrix} I_{\mathcal{E}} & 0 \\ 0 & P \end{bmatrix},$$

$$(1.7) \quad \begin{bmatrix} D^* & B^* \\ C^* & Z^* \end{bmatrix} \begin{bmatrix} I_{\mathcal{E}} & 0 \\ 0 & P^{-1} \end{bmatrix} \begin{bmatrix} D & C \\ B & Z \end{bmatrix} = \begin{bmatrix} I_{\mathcal{Y}} & 0 \\ 0 & P^{-1} \end{bmatrix}.$$

We shall call such a pair C and D an *admissible pair of complementary operators* determined by the data set $\{W, \widetilde{W}, Z\}$. In (1.6) and (1.7) the symbols $I_{\mathcal{E}}$ and $I_{\mathcal{Y}}$ denote the identity operators on the spaces \mathcal{E} and \mathcal{Y} , respectively. In general, when it is clear from the context on which space the identity operator is acting, the subscript is omitted and we simply write I .

An application of Lemma XXVIII.7.1 in [26] shows that admissible pairs exist and that such a pair is unique up to multiplication by a unitary operator from the left. There are various ways to construct admissible pairs in a concrete way, also in a multivariable setting (see, e.g., [10]). In this introduction we mention only one way to obtain such a pair of operators, namely as follows. Since $ZW = WS_{\mathcal{Y}}$, the space $\text{Ker } W$ is an invariant subspace for the forward shift $S_{\mathcal{Y}}$. But then, by the Beurling-Lax-Halmos theorem, there exists an inner function $\Theta \in \mathcal{S}(\mathcal{E}, \mathcal{U})$, for some Hilbert space \mathcal{E} , such that $\text{Ker } W = \text{Im } T_{\Theta}$. Now put

$$(1.8) \quad C = E_{\mathcal{E}}^* T_{\Theta}^* S_{\mathcal{Y}} W^* P^{-1} : \mathcal{Z} \rightarrow \mathcal{E} \quad \text{and} \quad D = \Theta(0)^* : \mathcal{Y} \rightarrow \mathcal{E}.$$

Then C and D form an admissible pair of complementary operators. Another method to construct admissible pairs of complementary operators, which has the advantage that it can be readily used in Matlab in the finite dimensional case, is given Section A.3. We are now ready to state our first main result.

Theorem 1.1. *Let $\{W, \widetilde{W}, Z\}$ be a data set for a LTONP interpolation problem. Assume $\Lambda = WW^* - \widetilde{W}\widetilde{W}^*$ is strictly positive. Then $P = WW^*$ is strictly positive and the operator $\Lambda^{-1} - P^{-1}$ is non-negative, the operator Z^* is pointwise stable and its spectral radius is less than or equal to one. Furthermore, all solutions to the LTONP interpolation problem are given by*

$$(1.9) \quad F(\lambda) = \left(\Upsilon_{11}(\lambda)X(\lambda) + \Upsilon_{12}(\lambda) \right) \left(\Upsilon_{21}(\lambda)X(\lambda) + \Upsilon_{22}(\lambda) \right)^{-1}, \quad \lambda \in \mathbb{D}$$

where the free parameter X is an arbitrary Schur class function, $X \in \mathcal{S}(\mathcal{U}, \mathcal{E})$, and the coefficients in (1.9) are the analytic functions on \mathbb{D} given by

$$(1.10) \quad \Upsilon_{11}(\lambda) = D^* Q_{\circ} + \lambda B^* (I - \lambda Z^*)^{-1} \Lambda^{-1} P C^* Q_{\circ},$$

$$(1.11) \quad \Upsilon_{12}(\lambda) = B^* (I - \lambda Z^*)^{-1} \Lambda^{-1} \widetilde{B} R_{\circ},$$

$$(1.12) \quad \Upsilon_{21}(\lambda) = \lambda \widetilde{B}^* (I - \lambda Z^*)^{-1} \Lambda^{-1} P C^* Q_{\circ},$$

$$(1.13) \quad \Upsilon_{22}(\lambda) = R_{\circ} + \widetilde{B}^* (I - \lambda Z^*)^{-1} \Lambda^{-1} \widetilde{B} R_{\circ}.$$

Here the operators B and \widetilde{B} are given by (1.4) and (1.5), respectively, the operators $C : \mathcal{Z} \rightarrow \mathcal{E}$ and $D : \mathcal{Y} \rightarrow \mathcal{E}$ form an admissible pair of complementary operators,

and Q_\circ and R_\circ are the strictly positive operators given by

$$(1.14) \quad \begin{aligned} Q_\circ &= (I_{\mathcal{E}} + CP(\Lambda^{-1} - P^{-1})PC^*)^{-\frac{1}{2}} : \mathcal{E} \rightarrow \mathcal{E}, \\ R_\circ &= (I_{\mathcal{U}} + \tilde{B}^*\Lambda^{-1}\tilde{B})^{-\frac{1}{2}} : \mathcal{U} \rightarrow \mathcal{U}. \end{aligned}$$

The parameterization given by (1.9) is proper, that is, the map $X \mapsto F$ is one-to-one.

Note that (1.9) implicitly contains the statement that the operator $\Upsilon_{21}(\lambda)X(\lambda) + \Upsilon_{22}(\lambda)$ is invertible for each $\lambda \in \mathbb{D}$. In particular, taking $X \equiv 0$ in (1.9), we see that under the conditions of the above theorem, the operator $\Upsilon_{22}(\lambda)$ is invertible for each $\lambda \in \mathbb{D}$.

Furthermore, setting $X \equiv 0$ in (1.9), we obtain the so-called central solution $F_\circ(\lambda) = \Upsilon_{12}(\lambda)\Upsilon_{22}(\lambda)^{-1}$, which is introduced, in a different way, in Remark 2.2. See also Theorem 4.2 and Proposition 6.2. In Section 7 we show that the central solution is the unique Schur class function that maximizes a notion of entropy among all solutions; see Theorem 7.1 below.

By Theorem 1.1 the set of all solutions is parameterised by the Schur class $\mathcal{S}(\mathcal{U}, \mathcal{E})$. Hence the LTONP interpolation problem has a single solution if and only if $\mathcal{E} = \{0\}$; [we assume that \mathcal{U} is non-trivial]. On the other hand we know that \mathcal{E} can be chosen in such a way that $\text{Ker } W = \text{Im } T_\Theta$, where $\Theta \in \mathcal{S}(\mathcal{E}, \mathcal{U})$ is an inner function. Thus $\mathcal{E} = \{0\}$ holds if and only if $\text{Ker } W = \{0\}$, i.e., W is one-to-one. On the other hand, since we assume Λ to be strictly positive, WW^* is also strictly positive. Thus there exists a single solution if and only if W is invertible.

In Section 9 we specialize Theorem 1.1 for the Leech problem case, yielding Theorem 9.1 below, which gives a generalization and a further improvement of the description of all solutions of the rational Leech problem given in [24].

The explicit formulas for the functions Υ_{ij} , $1 \leq i, j \leq 2$, given in (1.10)–(1.13) are new. The formulas are of the same complexity as the corresponding formulas for the coefficients appearing in the linear fractional representation of all solutions of the sub-optimal Nehari problem presented in the classical Adamjan-Arov-Kreĭn paper [1]. See also Theorem XXXV.4.1 in [26] where the linear fractional representation of all solutions of the sub-optimal Nehari problem in the Wiener class setting is obtained as an application of the band method [27] and [28]. The variation of the band method for solving extension problems presented in [34] and the related unpublished manuscript [33] inspired us to derive the formulas in Theorem 1.1.

When the inner function Θ determined by $\text{Ker } W = \text{Im } T_\Theta$ is bi-inner, then the LTONP interpolation problem is equivalent to a Nehari extension problem. But even in this special case, it requires some work to derive the formulas (1.10) – (1.13); cf., [26, Section XXXV.5].

The next theorem is an addition to Theorem 1.1 which will allow us to derive further properties for the coefficients Υ_{ij} , $1 \leq i, j \leq 2$, in the linear fractional representation (1.9); see Proposition 1.3 below and Section 6. The theorem also shows that the functions (1.10) – (1.13) are the natural analogs of the formulas appearing [26, Theorem XXXV.4.1] for the Nehari problem.

Theorem 1.2. *Let $\{W, \tilde{W}, Z\}$ be a data set for a LTONP interpolation problem. Assume $\Lambda = WW^* - \tilde{W}\tilde{W}^*$ is strictly positive. Then $P = WW^*$ is strictly positive, the operator*

$$(1.15) \quad A = W^*P^{-1}\tilde{W} : \ell_+^2(\mathcal{U}) \rightarrow \ell_+^2(\mathcal{Y})$$

is a strict contraction, and the functions defined by (1.10) – (1.13) are also given by

$$(1.16) \quad \Upsilon_{11}(\lambda) = D^*Q_\circ + \lambda E_{\mathcal{Y}}^*(I - \lambda S_{\mathcal{Y}}^*)^{-1}(I - AA^*)^{-1}W^*C^*Q_\circ,$$

$$(1.17) \quad \Upsilon_{12}(\lambda) = E_{\mathcal{Y}}^*(I - \lambda S_{\mathcal{Y}}^*)^{-1}A(I - A^*A)^{-1}E_{\mathcal{U}}R_\circ,$$

$$(1.18) \quad \Upsilon_{21}(\lambda) = \lambda E_{\mathcal{U}}^*(I - \lambda S_{\mathcal{U}}^*)^{-1}A^*(I - AA^*)^{-1}W^*C^*Q_\circ,$$

$$(1.19) \quad \Upsilon_{22}(\lambda) = E_{\mathcal{U}}^*(I - \lambda S_{\mathcal{U}}^*)^{-1}(I - A^*A)^{-1}E_{\mathcal{U}}R_\circ.$$

Here, as in the preceding theorem, $C : \mathcal{Z} \rightarrow \mathcal{E}$ and $D : \mathcal{Y} \rightarrow \mathcal{E}$ form an admissible pair of complementary operators determined by the data. Furthermore, the strictly positive operators Q_\circ and R_\circ defined by (1.14) are also given by

$$(1.20) \quad Q_\circ = \left(I_{\mathcal{E}} + CWA(I - A^*A)^{-1}A^*W^*C^* \right)^{-\frac{1}{2}},$$

$$(1.21) \quad R_\circ = \left(E_{\mathcal{U}}^*(I - A^*A)^{-1}E_{\mathcal{U}} \right)^{-\frac{1}{2}}.$$

In the following result we list a few properties of the coefficients of the linear fractional transformation (1.9).

Proposition 1.3. *Let $\{W, \widetilde{W}, Z\}$ be a data set for a LTONP interpolation problem. Assume $\Lambda = WW^* - \widetilde{W}\widetilde{W}^*$ is strictly positive. Then the functions Υ_{ij} , $1 \leq i, j \leq 2$, given by (1.10)–(1.13) are H^2 -functions. More precisely, we have*

$$(1.22) \quad \Upsilon_{11}(\cdot)x \in H^2(\mathcal{Y}) \quad \text{and} \quad \Upsilon_{21}(\cdot)x \in H^2(\mathcal{U}), \quad x \in \mathcal{E},$$

$$(1.23) \quad \Upsilon_{12}(\cdot)u \in H^2(\mathcal{Y}) \quad \text{and} \quad \Upsilon_{22}(\cdot)u \in H^2(\mathcal{U}), \quad u \in \mathcal{U}.$$

Moreover, the functions $\Upsilon_{i,j}$ form a 2×2 J -contractive operator function, that is, for all $\lambda \in \mathbb{D}$ we have

$$(1.24) \quad \begin{bmatrix} \Upsilon_{11}(\lambda)^* & \Upsilon_{21}(\lambda)^* \\ \Upsilon_{12}(\lambda)^* & \Upsilon_{22}(\lambda)^* \end{bmatrix} \begin{bmatrix} I_{\mathcal{Y}} & 0 \\ 0 & -I_{\mathcal{U}} \end{bmatrix} \begin{bmatrix} \Upsilon_{11}(\lambda) & \Upsilon_{12}(\lambda) \\ \Upsilon_{21}(\lambda) & \Upsilon_{22}(\lambda) \end{bmatrix} \leq \begin{bmatrix} I_{\mathcal{E}} & 0 \\ 0 & -I_{\mathcal{U}} \end{bmatrix},$$

with equality for each λ in the intersection of the resolvent set of Z and the unit circle \mathbb{T} . Furthermore, $\Upsilon_{22}(\lambda)$ is invertible for each $\lambda \in \mathbb{D}$ and $\Upsilon_{22}(\lambda)^{-1}$ is a Schur class function.

Here for any Hilbert space \mathcal{V} the symbol $H^2(\mathcal{V})$ stands for the Hardy space of \mathcal{V} -valued measurable functions on the unit circle \mathbb{T} that are square integrable and whose negative Fourier coefficients are equal to zero. Equivalently, $\varphi \in H^2(\mathcal{V})$ if and only if φ is an \mathcal{V} -valued analytic function on the unit \mathbb{D} and its Taylor coefficients $\varphi_0, \varphi_1, \varphi_2, \dots$ are square summable in norm.

Assume that $\Lambda = WW^* - \widetilde{W}\widetilde{W}^*$ is strictly positive. Then $A = W^*P^{-1}\widetilde{W}$ is a strict contraction. Because $P = WW^*$ is strictly positive, we see that $\text{Im } W$ is closed and $\text{Im } A \subset \text{Im } W^*$. Furthermore, $WA = \widetilde{W}$, and hence $W(T_F - A) = 0$ for any solution F to the LTONP interpolation problem. In other words, if F is a solution to the LTONP interpolation problem, then necessarily

$$T_F = \begin{bmatrix} A \\ \star \end{bmatrix} : \ell_+^2(\mathcal{U}) \rightarrow \begin{bmatrix} \text{Im } W^* \\ \text{Ker } W \end{bmatrix}.$$

The converse is also true. This observation enables us to rephrase the LTONP interpolation problem as a commutant lifting problem. On the other hand, as we shall see in Section 8, a large class of commutant lifting problems can be viewed as

LTONP interpolation problems, and hence Theorem 1.2 can be used to describe all solutions of a large class commutant lifting problems. This will lead to a commutant lifting version of Theorem 1.2; see Theorem 8.1 below.

CONTENTS. The paper consists of nine sections, including the present introduction, and an appendix. In Section 2 we develop our primary techniques that are used to prove the main results, namely observable, co-isometric realizations from system theory, and we show how solutions can be obtained from a specific class of observable, co-isometric realizations, referred to as Λ -preferable. The main result, Theorem 2.1, presents yet another description of the solutions to the LTONP interpolation problem. This description is less explicit, but on the other hand only requires the Pick operator to be non-negative. In Section 3 we prove the main result of Section 2, Theorem 2.1. Starting with Section 4 we add the assumption that the Pick operator is strictly positive. The main results, Theorems 1.1 and 1.2 are proven in Sections 4 and 5, respectively. The next section is devoted to the proof of Proposition 1.3. Here we also show that the central solution, introduced in Remark 2.2, is indeed given by the quotient formula mentioned in the first paragraph after Theorem 1.1; see Proposition 6.2. In Section 7 we introduce a notion of entropy associated with the LTONP interpolation problem and show that the central solution is the unique solution that maximizes the entropy. This result is in correspondence with similar results on metric constrained interpolation; cf., Section IV.7 in [19]. The new feature in the present paper is that we can rephrase the entropy of a solution in terms of its Λ -preferable, observable, co-isometric realizations. In the last two sections, Sections 8 and 9, we describe the connections with the commutant lifting problem and the Leech problem, respectively. Finally, the appendix consists of seven subsections containing various preliminary results that are used throughout the paper, with proofs often added for the sake of completeness.

TERMINOLOGY AND NOTATION. We conclude this introduction with a few words on terminology and notation. With the term *operator* we will always mean a bounded linear operator. Moreover, we say that an operator is *invertible* when it is both injective and surjective, and in that case its inverse is an operator, and hence bounded. An operator T on a Hilbert space \mathcal{H} is called *strictly positive* whenever it is non-negative ($T \geq 0$) and invertible; we denote this by $T \gg 0$. The unique non-negative square root of a non-negative operator T is denoted by $T^{\frac{1}{2}}$. Furthermore, an operator T on \mathcal{H} is said to be *exponentially stable* whenever its *spectrum* $\sigma(T)$ is inside the open unit disc \mathbb{D} , in other words, when the *spectral radius* $r_{\text{spec}}(T)$ of T is strictly less than one. Moreover, we say that T is *pointwise stable* whenever $T^n h \rightarrow 0$ for each $h \in \mathcal{H}$; by some authors (see, e.g., Definition 4.5 in [9]) this kind of stability is referred to as *strongly stable*. Clearly, a exponentially stable operator is also pointwise stable. A subspace \mathcal{M} of a Hilbert space \mathcal{H} is by definition a closed linear manifold in \mathcal{H} . Given a subspace \mathcal{M} of \mathcal{H} we write $P_{\mathcal{M}}$ for the orthogonal projection on \mathcal{H} along \mathcal{M} . We will also use the embedding operator $\tau_{\mathcal{M}} : \mathcal{M} \rightarrow \mathcal{H}$, which maps $m \in \mathcal{M}$ to $m \in \mathcal{H}$. Its adjoint $\tau_{\mathcal{M}}^* : \mathcal{H} \rightarrow \mathcal{M}$ will also be denoted by $\Pi_{\mathcal{M}}$, and thus $\Pi_{\mathcal{M}}^*$ is the embedding operator $\tau_{\mathcal{M}}$. Recall that $S_{\mathcal{U}}$ denotes the unilateral forward shift operator on $\ell_+^2(\mathcal{U})$, for a given Hilbert space \mathcal{U} . We will also need the operator $E_{\mathcal{U}} : \mathcal{U} \rightarrow \ell_+^2(\mathcal{U})$ which is the embedding operator that embeds \mathcal{U} into the first entry of $\ell_+^2(\mathcal{U})$, that is, $E_{\mathcal{U}}u = [u \ 0 \ 0 \ \cdots]^{\top} \in \ell_+^2(\mathcal{U})$. Here, and in the sequel, the symbol \top indicates the block transpose. Hence for a (finite

or infinite) sequence C_1, C_2, \dots of vectors or operators we have

$$\begin{bmatrix} C_1 & C_2 & \cdots \end{bmatrix}^\top = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} C_1 \\ C_2 \\ \vdots \end{bmatrix}^\top = \begin{bmatrix} C_1 & C_2 & \cdots \end{bmatrix}.$$

Finally, for any $y = \text{col}[y_j]_{j=0}^\infty$ in $\ell_+^2(\mathcal{Y})$ we have

$$(1.25) \quad \widehat{y}(\lambda) := E_{\mathcal{Y}}^*(I - \lambda S_{\mathcal{Y}}^*)^{-1} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \end{bmatrix} = \sum_{n=0}^{\infty} \lambda^n y_n, \quad \lambda \in \mathbb{D}.$$

It follows that \widehat{y} belongs to the Hardy space $H^2(\mathcal{Y})$, and any function in the Hardy space $H^2(\mathcal{Y})$ is obtained in this way. The map $y \mapsto \widehat{y}$ is the Fourier transform mapping $\ell_+^2(\mathcal{Y})$ onto the Hardy space $H^2(\mathcal{Y})$.

2. OPERATOR NEVANLINNA-PICK INTERPOLATION AND CO-ISOMETRIC REALIZATIONS

Throughout this section $\{W, \widetilde{W}, Z\}$ is a data set for a LTONP interpolation problem, and Λ is the associate Pick operator. We assume that Λ is a non-negative operator, but not necessarily strictly positive, and we define \mathcal{Z}_\circ to be the closure of the range of Λ . Thus

$$(2.1) \quad \mathcal{Z} = \mathcal{Z}_\circ \oplus \text{Ker } \Lambda.$$

The main result of this section, Theorem 2.1 below, provides a Redheffer type description of the set of all solutions of the LTONP interpolation problem with data set $\{W, \widetilde{W}, Z\}$. The proof of this result will be given in Section 3, but much of the preparatory work is done in the current section.

From the definition of the Pick operator and the two identities (1.4) and (1.5) it follows that

$$(2.2) \quad \Lambda - Z\Lambda Z^* = BB^* - \widetilde{B}\widetilde{B}^*.$$

Since $\Lambda \geq 0$, the identity (2.2) can be rewritten as $K_1 K_1^* = K_2 K_2^*$, where

$$(2.3) \quad K_1 = \begin{bmatrix} \widetilde{B} & \Lambda^{\frac{1}{2}} \end{bmatrix} : \begin{bmatrix} \mathcal{U} \\ \mathcal{Z} \end{bmatrix} \rightarrow \mathcal{Z} \quad \text{and} \quad K_2 = \begin{bmatrix} B & Z\Lambda^{\frac{1}{2}} \end{bmatrix} : \begin{bmatrix} \mathcal{Y} \\ \mathcal{Z} \end{bmatrix} \rightarrow \mathcal{Z}.$$

This allows to apply Lemma A.3. Let \mathcal{F} and \mathcal{F}' be the subspaces defined by

$$(2.4) \quad \mathcal{F} = \overline{\text{Im } K_1^*} \quad \text{and} \quad \mathcal{F}' = \overline{\text{Im } K_2^*}.$$

Notice that \mathcal{F} is a subspace of $\mathcal{U} \oplus \mathcal{Z}_\circ$ while \mathcal{F}' is a subspace of $\mathcal{Y} \oplus \mathcal{Z}_\circ$, where \mathcal{Z}_\circ is the subspace of \mathcal{Z} given by (2.1). Applying Lemma A.3 we see that there exists a unique operator ω from \mathcal{F} into \mathcal{F}' such that

$$(2.5) \quad \begin{bmatrix} B & Z\Lambda^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} B^* \\ \Lambda^{\frac{1}{2}} Z^* \end{bmatrix} = \begin{bmatrix} B & Z\Lambda^{\frac{1}{2}} \end{bmatrix} \omega \begin{bmatrix} \widetilde{B}^* \\ \Lambda^{\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} \widetilde{B} & \Lambda^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \widetilde{B}^* \\ \Lambda^{\frac{1}{2}} \end{bmatrix}.$$

Moreover, ω is a unitary operator mapping \mathcal{F} onto \mathcal{F}' . We shall refer to ω as the *unitary operator determined by the data set* $\{W, \widetilde{W}, Z\}$. Note that the two identities

in (2.5) imply that

$$(2.6) \quad \omega \begin{bmatrix} \tilde{B}^* \\ \Lambda^{\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} B^* \\ \Lambda^{\frac{1}{2}} Z^* \end{bmatrix} \quad \text{and} \quad [B \quad Z\Lambda^{\frac{1}{2}}] \omega = [\tilde{B} \quad \Lambda^{\frac{1}{2}}] \Big|_{\mathcal{F}}.$$

In fact each of the identities in (2.6) separately can be used as the definition of ω .

In the sequel \mathcal{G} and \mathcal{G}' will denote the orthogonal complements of \mathcal{F} and \mathcal{F}' in $\mathcal{U} \oplus \mathcal{Z}_\circ$ and $\mathcal{Y} \oplus \mathcal{Z}_\circ$, respectively, that is,

$$(2.7) \quad \mathcal{G} = (\mathcal{U} \oplus \mathcal{Z}_\circ) \ominus \mathcal{F} \quad \text{and} \quad \mathcal{G}' = (\mathcal{Y} \oplus \mathcal{Z}_\circ) \ominus \mathcal{F}'.$$

In particular, $\mathcal{F} \oplus \mathcal{G} = \mathcal{U} \oplus \mathcal{Z}_\circ$ and $\mathcal{F}' \oplus \mathcal{G}' = \mathcal{Y} \oplus \mathcal{Z}_\circ$. The fact that \mathcal{G} is perpendicular to \mathcal{F} and \mathcal{G}' is perpendicular to \mathcal{F}' implies that

$$(2.8) \quad K_1 \mathcal{G} = \{0\} \quad \text{and} \quad K_2 \mathcal{G}' = \{0\}.$$

The following result, which is the main theorem of this section, will be used in the later sections to derive our main theorems.

Theorem 2.1. *Let $\{W, \tilde{W}, Z\}$ be a data set for a LTONP interpolation problem with Z^* being pointwise stable, and assume that the Pick operator Λ is non-negative. Furthermore, let ω be the unitary operator determined by the data set. Then the LTONP interpolation problem is solvable and its solutions are given by*

$$(2.9) \quad F(\lambda) = G_{11}(\lambda) + \lambda G_{12}(\lambda) (I_{\mathcal{Z}_\circ} - \lambda G_{22}(\lambda))^{-1} G_{21}(\lambda), \quad \lambda \in \mathbb{D},$$

where

$$(2.10) \quad G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \in \mathcal{S}(\mathcal{U} \oplus \mathcal{Z}_\circ, \mathcal{Y} \oplus \mathcal{Z}_\circ) \text{ and } G(0)|_{\mathcal{F}} = \omega.$$

Moreover, there is a one-to-one correspondence between the set of all solutions F and the set of all Schur class functions G satisfying the two conditions in (2.10).

Remark 2.2. Let G_\circ be the function identically equal to $\omega P_{\mathcal{F}}$. Then G_\circ is a Schur class function, $G_\circ \in \mathcal{S}(\mathcal{U} \oplus \mathcal{Z}_\circ, \mathcal{Y} \oplus \mathcal{Z}_\circ)$, and $G_\circ(\lambda)|_{\mathcal{F}} = \omega$ for each $\lambda \in \mathbb{D}$. Thus the two conditions in (2.10) are satisfied for $G = G_\circ$. The corresponding solution F is denoted by F_\circ and referred to as the *central solution*. Note that $F_\circ \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$.

The following corollary is an immediate consequence of Theorem 2.1 using the definition of the central solution given above.

Corollary 2.3. *Let $\{W, \tilde{W}, Z\}$ be a data set for a LTONP interpolation problem with Z^* being pointwise stable, and assume that the Pick operator Λ is non-negative. Furthermore, let M_\circ be the operator mapping $\mathcal{U} \oplus \mathcal{Z}_\circ$ into $\mathcal{Y} \oplus \mathcal{Z}_\circ$ defined by $M_\circ = \omega P_{\mathcal{F}}$, where ω is the unitary operator determined by the data set. Write M_\circ as a 2×2 operator matrix as follows:*

$$M_\circ = \begin{bmatrix} \delta_\circ & \gamma_\circ \\ \beta_\circ & \alpha_\circ \end{bmatrix} : \begin{bmatrix} \mathcal{U} \\ \mathcal{Z}_\circ \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{Y} \\ \mathcal{Z}_\circ \end{bmatrix}.$$

Then the central solution F_\circ is given by

$$(2.11) \quad F_\circ(\lambda) = \delta_\circ + \lambda \gamma_\circ (I_{\mathcal{Z}_\circ} - \lambda \alpha_\circ)^{-1} \beta_\circ, \quad \lambda \in \mathbb{D}.$$

Since M_\circ is a contraction, one calls the right side of (2.11) a *contractive realization*. The next example is a trivial one to a certain extent, on the other hand it tells us how one can construct a contractive realization for any Schur class function.

Example 2.4. Let F be a Schur class function, $F \in S(\mathcal{U}, \mathcal{Y})$, and let T_F be the Toeplitz operator defined by F . Put $Z_1 = \ell_+^2(\mathcal{Y})$, and consider the operators

$$(2.12) \quad W_1 = I_{\ell_+^2(\mathcal{Y})}, \quad \widetilde{W}_1 = T_F, \quad Z_1 = S_{\mathcal{Y}}.$$

Then

$$Z_1 W_1 = S_{\mathcal{Y}} = W_1 S_{\mathcal{Y}}, \quad Z_1 \widetilde{W}_1 = S_{\mathcal{Y}} T_F = T_F S_{\mathcal{U}} = \widetilde{W}_1 S_{\mathcal{U}}.$$

Thus $\{W_1, \widetilde{W}_1, Z_1\}$ is a data set for a LTONP interpolation problem. Moreover, $Z_1^* = S_{\mathcal{Y}}^*$, and hence Z_1^* is pointwise stable. Note that $\Psi \in S(\mathcal{U}, \mathcal{Y})$ is a solution to the related LTONP interpolation problem if and only if $W_1 T_{\Phi} = \widetilde{W}_1$. But $W_1 T_{\Phi} = \widetilde{W}_1$ if and only if $T_{\Phi} = T_F$. It follows that the LTONP interpolation problem for the data set $\{W_1, \widetilde{W}_1, Z_1\}$ is solvable, and the solution is unique, namely $\Phi = F$. But then F is the central solution of the LTONP interpolation problem for the data set $\{W_1, \widetilde{W}_1, Z_1\}$, and Corollary 2.3 tells us that F admits a representation of the form

$$(2.13) \quad F(\lambda) = \delta_1 + \lambda \gamma_1 (I_{\mathcal{Z}_{1,\circ}} - \lambda \alpha_1)^{-1} \beta_1, \quad \lambda \in \mathbb{D}.$$

Moreover, the operator matrix M_1 defined by

$$M_1 = \begin{bmatrix} \delta_1 & \gamma_1 \\ \beta_1 & \alpha_1 \end{bmatrix} : \begin{bmatrix} \mathcal{U} \\ \mathcal{Z}_{1,\circ} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{Y} \\ \mathcal{Z}_{1,\circ} \end{bmatrix},$$

is given by $M_1 = \omega_1 P_{\mathcal{F}_1}$, where ω_1 is the unitary operator determined by the data set $\{W_1, \widetilde{W}_1, Z_1\}$. Since M_1 is a contraction, the right hand side is a contractive realization of F . Thus given any $F \in S(\mathcal{U}, \mathcal{Y})$ Corollary 2.3 provides a way to construct a contractive realization for F . Finally, it is noted that in this setting the corresponding subspace $\mathcal{G}'_1 := \mathcal{G}' = \{0\}$, and thus, $M_1 = \omega_1 P_{\mathcal{F}_1}$ is in fact a co-isometry. Indeed, to see that this is the case, note that $\mathcal{Z}_{1,\circ} := \mathcal{Z}_{\circ} = \overline{\text{Im}(I - T_F T_F^*)^{\frac{1}{2}}}$ and \mathcal{F}'_1 is the closure of the range of

$$\begin{bmatrix} E_{\mathcal{Y}}^* I_{\ell_+^2(\mathcal{Y})} \\ (I - T_F T_F^*)^{\frac{1}{2}} S_{\mathcal{Y}}^* \end{bmatrix} = \begin{bmatrix} I_{\mathcal{Y}} & 0 \\ 0 & (I - T_F T_F^*)^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} E_{\mathcal{Y}}^* \\ S_{\mathcal{Y}}^* \end{bmatrix}.$$

Since the block column operator on the right hand side is unitary it follows that \mathcal{F}'_1 is equal to the closure of the range of the 2×2 block operator on the right hand side, which equals $\mathcal{Y} \oplus \mathcal{Z}_{1,\circ}$. Therefore, $\mathcal{G}'_1 = (\mathcal{Y} \oplus \mathcal{Z}_{1,\circ}) \ominus \mathcal{F}'_1 = \{0\}$, as claimed. We shall come back to this construction in Subsection A.4 of the appendix.

Describing the solution set of an interpolation problem with a map of the form (2.9) with a restriction of G equal to a constant unitary operator is one of “standard” methods of parameterizing all solutions of interpolation problems. For instance, this type of formula is used in the description of all solutions to the commutant lifting theorem; see Section VI.5 in [19], where the unitary operator ω is defined by formula (2.2) on page 265, the analogs of the spaces \mathcal{F} , \mathcal{F}' , \mathcal{G} , \mathcal{G}' appear on page 266, and the analog of the function G is referred to as a Schur contraction. Such maps are also used to describe all solutions to the so-called abstract interpolation problem, cf., [36, 37], and these are only a few of many instances. The operator ω is also closely related to the “lurking isometry” used in [12], which has its roots in [45].

In the present paper the proof of Theorem 2.1 is based purely on state space methods, using the theory of co-isometric realizations. Therefore we first review some notation, terminology and standard facts from realization theory, including

the main theorem about observable, co-isometric realizations of Schur class functions. The reader familiar with system theory may skip this subsection.

2.1. Preliminaries from realization theory. We say that a quadruple of Hilbert space operators $\Sigma = \{\alpha, \beta, \gamma, \delta\}$,

$$\alpha : \mathcal{X} \rightarrow \mathcal{X}, \quad \beta : \mathcal{U} \rightarrow \mathcal{X}, \quad \gamma : \mathcal{X} \rightarrow \mathcal{Y}, \quad \delta : \mathcal{U} \rightarrow \mathcal{Y},$$

is a (*state space*) *realization* for a function F with values in $\mathcal{L}(\mathcal{U}, \mathcal{Y})$ if

$$(2.14) \quad F(\lambda) = \delta + \lambda\gamma(I - \lambda\alpha)^{-1}\beta$$

for all λ in some neighborhood of the origin. The space \mathcal{X} is called the *state space* while \mathcal{U} is the *input space* and \mathcal{Y} is the *output space*. In systems theory F is referred to as the *transfer function* of the system $\Sigma = \{\alpha, \beta, \gamma, \delta\}$. Note that $\{\alpha, \beta, \gamma, \delta\}$ is a realization for F implies that F is analytic in some neighborhood of the origin, and in that case the coefficients $\{F_n\}_{n=0}^{\infty}$ of the Taylor expansion of $F(\lambda) = \sum_{n=0}^{\infty} \lambda^n F_n$ at zero are given by

$$(2.15) \quad F_0 = F(0) = \delta \quad \text{and} \quad F_n = \gamma\alpha^{n-1}\beta \quad (n \geq 1).$$

The system $\Sigma = \{\alpha, \beta, \gamma, \delta\}$ or the pair $\{\gamma, \alpha\}$ is said to be *observable* if $\bigcap_{n \geq 0} \text{Ker } \gamma\alpha^n = \{0\}$. Two systems $\{\alpha_1, \beta_1, \gamma_1, \delta_1\}$ and $\{\alpha_2, \beta_2, \gamma_2, \delta_2\}$ with state spaces \mathcal{X}_1 and \mathcal{X}_2 , respectively, are called *unitarily equivalent* if $\delta_1 = \delta_2$ and there exists a unitary operator U mapping \mathcal{X}_1 onto \mathcal{X}_2 such that

$$\alpha_2 U = U \alpha_1, \quad \beta_2 = U \beta_1, \quad \gamma_2 U = \gamma_1.$$

Clearly, two unitarily equivalent systems both realize the same transfer function F . Given a system $\Sigma = \{\alpha, \beta, \gamma, \delta\}$ the 2×2 operator matrix M_Σ defined by

$$(2.16) \quad M_\Sigma = \begin{bmatrix} \delta & \gamma \\ \beta & \alpha \end{bmatrix} : \begin{bmatrix} \mathcal{U} \\ \mathcal{X} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{Y} \\ \mathcal{X} \end{bmatrix},$$

is called the *system matrix* defined by Σ . If the system matrix M_Σ is a contraction, then its transfer function is a *Schur class function*, $F \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$, that is, F is analytic on the open unit disc \mathbb{D} and $\sup_{\lambda \in \mathbb{D}} \|F(\lambda)\| \leq 1$. The converse is also true. More precisely, we have the following classical result.

Theorem 2.5. *A function F is in $\mathcal{S}(\mathcal{U}, \mathcal{Y})$ if and only if F admits an observable, co-isometric realization. Moreover, all observable, co-isometric realizations of F are unitarily equivalent.*

The “if part” of the above theorem is rather straight forward to prove, the “only if part” is much less trivial and has a long and interesting history, for example involving operator model theory (see [43] and the revised and enlarged edition [44]) or the theory of reproducing kernel Hilbert spaces (see, [14] and [15]). We also mention Ando’s Lecture Notes [4, Theorem 3.9 and formulas (3.16), (3.17)], and recent work in a multivariable setting due to J.A. Ball and co-authors [7]– [13].

An alternative new proof of Theorem 2.5 is given in Subsection A.4 in the Appendix.

If the system $\Sigma = \{\alpha, \beta, \gamma, \delta\}$ has a contractive system matrix, then

$$(2.17) \quad \Gamma := \text{col } [\gamma\alpha^j]_{j=0}^{\infty} = \begin{bmatrix} \gamma \\ \gamma\alpha \\ \gamma\alpha^2 \\ \vdots \end{bmatrix} : \mathcal{X} \rightarrow \ell_+^2(\mathcal{Y})$$

is a well defined operator and Γ is a contraction. This classical result is Lemma 3.1 in [20]; see also Lemma A.10 in the Appendix where the proof is given for completeness. We call Γ the *observability operator* defined by Σ , or simply by the pair $\{\gamma, \alpha\}$. Note that in this case Σ is observable if and only if Γ is one-to-one. We conclude with the following lemma.

Lemma 2.6. *If $F \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ has a co-isometric realization, then F has an observable, co-isometric realization.*

Proof. Assume that $\Sigma = \{\alpha, \beta, \gamma, \delta\}$ is a co-isometric realization of F . Let Γ be the observability operator defined (2.17), and let \mathcal{X}_0 be the closure of the range of Γ^* . Thus $\mathcal{X} = \mathcal{X}_0 \oplus \text{Ker } \Gamma$, and relative to this Hilbert space direct sum the operators α, β, γ admit the following partitions:

$$\alpha = \begin{bmatrix} \alpha_0 & 0 \\ \star & \star \end{bmatrix} : \begin{bmatrix} \mathcal{X}_0 \\ \text{Ker } \Gamma \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{X}_0 \\ \text{Ker } \Gamma \end{bmatrix}, \quad \beta = \begin{bmatrix} \beta_0 \\ \star \end{bmatrix} : \mathcal{U} \rightarrow \begin{bmatrix} \mathcal{X}_0 \\ \text{Ker } \Gamma \end{bmatrix},$$

$$\gamma = \begin{bmatrix} \gamma_0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{X}_0 \\ \text{Ker } \Gamma \end{bmatrix} \rightarrow \mathcal{Y}.$$

Then the system $\Sigma_0 = \{\alpha_0, \beta_0, \gamma_0, \delta\}$ is an observable realization of F .

The system matrix $M_0 = M_{\Sigma_0}$ for Σ_0 is also co-isometric. To see this notice that M_Σ admits a matrix representation of the form

$$M_\Sigma = \begin{bmatrix} \delta & \gamma_0 & 0 \\ \beta_0 & \alpha_0 & 0 \\ \star & \star & \star \end{bmatrix} = \begin{bmatrix} M_0 & 0 \\ \star & \star \end{bmatrix}, \quad \text{and hence } M_\Sigma M_\Sigma^* = \begin{bmatrix} M_0 M_0^* & \star \\ \star & \star \end{bmatrix}.$$

Since M_Σ is a co-isometry, $M_\Sigma M_\Sigma^*$ is the identity operator on the space $\mathcal{Y} \oplus \mathcal{X}_0 \oplus \text{Ker } \Gamma$, and thus $M_0 M_0^*$ is the identity operator on $\mathcal{Y} \oplus \mathcal{X}_0$. Therefore, $\Sigma_0 = \{\alpha_0, \beta_0, \gamma_0, \delta\}$ is an observable, co-isometric realization of F . \square

2.2. Solutions of the LTONP interpolation problem and Λ -preferable realizations. As before $\{W, \widetilde{W}, Z\}$ is a data set for a LTONP interpolation problem, and we assume that the Pick operator Λ is non-negative.

Let $\Sigma = \{\alpha, \beta, \gamma, \delta\}$ be a co-isometric realization of F with state space \mathcal{X} and system matrix $M = M_\Sigma$. We call the realization Λ -*preferable* if $\mathcal{X} = \mathcal{Z}_\circ \oplus \mathcal{V}$ for some Hilbert space \mathcal{V} and $M|_{\mathcal{F}} = \omega$. As before, \mathcal{Z}_\circ equals the closure of the range of Λ ; see (2.1), and ω is the unitary operator from \mathcal{F} onto \mathcal{F}' determined by the data set $\{W, \widetilde{W}, Z\}$. In particular, \mathcal{F} and \mathcal{F}' are the subspaces of $\mathcal{U} \oplus \mathcal{Z}_\circ$ and $\mathcal{Y} \oplus \mathcal{Z}_\circ$, respectively, defined by (2.4). Note that $\mathcal{X} = \mathcal{Z}_\circ \oplus \mathcal{V}$ implies that $\mathcal{F} \subset \mathcal{U} \oplus \mathcal{X}$, and thus $M|_{\mathcal{F}}$ is well defined. Furthermore, M partitions as

$$(2.18) \quad M = \begin{bmatrix} \delta & \gamma_1 & \gamma_2 \\ \beta_1 & \alpha_{11} & \alpha_{12} \\ \beta_2 & \alpha_{21} & \alpha_{22} \end{bmatrix} : \begin{bmatrix} \mathcal{U} \\ \mathcal{Z}_\circ \\ \mathcal{V} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{Y} \\ \mathcal{Z}_\circ \\ \mathcal{V} \end{bmatrix},$$

and the constraint $M|_{\mathcal{F}} = \omega$ is equivalent to

$$(2.19) \quad \omega = \left[\begin{array}{cc} \delta & \gamma_1 \\ \beta_1 & \alpha_{11} \end{array} \right] \Big|_{\mathcal{F}}.$$

To see the latter, observe that $M|_{\mathcal{F}} = \omega$ implies that $M\mathcal{F} = \omega\mathcal{F} = \mathcal{F}' \subset \mathcal{Y} \oplus \mathcal{Z}_\circ$, and hence $[\beta_2 \ \alpha_{21}]|_{\mathcal{F}} = \{0\}$. Conversely, if (2.19) holds, then the restriction of the first two block rows of M in (2.18) to \mathcal{F} is equal to ω . Since ω is unitary, the

restriction of the last block row to \mathcal{F} must be zero, for otherwise M would not be a contraction. Hence $M|_{\mathcal{F}} = \omega$.

The following theorem is the main result of the present subsection.

Theorem 2.7. *Let $\{W, \widetilde{W}, Z\}$ be a data set for a LTONP interpolation problem with Z^* being pointwise stable, and assume that the Pick operator Λ is non-negative. Then all solutions F of the LTONP interpolation problem are given by*

$$(2.20) \quad F(\lambda) = \delta + \lambda\gamma(I - \lambda\alpha)^{-1}\beta, \quad \lambda \in \mathbb{D},$$

where $\Sigma = \{\alpha, \beta, \gamma, \delta\}$ is an observable, co-isometric realization of F which is Λ -preferable. Moreover,

$$(2.21) \quad \Lambda = W\Gamma\Gamma^*W^*,$$

where Γ is the observability operator mapping \mathcal{X} into $\ell_+^2(\mathcal{Y})$ determined by $\{\gamma, \alpha\}$. Finally, up to unitary equivalence of realizations this parameterization of all solutions to the LTONP interpolation problem via Λ -preferable, observable, co-isometric realizations $\Sigma = \{\alpha, \beta, \gamma, \delta\}$ is one-to-one and onto.

Remark 2.8. If one specifies Theorem 2.7 for the case when the data set is the set $\{W_1, \widetilde{W}_1, Z_1\}$, where W_1, \widetilde{W}_1 and Z_1 are given by (2.12), then Theorem 2.5 is obtained. Note however that Theorem 2.5 is used in the proof of Theorem 2.7, and therefore Theorem 2.5 does not appear as a corollary of Theorem 2.7. On the other hand, if one uses the arguments in the proof of Theorem 2.7 for the data set $\{W_1, \widetilde{W}_1, Z_1\}$ only, then one obtains a new direct proof of the fact that any Schur class function admits an observable co-isometric realization. This proof is given in Subsection A.4; cf., Example 2.4.

The proof of Theorem 2.7 will be based on two lemmas.

Lemma 2.9. *Let $\{W, \widetilde{W}, Z\}$ be a data set for a LTONP interpolation problem with Z^* being pointwise stable, and assume that the Pick operator Λ is non-negative. Let $F \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$, and assume that $\Sigma = \{\alpha, \beta, \gamma, \delta\}$ is a Λ -preferable, co-isometric realization of F . Then F is a solution to the LTONP interpolation problem. Moreover,*

$$(2.22) \quad \Lambda^{\frac{1}{2}}\Pi_{\mathcal{Z}_o} = W\Gamma,$$

where Γ is the observability operator defined by $\{\gamma, \alpha\}$ and $\mathcal{Z}_o = \overline{\text{Im } \Lambda}$; see (2.1).

Proof. Using $ZW = WS_{\mathcal{Y}}$ and $S_{\mathcal{Y}}^*\Gamma = \Gamma\alpha$, we obtain

$$W\Gamma - ZW\Gamma\alpha = W(I - S_{\mathcal{Y}}S_{\mathcal{Y}}^*)\Gamma = WE_{\mathcal{Y}}E_{\mathcal{Y}}^*\Gamma = B\gamma.$$

In other words,

$$(2.23) \quad W\Gamma - ZW\Gamma\alpha = B\gamma.$$

Because Z^* is pointwise stable, it follows that $W\Gamma$ is the unique solution to the Stein equation $\Omega - Z\Omega\alpha = B\gamma$; see Lemma A.1 in the Appendix.

Since the system $\Sigma = \{\alpha, \beta, \gamma, \delta\}$ is Λ -preferable, we know that the state space \mathcal{X} is equal to $\mathcal{Z}_o \oplus \mathcal{V}$ for some Hilbert space \mathcal{V} , where $\mathcal{Z}_o = \overline{\text{Im } \Lambda}$. Let $\Pi_{\mathcal{Z}_o}$ be the orthogonal projection from $\mathcal{X} = \mathcal{Z}_o \oplus \mathcal{V}$ onto \mathcal{Z}_o . We shall prove that

$$(2.24) \quad \begin{bmatrix} B & Z\Lambda^{\frac{1}{2}}\Pi_{\mathcal{Z}_o} \end{bmatrix} \begin{bmatrix} \delta & \gamma \\ \beta & \alpha \end{bmatrix} = \begin{bmatrix} \widetilde{B} & \Lambda^{\frac{1}{2}}\Pi_{\mathcal{Z}_o} \end{bmatrix}.$$

Let $M = M_{\Sigma}$ be the system matrix of the realization Σ , i.e., the 2×2 operator matrix appearing in the left hand side of (2.24). To prove the identity (2.24) we

first note that the second identity in (2.6) and $M|_{\mathcal{F}} = \omega$ imply that the two sides of (2.24) are equal when restricted to \mathcal{F} . Next, consider the orthogonal complements

$$\mathcal{F}^\perp = (\mathcal{U} \oplus \mathcal{Z}_\circ \oplus \mathcal{V}) \ominus \mathcal{F} = \mathcal{G} \oplus \mathcal{V} \quad \mathcal{F}'^\perp = (\mathcal{Y} \oplus \mathcal{Z}_\circ \oplus \mathcal{V}) \ominus \mathcal{F}' = \mathcal{G}' \oplus \mathcal{V}.$$

Since M is a contraction with $M\mathcal{F} = \mathcal{F}'$ and $M|_{\mathcal{F}}$ is unitary, we have $M\mathcal{F}^\perp \subset \mathcal{F}'^\perp$. Therefore it remains to show that the two sides of (2.24) are also equal when restricted to \mathcal{F}^\perp . To do this, take $f = [u_0 \ z_0 \ v_0]^\top$ in \mathcal{F}^\perp . Here $u_0 \in \mathcal{U}$, $z_0 \in \mathcal{Z}_\circ$, and $v_0 \in \mathcal{V}$. Then

$$(2.25) \quad \begin{bmatrix} \tilde{B} & \Lambda^{\frac{1}{2}}\Pi_{\mathcal{Z}_\circ} \end{bmatrix} f = \begin{bmatrix} \tilde{B} & \Lambda^{\frac{1}{2}}|_{\mathcal{Z}_\circ} & 0 \end{bmatrix} \begin{bmatrix} u_0 \\ z_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} \tilde{B} & \Lambda^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} u_0 \\ z_0 \end{bmatrix}.$$

But the vector $[u_0 \ z_0]^\top$ belongs to the space \mathcal{G} . Thus the first identity in (2.8) shows that $\begin{bmatrix} \tilde{B} & \Lambda^{\frac{1}{2}}\Pi_{\mathcal{Z}_\circ} \end{bmatrix} f = 0$. Now consider $f' := Mf \in \mathcal{F}'^\perp$. Write $f' = [y_0 \ z'_0 \ v'_0]^\top$, where $y_0 \in \mathcal{Y}$, $z'_0 \in \mathcal{Z}_\circ$, and $v'_0 \in \mathcal{V}$. Then

$$\begin{bmatrix} B & Z\Lambda^{\frac{1}{2}}\Pi_{\mathcal{Z}_\circ} \end{bmatrix} Mf = \begin{bmatrix} B & Z\Lambda^{\frac{1}{2}}|_{\mathcal{Z}_\circ} & 0 \end{bmatrix} \begin{bmatrix} y_0 \\ z'_0 \\ v'_0 \end{bmatrix} = \begin{bmatrix} B & Z\Lambda^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} y_0 \\ z'_0 \end{bmatrix} = 0,$$

because $[y_0 \ z'_0]^\top$ belongs to \mathcal{G}' and using the second identity in (2.8). We conclude that $\begin{bmatrix} B & Z\Lambda^{\frac{1}{2}}\Pi_{\mathcal{Z}_\circ} \end{bmatrix} Mf = 0$. Hence when applied to f both sides of (2.24) are equal to zero, which completes the proof of (2.24).

Note that (2.24) is equivalent to the following two identities:

$$(2.26) \quad \Lambda^{\frac{1}{2}}\Pi_{\mathcal{Z}_\circ} = Z\Lambda^{\frac{1}{2}}\Pi_{\mathcal{Z}_\circ}\alpha + B\gamma \quad \text{and} \quad \tilde{B} = Z\Lambda^{\frac{1}{2}}\Pi_{\mathcal{Z}_\circ}\beta + B\delta.$$

Because WT is the unique solution to the Stein equation (2.23), as observed above, the first identity in (2.26) shows that $WT = \Lambda^{\frac{1}{2}}\Pi_{\mathcal{Z}_\circ}$, i.e., the identity (2.22) holds true.

By consulting the second equation in (2.26), we have $\tilde{B} = ZWT\beta + B\delta$. Using this we obtain

$$WT_F E_{\mathcal{U}} = [WE_{\mathcal{Y}} \ WS_{\mathcal{Y}}] \begin{bmatrix} F(0) \\ S_{\mathcal{Y}}^* T_F E_{\mathcal{U}} \end{bmatrix} = [B \ ZW] \begin{bmatrix} \delta \\ \Gamma\beta \end{bmatrix} = \tilde{B}.$$

Therefore $WT_F E_{\mathcal{U}} = \tilde{B} = \widetilde{W}E_{\mathcal{U}}$. So for any integer $n \geq 0$, we have

$$WT_F S_{\mathcal{U}}^n E_{\mathcal{U}} = WS_{\mathcal{Y}}^n T_F E_{\mathcal{U}} = Z^n WT_F E_{\mathcal{U}} = Z^n \widetilde{W}E_{\mathcal{U}} = \widetilde{W}S_{\mathcal{U}}^n E_{\mathcal{U}}.$$

Because $\{S_{\mathcal{U}}^n E_{\mathcal{U}}\}_{n=0}^\infty$ spans $\ell_+^2(\mathcal{U})$, we see that $WT_F = \widetilde{W}$. Hence, F is a solution to the LTONP interpolation problem. \square

Lemma 2.10. *Let F be a solution to the LTONP interpolation problem with data set $\{W, \widetilde{W}, Z\}$, and assume $\Sigma = \{\alpha, \beta, \gamma, \delta\}$ is a co-isometric realization of F . Then up to unitary equivalence the realization Σ is Λ -preferable.*

Proof. Throughout $F(\lambda) = \delta + \lambda\gamma(I - \lambda\alpha)^{-1}\beta$ is a co-isometric realization of the solution F for the LTONP interpolation problem with data set $\{W, \widetilde{W}, Z\}$. We split the proof into three parts.

PART 1. In this part we show that

$$(2.27) \quad [B \ ZW\Gamma] \begin{bmatrix} \delta & \gamma \\ \beta & \alpha \end{bmatrix} = [\tilde{B} \ W\Gamma].$$

To prove this equality, note that

$$\begin{aligned}
 [B \quad ZW\Gamma] \begin{bmatrix} \delta \\ \beta \end{bmatrix} &= B\delta + ZW\Gamma\beta = B\delta + WS_y\Gamma\beta \\
 &= WE_y\delta + WS_y\Gamma\beta = W(E_y\delta + S_y\Gamma\beta) \\
 &= W \begin{bmatrix} \delta \\ \gamma\beta \\ \gamma\alpha\beta \\ \gamma\alpha^2\beta \\ \vdots \end{bmatrix} = WT_F E_{\mathcal{U}} = \widetilde{W} E_{\mathcal{U}} = \widetilde{B}.
 \end{aligned}
 \tag{2.28}$$

Furthermore, we have

$$\begin{aligned}
 [B \quad ZW\Gamma] \begin{bmatrix} \gamma \\ \alpha \end{bmatrix} &= B\gamma + ZW\Gamma\alpha = WE_y\gamma + WS_y\Gamma\alpha \\
 &= W(E_y\gamma + S_y\Gamma\alpha) = W\Gamma.
 \end{aligned}
 \tag{2.29}$$

Together the identities (2.28) and (2.29) prove the identity (2.27).

PART 2. In this part we show that $W\Gamma^*W^*$ is equal to the Pick operator Λ . Since the realization $\{\alpha, \beta, \gamma, \delta\}$ is co-isometric, the corresponding system matrix is a co-isometry, and hence (2.27) implies that

$$\begin{bmatrix} \widetilde{B} & W\Gamma \end{bmatrix} \begin{bmatrix} \widetilde{B}^* \\ \Gamma^*W^* \end{bmatrix} = [B \quad ZW\Gamma] \begin{bmatrix} B^* \\ \Gamma^*W^*Z^* \end{bmatrix}.$$

Now put $\Omega = W\Gamma^*W^*$. Then the preceding identity is equivalent to

$$\Omega - Z\Omega Z^* = BB^* - \widetilde{B}\widetilde{B}^*.$$

Hence Ω is a solution to the Stein equation (2.2). Since Z^* is pointwise stable, the solution to this Stein equation is unique (see Lemma A.1), and thus, $\Omega = \Lambda$.

PART 3. In this part we show that up to unitary equivalence the system $\Sigma = \{\alpha, \beta, \gamma, \delta\}$ is Λ -preferable. Let \mathcal{X} be the state space of Σ , and decompose \mathcal{X} as $\mathcal{X} = \mathcal{X}_o \oplus \mathcal{V}$, where $\mathcal{V} = \text{Ker } W\Gamma$. Since

$$(W\Gamma)(W\Gamma)^* = \Lambda = \Lambda^{\frac{1}{2}}\Pi_{\mathcal{Z}_o}\Pi_{\mathcal{Z}_o}^*\Lambda^{\frac{1}{2}}$$

by the second part of the proof, the Douglas factorization lemma shows that there exists a unique unitary operator τ_o mapping \mathcal{Z}_o onto \mathcal{X}_o such that

$$(W\Gamma|_{\mathcal{X}_o})\tau_o = \Lambda^{\frac{1}{2}}|_{\mathcal{Z}_o}.
 \tag{2.30}$$

Now, put $\widetilde{\mathcal{X}} = \mathcal{Z}_o \oplus \mathcal{V}$, let U be the unitary operator from $\widetilde{\mathcal{X}}$ onto \mathcal{X} defined by

$$U = \begin{bmatrix} \tau_o & 0 \\ 0 & I_{\mathcal{V}} \end{bmatrix} : \begin{bmatrix} \mathcal{Z}_o \\ \mathcal{V} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{X}_o \\ \mathcal{V} \end{bmatrix},
 \tag{2.31}$$

and define the system $\widetilde{\Sigma} = \{\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}, \widetilde{\delta}\}$ by setting

$$\widetilde{\alpha} = U^{-1}\alpha U, \quad \widetilde{\beta} = U^{-1}\beta, \quad \widetilde{\gamma} = \gamma U, \quad \widetilde{\delta} = \delta.
 \tag{2.32}$$

Note that the systems Σ and $\widetilde{\Sigma}$ are unitarily equivalent. Thus $\widetilde{\Sigma}$ is a co-isometric realization of F . Furthermore, the space \mathcal{Z}_o is a subspace of $\widetilde{\mathcal{X}}$. Therefore in order to complete the proof it remains to show that the system matrix \widetilde{M} of the system $\widetilde{\Sigma}$ has the following property:

$$\widetilde{M}|_{\mathcal{F}} = \omega.$$

Here ω is the unitary operator determined by the given data set $\{W, \widetilde{W}, Z\}$. In particular, $\omega : \mathcal{F} \rightarrow \mathcal{F}'$, with \mathcal{F} and \mathcal{F}' being defined by (2.4).

Let M be the system matrix for Σ . Multiplying (2.27) from the right by M^* , using the fact that M is a co-isometry, and taking adjoints, we see that

$$(2.33) \quad M \begin{bmatrix} \widetilde{B}^* \\ (W\Gamma)^* \end{bmatrix} = \begin{bmatrix} B^* \\ (W\Gamma)^* Z^* \end{bmatrix}.$$

Note that $(W\Gamma)^*$ maps \mathcal{Z} into \mathcal{X}_\circ . Hence taking adjoints in (2.30) and using that τ_\circ is a unitary operator, we see that

$$(W\Gamma)^* z = \tau_\circ \Lambda^{\frac{1}{2}} z \quad \text{for each } z \in \mathcal{Z}.$$

But then, using the definition of U in (2.31), we obtain

$$(2.34) \quad \begin{bmatrix} I_{\mathcal{U}} & 0 \\ 0 & U^{-1} \end{bmatrix} \begin{bmatrix} \widetilde{B}^* \\ (W\Gamma)^* \end{bmatrix} = \begin{bmatrix} \widetilde{B}^* \\ \Lambda^{\frac{1}{2}} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} I_{\mathcal{Y}} & 0 \\ 0 & U^{-1} \end{bmatrix} \begin{bmatrix} B^* \\ (W\Gamma)^* Z^* \end{bmatrix} = \begin{bmatrix} B^* \\ \Lambda^{\frac{1}{2}} Z^* \end{bmatrix}.$$

From (2.32) it follows that

$$\widetilde{M} \begin{bmatrix} I_{\mathcal{U}} & 0 \\ 0 & U^{-1} \end{bmatrix} = \begin{bmatrix} I_{\mathcal{Y}} & 0 \\ 0 & U^{-1} \end{bmatrix} M.$$

Using the later identity and the ones in (2.33) and (2.34) we see that

$$\begin{aligned} \widetilde{M} \begin{bmatrix} \widetilde{B}^* \\ \Lambda^{\frac{1}{2}} \end{bmatrix} &= \widetilde{M} \begin{bmatrix} I_{\mathcal{U}} & 0 \\ 0 & U^{-1} \end{bmatrix} \begin{bmatrix} \widetilde{B}^* \\ (W\Gamma)^* \end{bmatrix} = \begin{bmatrix} I_{\mathcal{Y}} & 0 \\ 0 & U^{-1} \end{bmatrix} M \begin{bmatrix} \widetilde{B}^* \\ (W\Gamma)^* \end{bmatrix} \\ &= \begin{bmatrix} I_{\mathcal{Y}} & 0 \\ 0 & U^{-1} \end{bmatrix} \begin{bmatrix} B^* \\ (W\Gamma)^* Z^* \end{bmatrix} = \begin{bmatrix} B^* \\ \Lambda^{\frac{1}{2}} Z^* \end{bmatrix}. \end{aligned}$$

Now recall that ω is the unique operator satisfying the first identity in (2.6). Thus \widetilde{M} and ω coincide on \mathcal{F} , that is, $\widetilde{M}|_{\mathcal{F}} = \omega$. \square

Corollary 2.11. *If $F \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ has a Λ -preferable, co-isometric realization, then F has a Λ -preferable, observable, co-isometric realization.*

Proof. The fact that F has Λ -preferable, co-isometric realization implies (use Lemma 2.9) that F is a solution to the LTONP interpolation problem. Moreover, from Lemma 2.6 we know that F has an observable, co-isometric realization. Since observability is preserved under unitarily equivalence, Lemma 2.10 tells us that F has a Λ -preferable, observable, co-isometric realization. \square

Proof of Theorem 2.7. Let Σ be an observable, co-isometric system which is Λ -preferable, and let F be its transfer function. Then Lemma 2.9 tells us that F is a solution to the LTONP interpolation problem. Moreover, since \mathcal{Z}_\circ is the closure of the range of Λ , the identity (2.22) shows that

$$W\Gamma^* W^* = \Lambda^{\frac{1}{2}} \Pi_{\mathcal{Z}_\circ} \Pi_{\mathcal{Z}_\circ}^* \Lambda^{\frac{1}{2}} = \Lambda,$$

which proves (2.21). Conversely, by Theorem 2.5 and Lemma 2.10, if F is a solution to the LTONP interpolation problem, then F has a Λ -preferable, co-isometric realization. But then F also has a Λ -preferable, observable, co-isometric realization by Corollary 2.11. Finally, by Theorem 2.5, two observable, co-isometric realizations have the same transfer function F if and only if they are unitarily equivalent. This proves that up to unitary equivalence the parametrization is one-to-one and onto. \square

For later purposes, namely the proof of Theorem 2.1 in the next section, we conclude this subsection with the following corollary of Lemma 2.9.

Corollary 2.12. *Let $F \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$, and let the systems $\Sigma = \{\alpha, \beta, \gamma, \delta\}$ and $\Sigma' = \{\alpha', \beta', \gamma', \delta\}$ be Λ -preferable, co-isometric realizations of F with state spaces $\mathcal{X} = \mathcal{Z}_\circ \oplus \mathcal{V}$ and $\mathcal{X}' = \mathcal{Z}_\circ \oplus \mathcal{V}'$, respectively. If $U : \mathcal{X} \rightarrow \mathcal{X}'$ is a unitary operator such that*

$$(2.35) \quad \alpha'U = U\alpha, \quad \beta' = U\beta, \quad \gamma'U = \gamma.$$

Then $U|_{\mathcal{Z}_\circ}$ is the identity operator on \mathcal{Z}_\circ and $U\mathcal{V} = \mathcal{V}'$.

Proof. Let Γ and Γ' be the observability operators of Σ and Σ' , respectively. From (2.35) it follows that $\Gamma'U = \Gamma$. Furthermore, using the identity (2.22) for both Σ and Σ' we see that

$$\Lambda^{\frac{1}{2}}\Pi_{\mathcal{Z}_\circ} = W\Gamma \quad \text{and} \quad \Lambda^{\frac{1}{2}}\Pi_{\mathcal{Z}_\circ} = W\Gamma'.$$

Taking adjoints, it follows that $U\Pi_{\mathcal{Z}_\circ}^* \Lambda^{\frac{1}{2}} = U\Gamma^*W^* = \Gamma'^*W^* = \Pi_{\mathcal{Z}_\circ}^* \Lambda^{\frac{1}{2}}$. Since the range of $\Lambda^{\frac{1}{2}}$ is dense in \mathcal{Z}_\circ , we conclude that the operator U acts as the identity operator on \mathcal{Z}_\circ , i.e., $U|_{\mathcal{Z}_\circ} = I_{\mathcal{Z}_\circ}$. But then, using the fact that U is unitary, we see that $U\mathcal{V} = \mathcal{V}'$. \square

3. PROOF OF THEOREM 2.1

In this section we prove Theorem 2.1. Thus throughout $\{W, \widetilde{W}, Z\}$ is a data set for a LTONP interpolation problem with Z^* being pointwise stable, and we assume that the Pick operator Λ is non-negative. Furthermore, we use freely the notation and terminology introduced in the first three paragraphs of Section 2. In particular, ω is the unitary operator determined by the data set.

We begin with two lemmas. The first shows how Schur class functions F and G that satisfy (2.9) can be constructed from contractive realizations, and hence, in particular, from co-isometric realizations.

Lemma 3.1. *Let M be a contractive linear operator mapping $\mathcal{U} \oplus \mathcal{Z}_\circ \oplus \mathcal{V}$ into $\mathcal{Y} \oplus \mathcal{Z}_\circ \oplus \mathcal{V}$, for some Hilbert space \mathcal{V} , partitioned as in (2.18). Define*

$$(3.1) \quad \begin{aligned} F(\lambda) &= \delta + \lambda \begin{bmatrix} \gamma_1 & \gamma_2 \end{bmatrix} \left(I - \lambda \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \right)^{-1} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}, \\ G(\lambda) &= \begin{bmatrix} G_{11}(\lambda) & G_{12}(\lambda) \\ G_{21}(\lambda) & G_{22}(\lambda) \end{bmatrix} \\ &= \begin{bmatrix} \delta & \gamma_1 \\ \beta_1 & \alpha_{11} \end{bmatrix} + \lambda \begin{bmatrix} \gamma_2 \\ \alpha_{12} \end{bmatrix} (I - \lambda\alpha_{22})^{-1} \begin{bmatrix} \beta_2 & \alpha_{21} \end{bmatrix}. \end{aligned}$$

Then F , G and the functions G_{ij} , $1 \leq i, j \leq 2$, are Schur class functions, and

$$(3.2) \quad F(\lambda) = G_{11}(\lambda) + \lambda G_{12}(\lambda) (I_{\mathcal{Z}_\circ} - \lambda G_{22}(\lambda))^{-1} G_{21}(\lambda), \quad \lambda \in \mathbb{D}.$$

Proof. Since M is contractive, the system matrices of the realizations of F and G in (3.1) are also contractive, and hence F and G are Schur class functions. Note that the second identity in (3.1) tells us that

$$\begin{aligned} G_{11}(\lambda) &= \delta + \lambda\gamma_2 (I - \lambda\alpha_{22})^{-1} \beta_2, & G_{12}(\lambda) &= \gamma_1 + \lambda\gamma_2 (I - \lambda\alpha_{22})^{-1} \alpha_{21}, \\ G_{21}(\lambda) &= \beta_1 + \lambda\alpha_{12} (I - \lambda\alpha_{22})^{-1} \beta_2, & G_{22}(\lambda) &= \alpha_{11} + \lambda\alpha_{12} (I - \lambda\alpha_{22})^{-1} \alpha_{21}. \end{aligned}$$

Again using M is contractive, we see that the system matrices of the realizations of G_{ij} , $1 \leq i, j \leq 2$, are also contractive, and hence the functions G_{ij} , $1 \leq i, j \leq 2$, are also Schur class functions.

Now let F be given by the first identity in (3.1). Fix $\lambda \in \mathbb{D}$ and $u \in \mathcal{U}$. Put $y = F(\lambda)u$, and define

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} := \left(I_{\mathcal{Z}_\circ \oplus \mathcal{V}} - \lambda \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \right)^{-1} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} u.$$

Then the identity $F(\lambda)u = y$ is equivalent to the following three identities:

$$(3.3) \quad y = \delta u + \lambda \gamma_1 x_1 + \lambda \gamma_2 x_2,$$

$$(3.4) \quad x_1 = \beta_1 u + \lambda \alpha_{11} x_1 + \lambda \alpha_{12} x_2, \quad x_2 = \beta_2 u + \lambda \alpha_{21} x_1 + \lambda \alpha_{22} x_2.$$

The second identity in (3.4) implies that

$$(3.5) \quad x_2 = (I - \lambda \alpha_{22})^{-1} \beta_2 u + \lambda (I - \lambda \alpha_{22})^{-1} \alpha_{21} x_1.$$

Inserting this formula for x_2 into the first identity in (3.4) yields

$$\begin{aligned} x_1 &= \beta_1 u + \lambda \alpha_{11} x_1 + \lambda \alpha_{12} (I - \lambda \alpha_{22})^{-1} \beta_2 u + \lambda^2 \alpha_{12} (I - \lambda \alpha_{22})^{-1} \alpha_{21} x_1 \\ &= G_{21}(\lambda)u + \lambda \alpha_{11} x_1 + \lambda (G_{22}(\lambda)x_1 - \alpha_{11} x_1) \\ &= G_{21}(\lambda)u + \lambda G_{22}(\lambda)x_1, \end{aligned}$$

and thus

$$(3.6) \quad x_1 = (I - \lambda G_{22}(\lambda))^{-1} G_{21}(\lambda)u.$$

Using the identity (3.3) together with the identities (3.5) and (3.6) we obtain

$$\begin{aligned} F(\lambda)u &= \delta u + \lambda \gamma_1 x_1 + \lambda \gamma_2 x_2 \\ &= \delta u + \lambda \gamma_1 x_1 + \lambda \gamma_2 (I - \lambda \alpha_{22})^{-1} \beta_2 u + \lambda^2 \gamma_2 (I - \lambda \alpha_{22})^{-1} \alpha_{21} x_1 \\ &= G_{11}(\lambda)u + \lambda \left(\gamma_1 + \lambda \gamma_2 (I - \lambda \alpha_{22})^{-1} \alpha_{21} \right) x_1 \\ &= G_{11}(\lambda)u + \lambda G_{12}(\lambda) (I - \lambda G_{22}(\lambda))^{-1} G_{21}(\lambda)u. \end{aligned}$$

Hence (3.2) holds as claimed. \square

Lemma 3.2. *Let M be a contractive linear operator mapping $\mathcal{U} \oplus \mathcal{Z}_\circ \oplus \mathcal{V}$ into $\mathcal{Y} \oplus \mathcal{Z}_\circ \oplus \mathcal{V}$, for some Hilbert space \mathcal{V} , partitioned as in (2.18). Consider the systems*

$$(3.7) \quad \Sigma = \left\{ \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}, \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}, [\gamma_1 \quad \gamma_2], \delta \right\},$$

$$(3.8) \quad \tilde{\Sigma} = \left\{ \alpha_{22}, [\beta_2 \quad \alpha_{21}], \begin{bmatrix} \gamma_2 \\ \alpha_{12} \end{bmatrix}, \begin{bmatrix} \delta & \gamma_1 \\ \beta_1 & \alpha_{11} \end{bmatrix} \right\}.$$

Then Σ is observable if and only if $\tilde{\Sigma}$ is observable and

$$(3.9) \quad [\gamma_1 \quad \gamma_2] \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}^n \begin{bmatrix} z \\ v \end{bmatrix} = 0 \quad (n = 0, 1, 2, \dots) \implies z = 0.$$

Proof. We split the proof into two parts. In the first part we assume Σ is observable, and we prove that $\tilde{\Sigma}$ is observable and that (3.9) holds. The second part deals with the reverse implication.

PART 1. Let Σ be observable. In that case the identities on the left side of the arrow in (3.9) imply that $z = 0$ and $v = 0$. In particular, the implication in (3.9) holds. To see that $\tilde{\Sigma}$ is observable, fix a $v \in \mathcal{V}$, and assume that

$$\begin{bmatrix} \gamma_2 \\ \alpha_{12} \end{bmatrix} \alpha_{22}^n v = 0, \quad n = 0, 1, 2, \dots$$

In other words, we assume that

$$(3.10) \quad \gamma_2 \alpha_{22}^n v = 0 \quad \text{and} \quad \alpha_{12} \alpha_{22}^n v = 0, \quad n = 0, 1, 2, \dots$$

We want to show that $v = 0$. We first show that

$$(3.11) \quad \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}^n \begin{bmatrix} 0 \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha_{22}^n v \end{bmatrix}, \quad n = 0, 1, 2, \dots$$

For $n = 0$ the statement is trivially true. Assume that the identity in (3.11) holds for some integer $n \geq 0$. Then, using the second part of (3.11), we obtain

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}^{n+1} \begin{bmatrix} 0 \\ v \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \begin{bmatrix} 0 \\ \alpha_{22}^n v \end{bmatrix} = \begin{bmatrix} \alpha_{12} \alpha_{22}^n v \\ \alpha_{22}^{n+1} v \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha_{22}^{n+1} v \end{bmatrix}.$$

By induction (3.11) is proved. Using the second part of (3.11), we conclude that

$$[\gamma_1 \quad \gamma_2] \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}^n \begin{bmatrix} 0 \\ v \end{bmatrix} = [\gamma_1 \quad \gamma_2] \begin{bmatrix} 0 \\ \alpha_{22}^n v \end{bmatrix} = \gamma_2 \alpha_{22}^n v = 0, \quad n = 0, 1, 2, \dots$$

Since the system Σ is observable, we conclude that $v = 0$, and hence $\tilde{\Sigma}$ is observable.

PART 2. Assume that $\tilde{\Sigma}$ is observable and that (3.9) holds. Let Γ be the observability operator defined by Σ . Thus

$$\Gamma = \begin{bmatrix} \gamma \\ \gamma\alpha \\ \gamma\alpha^2 \\ \vdots \end{bmatrix} : \mathcal{X} \rightarrow \ell_+^2(\mathcal{Y}), \quad \text{where } \mathcal{X} = \mathcal{Z}_\circ \oplus \mathcal{V} \text{ and}$$

$$\gamma = [\gamma_1 \quad \gamma_2] : \begin{bmatrix} \mathcal{Z}_\circ \\ \mathcal{V} \end{bmatrix} \rightarrow \mathcal{Y}, \quad \alpha = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} : \begin{bmatrix} \mathcal{Z}_\circ \\ \mathcal{V} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{Z}_\circ \\ \mathcal{V} \end{bmatrix}.$$

Since M is a contraction, the operator Γ is a well defined contraction; see Lemma A.10. We want to prove that Γ is one-to-one.

Let $x = z \oplus v \in \text{Ker } \Gamma$. Then condition (3.9) tells us that $z = 0$. Thus $\text{Ker } \Gamma \subset \mathcal{V}$. It remains to prove that $v = 0$.

Observe that $S_{\mathcal{Y}}^* \Gamma = \Gamma \alpha$. Thus $\alpha^n x \in \text{Ker } \Gamma \subset \mathcal{V}$ for each $n = 0, 1, 2, \dots$ which, by induction, implies that

$$(3.12) \quad \alpha^n \begin{bmatrix} 0 \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha_{22}^n v \end{bmatrix}, \quad n = 0, 1, 2, \dots$$

We see that

$$0 = \gamma \alpha^n \begin{bmatrix} 0 \\ v \end{bmatrix} = [\gamma_1 \quad \gamma_2] \begin{bmatrix} 0 \\ \alpha_{22}^n v \end{bmatrix} = \gamma_2 \alpha_{22}^n v, \quad n = 0, 1, 2, \dots$$

Furthermore, again using (3.12), we have $\alpha_{12} \alpha_{22}^n v = 0$ for each $n \geq 0$. Thus

$$(3.13) \quad \begin{bmatrix} \gamma_2 \\ \alpha_{12} \end{bmatrix} \alpha_{22}^n v = 0, \quad n = 0, 1, 2, \dots$$

But, by assumption, $\widetilde{\Sigma}$ is observable. Thus (3.13) implies that $v = 0$, as desired. \square

Proof of Theorem 2.1. First assume $F \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ is a solution to the LTONP interpolation problem. By Theorem 2.5, the function F admits an observable co-isometric realization $\Sigma = \{\alpha, \beta, \gamma, \delta\}$. Since F is a solution of the LTONP interpolation problem, by Lemma 2.10, the realization Σ is Λ -preferable, up to unitary equivalence. Hence, we may assume Σ is Λ -preferable. This implies that the system matrix M of Σ has a decomposition as in (2.18) and $M|_{\mathcal{F}} = \omega$. Now define G as in (3.1). Then, by Lemma 3.1, the function F is given by (2.9). Moreover, since the constraint $M|_{\mathcal{F}} = \omega$ is equivalent to (2.19) the fact that $M|_{\mathcal{F}} = \omega$ implies $G(0)|_{\mathcal{F}} = \omega$.

Conversely, assume $G \in \mathcal{S}(\mathcal{U} \oplus \mathcal{Z}_o, \mathcal{Y} \oplus \mathcal{Z}_o)$ with $G(0)|_{\mathcal{F}} = \omega$. We show that F given by (2.9) is a solution to the LTONP interpolation problem. Let $\widetilde{\Sigma} = \{\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}, \widetilde{\delta}\}$ be an observable co-isometric realization of G with state space \mathcal{V} . Then $\widetilde{\delta}|_{\mathcal{F}} = G(0)|_{\mathcal{F}} = \omega$. Note that the system matrix \widetilde{M} of $\widetilde{\Sigma}$ admits a decomposition as in (2.18), that is,

$$\widetilde{M} = \begin{bmatrix} \widetilde{\delta} & \widetilde{\gamma} \\ \widetilde{\beta} & \widetilde{\alpha} \end{bmatrix} = \left[\begin{array}{cc|c} \delta & \gamma_1 & \gamma_2 \\ \beta_1 & \alpha_{11} & \alpha_{12} \\ \beta_2 & \alpha_{21} & \alpha_{22} \end{array} \right] : \begin{bmatrix} \mathcal{U} \\ \mathcal{Z}_o \\ \mathcal{V} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{Y} \\ \mathcal{Z}_o \\ \mathcal{V} \end{bmatrix}$$

By Lemma 3.1 we obtain that the system

$$(3.14) \quad \Sigma = \left\{ \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}, \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}, [\gamma_1 \quad \gamma_2], \delta \right\}$$

is a co-isometric realization for the function $F \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ given by (2.9). Furthermore, $\widetilde{\delta}|_{\mathcal{F}} = \omega$ together with the fact that ω is unitary and \widetilde{M} a co-isometry, implies that $\widetilde{M}|_{\mathcal{F}} = \omega$. Hence Σ is a Λ -preferable realization. Then, by Lemma 2.9, it follows that F given by (2.9) is a solution to the LTONP interpolation problem.

It remains to show that in the characterization of the solutions to the LTONP interpolation problem given in Theorem 2.1, the functions F and G determine each other uniquely. Clearly, F is uniquely determined by G via (2.9). Thus the proof is complete when we show that for each solution F there exists a unique G as in (2.10) such that (2.9) holds.

As in the second paragraph of the present proof, let G be in the Schur class $\mathcal{S}(\mathcal{U} \oplus \mathcal{Z}_o, \mathcal{Y} \oplus \mathcal{Z}_o)$ with $G(0)|_{\mathcal{F}} = \omega$, and let the system

$$\widetilde{\Sigma} = \left\{ \alpha_{22}, [\beta_2 \quad \alpha_{21}], \begin{bmatrix} \gamma_2 \\ \alpha_{12} \end{bmatrix}, \begin{bmatrix} \delta & \gamma_1 \\ \beta_1 & \alpha_{11} \end{bmatrix} \right\}$$

be an observable co-isometric realization of G . Define F by (2.9). Then the system (3.14) is a Λ -preferable co-isometric realization of F . We claim that this realization is also observable. To see this, we use the identity (2.22). Taking adjoints in (2.22) we see that $\mathcal{Z}_o \subset \overline{\text{Im } \Gamma^*}$, where Γ is the observability operator defined by the pair $\{\gamma, \alpha\}$, i.e., as in (2.17), and hence $\text{Ker } \Gamma \subset \mathcal{V}$. In other words, condition (3.9) in Lemma 3.2 is satisfied. But then, since $\widetilde{\Sigma}$ is observable, using Lemma 3.2, we conclude that the system Σ is also observable.

Now assume $G' \in \mathcal{S}(\mathcal{U} \oplus \mathcal{Z}_o, \mathcal{Y} \oplus \mathcal{Z}_o)$ with $G'(0)|_{\mathcal{F}} = \omega$ is such that F is also given by (2.9) with G replaced by G' . Let

$$\widetilde{\Sigma}' = \left\{ \alpha'_{22}, [\beta'_2 \quad \alpha'_{21}], \begin{bmatrix} \gamma'_2 \\ \alpha'_{12} \end{bmatrix}, \begin{bmatrix} \delta' & \gamma'_1 \\ \beta'_1 & \alpha'_{11} \end{bmatrix} \right\}$$

be an observable co-isometric realization for G' . Then

$$\Sigma' = \left\{ \begin{bmatrix} \alpha'_{11} & \alpha'_{12} \\ \alpha'_{21} & \alpha'_{22} \end{bmatrix}, \begin{bmatrix} \beta'_1 \\ \beta'_2 \end{bmatrix}, [\gamma'_1 \quad \gamma'_2], \delta' \right\}$$

is a Λ -preferable co-isometric realization for F , which is observable by the same argument as used for Σ . Since all observable, co-isometric realizations of F are unitarily equivalent, by Theorem 2.5, we obtain that there exists a unitary operator U from the state space $\mathcal{Z}_\circ \oplus \mathcal{V}$ of Σ to the state space $\mathcal{Z}_\circ \oplus \mathcal{V}'$ of Σ' such that (2.35) holds, where

$$\begin{aligned} \alpha &= \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}, \quad \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}, \quad \gamma = [\gamma_1 \quad \gamma_2], \\ \alpha' &= \begin{bmatrix} \alpha'_{11} & \alpha'_{12} \\ \alpha'_{21} & \alpha'_{22} \end{bmatrix}, \quad \beta' = \begin{bmatrix} \beta'_1 \\ \beta'_2 \end{bmatrix}, \quad \gamma' = [\gamma'_1 \quad \gamma'_2]. \end{aligned}$$

By Corollary 2.12, we obtain that $U|_{\mathcal{Z}_\circ} = I_{\mathcal{Z}_\circ}$ and U maps \mathcal{V} onto \mathcal{V}' . Let $\tilde{U} = U|_{\mathcal{V}} : \mathcal{V} \rightarrow \mathcal{V}'$. Then (2.35) takes the form

$$\begin{aligned} \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \tilde{U}\alpha_{21} & \tilde{U}\alpha_{22} \end{bmatrix} &= \begin{bmatrix} \alpha'_{11} & \alpha'_{12}\tilde{U} \\ \alpha'_{21} & \alpha'_{22}\tilde{U} \end{bmatrix}, \quad \begin{bmatrix} \beta'_1 \\ \beta'_2 \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \tilde{U}\beta_2 \end{bmatrix}, \\ \begin{bmatrix} \gamma'_1 & \gamma'_2\tilde{U} \end{bmatrix} &= [\gamma_1 \quad \gamma_2]. \end{aligned}$$

This yields

$$\tilde{U}\alpha_{22} = \alpha'_{22}\tilde{U}, \quad \tilde{U}[\beta_2 \quad \alpha_{21}] = [\beta_2 \quad \alpha_{21}], \quad \begin{bmatrix} \gamma_2 \\ \alpha_{12} \end{bmatrix} = \begin{bmatrix} \gamma'_2 \\ \alpha'_{12} \end{bmatrix}\tilde{U}.$$

However, this shows that the realizations $\tilde{\Sigma}$ and $\tilde{\Sigma}'$ of G and G' , respectively, are unitarily equivalent. Hence $G = G'$. We conclude that there exists only one $G \in \mathcal{S}(\mathcal{U} \oplus \mathcal{Z}_\circ, \mathcal{Y} \oplus \mathcal{Z}_\circ)$ with $G(0)|_{\mathcal{F}} = \omega$ such that F is also given by (2.9). \square

We conclude this section with the construction of an observable co-isometric realization of the central solution F_\circ introduced in Remark 2.2. Decompose $\omega P_{\mathcal{F}}$ as

$$(3.15) \quad \omega P_{\mathcal{F}} = \begin{bmatrix} \delta_\circ & \gamma_\circ \\ \beta_\circ & \alpha_\circ \end{bmatrix} : \begin{bmatrix} \mathcal{U} \\ \mathcal{Z}_\circ \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{Y} \\ \mathcal{Z}_\circ \end{bmatrix}.$$

Then we know from (2.11) in Corollary 2.3 that

$$(3.16) \quad F_\circ(\lambda) = \delta_\circ + \lambda\gamma_\circ(I - \lambda\alpha_\circ)^{-1}\beta_\circ.$$

However, (3.15) does not provide an observable co-isometric realization of F_\circ .

Lemma 3.3. *Assume that the Pick operator Λ is non-negative. Let $\omega P_{\mathcal{F}}$ decompose as in (3.15), and define*

$$(3.17) \quad M = \begin{bmatrix} \delta & \gamma \\ \beta & \alpha \end{bmatrix} = \begin{bmatrix} \delta_\circ & \gamma_\circ & \Pi_{\mathcal{Y}} E_{\mathcal{G}'}^* \\ \beta_\circ & \alpha_\circ & \Pi_{\mathcal{Z}_\circ} E_{\mathcal{G}'}^* \\ 0 & 0 & S_{\mathcal{G}'}^* \end{bmatrix} : \begin{bmatrix} \mathcal{U} \\ \mathcal{Z}_\circ \\ \ell_+^2(\mathcal{G}') \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{Y} \\ \mathcal{Z}_\circ \\ \ell_+^2(\mathcal{G}') \end{bmatrix}.$$

Here $\Pi_{\mathcal{Z}_\circ}$ and $\Pi_{\mathcal{Y}}$ are the orthogonal projections of $\mathcal{Y} \oplus \mathcal{Z}_\circ$ onto \mathcal{Z}_\circ and \mathcal{Y} respectively. Then $\{\alpha, \beta, \gamma, \delta\}$ is a Λ -preferable observable co-isometric realization of F_\circ . Moreover, $\text{Ker } M = \mathcal{G}$.

Proof. Since $\mathcal{F} \oplus \mathcal{G} = \mathcal{U} \oplus \mathcal{Z}_\circ$ and $\mathcal{F}' \oplus \mathcal{G}' = \mathcal{Y} \oplus \mathcal{Z}_\circ$, the system matrix M can be rewritten as

$$(3.18) \quad M = \begin{bmatrix} \omega & 0 & 0 \\ 0 & 0 & E_{\mathcal{G}'}^* \\ 0 & 0 & S_{\mathcal{G}'}^* \end{bmatrix} : \begin{bmatrix} \mathcal{F} \\ \mathcal{G} \\ \ell_+^2(\mathcal{G}') \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{F}' \\ \mathcal{G}' \\ \ell_+^2(\mathcal{G}') \end{bmatrix}.$$

The fact that

$$(3.19) \quad \omega : \mathcal{F} \rightarrow \mathcal{F}' \quad \text{and} \quad \begin{bmatrix} E_{\mathcal{G}'}^* \\ S_{\mathcal{G}'}^* \end{bmatrix} : \ell_+^2(\mathcal{G}') \rightarrow \begin{bmatrix} \mathcal{G}' \\ \ell_+^2(\mathcal{G}') \end{bmatrix}$$

are both unitary maps, implies that M is a co-isometry. Moreover, we have

$$\begin{aligned} \delta_\circ + \lambda [\gamma_\circ \quad \Pi_{\mathcal{Y}} E_{\mathcal{G}'}^*] \left(I - \lambda \begin{bmatrix} \alpha_\circ & \Pi_{\mathcal{Z}_\circ} E_{\mathcal{G}'}^* \\ 0 & S_{\mathcal{G}'}^* \end{bmatrix} \right)^{-1} \begin{bmatrix} \beta_\circ \\ 0 \end{bmatrix} &= \\ &= \delta_\circ + \lambda \gamma_\circ (I - \lambda \alpha_\circ)^{-1} \beta_\circ = F_\circ(\lambda). \end{aligned}$$

Here $\Pi_{\mathcal{Z}_\circ}$ is the orthogonal projection from $\mathcal{Y} \oplus \mathcal{Z}_\circ = \mathcal{F}' \oplus \mathcal{G}'$ onto the subspace \mathcal{Z}_\circ . Hence M is the system matrix of a co-isometric realization of F_\circ . It is also clear from (3.18) that the realization $\Sigma = \{\alpha, \beta, \gamma, \delta\}$ of F_\circ is Λ -preferable.

To prove that Σ is observable, let Γ be the observability operator for the pair $\{\gamma, \alpha\}$. Note that

$$\Gamma^* = [\gamma^* \quad \alpha^* \gamma^* \quad (\alpha^*)^2 \gamma^* \quad \dots] : \ell_+^2(\mathcal{Y}) \rightarrow \begin{bmatrix} \mathcal{Z}_\circ \\ \ell_+^2(\mathcal{G}') \end{bmatrix}.$$

Furthermore, we have

$$\gamma^* = \begin{bmatrix} \gamma_\circ^* \\ E_{\mathcal{G}'} \Pi_{\mathcal{G}'} \end{bmatrix} : \mathcal{Y} \rightarrow \begin{bmatrix} \mathcal{Z}_\circ \\ \ell_+^2(\mathcal{G}') \end{bmatrix}, \quad \alpha^* = \begin{bmatrix} \alpha_\circ^* & 0 \\ E_{\mathcal{G}'} \Pi_{\mathcal{G}'} & S_{\mathcal{G}'} \end{bmatrix} \quad \text{on} \quad \begin{bmatrix} \mathcal{Z}_\circ \\ \ell_+^2(\mathcal{G}') \end{bmatrix}.$$

Here $\Pi_{\mathcal{G}'}$ is the orthogonal projection from $\mathcal{Y} \oplus \mathcal{Z}_\circ = \mathcal{F}' \oplus \mathcal{G}'$ onto the subspace \mathcal{G}' . Let \mathcal{X}_{obs} be the closure of the range of Γ^* . We have to prove that $\mathcal{X}_{\text{obs}} = \mathcal{Z}_\circ \oplus \ell_+^2(\mathcal{G}')$.

Next observe that \mathcal{X}_{obs} is an invariant subspace for α^* . By Lemma 2.9 we have $\Pi_{\mathcal{Z}_\circ}^* \Lambda^{\frac{1}{2}} = \Gamma^* W^*$. From the latter identity together with the fact that the range of Λ is dense in \mathcal{Z}_\circ , we conclude that \mathcal{Z}_\circ is a subspace of \mathcal{X}_{obs} . It follows that

$$\gamma_\circ^* y \in \mathcal{Z}_\circ \subset \mathcal{X}_{\text{obs}} \quad \text{and} \quad \gamma_\circ^* y + E_{\mathcal{G}'} \Pi_{\mathcal{G}'} y = \gamma^* y \in \mathcal{X}_{\text{obs}}, \quad y \in \mathcal{Y}.$$

These inclusions show that $E_{\mathcal{G}'} \Pi_{\mathcal{G}'} \mathcal{Y}$ is a subset of \mathcal{X}_{obs} . Next we prove that $E_{\mathcal{G}'} \Pi_{\mathcal{G}'} \mathcal{Z}_\circ$ is a subset of \mathcal{X}_{obs} . To do this recall that \mathcal{X}_{obs} is invariant under the operator α^* . But then the relation $\mathcal{Z}_\circ \subset \mathcal{X}_{\text{obs}}$ implies that $\alpha^* \mathcal{Z}_\circ$ is a subset of \mathcal{X}_{obs} . Hence

$$(3.20) \quad \begin{bmatrix} \{0\} \\ E_{\mathcal{G}'} \Pi_{\mathcal{G}'} \mathcal{Z}_\circ \end{bmatrix} \subset \mathcal{X}_{\text{obs}} \vee \begin{bmatrix} \alpha_\circ^* \\ E_{\mathcal{G}'} \Pi_{\mathcal{G}'} \end{bmatrix} \mathcal{Z}_\circ \subset \mathcal{X}_{\text{obs}} \vee \alpha^* \mathcal{Z}_\circ \subset \mathcal{X}_{\text{obs}}.$$

Here $\mathcal{L} \vee \mathcal{K}$ denotes the closure of the linear hull of the linear spaces \mathcal{L} and \mathcal{K} . We know now that both $E_{\mathcal{G}'} \Pi_{\mathcal{G}'} \mathcal{Y}$ and $E_{\mathcal{G}'} \Pi_{\mathcal{G}'} \mathcal{Z}_\circ$ are contained in \mathcal{X}_{obs} . Hence $\{0\} \oplus E_{\mathcal{G}'} \mathcal{G}'$ is a subspace for \mathcal{X}_{obs} . But then

$$\mathcal{X}_{\text{obs}} \supset \bigvee_{n=0}^{\infty} \alpha^{*n} \begin{bmatrix} \{0\} \\ E_{\mathcal{G}'} \mathcal{G}' \end{bmatrix} = \bigvee_{n=0}^{\infty} \begin{bmatrix} \{0\} \\ S_{\mathcal{G}'}^n E_{\mathcal{G}'} \mathcal{G}' \end{bmatrix} = \bigvee_{n=0}^{\infty} \begin{bmatrix} \{0\} \\ \ell_+^2(\mathcal{G}') \end{bmatrix}.$$

So \mathcal{X}_{obs} contains the whole state space $\mathcal{Z}_\circ \oplus \ell_+^2(\mathcal{G}')$. Therefore $\{\gamma, \alpha\}$ is observable, and M is a Λ -preferable observable co-isometric systems matrix.

Finally, from (3.18) and the fact that the operators in (3.19) are unitary it follows that $\text{Ker } M = \mathcal{G}$. \square

4. THE CASE WHEN THE PICK OPERATOR IS STRICTLY POSITIVE AND THE PROOF OF THEOREM 1.1

In this section we prove Theorem 1.1. Throughout $\{W, \widetilde{W}, Z\}$ is a data set for a LTONP interpolation problem, and we assume that the Pick operator Λ is strictly positive. We start with a lemma that proves the first statements in Theorem 1.1 and presents a useful formula for the unitary operator ω determined by the data set $\{W, \widetilde{W}, Z\}$.

Lemma 4.1. *Let $\{W, \widetilde{W}, Z\}$ be a data set for a LTONP interpolation problem, and assume that the Pick operator Λ is strictly positive. Then*

- (i) P is strictly positive and $\Lambda^{-1} - P^{-1}$ is nonnegative,
- (ii) Z^* is pointwise stable, in particular, its spectral radius is less than or equal to one,
- (iii) $\widetilde{B}\widetilde{B}^* + \Lambda = BB^* + Z\Lambda Z^*$ and this operator is strictly positive.

Moreover, the unitary operator $\omega : \mathcal{F} \rightarrow \mathcal{F}'$ determined by the data set $\{W, \widetilde{W}, Z\}$ is given by

$$(4.1) \quad \omega P_{\mathcal{F}} = \begin{bmatrix} B^* \\ \Lambda^{\frac{1}{2}} Z^* \end{bmatrix} K \begin{bmatrix} \widetilde{B} & \Lambda^{\frac{1}{2}} \end{bmatrix} : \begin{bmatrix} \mathcal{U} \\ \mathcal{Z} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{Y} \\ \mathcal{Z} \end{bmatrix}$$

with $K = (BB^* + Z\Lambda Z^*)^{-1} = (\widetilde{B}\widetilde{B}^* + \Lambda)^{-1}$.

Proof. Since $\Lambda = P - \widetilde{P}$ is strictly positive and $\widetilde{P} \geq 0$, we have $P = \Lambda + \widetilde{P} \geq \Lambda$. Thus $P \geq \Lambda$, and the operator P is also strictly positive. But then $P \geq \Lambda$ implies $\Lambda^{-1} \geq P^{-1}$. To see this, note that $P \geq \Lambda$ yields $I - P^{-\frac{1}{2}}\Lambda P^{-\frac{1}{2}} \geq 0$, and hence $\Lambda^{\frac{1}{2}}P^{-\frac{1}{2}}$ is a contraction. Taking adjoints, we see that $P^{-\frac{1}{2}}\Lambda^{\frac{1}{2}}$ is also a contraction, and thus $I - \Lambda^{\frac{1}{2}}P^{-\frac{1}{2}}\Lambda^{\frac{1}{2}}$ is non-negative. Multiplying both sides with $\Lambda^{-\frac{1}{2}}$ we obtain $\Lambda^{-1} \geq P^{-1}$ as desired. Finally, note that $\Lambda^{-1} - P^{-1}$ is not necessarily strictly positive. For example, choose $\widetilde{W} = 0$, then $\Lambda = P$ and $\Lambda^{-1} - P^{-1} = 0$.

To see that Z^* is pointwise stable, note that $P = WW^*$ is strictly positive by item (i). From $ZW = WS_{\mathcal{Y}}$ it follows that $S_{\mathcal{Y}}^*W^* = W^*Z^*$. Because $P = WW^*$ is strictly positive, $\|W^*x\|^2 = (WW^*x, x) \geq \epsilon\|x\|^2$ for some $\epsilon > 0$ and all x in \mathcal{Z} . Thus the range \mathcal{H} of W^* is closed and W^* can be viewed as an invertible operator from \mathcal{Z} onto \mathcal{H} . In particular, the identity $S_{\mathcal{Y}}^*W^* = W^*Z^*$ shows that \mathcal{H} is an invariant subspace for the backward shift $S_{\mathcal{Y}}^*$ and Z^* is similar to $S_{\mathcal{Y}}^*|_{\mathcal{H}}$. So the spectral radius of Z^* is less than or equal to one. Since $S_{\mathcal{Y}}^*|_{\mathcal{H}}$ is pointwise stable, Z^* is also pointwise stable.

The identity in the first part of item (iii) follows from (2.2). Since $BB^* + \Lambda \geq \Lambda$ and Λ is strictly positive, the operator $BB^* + \Lambda$ is also strictly positive, which proves the second part of item (iii). Finally, formula (4.1) is a direct corollary of Lemma A.6 by applying this lemma with $K_1 = \begin{bmatrix} \widetilde{B} & \Lambda^{\frac{1}{2}} \end{bmatrix}$ and $K_2 = \begin{bmatrix} B & Z\Lambda^{\frac{1}{2}} \end{bmatrix}$, see (2.3), and with $N = \widetilde{B}\widetilde{B}^* + \Lambda = BB^* + Z\Lambda Z^*$. \square

Using formula (4.1) we obtain the following explicit formula for the central solution F_0 .

Theorem 4.2. *Let $\{W, \widetilde{W}, Z\}$ be a data set for a LTONP interpolation problem, and assume that the Pick operator Λ is strictly positive. Then the central solution F_\circ is given by*

$$(4.2) \quad F_\circ(\lambda) = B^*(\widetilde{B}\widetilde{B}^* + \Lambda)^{-1}(I_{\mathcal{Z}} - \lambda T)^{-1}\widetilde{B}, \quad \text{where } T = \Lambda Z^*(\widetilde{B}\widetilde{B}^* + \Lambda)^{-1}.$$

Moreover, the spectral radius $r_{\text{spec}}(T)$ of T is at most 1. Finally, if \mathcal{Z} is finite dimensional, then T is exponentially stable, that is, $r_{\text{spec}}(T) < 1$.

Proof. Because Λ is strictly positive, $\mathcal{Z}_\circ = \mathcal{Z}$. Let G_\circ be the function identically equal to $\omega P_{\mathcal{F}}$. Using (4.1) we see that

$$G_\circ(\lambda) = \begin{bmatrix} B^*K\widetilde{B} & B^*K\Lambda^{\frac{1}{2}} \\ \Lambda^{\frac{1}{2}}Z^*K\widetilde{B} & \Lambda^{\frac{1}{2}}Z^*K\Lambda^{\frac{1}{2}} \end{bmatrix} : \begin{bmatrix} \mathcal{U} \\ \mathcal{Z} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{Y} \\ \mathcal{Z} \end{bmatrix}.$$

Hence, by Theorem 2.1, the central solution F_\circ (see also Corollary 2.3 and Remark 2.2) is given by

$$F_\circ(\lambda) = B^*K\widetilde{B} + \lambda B^*K\Lambda^{\frac{1}{2}} \left(I - \lambda \Lambda^{\frac{1}{2}}Z^*K\Lambda^{\frac{1}{2}} \right)^{-1} \Lambda^{\frac{1}{2}}Z^*K\widetilde{B}.$$

Using $\Lambda^{\frac{1}{2}} \left(I - \lambda \Lambda^{\frac{1}{2}}Z^*K\Lambda^{\frac{1}{2}} \right)^{-1} = (I - \lambda \Lambda Z^*K)^{-1} \Lambda^{\frac{1}{2}}$, we have

$$\begin{aligned} F_\circ(\lambda) &= B^*K\widetilde{B} + \lambda B^*K(I - \lambda \Lambda Z^*K)^{-1} \Lambda Z^*K\widetilde{B} \\ &= B^*K\widetilde{B} + B^*K(I - \lambda \Lambda Z^*K)^{-1} \left(I - (I - \lambda \Lambda Z^*K) \right) \widetilde{B} \\ &= B^*K(I - \lambda \Lambda Z^*K)^{-1} \widetilde{B}. \end{aligned}$$

Since $K = (\widetilde{B}\widetilde{B}^* + \Lambda)^{-1}$, this proves (4.2).

Since $G_\circ(\lambda) = \omega P_{\mathcal{F}}$ is a contraction, its component $A = P_{\mathcal{Z}}\omega P_{\mathcal{F}}|_{\mathcal{Z}} = \Lambda^{\frac{1}{2}}Z^*K\Lambda^{\frac{1}{2}}$ is also a contraction. Because $T = \Lambda^{\frac{1}{2}}(\Lambda^{\frac{1}{2}}Z^*K\Lambda^{\frac{1}{2}})\Lambda^{-\frac{1}{2}}$ is similar to A , it follows that $r_{\text{spec}}(T) = r_{\text{spec}}(A) \leq 1$.

Now assume that \mathcal{Z} is finite dimensional, and λ is an eigenvalue for T on the unit circle. Because T is similar to A , it follows that $Ax = \lambda x$ for some nonzero x in \mathcal{Z} . In particular, $\|Ax\| = \|\lambda x\| = \|x\|$. Since A is contained in the lower right hand corner of $\omega P_{\mathcal{F}}$ and ω is unitary, we have $\omega P_{\mathcal{F}}(0 \oplus x) = 0 \oplus \lambda x$. To see this notice that

$$\|x\|^2 \geq \|\omega P_{\mathcal{F}}(0 \oplus x)\|^2 = \|P_{\mathcal{U}}\omega P_{\mathcal{F}}(0 \oplus x)\|^2 + \|Ax\|^2 = \|P_{\mathcal{U}}\omega P_{\mathcal{F}}(0 \oplus x)\|^2 + \|x\|^2.$$

Hence $P_{\mathcal{U}}\omega P_{\mathcal{F}}(0 \oplus x) = 0$ and $\omega P_{\mathcal{F}}(0 \oplus x) = 0 \oplus Ax = 0 \oplus \lambda x$. Since ω is a unitary operator, $0 \oplus x$ must be in \mathcal{F} . So $0 \oplus x = \widetilde{B}^*\xi \oplus \Lambda^{\frac{1}{2}}\xi$ for some nonzero ξ in \mathcal{Z} , that is, $x = \Lambda^{\frac{1}{2}}\xi$. This with the definition of ω in (2.6) readily implies that

$$\begin{bmatrix} 0 \\ \lambda \Lambda^{\frac{1}{2}}\xi \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda x \end{bmatrix} = \omega P_{\mathcal{F}} \begin{bmatrix} 0 \\ x \end{bmatrix} = \omega P_{\mathcal{F}} \begin{bmatrix} \widetilde{B}^* \\ \Lambda^{\frac{1}{2}} \end{bmatrix} \xi = \begin{bmatrix} B^* \\ \Lambda^{\frac{1}{2}}Z^* \end{bmatrix} \xi.$$

In other words, $\lambda \Lambda^{\frac{1}{2}}\xi = \Lambda^{\frac{1}{2}}Z^*\xi$, or equivalently, $\lambda \xi = Z^*\xi$. This says that Z^* has an eigenvalue on the unit circle. However, \mathcal{Z} is finite dimensional and Z^{*n} converges to zero. Hence Z^* is exponentially stable, and thus all the eigenvalues of Z^* are contained in the open unit disc. Therefore λ cannot be an eigenvalue for Z^* and T must be exponentially stable. \square

Lemma 4.3. *Let $\{W, \widetilde{W}, Z\}$ be a data set for a LTONP interpolation problem and assume the Pick operator Λ is strictly positive. Further, let $C : \mathcal{Z} \rightarrow \mathcal{E}$ and $D : \mathcal{Y} \rightarrow \mathcal{E}$ form an admissible pair of complementary operators, i.e., such that (1.6) and (1.7) holds. Then the operators τ_1 and τ_2 given by*

$$(4.3) \quad \tau_1 = \begin{bmatrix} I \\ -\Lambda^{-\frac{1}{2}} \widetilde{B} \end{bmatrix} R_\circ : \mathcal{U} \rightarrow \begin{bmatrix} \mathcal{U} \\ \mathcal{Z} \end{bmatrix}, \quad \tau_2 = \begin{bmatrix} D^* \\ \Lambda^{-\frac{1}{2}} PC^* \end{bmatrix} Q_\circ : \mathcal{E} \rightarrow \begin{bmatrix} \mathcal{Y} \\ \mathcal{Z} \end{bmatrix},$$

with R_\circ and Q_\circ given by (1.14), are isometries, the range of τ_1 is \mathcal{G} and the range of τ_2 is \mathcal{G}' .

Proof. We split the proof into two parts. In the first part we deal with τ_1 and in the second part with τ_2 .

PART 1. Using the definition of R_\circ in (1.14), we have

$$\tau_1^* \tau_1 = R_\circ \begin{bmatrix} I & -\widetilde{B}^* \Lambda^{-\frac{1}{2}} \\ -\Lambda^{-\frac{1}{2}} \widetilde{B} \end{bmatrix} \begin{bmatrix} I \\ -\Lambda^{-\frac{1}{2}} \widetilde{B} \end{bmatrix} R_\circ = R_\circ \left(I + \widetilde{B}^* \Lambda^{-1} \widetilde{B} \right) R_\circ = I_{\mathcal{U}}.$$

Thus τ_1 is an isometry. In particular, the range of τ_1 is closed. Furthermore, note that

$$\tau_1^* \begin{bmatrix} \widetilde{B}^* \\ \Lambda^{\frac{1}{2}} \end{bmatrix} = R_\circ \begin{bmatrix} I & -\widetilde{B}^* \Lambda^{-\frac{1}{2}} \\ -\Lambda^{-\frac{1}{2}} \widetilde{B} \end{bmatrix} \begin{bmatrix} \widetilde{B}^* \\ \Lambda^{\frac{1}{2}} \end{bmatrix} = R_\circ (\widetilde{B}^* - \widetilde{B}^*) = 0.$$

Recall that in the present case, when Λ is strictly positive, we have

$$\mathcal{F} = \text{Im} \begin{bmatrix} \widetilde{B}^* \\ \Lambda^{\frac{1}{2}} \end{bmatrix} \quad \text{and} \quad \mathcal{G} = \mathcal{F}^\perp = \text{Ker} \begin{bmatrix} \widetilde{B} & \Lambda^{\frac{1}{2}} \end{bmatrix}.$$

The fact that $\widetilde{B} \widetilde{B}^* + \Lambda$ is strictly positive, implies that the range of the previous 2×1 operator matrix is closed. It follows that $\mathcal{F} \subset \text{Ker} \tau_1^*$, and hence $\text{Im} \tau_1 \subset \mathcal{F}^\perp = \mathcal{G}$. To prove that $\text{Im} \tau_1 = \mathcal{G}$, consider the operator

$$(4.4) \quad N = \begin{bmatrix} R_\circ & \widetilde{B}^* \\ -\Lambda^{-\frac{1}{2}} \widetilde{B} R_\circ & \Lambda^{\frac{1}{2}} \end{bmatrix} : \begin{bmatrix} \mathcal{U} \\ \mathcal{Z} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{U} \\ \mathcal{Z} \end{bmatrix}.$$

This operator matrix is invertible because the operator $\Lambda^{\frac{1}{2}}$ and the Schur complement N^\times of $\Lambda^{\frac{1}{2}}$ in N are both invertible. To see that N^\times is invertible, note that

$$N^\times = R_\circ + \widetilde{B}^* \Lambda^{-\frac{1}{2}} \Lambda^{-\frac{1}{2}} \widetilde{B} R_\circ = (I + \widetilde{B}^* \Lambda^{-1} B) R_\circ = R_\circ^{-1}.$$

Next observe that the first column of N is the operator τ_1 while the range of the second column of N is \mathcal{F} . Since N is invertible, $\text{Im} N = \mathcal{U} \oplus \mathcal{Z} = \mathcal{G} \oplus \mathcal{F}$. It follows that \mathcal{G} must be included in the range of the first column of N , that is, $\mathcal{G} \subset \text{Im} \tau_1$. But then $\text{Im} \tau_1 = \mathcal{G}$.

PART 2. First observe that Q_\circ is also given by

$$(4.5) \quad Q_\circ = (DD^* + CPA^{-1}PC^*)^{-\frac{1}{2}}.$$

To see this, note that (1.6) implies that $DD^* + CPC^* = I_{\mathcal{E}}$, and thus

$$DD^* + CPA^{-1}PC^* = I_{\mathcal{E}} - CPC^* + CPA^{-1}PC^* = I_{\mathcal{E}} + CP(\Lambda^{-1} - P^{-1})PC^*.$$

Using the definition of τ_2 in (4.3) and the formula for Q_\circ in (4.5), we obtain

$$\tau_2^* \tau_2 = Q_\circ \begin{bmatrix} D & CPA^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} D^* \\ \Lambda^{-\frac{1}{2}} PC^* \end{bmatrix} Q_\circ = Q_\circ (DD^* + CPA^{-1}PC^*) Q_\circ = I_{\mathcal{E}}.$$

Thus τ_2 is an isometry. In particular, the range of τ_2 is closed. From the identity (1.6) we know that $BD^* + ZPC^* = 0$. This implies that

$$\tau_2^* \begin{bmatrix} B^* \\ \Lambda^{\frac{1}{2}} Z^* \end{bmatrix} = Q_\circ [D \quad CP\Lambda^{-\frac{1}{2}}] \begin{bmatrix} B^* \\ \Lambda^{\frac{1}{2}} Z^* \end{bmatrix} = Q_\circ (BD^* + CPZ^*) = 0.$$

Recall that in the present strictly positive case

$$\mathcal{F}' = \text{Im} \begin{bmatrix} B^* \\ \Lambda^{\frac{1}{2}} Z^* \end{bmatrix} \quad \text{so that} \quad \mathcal{G}' = \mathcal{F}'^\perp = \text{Ker} [B \quad Z\Lambda^{\frac{1}{2}}].$$

We conclude that $\mathcal{F}' \subset \text{Ker} \tau_2^*$, and hence $\text{Im} \tau_2 \subset \mathcal{F}'^\perp = \mathcal{G}'$. To prove $\text{Im} \tau_2 = \mathcal{G}'$ we take $y \in \mathcal{Y}$ and $z \in \mathcal{Z}$, and assume that $y \oplus z \in \mathcal{G}'$ and $y \oplus z \perp \text{Im} \tau_2$. In other words, we assume that

$$(4.6) \quad \begin{bmatrix} y \\ z \end{bmatrix} \perp \mathcal{F}' = \text{Im} \begin{bmatrix} B^* \\ \Lambda^{\frac{1}{2}} Z^* \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} y \\ z \end{bmatrix} \perp \text{Im} \tau_2 = \text{Im} \begin{bmatrix} D^* \\ \Lambda^{-\frac{1}{2}} PC^* \end{bmatrix}.$$

But then

$$\begin{bmatrix} B & Z \end{bmatrix} \begin{bmatrix} y \\ \Lambda^{\frac{1}{2}} z \end{bmatrix} = 0 \quad \text{and} \quad \begin{bmatrix} D & CP\Lambda^{-1} \end{bmatrix} \begin{bmatrix} y \\ \Lambda^{\frac{1}{2}} z \end{bmatrix} = 0.$$

In other words,

$$(4.7) \quad \begin{bmatrix} D & CP\Lambda^{-1} \\ B & Z \end{bmatrix} \begin{bmatrix} y \\ \Lambda^{\frac{1}{2}} z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Now observe that

$$(4.8) \quad \begin{bmatrix} D & CP\Lambda^{-1} \\ B & Z \end{bmatrix} \begin{bmatrix} D^* & B^* \\ PC^* & PZ^* \end{bmatrix} = \begin{bmatrix} Q_\circ^{-2} & \star \\ 0 & P \end{bmatrix}$$

where \star represents an unspecified entry. The identities (1.6) and (1.7) imply that the operator matrix

$$\begin{bmatrix} D^* & B^* \\ PC^* & PZ^* \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} D^* & B^* \\ C^* & Z^* \end{bmatrix}$$

is invertible. Because Q_\circ and P are both invertible, the matrix on the right hand side of (4.8) is invertible. So the operator matrix on the left hand side of (4.8) or (4.7) is invertible. Thus $y \oplus \Lambda^{\frac{1}{2}} z = 0$. Since $\Lambda^{\frac{1}{2}}$ is invertible, both y and z are zero. This can only happen when $\mathcal{G}' = \text{Im} \tau_2$. \square

Corollary 4.4. *Let $\{W, \widetilde{W}, Z\}$ be a data set for a LTONP interpolation problem and assume the Pick operator Λ is strictly positive. Then all functions G in $\mathcal{S}(\mathcal{U} \oplus \mathcal{Z}, \mathcal{Y} \oplus \mathcal{Z})$ with $G(0)|_{\mathcal{F}} = \omega$ are given by*

$$(4.9) \quad G(\lambda) = \begin{bmatrix} G_{11}(\lambda) & G_{12}(\lambda) \\ G_{21}(\lambda) & G_{22}(\lambda) \end{bmatrix} = \\ = \begin{bmatrix} B^* K \widetilde{B} + D^* Q_\circ X(\lambda) R_\circ & B^* K \Lambda^{\frac{1}{2}} - D^* Q_\circ X(\lambda) R_\circ \widetilde{B}^* \Lambda^{-\frac{1}{2}} \\ \Lambda^{\frac{1}{2}} Z^* K \widetilde{B} + \Lambda^{-\frac{1}{2}} PC^* Q_\circ X(\lambda) R_\circ & \Lambda^{\frac{1}{2}} Z^* K \Lambda^{\frac{1}{2}} - \Lambda^{-\frac{1}{2}} PC^* Q_\circ X(\lambda) R_\circ \widetilde{B}^* \Lambda^{-\frac{1}{2}} \end{bmatrix}$$

with X is an arbitrary Schur class function in $\mathcal{S}(\mathcal{U}, \mathcal{E})$. Moreover, G and X determine each other uniquely. Furthermore, we have

$$(4.10) \quad \begin{aligned} G_{12}(\lambda) &= (B^* - G_{11}(\lambda) \widetilde{B}^*) \Lambda^{-\frac{1}{2}}, \\ G_{22}(\lambda) &= \Lambda^{\frac{1}{2}} (Z^* - \Lambda^{-\frac{1}{2}} G_{21}(\lambda) \widetilde{B}^*) \Lambda^{-\frac{1}{2}}. \end{aligned}$$

Proof. The fact that $\omega : \mathcal{F} \rightarrow \mathcal{F}'$ is unitary implies that $G \in \mathcal{S}(\mathcal{U} \oplus \mathcal{Z}, \mathcal{Y} \oplus \mathcal{Z})$ satisfies $G(0)|_{\mathcal{F}} = \omega$ if and only if $G(\lambda) = \omega P_{\mathcal{F}} + \tilde{X}(\lambda)P_{\mathcal{G}}$, $\lambda \in \mathbb{D}$, for some $\tilde{X} \in \mathcal{S}(\mathcal{G}, \mathcal{G}')$. Since the operators τ_1 and τ_2 introduced in Lemma 4.3 are isometries with ranges equal to \mathcal{G} and \mathcal{G}' , respectively, it follows (see Lemma A.5) that \tilde{X} is in $\mathcal{S}(\mathcal{G}, \mathcal{G}')$ if and only if $\tilde{X}(\lambda) = \tau_2 X(\lambda) \tau_1^*$, $\lambda \in \mathbb{D}$, for a $X \in \mathcal{S}(\mathcal{U}, \mathcal{E})$, namely $X(\lambda) \equiv \tau_2^* \tilde{X}(\lambda) \tau_1$. Hence the Schur class functions $G \in \mathcal{S}(\mathcal{U} \oplus \mathcal{Z}, \mathcal{Y} \oplus \mathcal{Z})$ with $G(0)|_{\mathcal{F}} = \omega$ are characterized by $G(\lambda) = \omega P_{\mathcal{F}} + \tau_2 X(\lambda) \tau_1^*$ with $X \in \mathcal{S}(\mathcal{U}, \mathcal{E})$. It is clear from the above constructions that G and \tilde{X} determine each other uniquely, and that \tilde{X} and X determine each other uniquely. Hence G and X determine each other uniquely. Using the formulas for $\omega P_{\mathcal{F}}$ and τ_1 and τ_2 obtained in Lemmas 4.1 and 4.3 we see that $\omega P_{\mathcal{F}} + \tau_2 X(\lambda) \tau_1^*$ coincides with the right-hand side of (4.9).

It remains to derive (4.10). Note that

$$K = (\Lambda + \tilde{B}\tilde{B}^*)^{-1} = \Lambda^{-1} - \Lambda^{-1}\tilde{B}R_{\circ}^2\tilde{B}^*\Lambda^{-1}.$$

This implies that

$$\begin{aligned} K\tilde{B} &= \Lambda^{-1}\tilde{B}(I - R_{\circ}^2\tilde{B}^*\Lambda^{-1}\tilde{B}) = \Lambda^{-1}\tilde{B}R_{\circ}^2, \\ K\Lambda &= (\Lambda^{-1} - \Lambda^{-1}\tilde{B}R_{\circ}^2\tilde{B}^*\Lambda^{-1})\Lambda = I - \Lambda^{-1}\tilde{B}R_{\circ}^2\tilde{B}^* = I - K\tilde{B}\tilde{B}^*. \end{aligned}$$

Summarising we have

$$(4.11) \quad K\tilde{B} = \Lambda^{-1}\tilde{B}R_{\circ}^2 \quad \text{and} \quad K\Lambda = I - K\tilde{B}\tilde{B}^*.$$

We now obtain that

$$\begin{aligned} G_{12}(\lambda) &= \left(B^*K\Lambda - D^*Q_{\circ}X(\lambda)R_{\circ}\tilde{B}^* \right) \Lambda^{-\frac{1}{2}} \\ &= \left(B^* - B^*K\tilde{B}\tilde{B}^* - D^*Q_{\circ}X(\lambda)R_{\circ}\tilde{B}^* \right) \Lambda^{-\frac{1}{2}} \\ &= \left(B^* - \left(B^*K\tilde{B} + D^*Q_{\circ}X(\lambda)R_{\circ} \right) \tilde{B}^* \right) \Lambda^{-\frac{1}{2}} \\ &= \left(B^* - G_{11}(\lambda)\tilde{B}^* \right) \Lambda^{-\frac{1}{2}}, \end{aligned}$$

and

$$\begin{aligned} G_{22}(\lambda) &= \Lambda^{\frac{1}{2}} \left(Z^*K\Lambda - \Lambda^{-1}PC^*Q_{\circ}X(\lambda)R_{\circ}\tilde{B}^* \right) \Lambda^{-\frac{1}{2}} \\ &= \Lambda^{\frac{1}{2}} \left(Z^* - Z^*K\tilde{B}\tilde{B}^* - \Lambda^{-1}PC^*Q_{\circ}X(\lambda)R_{\circ}\tilde{B}^* \right) \Lambda^{-\frac{1}{2}} \\ &= \Lambda^{\frac{1}{2}} \left(Z^* - \left(Z^*K\tilde{B} + \Lambda^{-1}PC^*Q_{\circ}X(\lambda)R_{\circ} \right) \tilde{B}^* \right) \Lambda^{-\frac{1}{2}} \\ &= \Lambda^{\frac{1}{2}} \left(Z^* - \Lambda^{-\frac{1}{2}}G_{21}(\lambda)\tilde{B}^* \right) \Lambda^{-\frac{1}{2}}, \end{aligned}$$

as claimed. \square

Proof of Theorem 1.1. The first statements in Theorem 1.1 are covered by Lemma 4.1. Clearly the operators Q_{\circ} and R_{\circ} are well defined. Since the spectral radius of Z is at most one, the operator-valued functions Υ_{ij} , $i, j = 1, 2$, given by (1.10) – (1.13) are well defined and analytic on \mathbb{D} . Given these functions it remains to prove the main part of the theorem describing all solutions of the LTONP interpolation problem by (1.9).

Let $X \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ be an arbitrary Schur class function. Define G in $\mathcal{S}(\mathcal{U} \oplus \mathcal{Z}, \mathcal{Y} \oplus \mathcal{Z})$ by $G(\lambda) = \omega P_{\mathcal{F}} + \tau_2 X(\lambda) \tau_1^*$, $\lambda \in \mathbb{D}$, where τ_1 and τ_2 are given by (4.3). Hence G is given by (4.9) and we have (4.10). Set

$$F(\lambda) = G_{11}(\lambda) + \lambda G_{12}(\lambda) \left(I - \lambda G_{22}(\lambda) \right)^{-1} G_{21}(\lambda), \quad \lambda \in \mathbb{D}.$$

By item (ii) in Lemma 4.1 the spectral radius of Z is at most one, and hence the same holds true for spectral radius of Z^* . Thus $I - \lambda Z^*$ is invertible for each $\lambda \in \mathbb{D}$. Now fix a $\lambda \in \mathbb{D}$. Since $G \in \mathcal{S}(\mathcal{U} \oplus \mathcal{Z}, \mathcal{Y} \oplus \mathcal{Z})$, we have $G_{22} \in \mathcal{S}(\mathcal{Z}, \mathcal{Z})$ and thus $I - \lambda G_{22}(\lambda)$ is invertible. Notice that

$$\begin{aligned} I - \lambda G_{22}(\lambda) &= \Lambda^{\frac{1}{2}} \left(I - \lambda Z^* + \lambda \Lambda^{-\frac{1}{2}} G_{21}(\lambda) \tilde{B}^* \right) \Lambda^{-\frac{1}{2}} \\ &= \Lambda^{\frac{1}{2}} (I - \lambda Z^*) \left(I + \lambda (I - \lambda Z^*)^{-1} \Lambda^{-\frac{1}{2}} G_{21}(\lambda) \tilde{B}^* \right) \Lambda^{-\frac{1}{2}}. \end{aligned}$$

The above identity shows that $I + \lambda (I - \lambda Z^*)^{-1} \Lambda^{-\frac{1}{2}} G_{21}(\lambda) \tilde{B}^*$ is invertible. Applying the rule that $I + AB$ is invertible if and only if $I + BA$ is invertible, we obtain that the operator $I + \lambda \tilde{B}^* (I - \lambda Z^*)^{-1} \Lambda^{-\frac{1}{2}} G_{21}(\lambda)$ is invertible. Next, using the rule $(I + AB)^{-1} A = A(I + BA)^{-1}$ we obtain

$$\begin{aligned} (I - \lambda G_{22}(\lambda))^{-1} G_{21}(\lambda) &= \\ &= \Lambda^{\frac{1}{2}} \left(I + \lambda (I - \lambda Z^*)^{-1} \Lambda^{-\frac{1}{2}} G_{21}(\lambda) \tilde{B}^* \right)^{-1} (I - \lambda Z^*)^{-1} \Lambda^{-\frac{1}{2}} G_{21}(\lambda) \\ &= \Lambda^{\frac{1}{2}} (I - \lambda Z^*)^{-1} \Lambda^{-\frac{1}{2}} G_{21}(\lambda) \left(I + \lambda \tilde{B}^* (I - \lambda Z^*)^{-1} \Lambda^{-\frac{1}{2}} G_{21}(\lambda) \right)^{-1}. \end{aligned}$$

From the first identity in (4.10) we obtain

$$\begin{aligned} \lambda G_{12}(\lambda) \Lambda^{\frac{1}{2}} (I - \lambda Z^*)^{-1} \Lambda^{-\frac{1}{2}} G_{21}(\lambda) &= \\ &= \lambda \left(B^* - G_{11}(\lambda) \tilde{B}^* \right) (I - \lambda Z^*)^{-1} \Lambda^{-\frac{1}{2}} G_{21}(\lambda) \\ &= \lambda B^* (I - \lambda Z^*)^{-1} \Lambda^{-\frac{1}{2}} G_{21}(\lambda) - \lambda G_{11}(\lambda) \tilde{B}^* (I - \lambda Z^*)^{-1} \Lambda^{-\frac{1}{2}} G_{21}(\lambda) \\ &= \lambda B^* (I - \lambda Z^*)^{-1} \Lambda^{-\frac{1}{2}} G_{21}(\lambda) + G_{11}(\lambda) + \\ &\quad - G_{11}(\lambda) \left(I + \lambda \tilde{B}^* (I - \lambda Z^*)^{-1} \Lambda^{-\frac{1}{2}} G_{21}(\lambda) \right). \end{aligned}$$

Summarising we have shown that

$$\begin{aligned} (I - \lambda G_{22}(\lambda))^{-1} G_{21}(\lambda) &= \Lambda^{\frac{1}{2}} (I - \lambda Z^*)^{-1} \Lambda^{-\frac{1}{2}} G_{21}(\lambda) \Xi(\lambda) \\ \lambda G_{12}(\lambda) \Lambda^{\frac{1}{2}} (I - \lambda Z^*)^{-1} \Lambda^{-\frac{1}{2}} G_{21}(\lambda) &= \\ &= G_{11}(\lambda) + \lambda B^* (I - \lambda Z^*)^{-1} \Lambda^{-\frac{1}{2}} G_{21}(\lambda) - G_{11}(\lambda) \Xi(\lambda)^{-1}, \\ &\quad \text{where } \Xi(\lambda) = \left(I + \lambda \tilde{B}^* (I - \lambda Z^*)^{-1} \Lambda^{-\frac{1}{2}} G_{21}(\lambda) \right)^{-1}. \end{aligned}$$

It follows that

$$\begin{aligned}
 F(\lambda) &= G_{11}(\lambda) + \lambda G_{12}(\lambda) (I - \lambda G_{22}(\lambda))^{-1} G_{21}(\lambda) \\
 &= G_{11}(\lambda) + \lambda G_{12}(\lambda) \Lambda^{\frac{1}{2}} (I - \lambda Z^*)^{-1} \Lambda^{-\frac{1}{2}} G_{21}(\lambda) \Xi(\lambda) \\
 &= G_{11}(\lambda) + \left(G_{11}(\lambda) + \lambda B^* (I - \lambda Z^*)^{-1} \Lambda^{-\frac{1}{2}} G_{21}(\lambda) \right) \Xi(\lambda) - G_{11}(\lambda) \\
 &= \left(G_{11}(\lambda) + \lambda B^* (I - \lambda Z^*)^{-1} \Lambda^{-\frac{1}{2}} G_{21}(\lambda) \right) \times \\
 &\quad \times \left(I + \lambda \tilde{B}^* (I - \lambda Z^*)^{-1} \Lambda^{-\frac{1}{2}} G_{12}(\lambda) \right)^{-1}.
 \end{aligned}$$

To prove the parametrization of solutions through (1.9) it remains to show that

$$(4.12) \quad G_{11}(\lambda) + \lambda B^* (I - \lambda Z^*)^{-1} \Lambda^{-\frac{1}{2}} G_{21}(\lambda) = (\Upsilon_{12}(\lambda) + \Upsilon_{11}(\lambda) X(\lambda)) R_{\circ},$$

$$(4.13) \quad I + \lambda \tilde{B}^* (I - \lambda Z^*)^{-1} \Lambda^{-\frac{1}{2}} G_{12}(\lambda) = (\Upsilon_{22}(\lambda) + \Upsilon_{21}(\lambda) X(\lambda)) R_{\circ}.$$

Note that these two identities show that F is given by (1.9) and, combined with Theorem 2.1, this yields that all solutions to the LTONP interpolation problem are given by (1.9). Hence we have proved Theorem 1.1 once these two identities are established.

Using (4.11) we obtain that

$$\begin{aligned}
 (I - \lambda Z^*)^{-1} \Lambda^{-\frac{1}{2}} G_{21}(\lambda) &= \\
 &= (I - \lambda Z^*)^{-1} Z^* K \tilde{B} + (I - \lambda Z^*)^{-1} \Lambda^{-1} P C^* Q_{\circ} X(\lambda) R_{\circ} \\
 &= \left((I - \lambda Z^*)^{-1} Z^* \Lambda^{-1} \tilde{B} R_{\circ} + (I - \lambda Z^*)^{-1} \Lambda^{-1} P C^* Q_{\circ} X(\lambda) \right) R_{\circ}.
 \end{aligned}$$

Therefore, we have

$$\begin{aligned}
 I + \lambda \tilde{B}^* (I - \lambda Z^*)^{-1} \Lambda^{-\frac{1}{2}} G_{21}(\lambda) &= I + \\
 &+ \left(\lambda \tilde{B}^* (I - \lambda Z^*)^{-1} Z^* \Lambda^{-1} \tilde{B} R_{\circ} + \lambda \tilde{B}^* (I - \lambda Z^*)^{-1} \Lambda^{-1} P C^* Q_{\circ} X(\lambda) \right) R_{\circ} \\
 (4.14) \quad &= \left(R_{\circ}^{-1} + \lambda \tilde{B}^* (I - \lambda Z^*)^{-1} Z^* \Lambda^{-1} \tilde{B} R_{\circ} + \Upsilon_{21}(\lambda) X(\lambda) \right) R_{\circ}.
 \end{aligned}$$

From the definition R_{\circ} in (1.14) it follows that $R_{\circ}^{-2} - \tilde{B}^* \Lambda^{-1} \tilde{B} = I_{\mathcal{U}}$, and hence

$$\begin{aligned}
 R_{\circ}^{-1} + \lambda \tilde{B}^* (I - \lambda Z^*)^{-1} Z^* \Lambda^{-1} \tilde{B} R_{\circ} &= \\
 &= R_{\circ}^{-1} + \tilde{B}^* (I - \lambda Z^*)^{-1} (I - (I - \lambda Z^*)) \Lambda^{-1} \tilde{B} R_{\circ} \\
 &= R_{\circ}^{-1} - \tilde{B}^* \Lambda^{-1} \tilde{B} R_{\circ} + \tilde{B}^* (I - \lambda Z^*)^{-1} \Lambda^{-1} \tilde{B} R_{\circ} \\
 &= \left(R_{\circ}^{-2} - \tilde{B}^* \Lambda^{-1} \tilde{B} \right) R_{\circ} + \tilde{B}^* (I - \lambda Z^*)^{-1} \Lambda^{-1} \tilde{B} R_{\circ} \\
 (4.15) \quad &= R_{\circ} + \tilde{B}^* (I - \lambda Z^*)^{-1} \Lambda^{-1} \tilde{B} R_{\circ} = \Upsilon_{22}(\lambda).
 \end{aligned}$$

Inserting the identity (4.15) in (4.14) we obtain the identity (4.13).

We proceed with the left hand side of (4.12).

$$\begin{aligned}
G_{11}(\lambda) + \lambda B^*(I - \lambda Z^*)^{-1} \Lambda^{-\frac{1}{2}} G_{21}(\lambda) &= B^* K \tilde{B} + D^* Q_\circ X(\lambda) R_\circ + \\
&+ \lambda B^* \left((I - \lambda Z^*)^{-1} Z^* \Lambda^{-1} \tilde{B} R_\circ + (I - \lambda Z^*)^{-1} \Lambda^{-1} P C^* Q_\circ X(\lambda) \right) R_\circ \\
&= B^* \Lambda^{-1} \tilde{B} R_\circ^2 + \lambda B^* (I - \lambda Z^*)^{-1} Z^* \Lambda^{-1} \tilde{B} R_\circ^2 + \\
&\quad + \left(D^* Q_\circ + \lambda B^* (I - \lambda Z^*)^{-1} \Lambda^{-1} P C^* Q_\circ \right) X(\lambda) R_\circ \\
(4.16) \quad &= B^* \Lambda^{-1} \tilde{B} R_\circ^2 + \lambda B^* (I - \lambda Z^*)^{-1} Z^* \Lambda^{-1} \tilde{B} R_\circ^2 + \Upsilon_{11}(\lambda) X(\lambda) R_\circ.
\end{aligned}$$

Next we compute

$$\begin{aligned}
B^* \Lambda^{-1} \tilde{B} R_\circ^2 + \lambda B^* (I - \lambda Z^*)^{-1} Z^* \Lambda^{-1} \tilde{B} R_\circ^2 &= \\
&= B^* \Lambda^{-1} \tilde{B} R_\circ^2 + B^* (I - \lambda Z^*)^{-1} (I - (I - \lambda Z^*)) \Lambda^{-1} \tilde{B} R_\circ^2 \\
&= B^* \Lambda^{-1} \tilde{B} R_\circ^2 - B^* \Lambda^{-1} \tilde{B} R_\circ^2 + B^* (I - \lambda Z^*)^{-1} \Lambda^{-1} \tilde{B} R_\circ^2 \\
(4.17) \quad &= B^* (I - \lambda Z^*)^{-1} \Lambda^{-1} \tilde{B} R_\circ^2 = \Upsilon_{12}(\lambda) R_\circ.
\end{aligned}$$

Inserting the identity (4.17) in (4.16) we obtain the identity (4.12). Hence we have shown that all solutions are obtained through (1.9).

To complete the proof we show that the map $X \mapsto F$ given by (1.9) is one-to-one. This is a direct consequence of the uniqueness claims in Corollary (4.4) and Theorem 2.1. Indeed, by Corollary (4.4), the map $X \mapsto G$ from $\mathcal{S}(\mathcal{U}, \mathcal{E})$ to $\mathcal{S}(\mathcal{U} \oplus \mathcal{Z}_\circ, \mathcal{Y} \oplus \mathcal{Z}_\circ)$ given by (4.9) is one-to-one, and each G obtained in this way has $G(0)\mathcal{F} = \omega$. By Theorem 2.1, the map $G \mapsto F$ from the set of $G \in \mathcal{S}(\mathcal{U} \oplus \mathcal{Z}_\circ, \mathcal{Y} \oplus \mathcal{Z}_\circ)$ with $G(0)\mathcal{F} = \omega$ to the set of solutions in $\mathcal{S}(\mathcal{U}, \mathcal{Y})$ given by (2.9) is also one-to-one. Since the map $X \mapsto F$ defined here is the composition of these two maps, it follows that this map is one-to-one as well. \square

5. PROOF OF THEOREM 1.2

We begin with a general remark concerning the formulas for the functions Υ_{ij} , $1 \leq i, j \leq 2$, appearing in Theorem 1.1.

Let $\{W, \tilde{W}, Z\}$ be a LTONP data set, and assume that the associate Pick operator Λ is strictly positive. Then Z^* is pointwise stable. Using the definitions of $B = W E_{\mathcal{Y}}$ and $\tilde{B} = \tilde{W} E_{\mathcal{U}}$ (see in (1.4) and (1.5)) with the intertwining relations $S_{\mathcal{Y}}^* W^* = W^* Z^*$ and $S_{\mathcal{U}}^* \tilde{W}^* = \tilde{W}^* Z^*$ (see (1.1)), we obtain

$$\begin{aligned}
B^*(I - \lambda Z^*)^{-1} &= E_{\mathcal{Y}}^* W^*(I - \lambda Z^*)^{-1} = E_{\mathcal{Y}}^*(I - \lambda S_{\mathcal{Y}}^*)^{-1} W^* \quad (\lambda \in \mathbb{D}), \\
\tilde{B}^*(I - \lambda Z^*)^{-1} &= E_{\mathcal{U}}^* \tilde{W}^*(I - \lambda Z^*)^{-1} = E_{\mathcal{U}}^*(I - \lambda S_{\mathcal{U}}^*)^{-1} \tilde{W}^* \quad (\lambda \in \mathbb{D}).
\end{aligned}$$

It follows that the formulas (1.10) – (1.13) can be rewritten as follows:

$$(5.1) \quad \Upsilon_{11}(\lambda) = D^* Q_\circ + \lambda E_{\mathcal{Y}}^*(I - \lambda S_{\mathcal{Y}}^*)^{-1} W^* \Lambda^{-1} P C^* Q_\circ,$$

$$(5.2) \quad \Upsilon_{12}(\lambda) = E_{\mathcal{Y}}^*(I - \lambda S_{\mathcal{Y}}^*)^{-1} W^* \Lambda^{-1} \tilde{B} R_\circ,$$

$$(5.3) \quad \Upsilon_{21}(\lambda) = \lambda E_{\mathcal{U}}^*(I - \lambda S_{\mathcal{U}}^*)^{-1} \tilde{W}^* \Lambda^{-1} P C^* Q_\circ,$$

$$(5.4) \quad \Upsilon_{22}(\lambda) = R_\circ + E_{\mathcal{U}}^*(I - \lambda S_{\mathcal{U}}^*)^{-1} \tilde{W}^* \Lambda^{-1} \tilde{B} R_\circ.$$

Proof of Theorem 1.2. As before let $\{W, \widetilde{W}, Z\}$ be a LTONP data set, and assume that the associate Pick operator Λ is strictly positive. Note that

$$WW^* = \Lambda + \widetilde{W}\widetilde{W}^* \gg 0.$$

Hence $P = WW^*$ is also strictly positive. It follows that the operator $A = W^*P^{-1}\widetilde{W}$ in (1.15) is well-defined. Finally, it is noted that $WA = \widetilde{W}$.

We first show that A is strictly contractive following arguments similar to the ones used in [19, Remark II.1.4]. Note that

$$\begin{aligned} I - A^*A &= I - \widetilde{W}^*P^{-1}WW^*P^{-1}\widetilde{W} = I - \widetilde{W}^*P^{-1}\widetilde{W} \\ &= I - \left(\widetilde{W}^*P^{-\frac{1}{2}}\right) \left(P^{-\frac{1}{2}}\widetilde{W}\right). \end{aligned}$$

Put $W_0 = P^{-\frac{1}{2}}W$ and $\widetilde{W}_0 = P^{-\frac{1}{2}}\widetilde{W}$. Then $I - A^*A = I - \widetilde{W}_0^*\widetilde{W}_0$. Furthermore,

$$\begin{aligned} I - \widetilde{W}_0^*\widetilde{W}_0 &= I - P^{-\frac{1}{2}}\widetilde{W}\widetilde{W}^*P^{-\frac{1}{2}} = P^{-\frac{1}{2}} \left(P - \widetilde{W}\widetilde{W}^*\right) P^{-\frac{1}{2}} \\ &= P^{-\frac{1}{2}}\Lambda P^{-\frac{1}{2}} \gg 0. \end{aligned}$$

Thus \widetilde{W}_0^* is a strict contraction, and hence the same holds true for \widetilde{W}_0 . We conclude that

$$I - A^*A = I - \widetilde{W}_0^*\widetilde{W}_0 \gg 0,$$

and A is a strict contraction.

From the above calculations it follows that $I - A^*A$ is invertible and we can obtain the inverse of $I - A^*A$ by using the standard operator identity:

$$(5.5) \quad (I - ML)^{-1} = I + M(I - LM)^{-1}L.$$

Indeed, we have

$$\begin{aligned} (I - A^*A)^{-1} &= (I - \widetilde{W}_0^*\widetilde{W}_0)^{-1} = I + \widetilde{W}_0^* \left(I - \widetilde{W}_0^*\widetilde{W}_0\right)^{-1} \widetilde{W}_0 \\ &= I + \widetilde{W}^*P^{-\frac{1}{2}} \left(I - P^{-\frac{1}{2}}\widetilde{W}\widetilde{W}^*P^{-\frac{1}{2}}\right)^{-1} P^{-\frac{1}{2}}\widetilde{W} \\ &= I + \widetilde{W}^* \left(P - \widetilde{W}\widetilde{W}^*\right)^{-1} \widetilde{W} = I + \widetilde{W}^*\Lambda^{-1}\widetilde{W}. \end{aligned}$$

This readily implies that

$$(5.6) \quad (I - A^*A)^{-1} = I + \widetilde{W}^*\Lambda^{-1}\widetilde{W}.$$

Next we derive formulas (1.20) and (1.21). We begin with Q_\circ . Note that

$$\begin{aligned} A(I - A^*A)^{-1}A^* &= W^*P^{-1}\widetilde{W} \left(I + \widetilde{W}^*\Lambda^{-1}\widetilde{W}\right) \widetilde{W}^*P^{-1}W \\ &= W^*P^{-1}\widetilde{W}\widetilde{W}^*P^{-1}W + W^*P^{-1}\widetilde{W}\widetilde{W}^*\Lambda^{-1}\widetilde{W}\widetilde{W}^*P^{-1}W \\ &= W^*P^{-1}(P - \Lambda)P^{-1}W + W^*P^{-1}(P - \Lambda)\Lambda^{-1}(P - \Lambda)P^{-1}W \\ &= W^*P^{-1}(P - \Lambda)P^{-1}W + W^*P^{-1}(P - \Lambda)\Lambda^{-1}W + \\ &\quad - W^*P^{-1}(P - \Lambda)P^{-1}W \\ &= W^*P^{-1}(P - \Lambda)\Lambda^{-1}W = W^*\Lambda^{-1}W - W^*P^{-1}W. \end{aligned}$$

In other words,

$$(5.7) \quad A(I - A^*A)^{-1}A^* = W^*\Lambda^{-1}W - W^*P^{-1}W.$$

Thus

$$WA(I - A^*A)^{-1}A^*W^* = P\Lambda^{-1}P - P = P(\Lambda^{-1} - P^{-1})P.$$

Combining this with $Q_\circ = (I + CP(\Lambda^{-1} - P^{-1})PC^*)^{-\frac{1}{2}}$ (see (1.14)) yields the formula $Q_\circ = \left(I + CWA(I - A^*A)^{-1}A^*W^*C^*\right)^{-\frac{1}{2}}$ for Q_\circ in (1.20).

We proceed by deriving formula (1.21). According to the right hand side of (1.14) and using the identity (5.6) we have

$$\begin{aligned} R_\circ &= \left(I_{\mathcal{U}} + \widetilde{B}^*\Lambda^{-1}\widetilde{B}\right)^{-\frac{1}{2}} = \left(I_{\mathcal{U}} + E_{\mathcal{U}}^*\widetilde{W}^*\Lambda^{-1}\widetilde{W}E_{\mathcal{U}}\right)^{-\frac{1}{2}} \\ &= \left(E_{\mathcal{U}}^*\left(I + \widetilde{W}^*\Lambda^{-1}\widetilde{W}\right)E_{\mathcal{U}}\right)^{-\frac{1}{2}} = \left(E_{\mathcal{U}}^*(I - A^*A)^{-1}E_{\mathcal{U}}\right)^{-\frac{1}{2}}. \end{aligned}$$

We conclude that (1.21) is proved.

It remains to show that formulas (1.10) - (1.13) can be rewritten as (1.16) - (1.19), respectively. To do this we use the remark preceding the present proof. In other words we may assume that the functions Υ_{ij} , $1 \leq i, j \leq 2$, are given by (5.1) - (5.4). Then, to derive (1.16) - (1.19), it suffices to show that

$$(5.8) \quad W^*\Lambda^{-1}P = (I - AA^*)^{-1}W^*, \quad W^*\Lambda^{-1}\widetilde{B} = A(I - A^*A)^{-1}E_{\mathcal{U}},$$

$$(5.9) \quad \widetilde{W}^*\Lambda^{-1}P = A^*(I - AA^*)^{-1}W^*, \quad \widetilde{W}^*\Lambda^{-1}\widetilde{B} = (I - A^*A)^{-1}E_{\mathcal{U}} - E_{\mathcal{U}}.$$

Obviously, the first three identities are enough to derive formulas (1.16), (1.17), and (1.18) from the formulas (5.1), (5.2), and (5.3), respectively. To see that a similar result holds true for the second identity in (5.9), note that this second identity in (5.9) implies that

$$\begin{aligned} \Upsilon_{22}(\lambda) &= R_\circ + E_{\mathcal{U}}^*(I - \lambda S_{\mathcal{U}}^*)^{-1}\widetilde{W}^*\Lambda^{-1}\widetilde{B}R_\circ \\ &= R_\circ + E_{\mathcal{U}}^*(I - \lambda S_{\mathcal{U}}^*)^{-1}\left((I - A^*A)^{-1} - I\right)E_{\mathcal{U}}R_\circ \\ &= R_\circ + E_{\mathcal{U}}^*(I - \lambda S_{\mathcal{U}}^*)^{-1}(I - A^*A)^{-1}E_{\mathcal{U}}R_\circ - E_{\mathcal{U}}^*(I - \lambda S_{\mathcal{U}}^*)^{-1}E_{\mathcal{U}}R_\circ \\ &= R_\circ - E_{\mathcal{U}}^*E_{\mathcal{U}}R_\circ + E_{\mathcal{U}}^*(I - \lambda S_{\mathcal{U}}^*)^{-1}(I - A^*A)^{-1}E_{\mathcal{U}}R_\circ \\ &= E_{\mathcal{U}}^*(I - \lambda S_{\mathcal{U}}^*)^{-1}(I - A^*A)^{-1}E_{\mathcal{U}}R_\circ, \end{aligned}$$

which proves (1.19).

It remains to prove the four identities in (5.8) and (5.9). Note that the second identity in (5.9) follows from (5.6). Indeed,

$$\widetilde{W}^*\Lambda^{-1}\widetilde{B} = \widetilde{W}^*\Lambda^{-1}\widetilde{W}E_{\mathcal{U}} = \left((I - A^*A)^{-1} - I\right)E_{\mathcal{U}} = (I - A^*A)^{-1}E_{\mathcal{U}} - E_{\mathcal{U}}.$$

To prove the other identities we first use (5.6) to show that

$$\begin{aligned} A(I - A^*A)^{-1} &= W^*P^{-1}\widetilde{W}\left(I + \widetilde{W}^*\Lambda^{-1}\widetilde{W}\right) \\ &= W^*P^{-1}\widetilde{W} + W^*P^{-1}\widetilde{W}\widetilde{W}^*\Lambda^{-1}\widetilde{W} \\ &= W^*P^{-1}\widetilde{W} + W^*P^{-1}(P - \Lambda)\Lambda^{-1}\widetilde{W} \\ &= W^*P^{-1}\widetilde{W} + W^*\Lambda^{-1}\widetilde{W} - W^*P^{-1}\widetilde{W} \\ (5.10) \quad &= W^*\Lambda^{-1}\widetilde{W}. \end{aligned}$$

Since $W^*\Lambda^{-1}\widetilde{B} = W^*\Lambda^{-1}\widetilde{W}E_{\mathcal{U}}$, formula (5.10) yields the second identity in (5.8).

Next, using the general identity (5.5) and the identity (5.7), we see that

$$(5.11) \quad (I - AA^*)^{-1} = I + A(I - A^*A)^{-1}A^* = I + W^*\Lambda^{-1}W - W^*P^{-1}W.$$

It follows that

$$(5.12) \quad \begin{aligned} (I - AA^*)^{-1}W^* &= W^* + W^*\Lambda^{-1}WW^* - W^*P^{-1}WW^* \\ &= W^* + W^*\Lambda^{-1}P - W^*P^{-1}P \\ &= W^*\Lambda^{-1}P. \end{aligned}$$

This proves the first identity in (5.8). Finally, using (5.12), we have

$$A^*(I - AA^*)^{-1}W^* = A^*W^*\Lambda^{-1}P = \widetilde{W}^*P^{-1}WW^*\Lambda^{-1}P = \widetilde{W}^*\Lambda^{-1}P.$$

Hence the first identity in (5.9) is proved. \square

6. PROOF OF PROPOSITION 1.3 AND THE QUOTIENT FORMULA FOR THE CENTRAL SOLUTION

Throughout this section $\{W, \widetilde{W}, Z\}$ is a data set for a LTONP interpolation problem, and we assume that $\Lambda = WW^* - \widetilde{W}\widetilde{W}^*$ is strictly positive.

The section consists of three subsections. In the first subsection we show that the function Υ_{22} defined by (1.13) is outer, and we derive a quotient formula for the central solution. In the second subsection we prove our statement concerning the J -contractiveness of the coefficient matrix contained in Proposition 1.3. The final statement in Proposition 1.3 about Υ_{22}^{-1} being a Schur class function is covered by the final part of Proposition 6.3. The third subsection consists of a few remarks about the case when the operator Z is exponentially stable .

6.1. The quotient formula. First notice that the formulas (1.22) and (1.23) directly follow from the identities (5.1) – (5.4). Let us prove this for (1.22). Since W^* and \widetilde{W}^* are bounded linear operators from \mathcal{Z} into $\ell_+^2(\mathcal{Y})$ and $\ell_+^2(\mathcal{U})$, respectively, it follows that $W^*\Lambda^{-1}PC^*Q_{\circ}$ and $\widetilde{W}^*\Lambda^{-1}PC^*Q_{\circ}$ are bounded linear operators mapping \mathcal{E} into $\ell_+^2(\mathcal{Y})$ and $\ell_+^2(\mathcal{U})$, respectively. Thus

$$W^*\Lambda^{-1}PC^*Q_{\circ}x \in \ell_+^2(\mathcal{Y}) \quad \text{and} \quad \widetilde{W}^*\Lambda^{-1}PC^*Q_{\circ}x \in \ell_+^2(\mathcal{U}) \quad (x \in \mathcal{E}).$$

But then, applying (1.25) for \mathcal{Y} and for \mathcal{U} in place of \mathcal{Y} , we see that the inclusions in (1.22) are proved. Similar arguments prove (1.23).

Proposition 6.1. *The function Υ_{22} defined by (1.19) is outer and for each $\lambda \in \mathbb{D}$ the operator $\Upsilon_{22}(\lambda)$ is invertible and*

$$(6.1) \quad \Upsilon_{22}(\lambda)^{-1} = R_{\circ} - \lambda R_{\circ}\widetilde{B}^* \left(I - \lambda Z^*(\Lambda + \widetilde{B}\widetilde{B}^*)^{-1}\Lambda \right)^{-1} Z^*\Lambda^{-1}\widetilde{B}R_{\circ}^2.$$

In particular, the spectrum of $Z^(\Lambda + \widetilde{B}\widetilde{B}^*)^{-1}\Lambda$ is contained in the closed unit disc. Furthermore, the function $\Upsilon_{22}(\lambda)^{-1}$ belongs to $H^{\infty}(\mathcal{U}, \mathcal{U})$, that is, $\Upsilon_{22}(\lambda)^{-1}$ is uniformly bounded on the open unit disk. Finally, if \mathcal{Z} is finite dimensional, then both Z^* and $Z^*(\Lambda + \widetilde{B}\widetilde{B}^*)^{-1}\Lambda$ are exponentially stable, and $\Upsilon_{22}(\lambda)$ is an invertible outer function.*

Proof. From Theorem 4.2 we know that the operator $T = \Lambda Z^*(\Lambda + \tilde{B}\tilde{B}^*)^{-1}$ has spectral radius less than or equal to one. Since $Z^*(\Lambda + \tilde{B}\tilde{B}^*)^{-1}\Lambda = \Lambda^{-1}T\Lambda$ is similar to T , we see that the operator $Z^*(\Lambda + \tilde{B}\tilde{B}^*)^{-1}\Lambda$ also has spectral radius less than or equal to one. In particular, $I - \lambda Z^*(\Lambda + \tilde{B}\tilde{B}^*)^{-1}\Lambda$ is invertible for each $\lambda \in \mathbb{D}$. The remaining part of the proof is done in four steps.

STEP 1. In this part we show that for each $\lambda \in \mathbb{D}$ the operator $\Upsilon_{22}(\lambda)$ is invertible and that its inverse is given by (6.1). The invertibility of $\Upsilon_{22}(\lambda)$ we already know from Theorem 1.1; see the paragraph directly after Theorem 1.1. Here the main point is to prove the identity (6.1). To do this notice that

$$\begin{aligned} \Upsilon_{22}(\lambda)R_{\circ}^{-1} &= I + \tilde{B}^*(I - \lambda Z^*)^{-1}\Lambda^{-1}\tilde{B} \\ &= I + \tilde{B}^*\Lambda^{-1}\tilde{B} + \lambda\tilde{B}^*(I - \lambda Z^*)^{-1}Z^*\Lambda^{-1}\tilde{B}. \end{aligned}$$

Recall the following state space identity when D is invertible:

$$(D + \lambda C(I - \lambda A)^{-1}B)^{-1} = D^{-1} - \lambda D^{-1}C(I - \lambda(A - BD^{-1}C))^{-1}BD^{-1}.$$

Using this with $R_{\circ}^2 = (I + \tilde{B}^*\Lambda^{-1}\tilde{B})^{-1}$, we see that

$$(6.2) \quad R_{\circ}\Upsilon_{22}(\lambda)^{-1} = R_{\circ}^2 - \lambda R_{\circ}^2\tilde{B}^*Y(\lambda)^{-1}Z^*\Lambda^{-1}\tilde{B}R_{\circ}^2,$$

where

$$\begin{aligned} Y(\lambda) &= I - \lambda(Z^* - Z^*\Lambda^{-1}\tilde{B}R_{\circ}^2\tilde{B}^*) = I - \lambda Z^*(I - \Lambda^{-1}\tilde{B}R_{\circ}^2\tilde{B}^*) \\ &= I - \lambda Z^*(I - \Lambda^{-1}\tilde{B}(I + \tilde{B}^*\Lambda^{-1}\tilde{B})^{-1}\tilde{B}^*) \\ &= I - \lambda Z^*(I - \Lambda^{-1}\tilde{B}\tilde{B}^*(I + \Lambda^{-1}\tilde{B}\tilde{B}^*)^{-1}) \\ &= I - \lambda Z^*(I - [(I + \Lambda^{-1}\tilde{B}\tilde{B}^*) - I](I + \Lambda^{-1}\tilde{B}\tilde{B}^*)^{-1}) \\ &= I - \lambda Z^*(I + \Lambda^{-1}\tilde{B}\tilde{B}^*)^{-1} = I - \lambda Z^*(\Lambda + \tilde{B}\tilde{B}^*)^{-1}\Lambda. \end{aligned}$$

Inserting this formula for $Y(\lambda)$ into (6.2) we obtain the inverse formula for $\Upsilon_{22}(\lambda)$ in (6.1).

STEP 2. We proceed by proving that the function $\Upsilon_{22}(\lambda)$ is outer. To accomplish this we use that $\Upsilon_{22}(\lambda)$ is also given by (1.19), with $A = W^*P^{-1}\tilde{W}$ as in (1.15), and we apply Lemma A.11 in Subsection A.5 in the Appendix. Using $P = ZPZ^* + BB^*$ and the fact that P is strictly positive, we see that

$$I = P^{-\frac{1}{2}}ZP^{\frac{1}{2}}P^{\frac{1}{2}}Z^*P^{-\frac{1}{2}} + P^{-\frac{1}{2}}BB^*P^{-\frac{1}{2}}.$$

In particular, $P^{-\frac{1}{2}}ZP^{\frac{1}{2}}$ is a contraction. Hence

$$I \geq (P^{-\frac{1}{2}}ZP^{\frac{1}{2}})^*P^{-\frac{1}{2}}ZP^{\frac{1}{2}} = P^{\frac{1}{2}}Z^*P^{-1}ZP^{\frac{1}{2}}.$$

Multiplying both sides by $P^{-\frac{1}{2}}$, we see that

$$(6.3) \quad Z^*P^{-1}Z \leq P^{-1}.$$

Using this with $A^*A = \tilde{W}^*P^{-1}\tilde{W}$ and $\tilde{W}S_{\mathcal{U}} = Z\tilde{W}$, we obtain

$$S_{\mathcal{U}}^*A^*AS_{\mathcal{U}} = S_{\mathcal{U}}^*\tilde{W}^*P^{-1}\tilde{W}S_{\mathcal{U}} = \tilde{W}^*Z^*P^{-1}Z\tilde{W} \leq \tilde{W}^*P^{-1}\tilde{W} = A^*A.$$

Therefore $S_{\mathcal{U}}^* A^* A S_{\mathcal{U}} \leq A^* A$. But then, according to Lemma A.11 in Subsection A.5, the function

$$(6.4) \quad \Phi(\lambda) := E_{\mathcal{U}}^* (I - \lambda S_{\mathcal{U}}^*)^{-1} (I - A^* A)^{-1} E_{\mathcal{U}}, \quad \lambda \in \mathbb{D},$$

is outer. Because R_{\circ} is invertible, it follows that the function $\Upsilon_{22}(\lambda) = \Phi(\lambda) R_{\circ}$ is outer too.

STEP 3. Let Φ be given by (6.4). Since $\Upsilon_{22}(\lambda)$ is invertible for each $\lambda \in \mathbb{D}$ and R_{\circ} is invertible, the operator $\Phi(\lambda)$ is also invertible for each $\lambda \in \mathbb{D}$. But then the final part of Lemma A.11 tells us that the function $\Phi(\lambda)^{-1}$ belongs to $H^{\infty}(\mathcal{U}, \mathcal{U})$. But then $\Upsilon_{22}(\lambda)^{-1} = R_{\circ}^{-1} \Phi(\lambda)^{-1}$ also belongs to $H^{\infty}(\mathcal{U}, \mathcal{U})$.

STEP 4. Finally, assume \mathcal{Z} is finite dimensional. Since $Z^*(\Lambda + \tilde{B}\tilde{B}^*)^{-1}\Lambda$ is similar to $T = \Lambda Z^*(\Lambda + \tilde{B}\tilde{B}^*)^{-1}$, we have $r_{\text{spec}}(Z^*(\Lambda + \tilde{B}\tilde{B}^*)^{-1}\Lambda) = r_{\text{spec}}(T) < 1$; note that $r_{\text{spec}}(T) < 1$ follows from Theorem 4.2. Furthermore, Z^* is pointwise stable, by part (ii) of Lemma 4.1, which implies all eigenvalues of Z^* are contained in \mathbb{D} . Hence $r_{\text{spec}}(Z) = r_{\text{spec}}(Z^*) < 1$. This yields that Υ_{22} is an invertible outer function. \square

The next proposition shows that for the strictly positive case the definition of the central solution F_{\circ} to the LTONP interpolation problem given in Remark 2.2 coincides with the one given in the paragraph directly after Theorem 1.1. The proposition also justifies the title of this subsection.

Proposition 6.2. *Let F_{\circ} be the central solution of the LTONP problem with data set $\{W, \tilde{W}, Z\}$. If the Pick operator Λ is strictly positive, then F_{\circ} is given by the quotient formula:*

$$(6.5) \quad F_{\circ}(\lambda) = \Upsilon_{12}(\lambda) \Upsilon_{22}(\lambda)^{-1}, \quad \lambda \in \mathbb{D}.$$

In other words, when the free parameter X in (1.9) is zero, then the resulting function is the central solution.

Proof. By using (1.11) and (1.13), we obtain

$$\begin{aligned} \Upsilon_{12}(\lambda) \Upsilon_{22}(\lambda)^{-1} &= B^* (I - \lambda Z^*)^{-1} \Lambda^{-1} \tilde{B} \left(I + \tilde{B}^* (I - \lambda Z^*)^{-1} \Lambda^{-1} \tilde{B} \right)^{-1} \\ &= B^* \left(I + (I - \lambda Z^*)^{-1} \Lambda^{-1} \tilde{B} \tilde{B}^* \right)^{-1} (I - \lambda Z^*)^{-1} \Lambda^{-1} \tilde{B} \\ &= B^* \left(\Lambda - \lambda \Lambda Z^* + \tilde{B} \tilde{B}^* \right)^{-1} \tilde{B} \\ &= B^* (\Lambda + B B^*)^{-1} (I - \lambda \Lambda Z^* (\Lambda + B B^*)^{-1})^{-1} \tilde{B} \\ &= F_{\circ}(\lambda). \end{aligned}$$

The last equality follows from formula (4.2) for the central solution $F_{\circ}(\lambda)$ in Theorem 4.2. \square

Proposition 6.3. *Let F_{\circ} be the central solution of the LTONP problem with data set $\{W, \tilde{W}, Z\}$, with the Pick operator Λ being strictly positive, and let Υ_{22}^{-1} be given by (1.19). Then the functions F_{\circ} and Υ_{22}^{-1} are both uniformly bounded on \mathbb{D} in operator norm, and the corresponding Toeplitz operators satisfy the following identity:*

$$(6.6) \quad I - T_{F_{\circ}}^* T_{F_{\circ}} = T_{\Upsilon_{22}^{-1}}^* T_{\Upsilon_{22}^{-1}}$$

Furthermore, both F_\circ and Υ_{22}^{-1} are Schur class functions.

Proof. Since F_\circ is a solution to the LTONP interpolation problem, F_\circ is a Schur class function. In particular, the function F_\circ is uniformly bounded on \mathbb{D} in operator norm. The latter also holds true for Υ_{22}^{-1} by Proposition 6.1.

Let us assume that (6.6) is proved. Since F_\circ is a Schur class function, it follows that T_{F_\circ} is a contraction. But then the identity (6.6) implies that $\|T_{\Upsilon_{22}^{-1}}^* T_{\Upsilon_{22}^{-1}}\| \leq 1$. Hence the Toeplitz operator $T_{\Upsilon_{22}^{-1}}$ is a contraction too. The latter implies that Υ_{22}^{-1} is a Schur class function. Thus the final statement of the proposition is proved.

It remains to prove (6.6). Recall that $\Upsilon_{22} = \Phi R_\circ$, where the function Φ is given by (6.4) and $R_\circ = (E_{\mathcal{U}}^*(I - A^*A)^{-1}E_{\mathcal{U}})^{-\frac{1}{2}}$. Here $A = W^*P^{-1}\widetilde{W}$, and hence $WA = \widetilde{W}$. We claim that

$$(6.7) \quad \langle S_{\mathcal{Y}}Ah, Af \rangle = \langle AS_{\mathcal{U}}h, Af \rangle, \quad h, f \in \ell_+^2(\mathcal{U}).$$

Using $ZW = WS_{\mathcal{Y}}$ and $Z\widetilde{W} = \widetilde{W}S_{\mathcal{U}}$, we obtain

$$\begin{aligned} \langle S_{\mathcal{Y}}Ah, Af \rangle &= \langle WS_{\mathcal{Y}}Ah, P^{-1}\widetilde{W}f \rangle = \langle ZWAh, P^{-1}\widetilde{W}f \rangle \\ &= \langle Z\widetilde{W}h, P^{-1}\widetilde{W}f \rangle = \langle \widetilde{W}S_{\mathcal{U}}h, P^{-1}\widetilde{W}f \rangle \\ &= \langle WAS_{\mathcal{U}}h, P^{-1}\widetilde{W}f \rangle = \langle AS_{\mathcal{U}}h, Af \rangle. \end{aligned}$$

This yields (6.7).

Next, let $x \in \ell_+^2(\mathcal{U})$ be of compact support, that is, x has only a finite number of non-zero entries. We shall show that for any such x we have

$$(6.8) \quad \|T_{\Phi}x\|^2 - \|T_{F_\circ}T_{\Phi}x\|^2 = \|T_{\Upsilon_{22}^{-1}}T_{\Phi}x\|^2.$$

Recall that the central solution F_\circ is given by the quotient formula (6.5) $F_\circ(\lambda) = \Upsilon_{12}(\lambda)\Upsilon_{22}(\lambda)^{-1}$, where Υ_{12} and Υ_{22} are defined in (1.17) and (1.19), respectively. Thus $F_\circ(\lambda)\Upsilon_{22}(\lambda) = \Upsilon_{12}(\lambda)$ for each $\lambda \in \mathbb{D}$. By eliminating R_\circ in the definitions of Υ_{12} and Υ_{22} , we see that

$$F_\circ(\lambda)E_{\mathcal{U}}^*(I - S_{\mathcal{U}}^*)^{-1}D_A^{-2}E_{\mathcal{U}} = E_{\mathcal{Y}}^*(I - S_{\mathcal{Y}}^*)^{-1}AD_A^{-2}E_{\mathcal{U}},$$

where $D_A = (I - A^*A)^{\frac{1}{2}}$. So for $x = \{x_n\}_{n=0}^\infty$ in $\ell_+^2(\mathcal{U})$ with compact support, we have

$$\begin{aligned} \|T_{F_\circ}T_{\Phi}x\|^2 &= \|T_{F_\circ} \sum_{n=0}^\infty S_{\mathcal{U}}^n D_A^{-2} E_{\mathcal{U}} x_n\|^2 = \left\| \sum_{n=0}^\infty S_{\mathcal{Y}}^n A D_A^{-2} E_{\mathcal{U}} x_n \right\|^2 \\ &= \left\langle \sum_{n=0}^\infty S_{\mathcal{Y}}^n A D_A^{-2} E_{\mathcal{U}} x_n, \sum_{m=0}^\infty S_{\mathcal{Y}}^m A D_A^{-2} E_{\mathcal{U}} x_m \right\rangle \\ &= \sum_{n \geq m} \langle S_{\mathcal{Y}}^n A D_A^{-2} E_{\mathcal{U}} x_n, S_{\mathcal{Y}}^m A D_A^{-2} E_{\mathcal{U}} x_m \rangle + \\ &\quad + \sum_{n < m} \langle S_{\mathcal{Y}}^n A D_A^{-2} E_{\mathcal{U}} x_n, S_{\mathcal{Y}}^m A D_A^{-2} E_{\mathcal{U}} x_m \rangle. \\ &= \sum_{n \geq m} \langle S_{\mathcal{Y}}^{n-m} A D_A^{-2} E_{\mathcal{U}} x_n, A D_A^{-2} E_{\mathcal{U}} x_m \rangle + \\ &\quad + \sum_{n < m} \langle A D_A^{-2} E_{\mathcal{U}} x_n, S_{\mathcal{Y}}^{m-n} A D_A^{-2} E_{\mathcal{U}} x_m \rangle. \end{aligned}$$

Using the fact that $A^*AD_A^{-2} = (D_A^{-2} - I)$ we obtain

$$\begin{aligned}
 \sum_{n>m} \langle S_y^{n-m} AD_A^{-2} E_U x_n, AD_A^{-2} E_U x_m \rangle &= \sum_{n>m} \langle AS_U^{n-m} D_A^{-2} E_U x_n, AD_A^{-2} E_U x_m \rangle \\
 &= \sum_{n>m} \langle AS_U^{n-m} D_A^{-2} E_U x_n, AD_A^{-2} E_U x_m \rangle \\
 &= \sum_{n>m} \langle S_U^{n-m} D_A^{-2} E_U x_n, (D_A^{-2} - I) E_U x_m \rangle \\
 &= \sum_{n>m} \langle S_U^{n-m} D_A^{-2} E_U x_n, D_A^{-2} E_U x_m \rangle = \sum_{n>m} \langle S_U^n D_A^{-2} E_U x_n, S_U^m D_A^{-2} E_U x_m \rangle.
 \end{aligned}$$

A similar computation gives

$$\sum_{n<m} \langle S_y^n AD_A^{-2} E_U x_n, S_y^m AD_A^{-2} E_U x_m \rangle = \sum_{n<m} \langle S_U^n D_A^{-2} E_U x_n, S_U^m D_A^{-2} E_U x_m \rangle.$$

For $m = n$ we have

$$\begin{aligned}
 \langle AD_A^{-2} E_U x_n, AD_A^{-2} E_U x_n \rangle &= \langle AD_A^{-2} E_U x_n, AD_A^{-2} E_U x_n \rangle \\
 &= \langle D_A^{-2} E_U x_n, (D_A^{-2} - I) E_U x_n \rangle \\
 &= \langle D_A^{-2} E_U x_n, D_A^{-2} E_U x_n \rangle - \langle D_A^{-2} E_U x_n, E_U x_n \rangle \\
 &= \langle D_A^{-2} E_U x_n, D_A^{-2} E_U x_n \rangle - \langle R_o^{-1} x_n, x_n \rangle.
 \end{aligned}$$

Putting the above computations together gives

$$\begin{aligned}
 \|T_{F_o} T_\Phi x\|^2 &= \sum_{n,m=0}^{\infty} \langle S_U^n D_A^{-2} E_U x_n, S_U^m D_A^{-2} E_U x_m \rangle - \sum_{n=0}^{\infty} \langle R_o^{-2} x_n, x_n \rangle \\
 &= \langle T_\Phi x, T_\Phi x \rangle - \sum_{n=0}^{\infty} \|R_o^{-1} x_n\|^2 = \|T_\Phi x\|^2 - \|T_{R_o^{-1}} x\|^2 \\
 &= \|T_\Phi x\|^2 - \|T_{R_o^{-1}} T_{\Phi^{-1}} T_\Phi x\|^2 = \|T_\Phi x\|^2 - \|T_{\Upsilon_{22}^{-1}} T_\Phi x\|^2.
 \end{aligned}$$

Here $T_{R_o^{-1}}$ denotes the diagonal Toeplitz operator with the operator R_o^{-1} on the main diagonal. We proved (6.8) for all x in $\ell_+^2(\mathcal{U})$ with compact support. The fact that Φ is outer implies that T_Φ maps the compact support sequences in $\ell_+^2(\mathcal{U})$ to a dense subset of $\ell_+^2(\mathcal{U})$. Therefore

$$\|v\|^2 - \|T_{F_o} v\|^2 = \|T_{\Upsilon_{22}^{-1}} v\|^2, \quad v \in \ell_+^2(\mathcal{U}).$$

In other words, $I - T_{F_o}^* T_{F_o} = T_{\Upsilon_{22}^{-1}}^* T_{\Upsilon_{22}^{-1}}$, and (6.6). \square

6.2. J -contractiveness of the coefficient matrix. Throughout this section let $\{W, \widetilde{W}, Z\}$ be a data set for a LTONP interpolation problem. Assume $\Lambda = P - \widetilde{P}$ is strictly positive. Define Υ_{ij} , $i, j = 1, 2$, as in (1.10)–(1.13). Now set

$$(6.9) \quad \Upsilon(\lambda) = \begin{bmatrix} \Upsilon_{11}(\lambda) & \Upsilon_{12}(\lambda) \\ \Upsilon_{21}(\lambda) & \Upsilon_{22}(\lambda) \end{bmatrix} \quad (\lambda \in \mathbb{D}).$$

Furthermore, set

$$J_1 = \begin{bmatrix} I_y & 0 \\ 0 & -I_u \end{bmatrix} \quad \text{and} \quad J_2 = \begin{bmatrix} I_\mathcal{E} & 0 \\ 0 & -I_u \end{bmatrix}.$$

The following theorem is the main result of this section.

Theorem 6.4. *Let $\{W, \widetilde{W}, Z\}$ be a data set for a LTONP interpolation problem. Assume $\Lambda = P - \widetilde{P}$ is strictly positive. Then for each $\lambda \in \mathbb{D}$ the operator $\Upsilon(\lambda)$ is J -contractive, that is, $\Upsilon(\lambda)^* J_1 \Upsilon(\lambda) \leq J_2$. More precisely, for each $\lambda \in \mathbb{D}$ we have*

$$(6.10) \quad \begin{aligned} \Upsilon(\lambda)^* J_1 \Upsilon(\lambda) &= J_2 + \\ &- (1 - |\lambda|^2) \begin{bmatrix} Q_\circ C P \Lambda^{-1} \\ R_\circ \widetilde{B}^* \Lambda^{-1} Z \end{bmatrix} (I - \lambda Z^*)^{-*} \Lambda (I - \lambda Z^*)^{-1} \times \\ &\quad \times \begin{bmatrix} \Lambda^{-1} P C^* Q_\circ & Z^* \Lambda^{-1} \widetilde{B} R_\circ \end{bmatrix}. \end{aligned}$$

Furthermore, for each λ on the unit circle that is not in the spectrum of Z the operator $\Upsilon(\lambda)$ is J -unitary, that is, $\Upsilon(\lambda)^* J_1 \Upsilon(\lambda) = J_2$.

Remark 6.5. *Theorem 6.4 can be also used to show that Υ_{22}^{-1} is a function in $\mathcal{S}(\mathcal{U}, \mathcal{U})$. Indeed, the inequality $\Upsilon(\lambda)^* J_1 \Upsilon(\lambda) \leq J_2$, implies that*

$$\Upsilon_{12}(\lambda)^* \Upsilon_{12}(\lambda) - \Upsilon_{22}(\lambda)^* \Upsilon_{22}(\lambda) \leq -I \quad (\lambda \in \mathbb{D}).$$

Thus $I \leq \Upsilon_{22}(\lambda)^* \Upsilon_{22}(\lambda)$ for each λ in \mathbb{D} . Proposition 6.1 shows that $\Upsilon_{22}(\lambda)$ is invertible for λ in \mathbb{D} . Hence $\Upsilon_{22}(\lambda)^{-*} \Upsilon_{22}(\lambda)^{-1} \leq I$ for λ in \mathbb{D} . Therefore $\Upsilon_{22}(\lambda)^{-1}$ is a contraction for all λ in \mathbb{D} . In other words, $\Upsilon_{22}(\lambda)^{-1}$ is a function in $\mathcal{S}(\mathcal{U}, \mathcal{U})$.

Before we prove this result it is useful to first derive the following two lemmas. The first lemma provides a state space realization for the coefficient matrix-function Υ , the second lemma derives a number of useful identities of the operators involved in the realization.

Lemma 6.6. *The function Υ in (6.9) is given by*

$$(6.11) \quad \Upsilon(\lambda) = \left(\widehat{D} + \lambda \widehat{C} (I - \lambda Z^*)^{-1} \widehat{B} \right) \begin{bmatrix} Q_\circ & 0 \\ 0 & R_\circ \end{bmatrix}, \quad \lambda \in \mathbb{D},$$

where \widehat{B} , \widehat{C} and \widehat{D} are the operators given by

$$\widehat{B} = \begin{bmatrix} \Lambda^{-1} P C^* & Z^* \Lambda^{-1} \widetilde{B} \end{bmatrix}, \quad \widehat{C} = \begin{bmatrix} B^* \\ \widetilde{B}^* \end{bmatrix}, \quad \widehat{D} = \begin{bmatrix} D^* & B^* \Lambda^{-1} \widetilde{B} \\ 0 & I + \widetilde{B}^* \Lambda^{-1} \widetilde{B} \end{bmatrix}.$$

Proof. By writing out the right-hand side of (6.11) in 2×2 block matrix form, we see that the left upper block and left lower block coincide with Υ_{11} and Υ_{21} in (1.10) and (1.12), respectively. It remains to show that Υ_{12} in (1.11) and Υ_{22} in (1.13) can be written as

$$\begin{aligned} \Upsilon_{12}(\lambda) &= (B^* \Lambda^{-1} \widetilde{B} + \lambda B^* (I - \lambda Z^*)^{-1} Z^* \Lambda^{-1} \widetilde{B}) R_\circ; \\ \Upsilon_{22}(\lambda) &= (I + \widetilde{B}^* \Lambda^{-1} \widetilde{B} + \lambda \widetilde{B}^* (I - \lambda Z^*)^{-1} Z^* \Lambda^{-1} \widetilde{B}) R_\circ. \end{aligned}$$

In both cases this is a direct consequence of the fact that

$$(I - \lambda Z^*)^{-1} = I + \lambda (I - \lambda Z^*)^{-1} Z^*. \quad \square$$

Lemma 6.7. *With \widehat{B} , \widehat{C} , \widehat{D} and J_1 defined as above, we have the following identities:*

$$(6.12) \quad \widehat{D}^* J_1 \widehat{D} = \begin{bmatrix} Q_\circ^{-2} & 0 \\ 0 & -R_\circ^{-2} \end{bmatrix} - \widehat{B}^* \Lambda \widehat{B}$$

$$(6.13) \quad \widehat{C}^* J_1 \widehat{C} = \Lambda - Z \Lambda Z^* \quad \text{and} \quad \widehat{D}^* J_1 \widehat{C} = -\widehat{B}^* \Lambda Z^*,$$

Proof. Recall that

$$BB^* - \tilde{B}\tilde{B}^* = \Lambda - Z\Lambda Z^*.$$

The identities in (6.13) follow from this identity and the following straightforward computations:

$$\begin{aligned}
 (6.14) \quad \widehat{C}^* J_1 \widehat{C} &= BB^* - \tilde{B}\tilde{B}^* = \Lambda - Z\Lambda Z^*, \\
 \widehat{D}^* J_1 \widehat{C} &= \begin{bmatrix} D & 0 \\ \tilde{B}^* \Lambda^{-1} B & I + \tilde{B}^* \Lambda^{-1} \tilde{B} \end{bmatrix} \begin{bmatrix} B^* \\ -\tilde{B}^* \end{bmatrix} \\
 &= \begin{bmatrix} DB^* \\ \tilde{B}^* \Lambda^{-1} (BB^* - \Lambda - \tilde{B}\tilde{B}^*) \end{bmatrix} \\
 (6.15) \quad &= \begin{bmatrix} -CPZ^* \\ -\tilde{B}^* \Lambda^{-1} Z\Lambda Z^* \end{bmatrix} = - \begin{bmatrix} CP\Lambda^{-1} \\ \tilde{B}^* \Lambda^{-1} Z \end{bmatrix} \Lambda Z^* = -\widehat{B}^* \Lambda Z^*.
 \end{aligned}$$

In establishing the first identity on the last line we used $DB^* + CPZ^* = 0$, which follows from (1.6). Using $DD^* + CPC^* = I$ from (1.6), we have

$$\begin{aligned}
 \widehat{D}^* J_1 \widehat{D} &= \begin{bmatrix} D & 0 \\ \tilde{B}^* \Lambda^{-1} B & I + \tilde{B}^* \Lambda^{-1} \tilde{B} \end{bmatrix} \begin{bmatrix} D^* & B^* \Lambda^{-1} \tilde{B} \\ 0 & -(I + \tilde{B}^* \Lambda^{-1} \tilde{B}) \end{bmatrix} \\
 &= \begin{bmatrix} DD^* & DB^* \Lambda^{-1} \tilde{B} \\ \tilde{B}^* \Lambda^{-1} BD^* & \tilde{B}^* \Lambda^{-1} BB^* \Lambda^{-1} \tilde{B} - (I + \tilde{B}^* \Lambda^{-1} \tilde{B})^2 \end{bmatrix} \\
 &= \begin{bmatrix} I - CPC^* & -CPZ^* \Lambda^{-1} \tilde{B} \\ -\tilde{B}^* \Lambda^{-1} ZPC^* & \tilde{B}^* \Lambda^{-1} BB^* \Lambda^{-1} \tilde{B} - (I + \tilde{B}^* \Lambda^{-1} \tilde{B})^2 \end{bmatrix}.
 \end{aligned}$$

Next observe that

$$I - CPC^* = I + CP(\Lambda^{-1} - P^{-1})PC^* - CP\Lambda^{-1}PC^* = Q_{\circ}^{-2} - CP\Lambda^{-1}PC^*$$

and

$$\begin{aligned}
 (I + \tilde{B}^* \Lambda^{-1} \tilde{B})^2 &= (I + \tilde{B}^* \Lambda^{-1} \tilde{B}) + \tilde{B}^* \Lambda^{-1} \tilde{B} (I + \tilde{B}^* \Lambda^{-1} \tilde{B}) \\
 &= R_{\circ}^{-2} + \tilde{B}^* \Lambda^{-1} (\Lambda + \tilde{B}\tilde{B}^*) \Lambda^{-1} \tilde{B} \\
 &= R_{\circ}^{-2} + \tilde{B}^* \Lambda^{-1} (Z\Lambda Z^* + BB^*) \Lambda^{-1} \tilde{B} \\
 &= R_{\circ}^{-2} + \tilde{B}^* \Lambda^{-1} Z\Lambda Z^* \Lambda^{-1} \tilde{B} + \tilde{B}^* \Lambda^{-1} BB^* \Lambda^{-1} \tilde{B}.
 \end{aligned}$$

Hence

$$(I + \tilde{B}^* \Lambda^{-1} \tilde{B})^2 - \tilde{B}^* \Lambda^{-1} BB^* \Lambda^{-1} \tilde{B} = R_{\circ}^{-2} + \tilde{B}^* \Lambda^{-1} Z\Lambda Z^* \Lambda^{-1} \tilde{B}.$$

Using these identities we obtain that

$$\begin{aligned}
 \widehat{D}^* J_1 \widehat{D} &= \begin{bmatrix} Q_{\circ}^{-1} & 0 \\ 0 & -R_{\circ}^{-2} \end{bmatrix} - \begin{bmatrix} CP\Lambda^{-1}PC^* & CPZ^* \Lambda^{-1} \tilde{B} \\ \tilde{B}^* \Lambda^{-1} ZPC^* & \tilde{B}^* \Lambda^{-1} Z\Lambda Z^* \Lambda^{-1} \tilde{B} \end{bmatrix} \\
 &= \begin{bmatrix} Q_{\circ}^{-1} & 0 \\ 0 & -R_{\circ}^{-2} \end{bmatrix} - \begin{bmatrix} CP\Lambda^{-1} \\ \tilde{B}^* \Lambda^{-1} Z \end{bmatrix} \Lambda \begin{bmatrix} \Lambda^{-1} PC^* & Z^* \Lambda^{-1} \tilde{B} \end{bmatrix}.
 \end{aligned}$$

This shows that (6.12) holds as well. \square

Proof of Theorem 6.4. Fix a $\lambda \in \mathbb{D}$. In order to prove (6.10), we multiply the left hand side of (6.10) from both sides by $\begin{bmatrix} Q_\circ & 0 \\ 0 & R_\circ \end{bmatrix}^{-1}$. Then, by using (6.11), we obtain

$$\begin{aligned}
& \begin{bmatrix} Q_\circ & 0 \\ 0 & R_\circ \end{bmatrix}^{-1} \Upsilon(\lambda)^* J_1 \Upsilon(\lambda) \begin{bmatrix} Q_\circ & 0 \\ 0 & R_\circ \end{bmatrix}^{-1} = \\
& = (\widehat{D}^* + \bar{\lambda} \widehat{B}^* (I - \bar{\lambda} Z)^{-1} \widehat{C}^*) J_1 (\widehat{D} + \lambda \widehat{C} (I - \lambda Z^*)^{-1} \widehat{B}) \\
& = \widehat{D}^* J_1 \widehat{D} + \bar{\lambda} \widehat{B}^* (I - \bar{\lambda} Z)^{-1} \widehat{C}^* J_1 \widehat{D} + \lambda \widehat{D}^* J_1 \widehat{C} (I - \lambda Z^*)^{-1} \widehat{B} + \\
& \quad + |\lambda|^2 \widehat{B}^* (I - \bar{\lambda} Z)^{-1} \widehat{C}^* J_1 \widehat{C} (I - \lambda Z^*)^{-1} \widehat{B} \\
& = \widehat{D}^* J_1 \widehat{D} - \bar{\lambda} \widehat{B}^* (I - \bar{\lambda} Z)^{-1} Z \Lambda \widehat{B} - \lambda \widehat{B}^* \Lambda Z^* (I - \lambda Z^*)^{-1} \widehat{B} + \\
& \quad + |\lambda|^2 \widehat{B}^* (I - \bar{\lambda} Z)^{-1} (\Lambda - Z \Lambda Z^*) (I - \lambda Z^*)^{-1} \widehat{B} \\
& = \widehat{D}^* J_1 \widehat{D} - \widehat{B}^* (I - \bar{\lambda} Z)^{-1} \times \\
& \quad \times \left(\bar{\lambda} Z \Lambda (I - \lambda Z^*) + \lambda (I - \bar{\lambda} Z) \Lambda Z^* - |\lambda|^2 (\Lambda - Z \Lambda Z^*) \right) (I - \lambda Z^*)^{-1} \widehat{B}.
\end{aligned}$$

Note that

$$\begin{aligned}
& \bar{\lambda} Z \Lambda (I - \lambda Z^*) + \lambda (I - \bar{\lambda} Z) \Lambda Z^* - |\lambda|^2 (\Lambda - Z \Lambda Z^*) = \\
& = -(|\lambda|^2 \Lambda - \bar{\lambda} Z \Lambda - \lambda \Lambda Z^* + |\lambda|^2 Z \Lambda Z^*) \\
& = -(I - \bar{\lambda} Z) \Lambda (I - \lambda Z^*) + (I - |\lambda|^2) \Lambda.
\end{aligned}$$

Inserting this identity into the above computation yields

$$\begin{aligned}
& \begin{bmatrix} Q_\circ & 0 \\ 0 & R_\circ \end{bmatrix}^{-1} \Upsilon(\lambda)^* J_1 \Upsilon(\lambda) \begin{bmatrix} Q_\circ & 0 \\ 0 & R_\circ \end{bmatrix}^{-1} = \\
& = \widehat{D}^* J_1 \widehat{D} + \widehat{B}^* \Lambda \widehat{B} - (1 - |\lambda|^2) \widehat{B}^* (I - \bar{\lambda} Z)^{-1} \Lambda (I - \lambda Z^*)^{-1} \widehat{B} \\
& = \begin{bmatrix} Q_\circ^{-2} & 0 \\ 0 & -R_\circ^{-2} \end{bmatrix} - (1 - |\lambda|^2) \widehat{B}^* (I - \bar{\lambda} Z)^{-1} \Lambda (I - \lambda Z^*)^{-1} \widehat{B}.
\end{aligned}$$

Multiplying the resulting identity from both sides by $\begin{bmatrix} Q_\circ & 0 \\ 0 & R_\circ \end{bmatrix}$ yields (6.10).

By taking limits the final statement directly follows from (6.10). \square

6.3. The case when Z is exponentially stable. We conclude this section with a few remarks about the case when Z is exponentially stable. Note that this happens when Z is finite dimensional. Recall that Z is exponentially stable if $r_{\text{spec}}(Z)$, the spectral radius of Z , is strictly less than one.

Proposition 6.8. *Assume that the operator Z is exponentially stable. Then the operator $Z^*(\Lambda + \widetilde{B} \widetilde{B}^*)^{-1} \Lambda$ is also exponentially stable, and therefore the functions $\Upsilon_{ij}(\lambda)$, $i, j = 1, 2$, the central solutions F_\circ and the function $\Upsilon_{22}(\lambda)^{-1}$ are analytic on $|\lambda| < 1 + \epsilon$ for some $\epsilon > 0$. Furthermore,*

$$(6.16) \quad I - F_\circ(\lambda)^* F_\circ(\lambda) = \Upsilon_{22}(\lambda)^{-*} \Upsilon_{22}(\lambda)^{-1}, \quad |\lambda| = 1.$$

Finally, T_{F_\circ} is a strict contraction.

Proof. We first show that $Z^*(\Lambda + \tilde{B}\tilde{B}^*)^{-1}\Lambda$ is exponentially stable. Notice that

$$\begin{aligned} \Lambda^{\frac{1}{2}} \left(Z^*(\Lambda + \tilde{B}\tilde{B}^*)^{-1}\Lambda \right) \Lambda^{-\frac{1}{2}} &= \Lambda^{\frac{1}{2}} Z^* \Lambda^{-\frac{1}{2}} \Lambda^{\frac{1}{2}} (\Lambda + \tilde{B}\tilde{B}^*)^{-1} \Lambda^{\frac{1}{2}} \\ &= \Lambda^{\frac{1}{2}} Z^* \Lambda^{-\frac{1}{2}} \left(I + \Lambda^{-\frac{1}{2}} \tilde{B}\tilde{B}^* \Lambda^{-\frac{1}{2}} \right)^{-1}. \end{aligned}$$

Hence $\Lambda^{\frac{1}{2}} \left(Z^*(\Lambda + \tilde{B}\tilde{B}^*)^{-1}\Lambda \right) \Lambda^{-\frac{1}{2}}$ and $\Lambda^{\frac{1}{2}} Z^* \Lambda^{-\frac{1}{2}} \left(I + \Lambda^{-\frac{1}{2}} \tilde{B}\tilde{B}^* \Lambda^{-\frac{1}{2}} \right)^{-1}$ are similar. In particular, they have the same spectrum. Furthermore, $\Lambda - Z\Lambda Z^* = BB^* - \tilde{B}\tilde{B}^*$ can be rewritten as

$$I - \Lambda^{-\frac{1}{2}} Z \Lambda^{\frac{1}{2}} \Lambda^{\frac{1}{2}} Z^* \Lambda^{-\frac{1}{2}} = \Lambda^{-\frac{1}{2}} BB^* \Lambda^{-\frac{1}{2}} - \Lambda^{-\frac{1}{2}} \tilde{B}\tilde{B}^* \Lambda^{-\frac{1}{2}}.$$

Replacing $\Lambda^{-\frac{1}{2}} Z \Lambda^{\frac{1}{2}}$ by Z and $\Lambda^{-\frac{1}{2}} B$ by B and $\Lambda^{-\frac{1}{2}} \tilde{B}$ by \tilde{B} , we see that without loss of generality we may assume that $\Lambda = I$.

So we assume that

$$(6.17) \quad r_{\text{spec}}(Z) < 1 \quad \text{and} \quad I - ZZ^* + \tilde{B}\tilde{B}^* = BB^* \geq 0.$$

We have to show that $Z^*(I + \tilde{B}\tilde{B}^*)^{-1}$ is exponentially stable. By consulting (4.1) with $\Lambda = I$, we see that $\Pi_{\mathcal{Z}} \omega P_{\mathcal{F}}|_{\mathcal{Z}} = Z^*(I + \tilde{B}\tilde{B}^*)^{-1}$. Hence $Z^*(I + \tilde{B}\tilde{B}^*)^{-1}$ is a contraction, and thus,

$$(6.18) \quad r_{\text{spec}}(Z^*(I + \tilde{B}\tilde{B}^*)^{-1}) \leq 1.$$

Next consider the auxiliary operator

$$(6.19) \quad Y = I + \tilde{B}^*(I - Z^*)^{-1}\tilde{B} : \mathcal{U} \rightarrow \mathcal{U}.$$

We shall show that Y is invertible. The idea of the proof is taken from [38], page 128. One computes that

$$\begin{aligned} YY^* &= \{I + \tilde{B}^*(I - Z^*)^{-1}\tilde{B}\} \{I + \tilde{B}^*(I - Z)^{-1}\tilde{B}\} \\ &= I + \tilde{B}^*(I - Z^*)^{-1}\tilde{B} + \tilde{B}^*(I - Z)^{-1}\tilde{B} + \\ &\quad + \tilde{B}^*(I - Z^*)^{-1}\tilde{B}\tilde{B}^*(I - Z)^{-1}\tilde{B} \\ &= I + \tilde{B}^*(I - Z^*)^{-1} \{ (I - Z) + (I - Z^*) + \tilde{B}\tilde{B}^* \} (I - Z)^{-1}\tilde{B}. \end{aligned}$$

Now use the second part of (6.17) and

$$ZZ^* - I = (I - Z)(I - Z^*) - (I - Z^*) - (I - Z).$$

It follows that

$$\begin{aligned} (I - Z) + (I - Z^*) + \tilde{B}\tilde{B}^* &= (I - Z)(I - Z^*) + I - ZZ^* + \tilde{B}\tilde{B}^* \\ &= (I - Z)(I - Z^*) + BB^* \geq 0. \end{aligned}$$

Hence $YY^* \geq I$, and YY^* is strictly positive. In a similar fashion one computes that

$$\begin{aligned} Y^*Y &= I + \tilde{B}^*(I - Z)^{-1} \{ (I - Z^*) + (I - Z) + \tilde{B}\tilde{B}^* \} (I - Z^*)^{-1}\tilde{B} \\ &= I + \tilde{B}^*(I - Z)^{-1} \{ (I - Z)(I - Z^*) + BB^* \} (I - Z^*)^{-1}\tilde{B} \geq I. \end{aligned}$$

Thus Y^*Y is also strictly positive. Since both Y^*Y and YY^* are strictly positive, we conclude that the operator Y defined by (6.19) is invertible.

Since Y defined by (6.19) is invertible, it follows that $I + (I - Z^*)^{-1}\tilde{B}\tilde{B}^*$ is invertible. Here we used the fact that the nonzero spectrum of the product of two operators are the same. Multiplying by $I - Z^*$ on the left shows that $I - Z^* + \tilde{B}\tilde{B}^*$

is also invertible. Multiplying by $(I + \widetilde{B}\widetilde{B}^*)^{-1}$ on the right implies that $I - Z^*(I + \widetilde{B}\widetilde{B}^*)^{-1}$ is invertible. In other words,

$$(6.20) \quad 1 \notin \sigma \left(Z^*(I + \widetilde{B}\widetilde{B}^*)^{-1} \right).$$

Recall that $\sigma(A)$ denotes the spectrum of an operator A . Now take $\lambda \in \mathbb{T}$, and notice that the conditions in (6.17) remain valid if Z is replaced by λZ . Thus (6.20) yields

$$1 \notin \sigma \left(\lambda^{-1} Z^*(I + \widetilde{B}\widetilde{B}^*)^{-1} \right).$$

It follows that $\lambda \notin \sigma(Z^*(I + \widetilde{B}\widetilde{B}^*)^{-1})$. Since λ is an arbitrary element of \mathbb{T} , we conclude that $\sigma(Z^*(I + \widetilde{B}\widetilde{B}^*)^{-1}) \cap \mathbb{T}$ is empty, and hence using (6.18) we obtain that the spectral radius of $Z^*(I + \widetilde{B}\widetilde{B}^*)^{-1}$ is strictly less than one.

Since both Z and $Z^*(I + \widetilde{B}\widetilde{B}^*)^{-1}I$ are exponentially stable, it is clear from (1.10) to (1.13), (4.2) and (6.1) that the functions $\Upsilon_{ij}(\lambda)$, $i, j = 1, 2$, the central solutions F_\circ and the function $\Upsilon_{22}(\lambda)^{-1}$ are analytic on $|\lambda| < 1 + \epsilon$ for some $\epsilon > 0$.

Next we prove (6.16). Fix $\lambda \in \mathbb{T}$. Since $r_{\text{spec}}(Z) < 1$, the final statement of Theorem 6.4 tells is that

$$\Upsilon_{12}(\lambda)^* \Upsilon_{12}(\lambda) - \Upsilon_{22}(\lambda)^* \Upsilon_{22}(\lambda) = -I_{\mathcal{U}}.$$

Multiplying the latter identity from the right by $\Upsilon_{22}(\lambda)^{-1}$ and from the left by $\Upsilon_{22}(\lambda)^{-*}$ and using the quotient formula (6.5) we see that

$$F_\circ(\lambda)^* F_\circ(\lambda) - I = -\Upsilon_{22}(\lambda)^{-*} \Upsilon_{22}(\lambda)^{-1},$$

which proves (6.16).

Finally, using (6.16), we see that $\|T_{F_\circ}\| = \sup_{\lambda \in \mathbb{D}} \|F_\circ(\lambda)\| < 1$, and hence T_{F_\circ} is a strict contraction. \square

Corollary 6.9. *Let $\{W, \widetilde{W}, Z\}$ be a data set for a LTONP interpolation problem, and let $\Lambda = P - \widetilde{P}$ be strictly positive. If in addition \mathcal{Z} is finite dimensional, then the operator Z is exponentially stable, and the functions $\Upsilon_{ij}(\lambda)$, $i, j = 1, 2$, the central solutions F_\circ , and the function $\Upsilon_{22}(\lambda)^{-1}$ are rational operator functions with no poles on the closed unit disk and the factorization in (6.16) is a right canonical factorization. in the sense of [25, Section XXIV3]. In other words, Υ_{22} is invertible outer, that is, $T_{\Upsilon_{22}}$ is invertible and its inverse is $T_{\Upsilon_{22}^{-1}}$.*

Proof. From Theorem 1.1 we know that Z is exponentially stable, But for a finite dimensional space pointwise stable is equivalent to exponentially stable. Furthermore, since \mathcal{Z} is finite dimensional, formulas (1.10) – (1.13) imply that the functions $\Upsilon_{ij}(\lambda)$, $i, j = 1, 2$, are rational. Similarly, (4.2) and (6.1) show that F_\circ and $\Upsilon_{22}(\lambda)^{-1}$ are rational operator functions. Recall (see Proposition 6.8) that $\Upsilon_{22}(\lambda)$ and $\Upsilon_{22}(\lambda)^{-1}$ are both analytic at each point of the closed unit disc, which implies that the factorization in (6.16) is a right canonical factorization and Υ_{22} is invertible outer. \square

7. MAXIMAL ENTROPY PRINCIPLE

For a function $F \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ we define the *entropy* to be the cost function σ_F defined by the following optimization problem:

$$(7.1) \quad \sigma_F(u) = \inf \left\{ \|u - E_{\mathcal{U}}^* T_F^* h\|^2 + \langle (I - T_F T_F^*) h, h \rangle \mid h \in \ell_+^2(\mathcal{Y}) \right\},$$

where u is a vector in \mathcal{U} . Note that the above problem is precisely the optimization problem in (A.45) with $C = T_F$. Due to the equivalence of the optimization problems in (A.45) and (A.48), the entropy σ_F is also given by

$$(7.2) \quad \sigma_F(u) = \inf \left\{ \|D_{T_F}(E_{\mathcal{U}}u - S_{\mathcal{U}}e)\|^2 \mid e \in \ell_+^2(\mathcal{U}) \right\}, \quad u \in \mathcal{U}.$$

This is precisely the notion of entropy that is used in the commutant lifting setting presented in [19, Section IV.7]. Furthermore, if $\|F\|_{\infty} = \|T_F\| < 1$, then by (A.47) the entropy for F is determined by

$$(7.3) \quad \sigma_F(u) = \left\langle (E_{\mathcal{U}}^*(I - T_F^*T_F)^{-1}E_{\mathcal{U}})^{-1}u, u \right\rangle$$

In the band method theory on the maximal entropy principle the operator $E_{\mathcal{U}}^*(I - T_F^*T_F)^{-1}E_{\mathcal{U}}$ appears as the multiplicative diagonal of the function $I - F(\lambda)^*F(\lambda)$, $\lambda \in \mathbb{T}$, assuming the Fourier coefficients of F are summable in operator norm; see Sections I.3 and II.3 in [29], and Section XXXIV.4 in [26]. For further information on the multiplicative diagonal we refer to Subsection A.7.

In this section the function F is assumed to belong to the set of all solutions to a LTONP interpolation problem. The following theorem is the maximal entropy principle for this set of F 's.

Theorem 7.1. *Assume that the LTONP interpolation problem with given data set $\{W, \widetilde{W}, Z\}$ is solvable, i.e., the Pick matrix Λ is nonnegative. Let F_{\circ} in $\mathcal{S}(\mathcal{U}, \mathcal{Y})$ be the central solution to this LTONP interpolation problem. Then F_{\circ} is the unique maximal entropy solution, that is, if $F \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ is any other solution to the LTONP interpolation problem, then*

$$(7.4) \quad \sigma_F(u) \leq \sigma_{F_{\circ}}(u) \quad (u \in \mathcal{U}).$$

Moreover, we have $\sigma_F(u) = \sigma_{F_{\circ}}(u)$ for all $u \in \mathcal{U}$ if and only if $F = F_{\circ}$, and the entropy for the central solution is given by

$$(7.5) \quad \sigma_{F_{\circ}}(u) = \langle P_{\mathcal{G}}(u \oplus 0), (u \oplus 0) \rangle \quad (u \in \mathcal{U}),$$

where \mathcal{G} is the Hilbert space given by the first part of (2.7). Finally, if Λ is strictly positive, then the entropy for the central solution is also determined by

$$(7.6) \quad \sigma_{F_{\circ}}(u) = \left\langle \left(I + \widetilde{B}^* \Lambda^{-1} \widetilde{B} \right)^{-1} u, u \right\rangle \quad (u \in \mathcal{U}).$$

The above theorem is a more detailed version of Theorem IV.7.1 in [19] specialized for the LTONP interpolation problem. For related earlier results see [29] [26] and Section XXXV in [26].

The proof of Theorem 7.1 is new. It will be given after the next result, which characterizes the entropy function σ_F of any $F \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ in terms of an observable co-isometric realization.

Lemma 7.2. *Let $\Sigma = \{\alpha, \beta, \gamma, \delta\}$ be an observable co-isometric realization of $F \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$, and let M_{Σ} be the associated system matrix. Set $\mathcal{M}_{\Sigma} = \text{Im } M_{\Sigma}^*$. Then*

$$(7.7) \quad \sigma_F(u) = \langle P_{\mathcal{M}_{\Sigma}^{\perp}} \tau_{\mathcal{U}} u, \tau_{\mathcal{U}} u \rangle \quad (u \in \mathcal{U}).$$

Here $\tau_{\mathcal{U}}$ is the embedding operator of \mathcal{U} into $\mathcal{U} \oplus \mathcal{X}$.

Proof. Fix $F \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$, and let $\Sigma = \{\alpha, \beta, \gamma, \delta\}$ be an observable co-isometric realization of F with system matrix M_Σ , and put $\mathcal{M} = \text{Im } M_\Sigma^*$ where M_Σ is given by (2.16). Since M_Σ is a co-isometry, the range of M_Σ^* is closed. Thus \mathcal{M} is a subspace of $\mathcal{U} \oplus \mathcal{X}$. We set

$$(7.8) \quad \rho_F(u) = \langle P_{\mathcal{M}^\perp} \tau_{\mathcal{U}} u, \tau_{\mathcal{U}} u \rangle \quad (u \in \mathcal{U}).$$

We have to prove $\sigma_F = \rho_F$. Since all observable co-isometric realizations of F are unitarily equivalent, see Theorem 2.5, the definition of ρ_F is independent of the choice of the observable co-isometric realization of F . Hence it suffices to show $\sigma_F = \rho_F$ for a particular choice of Σ .

Observe that F is a solution to the LTONP interpolation problem with data set $\{I_{\ell_+^2(\mathcal{Y})}, T_F, S_{\mathcal{Y}}\}$. Indeed, with

$$W = I_{\ell_+^2(\mathcal{Y})}, \quad \widetilde{W} = T_F, \quad Z = S_{\mathcal{Y}}$$

the identities (1.1) and (1.2) are automatically fulfilled. Moreover, in this case F is the unique solution, and hence F is the central solution associated with the data set $\{I_{\ell_+^2(\mathcal{Y})}, T_F, S_{\mathcal{Y}}\}$. But then we can apply Lemma 3.3 to obtain a special observable co-isometric realization of F . To do this let us denote the subspaces \mathcal{F} in (2.4) and \mathcal{G} in (2.7) associated with our data set $\{I_{\ell_+^2(\mathcal{Y})}, T_F, S_{\mathcal{Y}}\}$ by $\widehat{\mathcal{F}}$ and $\widehat{\mathcal{G}}$, respectively. In this case the associate Pick operator $\widehat{\Lambda}$ is given by $\widehat{\Lambda} = I - T_F T_F^* = D_{T_F}^2$. Note that $\widehat{\mathcal{F}}$ is given by

$$(7.9) \quad \widehat{\mathcal{F}} = \overline{\text{Im} \begin{bmatrix} E_{\mathcal{U}}^* T_F^* \\ D_{T_F^*} \end{bmatrix}}.$$

Now let $\widehat{\Sigma}$ be the observable co-isometric realization obtained by applying Lemma 3.3. Then (3.18) tells us that $(M_{\widehat{\Sigma}}^*)^\perp = \text{Ker } M_{\widehat{\Sigma}} = \widehat{\mathcal{G}}$. Thus $\rho_F(u) = \langle P_{\widehat{\mathcal{G}}} \tau_u, \tau_u \rangle$. Using (7.9) and the projection theorem we then obtain for each $u \in \mathcal{U}$ that

$$\begin{aligned} \rho_F(u) &= \langle P_{\widehat{\mathcal{G}}} \tau_{\mathcal{U}} u, \tau_{\mathcal{U}} u \rangle = \inf \left\{ \|\tau_{\mathcal{U}} u - f\| \mid f \in \widehat{\mathcal{F}} \right\} \\ &= \inf \left\{ \left\| \begin{bmatrix} u \\ 0 \end{bmatrix} - \begin{bmatrix} E_{\mathcal{U}}^* T_F^* \\ D_{T_F^*} \end{bmatrix} h \right\| \mid h \in \ell_+^2(\mathcal{Y}) \right\} \\ &= \inf \left\{ \|u - E_{\mathcal{U}}^* T_F^* h\|^2 + \langle (I - T_F T_F^*) h, h \rangle \mid h \in \ell_+^2(\mathcal{Y}) \right\} = \sigma_F(u). \end{aligned}$$

Thus we proved $\sigma_F = \rho_F$ for a particular choice of Σ , which completes the proof. \square

Remark 7.3. Note that the formula for σ_F in (7.8) can be rewritten directly in terms of the system matrix M_Σ as

$$\sigma_F(u) = \inf \left\{ \left\| \begin{bmatrix} u \\ 0 \end{bmatrix} - M_\Sigma^* h \right\|^2 \mid h \in \mathcal{Y} \oplus \mathcal{X} \right\} \quad (u \in \mathcal{U}).$$

Proof of Theorem 7.1. We shall prove Theorem 7.1 using the formula for σ_F given in Lemma 7.2.

First we derive the formula (7.3) for the central solution. From the proof of Lemma 7.2, using Lemma 3.3, we know that

$$\sigma_{F_\circ}(u) = \langle P_{\mathcal{G}} \tau_{\mathcal{U}} u, \tau_{\mathcal{U}} u \rangle = \langle P_{\mathcal{G}} \begin{bmatrix} u \\ 0 \end{bmatrix}, \begin{bmatrix} u \\ 0 \end{bmatrix} \rangle, \quad u \in \mathcal{U},$$

which yields (7.3).

Let $F \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ be a solution to the LTONP interpolation problem with data $\{W, \widetilde{W}, Z\}$, and let $\Sigma = \{\alpha, \beta, \gamma, \delta\}$ be a Λ -preferable, observable, co-isometric realization of F . Then σ_F is given by (7.7) with $\mathcal{M}_\Sigma^\perp = \text{Ker } M_\Sigma$, the null space of the system matrix M_Σ . The fact that Σ is Λ -preferable implies that $M_\Sigma^*|\mathcal{F}' = \omega^*$. Hence $\mathcal{F} = \text{Im } \omega^* \subset \text{Im } M_\Sigma^*$, so that $\mathcal{M}^\perp \subset \mathcal{F}^\perp = \mathcal{G} \oplus \mathcal{V}$ with $\mathcal{V} = \mathcal{X} \ominus \mathcal{Z}_\circ$. Hence $P_{\mathcal{M}^\perp} \leq P_{\mathcal{G} \oplus \mathcal{V}}$. Since $\mathcal{U} \perp \mathcal{V}$, both seen as subspaces of $\mathcal{U} \oplus \mathcal{X}$, we have

$$\begin{aligned} \sigma_F(u) &= \left\langle P_{\mathcal{M}_\Sigma^\perp} \tau_{\mathcal{U}} u, \tau_{\mathcal{U}} u \right\rangle \leq \left\langle P_{\mathcal{G} \oplus \mathcal{V}} \tau_{\mathcal{U}} u, \tau_{\mathcal{U}} u \right\rangle = \\ &= \left\langle P_{\mathcal{G}} \tau_{\mathcal{U}} u, \tau_{\mathcal{U}} u \right\rangle = \sigma_{F_\circ}(u) \quad (u \in \mathcal{U}). \end{aligned}$$

Hence the entropy $\sigma_{F_\circ}(u)$ of the central solution F_\circ is maximal among all solutions to the LTONP interpolation problem for the data set $\{W, \widetilde{W}, Z\}$.

Next we show that F_\circ is the unique solution to the LTONP interpolation problem for the data set $\{W, \widetilde{W}, Z\}$ that maximizes the entropy. Hence, assume that the entropy of the solution F is maximal, that is, $\sigma_F(u) = \langle P_{\mathcal{G}} \tau_{\mathcal{U}} u, \tau_{\mathcal{U}} u \rangle$ for each $u \in \mathcal{U}$. Then

$$\begin{aligned} \|P_{\mathcal{M}_\Sigma^\perp} \tau_{\mathcal{U}} u\|^2 &= \langle P_{\mathcal{M}_\Sigma^\perp} \tau_{\mathcal{U}} u, \tau_{\mathcal{U}} u \rangle = \sigma_F(u) = \\ &= \langle P_{\mathcal{G}} \tau_{\mathcal{U}} u, \tau_{\mathcal{U}} u \rangle = \|P_{\mathcal{G}} \tau_{\mathcal{U}} u\|^2 \quad (u \in \mathcal{U}). \end{aligned}$$

We will first show that $\text{Ker } M_\Sigma = \mathcal{M}_\Sigma^\perp = \mathcal{G}$. Observe that for u in \mathcal{U} we have

$$\|P_{\mathcal{F}} \tau_{\mathcal{U}} u\|^2 = \|u\|^2 - \|P_{\mathcal{G}} \tau_{\mathcal{U}} u\|^2 = \|u\|^2 - \|P_{\mathcal{M}_\Sigma^\perp} \tau_{\mathcal{U}} u\|^2 = \|P_{\mathcal{M}_\Sigma} \tau_{\mathcal{U}} u\|^2.$$

Because $M_\Sigma^*|\mathcal{F} = \omega$, it follows that \mathcal{F} is a subspace of $\text{Im } M_\Sigma^* = \mathcal{M}_\Sigma$. This yields

$$\|P_{\mathcal{L}^\perp} \tau_{\mathcal{U}} u\|^2 = \|P_{\mathcal{F}} \tau_{\mathcal{U}} u\|^2 + \|P_{\mathcal{L}^\perp \ominus \mathcal{F}} \tau_{\mathcal{U}} u\|^2.$$

Thus $P_{\mathcal{L}^\perp \ominus \mathcal{F}} \tau_{\mathcal{U}} u = 0$. Hence $P_{\mathcal{F}} \tau_{\mathcal{U}} u = P_{\mathcal{L}^\perp} \tau_{\mathcal{U}} u$ holds for all $u \in \mathcal{U}$. Then

$$P_{\mathcal{G}} \tau_{\mathcal{U}} u = \tau_{\mathcal{U}} u - P_{\mathcal{F}} \tau_{\mathcal{U}} u = \tau_{\mathcal{U}} u - P_{\mathcal{M}_\Sigma} \tau_{\mathcal{U}} u = P_{\mathcal{M}_\Sigma^\perp} \tau_{\mathcal{U}} u \quad (u \in \mathcal{U}).$$

In what follows the symbol $\mathcal{H} \vee \mathcal{K}$ stands for closed linear hull of the spaces \mathcal{H} and \mathcal{K} . By consulting (2.4) and noting that \mathcal{Z}_\circ is the closure of $\text{Im } \Lambda^{\frac{1}{2}}$, we see that

$$\mathcal{U} \vee \mathcal{F} = \begin{bmatrix} \mathcal{U} \\ 0 \end{bmatrix} \vee \begin{bmatrix} \widetilde{B}^* \\ \Lambda^{\frac{1}{2}} \end{bmatrix} \mathcal{Z} = \mathcal{U} \oplus \mathcal{Z}_\circ.$$

Hence $\mathcal{F} \oplus \mathcal{G} = \mathcal{U} \oplus \mathcal{Z}_\circ = \mathcal{U} \vee \mathcal{F}$ and we obtain that

$$\mathcal{G} = P_{\mathcal{G}}(\mathcal{F} \oplus \mathcal{G}) = \overline{P_{\mathcal{G}} \mathcal{U}} = \overline{P_{\mathcal{M}_\Sigma^\perp} \mathcal{U}} \subset \mathcal{M}_\Sigma^\perp.$$

Therefore \mathcal{G} is a subset of \mathcal{M}_Σ^\perp . Set $\mathcal{V} = \mathcal{X} \ominus \mathcal{Z}_\circ$, with \mathcal{X} being the state space of Σ . Write $\mathcal{M}_\Sigma^\perp = \mathcal{G} \oplus \mathcal{L}$. Since $\mathcal{F} \perp \mathcal{M}_\Sigma^\perp$, we have

$$\mathcal{L} \subset (\mathcal{U} \oplus \mathcal{X}) \ominus (\mathcal{F} \oplus \mathcal{G}) = (\mathcal{U} \oplus \mathcal{X}) \ominus (\mathcal{U} \oplus \mathcal{Z}_\circ) = \mathcal{V}.$$

Because $\mathcal{G} \subset \mathcal{M}_\Sigma^\perp = \text{Ker } M_\Sigma$, we have

$$M_\Sigma|(\mathcal{U} \oplus \mathcal{Z}_\circ) = M_\Sigma|(\mathcal{F} \oplus \mathcal{G}) = \omega P_{\mathcal{F}}.$$

Therefore, M_Σ has a block operator decomposition of the form

$$M_\Sigma = \begin{bmatrix} \delta & \gamma \\ \beta & \alpha \end{bmatrix} = \left[\begin{array}{c|cc} \delta_\circ & \gamma_\circ & M_1 \\ \beta_\circ & \alpha_\circ & M_2 \\ \hline 0 & 0 & M_3 \end{array} \right] : \begin{bmatrix} \mathcal{U} \\ \mathcal{Z}_\circ \\ \mathcal{V} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{Y} \\ \mathcal{Z}_\circ \\ \mathcal{V} \end{bmatrix}$$

where $\{\alpha_\circ, \beta_\circ, \gamma_\circ, \delta_\circ\}$ form the system matrix for $\omega P_{\mathcal{F}}$; see (3.15). Let $x \in \mathcal{L} \subset \mathcal{V}$. We have $M_\Sigma x = 0$, and thus, $M_j x = 0$ for $j = 1, 2, 3$. But then $\alpha x = 0$ and

$\gamma x = 0$. Hence $\gamma \alpha^k x = 0$ for each k . The fact that Σ is an observable realization then implies that $x = 0$. Thus $\mathcal{L} = \{0\}$ and we obtain that $\text{Ker } M_\Sigma = \mathcal{M}_\Sigma^\perp = \mathcal{G}$.

Using the fact that M_Σ^* is an isometry with $M_\Sigma^*|_{\mathcal{F}'} = \omega^*$ and $\mathcal{G} = \mathcal{M}_\Sigma^\perp = \text{Ker } M_\Sigma$, we see that M_Σ^* admits a matrix decomposition of the form

$$M_\Sigma^* = \begin{bmatrix} \omega^* P_{\mathcal{F}'} & 0 \\ P_{\mathcal{G}'} & U_+ \end{bmatrix} : \begin{bmatrix} \mathcal{Y} \oplus \mathcal{Z}_\circ \\ \mathcal{V} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{U} \oplus \mathcal{Z}_\circ \\ \mathcal{V} \end{bmatrix}.$$

Because $M_\Sigma^*|_{(\mathcal{Y} \oplus \mathcal{Z}_\circ)}$ is an isometry, without loss of generality we can assume that the lower left hand corner of M_Σ^* is given by $P_{\mathcal{G}'}$. Moreover, U_+ is an isometry on \mathcal{V} . Since M_Σ^* is an isometry and $\mathcal{G} = \text{Ker } M_\Sigma$, we have

$$(7.10) \quad \mathcal{V} = \mathcal{G}' \oplus \text{Im}(U_+).$$

In particular, \mathcal{G}' is a wandering subspace for the isometry U_+ and we have $\bigoplus_{n=0}^{\infty} U_+^n \mathcal{G}' \subset \mathcal{V}$. Because the systems matrix M_Σ is observable, $\mathcal{Z}_\circ \oplus \mathcal{V} = \bigvee_{n=0}^{\infty} \alpha^{*n} \gamma^* \mathcal{Y}$. Observe that α^* admits a lower triangular matrix decomposition of the form:

$$\alpha^* = \begin{bmatrix} \star & 0 \\ P_{\mathcal{G}'} & U_+ \end{bmatrix} \text{ on } \begin{bmatrix} \mathcal{Z}_\circ \\ \mathcal{V} \end{bmatrix}.$$

Furthermore, $\gamma^* \mathcal{Y}$ is a subset of $\mathcal{Z}_\circ \oplus \mathcal{G}'$. For y in \mathcal{Y} , we have

$$\alpha^{*n} \gamma^* y = \begin{bmatrix} \star \\ \sum_{k=0}^{n-2} U_+^k P_{\mathcal{G}'} \star + U_+^{n-1} P_{\mathcal{G}'} \gamma^* y \end{bmatrix}.$$

The observability condition implies that $\mathcal{V} = \bigoplus_{n=0}^{\infty} U_+^n \mathcal{G}'$. Therefore U_+ can be viewed as the unilateral shift $S_{\mathcal{G}'}$. In other words, the realization Σ of F is unitarily equivalent to the realization of the central solution obtained in Lemma 3.3. Hence $F = F_\circ$. So the maximal solution is unique.

To conclude the proof it remains to show that (7.6) holds. Assume that Λ is strictly positive. Recall that the operator τ_1 in (4.3) is an isometry from \mathcal{U} into $\mathcal{U} \oplus \mathcal{Z}$ whose range equals \mathcal{G} . Hence $\tau_1 \tau_1^* = P_{\mathcal{G}}$ is the orthogonal projection onto \mathcal{G} . In other words,

$$(7.11) \quad P_{\mathcal{G}} = \tau_1 \tau_1^* = \begin{bmatrix} I \\ -\Lambda^{-\frac{1}{2}} \tilde{B} \end{bmatrix} R_\circ^2 \begin{bmatrix} I & -\tilde{B} \Lambda^{-\frac{1}{2}} \end{bmatrix}, \text{ where } R_\circ^2 = (I + \tilde{B}^* \Lambda \tilde{B})^{-1}.$$

So for u in \mathcal{U} , we have

$$\sigma_{F_\circ}(u) = \langle P_{\mathcal{G}} \tau_1 u, \tau_1 u \rangle = \langle \tau_1 \tau_1^*(u \oplus 0), (u \oplus 0) \rangle = \langle R_\circ^2 u, u \rangle.$$

In other words, (7.6) holds. \square

Remark 7.4. Consider the LTONP interpolation problem with data $\{W, \tilde{W}, Z\}$. Moreover, assume that Λ is strictly positive and Z is exponentially stable. Let F_\circ be the central solution. Then, by Proposition 6.8, the operator T_{F_\circ} is a strict contraction, and thus (7.3) holds with T_{F_\circ} in place of T_F . Using (6.16) in Proposition 6.8 we see that

$$\begin{aligned} \sigma_{F_\circ}(u) &= \left\langle (E_{\mathcal{U}}^* T_{\Upsilon_{22}^{-1}} T_{\Upsilon_{22}^{-1}} E_{\mathcal{U}}^{-1})^{-1} u, u \right\rangle \\ &= \left\langle (\Upsilon_{22}^{-1}(0))^* \Upsilon_{22}^{-1}(0) u, u \right\rangle = \|\Upsilon_{22}^{-1}(0) u\|^2, \quad u \in \mathbb{D} \end{aligned}$$

On the other hand, according to (7.6), we have

$$\sigma_{F_\circ}(u) = \left\langle (I_{\mathcal{U}} + \tilde{B}^* \Lambda^{-1} \tilde{B})^{-1} u, u \right\rangle, \quad u \in \mathbb{D}.$$

Hence

$$s_{F_o}(u) = \|\Upsilon_{22}(0)^{-1}u\|^2 = \left\langle (I_{\mathcal{U}} + \tilde{B}^* \Lambda^{-1} \tilde{B})^{-1}u, u \right\rangle, \quad u \in \mathbb{D}.$$

If \mathcal{U} is finite dimensional, then the later identity can be rewritten as

$$\det[(I_{\mathcal{U}} + \tilde{B}^* \Lambda^{-1} \tilde{B})^{-1}] = \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \ln \det[I - F_o(e^{i\theta})^* F_o(e^{i\theta})] d\theta\right).$$

For more details, in particular concerning the connections with spectral factorization, we refer to Subsection A.7.

8. COMMUTANT LIFTING AS LTONP INTERPOLATION

In the second paragraph after Proposition 1.3 we have seen that in the strictly positive case the LTONP interpolation problem is a commutant lifting problem. In this section we go in the reverse direction. We consider a large subclass of commutant lifting problems, and we show that this class of problems is equivalent to the class of LTONP interpolation problems. This equivalence will allow us to reformulate Theorem 1.2 as a theorem describing all solutions of a suboptimal commutant lifting problem (see Theorem 8.1 below).

Our starting point is the quadruple $\{A_o, S_{\mathcal{U}}, T', S_{\mathcal{Y}}\}$ as the given commutant lifting data set. Here A_o is an operator mapping $\ell_+^2(\mathcal{U})$ into \mathcal{H}' , where \mathcal{H}' is an invariant subspace for $S_{\mathcal{Y}}^*$. In particular, \mathcal{H}' is a subspace of $\ell_+^2(\mathcal{Y})$, and $\ell_+^2(\mathcal{Y}) \ominus \mathcal{H}'$ is invariant under $S_{\mathcal{Y}}$. Furthermore, T' is the compression of $S_{\mathcal{Y}}$ to \mathcal{H}' , that is, $T' = \Pi_{\mathcal{H}'} S_{\mathcal{Y}} \Pi_{\mathcal{H}'}^*$, where $\Pi_{\mathcal{H}'}$ is the orthogonal projection of $\ell_+^2(\mathcal{Y})$ onto \mathcal{H}' . The data set satisfies the intertwining relation $A_o S_{\mathcal{U}} = T' A_o$. Note that we do not assume the minimality condition $\bigvee_{n \geq 0} S_{\mathcal{Y}}^n \mathcal{H}' = \ell_+^2(\mathcal{Y})$, which often plays a simplifying role in proofs.

Given the lifting data set $\{A_o, S_{\mathcal{U}}, T', S_{\mathcal{Y}}\}$, the commutant lifting problem is to find all $F \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ such that

$$T_F = \begin{bmatrix} A_o \\ \star \end{bmatrix} : \ell_+^2(\mathcal{U}) \rightarrow \begin{bmatrix} \mathcal{H}' \\ \ell_+^2(\mathcal{Y}) \ominus \mathcal{H}' \end{bmatrix}.$$

If the problem is solvable, then necessarily A_o is a contraction.

To reformulate this commutant lifting problem as a LTONP interpolation problem, put

$$(8.1) \quad \mathcal{Z} = \mathcal{H}', \quad Z = T', \quad W = \Pi_{\mathcal{Z}} : \ell_+^2(\mathcal{Y}) \rightarrow \mathcal{Z}, \quad \tilde{W} = A_o : \ell_+^2(\mathcal{U}) \rightarrow \mathcal{Z}.$$

Here $\Pi_{\mathcal{Z}}$ is the orthogonal projection of $\ell_+^2(\mathcal{Y})$ onto $\mathcal{Z} = \mathcal{H}'$. With W, \tilde{W} and Z given by (8.1) it is straightforward to check that $ZW = W S_{\mathcal{Y}}$ and $Z\tilde{W} = \tilde{W} S_{\mathcal{U}}$. Thus the conditions in (1.1) are satisfied. Moreover, the solutions to the LTONP interpolation problem with this data set with data $\{W, W, Z\}$ are precisely the solutions to the commutant lifting problem with data set with data $\{W, W, Z\}$; see the second paragraph after Proposition 1.3. Since $S_{\mathcal{Y}}^*$ is pointwise stable, it is also clear that Z^* is pointwise stable. Note that in this case

$$(8.2) \quad P = \Pi_{\mathcal{Z}} \Pi_{\mathcal{Z}}^* = I_{\mathcal{Z}}, \quad \tilde{P} = A_o A_o^* \quad \text{and} \quad \Lambda = P - \tilde{P} = I - A_o A_o^*,$$

$$(8.3) \quad B = \Pi_{\mathcal{Z}} E_{\mathcal{Y}} \quad \text{and} \quad \tilde{B} = A_o E_{\mathcal{U}}.$$

So the commutant lifting problem with data $\{A_o, S_{\mathcal{U}}, T', S_{\mathcal{Y}}\}$ is solvable if and only if Λ is positive, or equivalently, A_o is a contraction. Finally, it is noted that one

can use Theorem 2.1 to find all solutions to this commutant lifting problem when $\|A_\circ\| \leq 1$.

Notice that $\text{Ker } W = \ell_+^2(\mathcal{Y}) \ominus \mathcal{H}'$. By the Beurling-Lax-Halmos theorem there exists an inner function $\Theta \in \mathcal{S}(\mathcal{E}, \mathcal{Y})$ such that $\ell_+^2(\mathcal{Y}) \ominus \mathcal{H}' = \text{Ker } W = \text{Im } T_\Theta$, which allows us to define:

$$(8.4) \quad C = E_{\mathcal{E}}^* T_\Theta^* S_{\mathcal{Y}} \Pi_{\mathcal{Z}}^* : \mathcal{Z} \rightarrow \mathcal{E} \quad \text{and} \quad D = \Theta(0)^* : \mathcal{Y} \rightarrow \mathcal{E}.$$

Note C and D defined above are precisely equal to the operators C and D defined by (1.8) provided the data set $\{W, \widetilde{W}, Z\}$ is the one defined by the commutant lifting setting (8.1). It follows that the operators C and D in (8.4) is an admissible pair of complementary operators determined by the data set $\{W, \widetilde{W}, Z\}$ defined by (8.1).

Using the above connections we can apply Theorem 1.2 to obtain the following theorem which describes all solutions of the commutant lifting problem with data $\{A_\circ, S_{\mathcal{U}}, T', S_{\mathcal{Y}}\}$ for the case when the operator A_\circ is a strict contraction. Note that in this case the operator A defined by (1.15) is equal to the operator

$$(8.5) \quad A = \Pi_{\mathcal{H}'}^* A_\circ = \Pi_{\mathcal{Z}}^* A_\circ : \ell_+^2(\mathcal{U}) \rightarrow \ell_+^2(\mathcal{Y}).$$

Hence using $\Pi_{\mathcal{Z}} \Pi_{\mathcal{Z}}^* = I_{\mathcal{Z}}$, we also have $\Pi_{\mathcal{Z}} A = A_\circ$.

Theorem 8.1. *Let $\{A_\circ, S_{\mathcal{U}}, T', S_{\mathcal{Y}}\}$ be a commutant lifting data set. Assume A_\circ is a strict contraction. Then all solutions F to the commutant lifting problem for the data set $\{A_\circ, S_{\mathcal{U}}, T', S_{\mathcal{Y}}\}$ are given by*

$$(8.6) \quad F(\lambda) = (\Upsilon_{11}(\lambda)X(\lambda) + \Upsilon_{12}(\lambda))(\Upsilon_{21}(\lambda)X(\lambda) + \Upsilon_{22}(\lambda))^{-1}, \quad \lambda \in \mathbb{D},$$

where the free parameter X is an arbitrary Schur class function, $X \in \mathcal{S}(\mathcal{U}, \mathcal{E})$, and the coefficients $\Upsilon_{i,j}$, $1 \leq i, j \leq 2$, are the analytic functions on \mathbb{D} defined by

$$(8.7) \quad \Upsilon_{11}(\lambda) = D^* Q_\circ + \lambda E_{\mathcal{Y}}^* (I - \lambda S_{\mathcal{Y}}^*)^{-1} \Pi_{\mathcal{H}'}^* (I - A_\circ A_\circ^*)^{-1} C^* Q_\circ,$$

$$(8.8) \quad \Upsilon_{12}(\lambda) = E_{\mathcal{Y}}^* (I - \lambda S_{\mathcal{Y}}^*)^{-1} \Pi_{\mathcal{H}'}^* A_\circ^* (I - A_\circ^* A_\circ)^{-1} E_{\mathcal{U}} R_\circ,$$

$$(8.9) \quad \Upsilon_{21}(\lambda) = \lambda E_{\mathcal{U}}^* (I - \lambda S_{\mathcal{U}}^*)^{-1} A_\circ^* (I - A_\circ A_\circ^*)^{-1} C^* Q_\circ,$$

$$(8.10) \quad \Upsilon_{22}(\lambda) = E_{\mathcal{U}}^* (I - \lambda S_{\mathcal{U}}^*)^{-1} (I - A_\circ^* A_\circ)^{-1} E_{\mathcal{U}} R_\circ.$$

Here C and D are the operators defined by (8.4), and

$$(8.11) \quad Q_\circ = \left(I_{\mathcal{E}} + C A_\circ (I - A_\circ^* A_\circ)^{-1} A_\circ^* C^* \right)^{-\frac{1}{2}},$$

$$(8.12) \quad R_\circ = \left(E_{\mathcal{U}}^* (I - A_\circ^* A_\circ)^{-1} E_{\mathcal{U}} \right)^{-\frac{1}{2}},$$

and these operators are strictly positive.

Proof. The above theorem is a direct corollary of Theorems 1.1 and 1.2. Indeed, in the present setting $A = \Pi_{\mathcal{Z}}^* A_\circ$ and $\Pi_{\mathcal{Z}} A = A_\circ$ while the operator $W^* = \Pi_{\mathcal{Z}}^* = \Pi_{\mathcal{H}'}^*$. This implies that

$$(I - A^* A)^{-1} = (I - A_\circ^* A_\circ)^{-1}, \quad W A (I - A^* A)^{-1} A^* W^* = A_\circ (I - A_\circ^* A_\circ)^{-1} A_\circ^*.$$

It follows that in this case the operators Q_\circ and R_\circ in Theorem 1.2 are given by (8.11) and (8.12), respectively. Furthermore,

$$\begin{aligned} A(I - A^*A)^{-1} &= \Pi_{\mathcal{H}'}^* A_\circ (I - A_\circ^* A_\circ)^{-1}, \\ (I - AA^*)^{-1} W^* &= (I - AA^*)^{-1} \Pi_{\mathcal{H}'}^* = \Pi_{\mathcal{H}'}^* (I - A_\circ A_\circ^*)^{-1}, \\ A^* (I - AA^*)^{-1} W^* &= A_\circ^* \Pi_{\mathcal{H}'} \Pi_{\mathcal{H}'}^* (I - A_\circ A_\circ^*)^{-1} = A_\circ^* (I - A_\circ A_\circ^*)^{-1}. \end{aligned}$$

The latter identities show that in this case the formulas for the function Υ_{ij} , $1 \leq i, j \leq 2$, in Theorem 1.2 can be rewritten as in (8.7) – (8.10), which completes the proof. \square

9. THE LEECH PROBLEM REVISITED

In this section we discuss the Leech problem and show how it appears as a special case of our LTONP interpolation problem. We will also show that our first main result, Theorem 1.1, after some minor computations, provides the ‘infinite dimensional state space’ characterization of the solutions to the Leech problem given in Theorem 3.1 in [24], without any ‘minimality’ condition. It is noted that in [24] these formulas are used to derive algorithms in the rational case. The paper by R.B. Leech [40] where this problem originated from was eventually published in 2014; see [35] for some background on the history of this paper.

The data set for the Leech problem consists of two functions $G \in H^\infty(\mathcal{Y}, \mathcal{V})$ and $K \in H^\infty(\mathcal{U}, \mathcal{V})$, for Hilbert spaces \mathcal{U} , \mathcal{Y} and \mathcal{V} , and the aim is to find Schur class functions $F \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ such that $GF = K$. In terms of Toeplitz operators, we seek $F \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ such that $T_G T_F = T_K$. To convert the Leech problem to a LTONP interpolation problem, set $\mathcal{Z} = \ell_+^2(\mathcal{V})$ and define

$$(9.1) \quad W = T_G : \ell_+^2(\mathcal{Y}) \rightarrow \mathcal{Z}, \quad \widetilde{W} = T_K : \ell^2(\mathcal{U}) \rightarrow \mathcal{Z}, \quad Z = S_{\mathcal{V}} : \mathcal{Z} \rightarrow \mathcal{Z}.$$

In this setting,

$$(9.2) \quad P = T_G T_G^* \quad \text{and} \quad \widetilde{P} = T_K T_K^*.$$

Since T_G and T_K are analytic Toeplitz operators they intertwine the unilateral forward shifts on the appropriate ℓ_+^2 -spaces. This shows that the triple $\{W, \widetilde{W}, Z\}$ satisfies the conditions of being a LTONP data set; see (1.1). Moreover, the solutions to the LTONP interpolation problem associated with the data set $\{W, \widetilde{W}, Z\}$ coincide with the solutions to the Leech problem for the functions G and K . Furthermore, note that $Z^* = S_{\mathcal{V}}^*$ is pointwise stable, but does not have spectral radius less than one, as required in Section 1.4 of [19]. The solution criterion $WW^* - \widetilde{W}\widetilde{W}^* \geq 0$ from the LTONP interpolation problem translates to the known solution criterion for the Leech problem, namely $T_G T_G^* - T_K T_K^* \geq 0$.

Note that in this setting $B = T_G E_{\mathcal{Y}}$ and $\widetilde{B} = T_K E_{\mathcal{U}}$. One can use Theorem 2.1 to find a parametrization of all solutions to the Leech problem when $\Lambda = T_G T_G^* - T_K T_K^* \geq 0$. From Theorem 1.1, we now obtain the following characterization of the solutions to the Leech problem under the condition that $\Lambda = T_G T_G^* - T_K T_K^*$ is strictly positive.

Theorem 9.1. *Let $G \in H^\infty(\mathcal{Y}, \mathcal{V})$ and $K \in H^\infty(\mathcal{U}, \mathcal{V})$, and assume that $T_G T_G^* - T_K T_K^*$ is strictly positive. Let $\Theta \in \mathcal{S}(\mathcal{E}, \mathcal{Y})$, for some Hilbert space \mathcal{E} , be the inner*

function such that $\text{Im } T_\Theta = \text{Ker } T_G$. Then the solutions F to the Leech problem associated with G and K are given by

$$(9.3) \quad F(\lambda) = \left(\Upsilon_{11}(\lambda)X(\lambda) + \Upsilon_{12}(\lambda) \right) \left(\Upsilon_{21}(\lambda)X(\lambda) + \Upsilon_{22}(\lambda) \right)^{-1},$$

where the free parameter X is an arbitrary Schur class function, $X \in \mathcal{S}(\mathcal{U}, \mathcal{E})$, and the coefficients in (9.3) are the analytic functions on \mathbb{D} given by

$$\begin{aligned} \Upsilon_{11}(\lambda) &= \Theta(0)^* Q_\circ - \lambda E_{\mathcal{Y}}^* (I - \lambda S_{\mathcal{Y}}^*)^{-1} T_G^* (T_G T_G^* - T_K T_K^*)^{-1} N Q_\circ, \\ \Upsilon_{12}(\lambda) &= E_{\mathcal{Y}}^* (I - \lambda S_{\mathcal{Y}}^*)^{-1} T_G^* (T_G T_G^* - T_K T_K^*)^{-1} T_K E_{\mathcal{U}} R_\circ, \\ \Upsilon_{21}(\lambda) &= -\lambda E_{\mathcal{U}}^* (I - \lambda S_{\mathcal{U}}^*)^{-1} T_K^* (T_G T_G^* - T_K T_K^*)^{-1} N Q_\circ, \\ \Upsilon_{22}(\lambda) &= R_\circ + E_{\mathcal{U}}^* (I - \lambda S_{\mathcal{U}}^*)^{-1} T_K^* (T_G T_G^* - T_K T_K^*)^{-1} T_K E_{\mathcal{U}} R_\circ. \end{aligned}$$

Here $N = -T_G S_{\mathcal{Y}}^* T_\Theta E_{\mathcal{E}} = S_{\mathcal{Y}}^* T_G E_{\mathcal{Y}} \Theta(0)$ and Q_\circ and R_\circ are the strictly positive operators given by

$$\begin{aligned} Q_\circ &= (I_{\mathcal{E}} + N^* ((T_G T_G^* - T_K T_K^*)^{-1} - (T_G T_G^*)^{-1}) N)^{-\frac{1}{2}} : \mathcal{E} \rightarrow \mathcal{E}, \\ R_\circ &= (I_{\mathcal{U}} + E_{\mathcal{U}}^* T_K^* (T_G T_G^* - T_K T_K^*)^{-1} T_K E_{\mathcal{U}})^{-\frac{1}{2}} : \mathcal{U} \rightarrow \mathcal{U}. \end{aligned}$$

Moreover, the parametrization given by (1.9) is proper, that is, the map $X \mapsto F$ is one-to-one.

Proof. The formulas for $\Upsilon_{i,j}$, $1 \leq i, j \leq 2$, follow directly from those in (1.10)–(1.13) after translation to the current setting, that is, using $\Lambda = T_G T_G^* - T_K T_K^*$ with $P = T_G T_G^*$ and $B = T_G E_{\mathcal{Y}}$ and $\tilde{B} = T_K E_{\mathcal{U}}$. Using (A.28) with $W = T_G$ we arrive at $PC^* = T_G S_{\mathcal{Y}}^* T_\Theta E_{\mathcal{E}} = -N$. For the second formula for N , namely $N = S_{\mathcal{Y}}^* T_G E_{\mathcal{Y}} \Theta(0)$, see Lemma 2.1 in [24]. \square

This characterization of the solutions to the Leech problem is almost identical to that obtained in Theorem 3.1 in [24], for the case $\mathcal{U} = \mathbb{C}^p$, $\mathcal{Y} = \mathbb{C}^p$, $\mathcal{V} = \mathbb{C}^m$ and under the ‘minimality’ condition that for no nonzero $x \in \mathbb{C}^p$ the function $z \mapsto G(z)x$ is identically equal to zero. Note that the operators Q_\circ and R_\circ above coincide with Δ_1^{-1} and Δ_0^{-1} of Theorem 3.1 in [24], respectively. However, in the definition of Δ_1 in [24, eqn. (3.7)] it should have been $((T_G T_G^* - T_K T_K^*)^{-1} - (T_G T_G^*)^{-1})$ rather than $((T_G T_G^* - T_K T_K^*)^{-1} - (T_G T_G^*)^{-1})^{-1}$. To see that Υ_{12} and Υ_{22} in Theorem 9.1 indeed coincide with those in Theorem 3.1 in [24], use that $(I - \lambda S_{\mathcal{Y}}^*)^{-1} = I + \lambda(I - \lambda S_{\mathcal{Y}}^*)^{-1} S_{\mathcal{Y}}^*$, so that

$$\begin{aligned} \Upsilon_{12}(\lambda) &= E_{\mathcal{Y}}^* T_G^* (T_G T_G^* - T_K T_K^*)^{-1} T_K E_{\mathcal{U}} R_\circ + \\ &\quad \lambda E_{\mathcal{Y}}^* T_G^* (I - \lambda S_{\mathcal{Y}}^*)^{-1} S_{\mathcal{Y}}^* (T_G T_G^* - T_K T_K^*)^{-1} T_K E_{\mathcal{U}} R_\circ \\ \Upsilon_{22}(\lambda) &= R_\circ + E_{\mathcal{U}}^* T_K^* (T_G T_G^* - T_K T_K^*)^{-1} T_K E_{\mathcal{U}} R_\circ \\ &\quad + \lambda E_{\mathcal{U}}^* T_K^* (I - \lambda S_{\mathcal{Y}}^*)^{-1} S_{\mathcal{Y}}^* (T_G T_G^* - T_K T_K^*)^{-1} T_K E_{\mathcal{U}} R_\circ \\ &= R_\circ^{-1} + \lambda E_{\mathcal{U}}^* T_K^* (I - \lambda S_{\mathcal{Y}}^*)^{-1} S_{\mathcal{Y}}^* (T_G T_G^* - T_K T_K^*)^{-1} T_K E_{\mathcal{U}} R_\circ, \end{aligned}$$

where the last identity follows because

$$(I_{\mathcal{U}} + E_{\mathcal{U}}^* T_K^* (T_G T_G^* - T_K T_K^*)^{-1} T_K E_{\mathcal{U}}) R_\circ = R_\circ^{-2} R_\circ = R_\circ^{-1}.$$

The Toeplitz-corona problem corresponds to the special case of the Leech problem where $\mathcal{U} = \mathcal{V}$ and $K = I_{\mathcal{U}}$ is identically equal to the identity operator on \mathcal{U} . In view of the connection made between the LTONP interpolation problem and the

commutant lifting problem in Section 8, we refer to Proposition A.5 in [24], where the Toeplitz-corona is identified as a special case of the commutant lifting problem discussed in Section 8. Although Proposition A.5 in [24] is proven only for the case where \mathcal{U} and \mathcal{Y} are finite dimensional, one easily sees that the result carries over to the infinite dimensional case. We present the result here rephrased in terms of the LTONP interpolation problem, and add a proof for completeness. Note that with K is identically equal to $I_{\mathcal{U}}$ we have $\widetilde{W} = I_{\ell_+^2(\mathcal{U})}$. Hence \widetilde{W} is invertible. The converse is also true.

Proposition 9.2. *Let $\{W, \widetilde{W}, Z\}$ as in (1.1) be a data set for a LTONP interpolation problem where \widetilde{W} is invertible. Then there exists a function $G \in H^\infty(\mathcal{Y}, \mathcal{U})$ such that with $K \equiv I_{\mathcal{Y}}$ the operators W , \widetilde{W} and Z are given by (9.1), with $\mathcal{V} = \mathcal{Y}$, up to multiplication with an invertible operator from \mathcal{Z} to $\ell_+^2(\mathcal{U})$. In fact, G is defined by $T_G = \widetilde{W}^{-1}W$, or equivalently, $W = \widetilde{W}T_G$ and $\widetilde{W} = \widetilde{W}T_I$ and $Z = \widetilde{W}S_{\mathcal{U}}\widetilde{W}^{-1}$.*

Proof. Let $\{W, \widetilde{W}, Z\}$ be a data set for a LTONP interpolation problem with \widetilde{W} invertible. Then $ZW = WS_{\mathcal{Y}}$ and $S_{\mathcal{U}}\widetilde{W}^{-1} = \widetilde{W}^{-1}Z$, so that

$$S_{\mathcal{U}}\widetilde{W}^{-1}W = \widetilde{W}^{-1}ZW = \widetilde{W}^{-1}WS_{\mathcal{Y}}.$$

This shows $\widetilde{W}^{-1}W$ is a Toeplitz operator T_G with defining function $G \in H^\infty(\mathcal{Y}, \mathcal{U})$. It is also clear that for $K \equiv I_{\mathcal{Y}}$ we have

$$T_K = I_{\ell_+^2(\mathcal{U})} = \widetilde{W}^{-1}\widetilde{W} \quad \text{and} \quad \widetilde{W}^{-1}Z\widetilde{W} = \widetilde{W}^{-1}\widetilde{W}S_{\mathcal{U}} = S_{\mathcal{U}}. \quad \square$$

APPENDIX A.

This appendix consists of seven subsections containing standard background material that is used throughout the paper. Often we added proofs for the sake of completeness.

A.1. Stein equation. In this section, we present some standard results concerning discrete time Stein equations.

Lemma A.1. *Let Z be an operator on \mathcal{Z} such that Z^* is pointwise stable. Let α be an operator on \mathcal{X} such that $\sup_{n \geq 0} \|\alpha^n\| < \infty$ while Ξ is an operator mapping \mathcal{X} into \mathcal{Z} . Assume that the Stein equation*

$$(A.1) \quad \Omega - Z\Omega\alpha = \Xi$$

has a solution Ω mapping \mathcal{X} into \mathcal{Z} . Then the solution to this Stein equation is unique.

Proof. If Ω_1 is another operator satisfying $\Omega_1 - Z\Omega_1\alpha = \Xi$, then subtracting these two Stein equations yields

$$\Omega - \Omega_1 = Z(\Omega - \Omega_1)\alpha.$$

Applying this identity recursively, we have $\Omega - \Omega_1 = Z^n(\Omega - \Omega_1)\alpha^n$ for all integers $n \geq 0$. By taking the adjoint, we obtain $\Omega^* - \Omega_1^* = \alpha^{*n}(\Omega^* - \Omega_1^*)Z^{*n}$. Since Z^* is pointwise stable and $\sup_{n \geq 0} \|\alpha^n\| < \infty$, for each $z \in \mathcal{Z}$ we have

$$\|(\Omega^* - \Omega_1^*)z\| = \|\alpha^{*n}(\Omega^* - \Omega_1^*)Z^{*n}z\| \leq \|\alpha^{*n}\| \|(\Omega^* - \Omega_1^*)\| \|Z^{*n}z\| \rightarrow 0.$$

Hence $\Omega^* = \Omega_1^*$, or equivalently, $\Omega = \Omega_1$. Therefore the solution to the Stein equation $\Omega = Z\Omega\alpha + \Xi$ is unique. \square

Let Z be an operator on \mathcal{Z} such that Z^* is pointwise stable. Assume that W is an operator mapping $\ell_+^2(\mathcal{Y})$ into \mathcal{Z} such that $ZW = WS_{\mathcal{Y}}$. Let B be the operator mapping \mathcal{Y} into \mathcal{Z} defined by $B = WE_{\mathcal{Y}}$. Then $P = WW^*$ is the unique solution to the Stein equation

$$(A.2) \quad P = ZPZ^* + BB^*.$$

Lemma A.1 guarantees that the solution to this Stein equation is unique. Moreover, using $ZW = WS_{\mathcal{Y}}$, we obtain

$$P = WW^* = W(S_{\mathcal{Y}}S_{\mathcal{Y}}^* + E_{\mathcal{Y}}E_{\mathcal{Y}}^*)W^* = ZWW^*Z^* + BB^* = ZPZ^* + BB^*.$$

Hence $P = WW^*$ satisfies the Stein equation (A.2). Notice that

$$\begin{bmatrix} E_{\mathcal{Y}} & S_{\mathcal{Y}}E_{\mathcal{Y}} & S_{\mathcal{Y}}^2E_{\mathcal{Y}} & S_{\mathcal{Y}}^3E_{\mathcal{Y}} & \cdots \end{bmatrix} = I,$$

the identity operator on $\ell_+^2(\mathcal{Y})$. Using this with $ZW = WS_{\mathcal{Y}}$, we see that

$$W = W \begin{bmatrix} E_{\mathcal{Y}} & S_{\mathcal{Y}}E_{\mathcal{Y}} & S_{\mathcal{Y}}^2E_{\mathcal{Y}} & \cdots \end{bmatrix} = \begin{bmatrix} B & ZB & Z^2B & \cdots \end{bmatrix}.$$

In particular, $P = WW^* = \sum_{n=0}^{\infty} Z^n BB^* Z^{*n}$. Motivated by this analysis we present the following result.

Lemma A.2. *Let Z be an operator on \mathcal{Z} such that Z^* is pointwise stable. Let B be an operator mapping \mathcal{Y} into \mathcal{Z} . If P is a solution to the Stein equation $P = ZPZ^* + BB^*$, then P is the only solution to this Stein equation. Moreover, $P = WW^*$ where W is the operator mapping $\ell_+^2(\mathcal{Y})$ into \mathcal{Z} given by*

$$(A.3) \quad W = \begin{bmatrix} B & ZB & Z^2B & \cdots \end{bmatrix} : \ell_+^2(\mathcal{Y}) \rightarrow \mathcal{Z}.$$

Finally, $ZW = WS_{\mathcal{Y}}$ and $WE_{\mathcal{Y}} = B$.

Proof. By recursively using $P = ZPZ^* + BB^*$, we obtain

$$\begin{aligned} P &= BB^* + ZPZ^* = BB^* + Z(BB^* + ZPZ^*)Z^* \\ &= BB^* + ZBB^*Z^* + Z^2(BB^* + ZPZ^*)Z^{*2} + \cdots \\ &= \sum_{j=0}^n Z^j BB^* Z^{*j} + Z^{n+1} P Z^{*n+1}, \end{aligned}$$

where n is any positive integer. Because Z^* is pointwise stable, the uniform boundedness principle implies that $\sup\{\|Z^n\| : n \geq 0\} < \infty$. Thus $Z^{n+1} P Z^{*n+1}$ converges to zero pointwise as n tends to infinity. Therefore $P = \sum_{j=0}^{\infty} Z^j BB^* Z^{*j}$ with pointwise convergence. Moreover, W in (A.3) is a well defined bounded operator and $P = WW^*$. Clearly, $ZW = WS_{\mathcal{Y}}$ and $B = WE_{\mathcal{Y}}$. \square

A.2. The Douglas factorization lemma for $K_1 K_1^* = K_2 K_2^*$. In this subsection we review a variant of the Douglas factorization lemma; for the full lemma see, e.g., [26, Lemma XVII.5.2]. The results presented are used in Sections 2 and 4. Consider the two Hilbert space operators and related subspaces given by:

$$(A.4) \quad K_1 : \mathcal{H}_1 \rightarrow \mathcal{Z} \quad \text{and} \quad \mathcal{F} = \overline{\text{Im } K_1^*} \subset \mathcal{H}_1,$$

$$(A.5) \quad K_2 : \mathcal{H}_2 \rightarrow \mathcal{Z} \quad \text{and} \quad \mathcal{F}' = \overline{\text{Im } K_2^*} \subset \mathcal{H}_2.$$

The following two lemmas are direct corollaries of the Douglas factorisation lemma.

Lemma A.3. *Let K_1 and K_2 be two operators of the form (A.4) and (A.5). Then the following are equivalent.*

- (i) The operators $K_1K_1^* = K_2K_2^*$.
(ii) There exists a unitary operator $\omega : \mathcal{F} \rightarrow \mathcal{F}'$ such that
- $$(A.6) \quad \omega K_1^* = K_2^* \quad \text{or equivalently} \quad K_2\omega = K_1|_{\mathcal{F}}.$$
- (iii) There exists an operator $\omega : \mathcal{F} \rightarrow \mathcal{F}'$ such that
- $$(A.7) \quad K_2K_2^* = K_2\omega K_1^* \quad \text{and} \quad K_2\omega K_1^* = K_1K_1^*.$$

In this case ω is unitary.

If Part (ii) or (iii) holds, then the operator ω is uniquely determined. Finally, each of the identities in (A.6) separately can be used as the definition of ω .

Remark A.4. The operator products in (A.6) and (A.7) have to be understood pointwise. For instance, the first identity in (A.6) just means that $\omega K_1^*x = K_2^*x$ for each $x \in \mathcal{Z}$. Note that for each $x \in \mathcal{Z}$ we have $K_1^*x \in \mathcal{F}$, and thus ωK_1^*x is well defined and belongs to \mathcal{F}' . On the other hand, K_2^*x also belongs to \mathcal{F}' , and hence $\omega K_1^*x = K_2^*x$ makes sense. This remark also implies to the other identities in this subsection.

Let us sketch a proof of Lemma A.3. One part of the Douglas factorization lemma says that if A and B are two operator acting between the appropriate spaces, then $AA^* \leq BB^*$ if and only if there exists a contraction C from the closure of the range of B^* to the closure of the range of A^* satisfying $A^* = CB^*$. Moreover, in this case, the operator C is unique. If $K_1K_1^* = K_2K_2^*$, then there exists a contraction ω such that $K_2^* = \omega K_1^*$. Because $K_1K_1^* = K_2K_2^*$, it follows that ω is an isometry from \mathcal{F} onto \mathcal{F}' . Since ω is onto, ω is unitary. On the other hand, if $K_2^* = \omega K_1^*$ where ω is unitary, then $K_1K_1^* = K_2K_2^*$. Therefore Parts (i) and (ii) are equivalent.

Clearly, Part (ii) implies that Part (iii) holds. Assume that Part (iii) holds. Then by the first identity in (A.7) and the fact that K_2 is zero on $\mathcal{H}_2 \ominus \mathcal{F}'$, we see that $\omega K_1^* = K_2^*$. Similarly, using the second identity in (A.7) and $\mathcal{F} = \overline{\text{Im } K_1^*}$, we obtain $K_2\omega = K_1|_{\mathcal{F}}$. This yields Part (ii). Therefore Parts (i) to (iii) are equivalent.

Lemma A.5. Let K_1 and K_2 be two operators of the form (A.4) and (A.5). Assume $K_1K_1^* = K_2K_2^*$ and let $\omega : \mathcal{F} \rightarrow \mathcal{F}'$ be the unitary map uniquely determined by (A.7). Let $\tau_1 : \mathcal{U}_1 \rightarrow \mathcal{H}_1$ and $\tau_2 : \mathcal{U}_2 \rightarrow \mathcal{H}_2$ be isometries such that $\text{Im } \tau_1 = \mathcal{H}_1 \ominus \mathcal{F}$ and $\text{Im } \tau_2 = \mathcal{H}_2 \ominus \mathcal{F}'$. Then all contractions $Y : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ such that

$$(A.8) \quad K_2K_2^* = K_2YK_1^* \quad \text{and} \quad K_2YK_1^* = K_1K_1^*$$

are given by $Y = \tau_2X\tau_1^* + \Pi_{\mathcal{F}'}^*\omega\Pi_{\mathcal{F}}$ where X is any contraction mapping \mathcal{U}_1 into \mathcal{U}_2 . Moreover, the map $X \rightarrow Y$ is one-to-one.

Recall that V is a right inverse of U if $UV = I$. Next we assume that $N := K_1K_1^* = K_2K_2^*$ is strictly positive. Then both K_1 and K_2 are right invertible, the operator $K_1^*N^{-1}$ is a right inverse of K_1 and the operator $K_2^*N^{-1}$ is a right inverse of K_2 . Indeed, we have

$$\begin{aligned} K_1K_1^*N^{-1} &= K_1K_1^*(K_1K_1^*)^{-1} = I_{\mathcal{Z}}, \\ K_2K_2^*N^{-1} &= K_2K_2^*(K_2K_2^*)^{-1} = I_{\mathcal{Z}}. \end{aligned}$$

Furthermore, a direct computation shows that the orthogonal projections $P_{\mathcal{F}}$ onto \mathcal{F} and $P_{\mathcal{F}'}$ onto \mathcal{F}' are respectively given by

$$(A.9) \quad P_{\mathcal{F}} = K_1^*N^{-1}K_1 \quad \text{and} \quad P_{\mathcal{F}'} = K_2^*N^{-1}K_2.$$

Lemma A.6. *Let K_1 and K_2 be two operators of the form (A.4) and (A.5). Assume that $K_1K_1^* = K_2K_2^*$ and $N = K_1K_1^* = K_2K_2^*$ is strictly positive. Then the unique operator $\omega : \mathcal{F} \rightarrow \mathcal{F}'$ satisfying (A.7) is given by*

$$(A.10) \quad \omega P_{\mathcal{F}} = K_2^* N^{-1} K_1.$$

Proof. Using the first identity in (A.9) and next the first identity in (A.6) we see that

$$\omega P_{\mathcal{F}} h = \omega(K_1^* N^{-1} K_1) h = (\omega K_1^*) N^{-1} K_1 h = K_2^* N^{-1} K_1 h, \quad h \in \mathcal{H}_1,$$

and (A.10) is proved. \square

A.3. Construction of complementary operators. This subsection deals with the construction of operators C and D satisfying (1.6) and (1.7) assuming the operators Z and B are given. As in Section 1 the operators Z and B are Hilbert space operators, $Z : \mathcal{Z} \rightarrow \mathcal{Z}$ and $B : \mathcal{Y} \rightarrow \mathcal{Z}$. Moreover, we assume that Z^* is pointwise stable, and P is a strictly positive operator on \mathcal{Z} satisfying the Stein equation

$$(A.11) \quad P - ZPZ^* = BB^*.$$

The fact that P is strictly positive, Z^* is pointwise stable and satisfies (A.11) implies that

$$W = \begin{bmatrix} B & ZB & Z^2B & \cdots \end{bmatrix} : \ell_+^2(\mathcal{Y}) \rightarrow \mathcal{Z}$$

defines a bounded linear operator and $P = WW^*$. Moreover, as in Section 1, we have

$$(A.12) \quad ZW = WS_{\mathcal{Y}} \quad \text{and} \quad B = WE_{\mathcal{Y}} : \mathcal{Y} \rightarrow \mathcal{Z}.$$

Finally, note that P is not necessarily equal to WW^* when Z^* is not pointwise stable. For example, if Z is unitary, and $P = I$, then $B = 0$ and $W = 0$.

To see that W is a well-defined operator, consider the auxiliary operators

$$(A.13) \quad B_1 = P^{-\frac{1}{2}} B : \mathcal{Y} \rightarrow \mathcal{Z} \quad \text{and} \quad Z_1 = P^{-\frac{1}{2}} Z P^{\frac{1}{2}} : \mathcal{Z} \rightarrow \mathcal{Z}.$$

Multiplying the Stein equation $P - ZPZ^* = BB^*$ by $P^{-\frac{1}{2}}$ on the left and right yields $I - Z_1 Z_1^* = B_1 B_1^*$, and hence

$$(A.14) \quad \begin{bmatrix} B_1^* \\ Z_1^* \end{bmatrix} : \mathcal{Z} \rightarrow \begin{bmatrix} \mathcal{Y} \\ \mathcal{Z} \end{bmatrix}$$

is an isometry. In particular, the operator in (A.14) is a contraction. But then we can apply Lemma A.10 to show that

$$(A.15) \quad K := \begin{bmatrix} B_1^* \\ B_1^* Z_1^* \\ B_1^* (Z_1^*)^2 \\ \vdots \end{bmatrix} : \ell_+^2(\mathcal{Y}) \rightarrow \mathcal{Z}$$

is a well defined bounded linear operator and $\|K\| \leq 1$. Note that the adjoint of K is the operator K^* given by

$$K^* = \begin{bmatrix} B_1 & Z_1 B_1 & Z_1^2 B_1 & \cdots \end{bmatrix} : \ell_+^2(\mathcal{Y}) \rightarrow \mathcal{Z}.$$

Using the definitions of B_1 and Z_1 in (A.13) we see that

$$Z_1^n B_1 = \left(P^{-\frac{1}{2}} Z P^{\frac{1}{2}} \right)^n P^{-\frac{1}{2}} B = \left(P^{-\frac{1}{2}} Z^n P^{\frac{1}{2}} \right) P^{-\frac{1}{2}} B = P^{-\frac{1}{2}} Z^n B.$$

Thus $P^{\frac{1}{2}}K^* = W$, and hence W is a well defined operator from $\ell_+^2(\mathcal{Y})$ into \mathcal{Z} . It is emphasized that because P is strictly positive, the operator Z^* must be pointwise stable; see the first part of the proof of Lemma 3.1. The latter implies that the solution of the Stein equation $P = ZPZ^* + BB^*$ is unique (see Lemma A.1), and thus $P = WW^*$.

As mentioned in the Introduction (in the paragraph after formulas (1.6) and (1.7)) there are various ways to construct admissible pairs of complementary operators. One such construction, using the Beurling-Lax-Halmos theorem, was given in the Introduction. The next proposition provides an alternative method which has the advantage that it can be readily used in Matlab in the finite dimensional case.

Proposition A.7. *Let $Z : \mathcal{Z} \rightarrow \mathcal{Z}$ and $B : \mathcal{Y} \rightarrow \mathcal{Z}$ be Hilbert space operators, where Z^* is pointwise stable. Moreover, assume that P is strictly positive operator satisfying the Stein equation $P = ZPZ^* + BB^*$. Then there exists a Hilbert space \mathcal{E} and Hilbert space operators $C : \mathcal{Z} \rightarrow \mathcal{E}$ and $D : \mathcal{Y} \rightarrow \mathcal{E}$ such that*

$$(A.16) \quad \begin{bmatrix} D & C \\ B & Z \end{bmatrix} \begin{bmatrix} I_{\mathcal{Y}} & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} D^* & B^* \\ C^* & Z^* \end{bmatrix} = \begin{bmatrix} I_{\mathcal{E}} & 0 \\ 0 & P \end{bmatrix},$$

$$(A.17) \quad \begin{bmatrix} D^* & B^* \\ C^* & Z^* \end{bmatrix} \begin{bmatrix} I_{\mathcal{E}} & 0 \\ 0 & P^{-1} \end{bmatrix} \begin{bmatrix} D & C \\ B & Z \end{bmatrix} = \begin{bmatrix} I_{\mathcal{Y}} & 0 \\ 0 & P^{-1} \end{bmatrix}.$$

One such a pair of operators can be constructed in the following way. Let φ be any isometry from some space \mathcal{E}_0 onto the null space of $\begin{bmatrix} B & ZP^{\frac{1}{2}} \end{bmatrix}$ of the form

$$(A.18) \quad \varphi = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} : \mathcal{E}_0 \rightarrow \begin{bmatrix} \mathcal{Y} \\ \mathcal{Z} \end{bmatrix}.$$

Define the operators C_0 and D_0 by

$$(A.19) \quad C_0 = \varphi_2^* P^{-\frac{1}{2}} : \mathcal{Z} \rightarrow \mathcal{E}_0 \quad \text{and} \quad D_0 = \varphi_1^* : \mathcal{Y} \rightarrow \mathcal{E}_0.$$

Finally, all operators $C : \mathcal{Z} \rightarrow \mathcal{E}$ and $D : \mathcal{Y} \rightarrow \mathcal{E}$ satisfying (A.16) and (A.17) are given by

$$(A.20) \quad C = UC_0 \quad \text{and} \quad D = UD_0 \quad \text{with} \quad U : \mathcal{E}_0 \rightarrow \mathcal{E} \quad \text{any unitary operator.}$$

Proof. Let Z_1 and B_1 be the operators defined by equation (A.13). Note that $Z_1 Z_1^* + B_1 B_1^* = I$, the identity operator on \mathcal{Z} . Furthermore, the two identities (A.16) and (A.17) are equivalent to the statement that the operator

$$(A.21) \quad M := \begin{bmatrix} D & CP^{\frac{1}{2}} \\ B_1 & Z_1 \end{bmatrix} : \begin{bmatrix} \mathcal{Y} \\ \mathcal{Z} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{E} \\ \mathcal{Z} \end{bmatrix}$$

is unitary. Notice that $\begin{bmatrix} B & ZP^{\frac{1}{2}} \end{bmatrix}$ and $\begin{bmatrix} B_1 & Z_1 \end{bmatrix}$ have the same null space. By construction the operator

$$(A.22) \quad \begin{bmatrix} \varphi_1^* & \varphi_2^* \\ B_1 & Z_1 \end{bmatrix} : \begin{bmatrix} \mathcal{Y} \\ \mathcal{Z} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{E}_0 \\ \mathcal{Z} \end{bmatrix}$$

is unitary. So choosing $D = \varphi_1^*$ and $C = \varphi_2^* P^{-\frac{1}{2}}$ yields a system $\{Z, B, C, D\}$ satisfying (A.16) and (A.17). It easily follows that (A.16) and (A.17) remain true when C and D are multiplied with a unitary operator on the left side. Hence (A.16) and (A.17) holds for C and D as in (A.20).

Let $\{Z, B, C, D\}$ be any system satisfying (A.16) and (A.17). Because M is unitary the two operators

$$\varphi = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} : \mathcal{E}_0 \rightarrow \begin{bmatrix} \mathcal{Y} \\ \mathcal{Z} \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} D^* \\ P^{\frac{1}{2}}C^* \end{bmatrix} : \mathcal{E} \rightarrow \begin{bmatrix} \mathcal{Y} \\ \mathcal{Z} \end{bmatrix}$$

are isometries whose ranges are equal to the null space of $\begin{bmatrix} B & ZP^{\frac{1}{2}} \end{bmatrix}$. Therefore, $\varphi\varphi^* = VV^*$ is equal to the orthogonal projection onto the null space of $\begin{bmatrix} B & ZP^{\frac{1}{2}} \end{bmatrix}$. Hence there exists a unitary operator U from \mathcal{E}_0 onto \mathcal{E} satisfying

$$\begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = \begin{bmatrix} D^* \\ P^{\frac{1}{2}}C^* \end{bmatrix} U;$$

use the special case of the Douglas factorization presented in Lemma A.3. Thus, $U\varphi_1^* = D$ and $U\varphi_2^*P^{-\frac{1}{2}} = C$. \square

Proposition A.8. *Let $Z : \mathcal{Z} \rightarrow \mathcal{Z}$ and $B : \mathcal{Y} \rightarrow \mathcal{Z}$ be Hilbert space operators where Z^* is pointwise stable. Moreover, assume that P is strictly positive operator satisfying the Stein equation $P = ZPZ^* + BB^*$. Let $C : \mathcal{Z} \rightarrow \mathcal{E}$ and $D : \mathcal{Y} \rightarrow \mathcal{E}$ be Hilbert space operators such that (A.16) and (A.17) are satisfied. Put*

$$(A.23) \quad \Theta(\lambda) = D^* + \lambda B^*(I - \lambda Z^*)^{-1}C^*.$$

Then $\Theta \in \mathcal{S}(\mathcal{E}, \mathcal{Y})$ and Θ is inner. Moreover,

$$(A.24) \quad \text{Ker } W = \text{Im } T_\Theta, \quad C = E_{\mathcal{E}}^* T_\Theta^* S_{\mathcal{Y}} W^* P^{-1}, \quad D = \Theta(0)^*,$$

where $W = \begin{bmatrix} B & ZB & Z^2B & \dots \end{bmatrix}$ mapping $\ell_+^2(\mathcal{Y})$ into \mathcal{Z} is the operator determined by (A.12).

Proof. The fact $\Theta \in \mathcal{S}(\mathcal{E}, \mathcal{Y})$ and Θ is inner is a direct consequence of (A.16) and the pointwise stability of Z^* . Indeed, from (A.16) we obtain that the realization of Θ given by the system matrix M^* , with M as in (A.21), has an isometric system matrix and a pointwise stable state matrix $Z_1^* = P^{\frac{1}{2}}Z^*P^{-\frac{1}{2}}$, so that the claim follows from Theorem III.10.1 in [19]. For completeness, we present a proof. Let $\Theta(\lambda) = \sum_{n=0}^{\infty} \lambda^n \Theta_n$ be the Taylor series expansion for Θ . Note that $\Theta(0) = D^*$ and $\Theta_n = B^*(Z^*)^{n-1}C^*$ for all integers $n \geq 1$. Let Φ be the operator defined by

$$(A.25) \quad \Phi = \begin{bmatrix} \Theta_0 \\ \Theta_1 \\ \Theta_2 \\ \Theta_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} D^* \\ W^*C^* \end{bmatrix} : \mathcal{E} \rightarrow \begin{bmatrix} \mathcal{Y} \\ \ell_+^2(\mathcal{Y}) \end{bmatrix}.$$

Because W is a bounded operator mapping $\ell_+^2(\mathcal{Y})$ into \mathcal{Z} , it follows that Φ is a well defined operator. In fact, Φ is an isometry. To see this observe that (A.16) yields,

$$\Phi^*\Phi = DD^* + CWW^*C^* = DD^* + CPC^* = I.$$

Hence Φ is an isometry. Moreover, $\Phi\mathcal{E}$ is a wandering subspace for the unilateral shift $S_{\mathcal{Y}}$, that is, $\{S_{\mathcal{Y}}^n\Phi\mathcal{E}\}_{n=0}^{\infty}$ forms a set of orthogonal subspaces. To see this it is sufficient to show that $\Phi\mathcal{E}$ is orthogonal to $S_{\mathcal{Y}}^n\Phi\mathcal{E}$ for all integers $n \geq 1$. Using

$S_{\mathcal{Y}}^*W^* = W^*Z^*$, with $n \geq 1$, we obtain

$$\begin{aligned} (S_{\mathcal{Y}}^n\Phi)^*\Phi &= \Phi^*(S_{\mathcal{Y}}^*)^n\Phi = [D \quad CW] (S_{\mathcal{Y}}^*)^n \begin{bmatrix} D^* \\ W^*C^* \end{bmatrix} \\ &= [D \quad CW] \begin{bmatrix} B^*(Z^*)^{n-1}C^* \\ W^*(Z^*)^nC^* \end{bmatrix} \\ &= DB^*(Z^*)^{n-1}C^* + CWW^*(Z^*)^nC^* \\ &= (DB^* + CPZ^*)(Z^*)^{n-1}C^* = 0. \end{aligned}$$

The last equality follows from (A.16). Therefore $\{S_{\mathcal{Y}}^n\Phi\mathcal{E}\}_0^\infty$ forms a set of orthogonal subspaces.

The Toeplitz matrix T_Θ is determined by

$$T_\Theta = [\Phi \quad S_{\mathcal{Y}}\Phi \quad S_{\mathcal{Y}}^2\Phi \quad \dots].$$

Because Φ is an isometry and $\Phi\mathcal{E}$ is a wandering subspace for $S_{\mathcal{Y}}$, it follow that all the columns $\{S_{\mathcal{Y}}^n\Phi\}_0^\infty$ are isometric and orthogonal. Therefore $T_\Theta^*T_\Theta = I$ and Θ is an inner function.

Now let us show that $\text{Ker } W = \text{Im } T_\Theta$. To this end, note that

$$(A.26) \quad T_\Theta E_{\mathcal{E}} = \Phi = \begin{bmatrix} D^* \\ W^*C^* \end{bmatrix} : \mathcal{E} \rightarrow \begin{bmatrix} \mathcal{Y} \\ \ell_+^2(\mathcal{Y}) \end{bmatrix}.$$

Because $P = WW^*$ is strictly positive the range of W^* is closed. Moreover, one can directly verify that $W^*P^{-1}W$ is the orthogonal projection onto the range of W^* . Hence $I - W^*P^{-1}W$ is the orthogonal projection onto $\text{Ker } W$. Since T_Θ is an isometry $T_\Theta T_\Theta^*$ is an orthogonal projection. We claim that $I - W^*P^{-1}W = T_\Theta T_\Theta^*$, and thus, $\text{Ker } W = \text{Im } T_\Theta$. To this end, notice that $T_\Theta T_\Theta^*$ is the unique solution to the Stein equation

$$(A.27) \quad T_\Theta T_\Theta^* = S_{\mathcal{Y}}T_\Theta T_\Theta^* S_{\mathcal{Y}}^* + T_\Theta E_{\mathcal{E}} E_{\mathcal{E}}^* T_\Theta^*.$$

Because $S_{\mathcal{Y}}^*$ is pointwise stable, the solution $T_\Theta T_\Theta^*$ to this Stein equation is unique; see Lemma A.1. Moreover, using $W = \begin{bmatrix} B & ZW \end{bmatrix}$ with (A.17), we have

$$\begin{aligned} I - W^*P^{-1}W - S_{\mathcal{Y}}(I - W^*P^{-1}W)S_{\mathcal{Y}}^* &= E_{\mathcal{Y}}E_{\mathcal{Y}}^* + S_{\mathcal{Y}}W^*P^{-1}WS_{\mathcal{Y}}^* - W^*P^{-1}W \\ &= \begin{bmatrix} I & 0 \\ 0 & W^*P^{-1}W \end{bmatrix} - \begin{bmatrix} B^* \\ W^*Z^* \end{bmatrix} P^{-1} \begin{bmatrix} B & ZW \end{bmatrix} \\ &= \begin{bmatrix} I - B^*P^{-1}B & -B^*P^{-1}ZW \\ -W^*Z^*P^{-1}B & W^*P^{-1}W - W^*Z^*P^{-1}ZW \end{bmatrix} \\ &= \begin{bmatrix} D^*D & D^*CW \\ W^*C^*D & W^*C^*CW \end{bmatrix} = T_\Theta E_{\mathcal{E}} E_{\mathcal{E}}^* T_\Theta^*. \end{aligned}$$

So $I - W^*P^{-1}W$ is also the solution to the Stein equation (A.27). Because $S_{\mathcal{Y}}^*$ is pointwise stable, the solution to this Stein equation is unique. Therefore $T_\Theta T_\Theta^* = I - W^*P^{-1}W$ and $\text{Ker } W = \text{Im } T_\Theta$.

It remains to prove the second and third identity in (A.24). Using (A.26) we see that

$$E_{\mathcal{E}}^* T_\Theta^* S_{\mathcal{Y}} W^* P^{-1} = [D \quad CW] \begin{bmatrix} 0 \\ W^* P^{-1} \end{bmatrix} C W W^* P^{-1} = C P P^{-1} = C.$$

This proves the second identity in (A.24). The third follows by taking $\lambda = 0$ in (A.23). \square

Proposition A.9. *Let $Z : \mathcal{Z} \rightarrow \mathcal{Z}$ and $B : \mathcal{Y} \rightarrow \mathcal{Z}$ be Hilbert space operators where Z^* is pointwise stable. Moreover, assume that P is strictly positive operator satisfying the Stein equation $P = ZPZ^* + BB^*$. Let $\Theta \in \mathcal{S}(\mathcal{E}, \mathcal{Y})$ be any inner function such that $\text{Ker } W = \text{Im } T_\Theta$, where W is the operator appearing in (A.12). Then the operators*

$$(A.28) \quad C := E_\mathcal{E}^* T_\Theta^* S_\mathcal{Y} W^* P^{-1} : \mathcal{Z} \rightarrow \mathcal{E} \quad \text{and} \quad D := \Theta(0)^* : \mathcal{Y} \rightarrow \mathcal{E}.$$

form an admissible pair of complementary operators determined by $\{B, Z\}$, that is, with this choice of C and D the identities (A.16) and (A.17) are satisfied.

Proof. Notice that $S_\mathcal{Y}^* T_\Theta E_\mathcal{E}$ is orthogonal to $\text{Im } T_\Theta$. To see this simply observe that

$$T_\Theta^* S_\mathcal{Y}^* T_\Theta E_\mathcal{E} = S_\mathcal{E}^* T_\Theta^* T_\Theta E_\mathcal{E} = S_\mathcal{E}^* E_\mathcal{E} = 0.$$

Because $\text{Im } T_\Theta = \text{Ker } W$, we see that the range of $S_\mathcal{Y}^* T_\Theta E_\mathcal{E}$ is contained in the range of W^* . Since $P = WW^*$ is strictly positive, the range of W^* is closed and W^* is one to one. Hence $\text{Ker } W^* = \{0\}$. By another implication of the Douglas factorization lemma, see e.g., [26, Lemma XVII.5.2], we obtain that there exists a unique operator C mapping \mathcal{Z} into \mathcal{E} such that $S_\mathcal{Y}^* T_\Theta E_\mathcal{E} = W^* C^*$. By taking the adjoint we have $CW = E_\mathcal{E}^* T_\Theta^* S_\mathcal{Y}$. Hence

$$C = CWW^* P^{-1} = E_\mathcal{E}^* T_\Theta^* S_\mathcal{Y} W^* P^{-1}.$$

In other words, C is determined by the first equation in (A.28). By taking the Fourier transform we get

$$\begin{aligned} \Theta(\lambda) &= E_\mathcal{Y}^* (I - \lambda S_\mathcal{Y}^*)^{-1} T_\Theta E_\mathcal{E} = \Theta(0) + \lambda E_\mathcal{Y}^* (I - \lambda S_\mathcal{Y}^*)^{-1} S_\mathcal{Y}^* T_\Theta E_\mathcal{E} \\ &= D^* + \lambda E_\mathcal{Y}^* (I - \lambda S_\mathcal{Y}^*)^{-1} W^* C^* = D^* + \lambda B^* (I - \lambda Z^*)^{-1} C^*. \end{aligned}$$

In other words, $\Theta(\lambda) = D^* + \lambda B^* (I - \lambda Z^*)^{-1} C^*$ and (A.24) holds.

To derive (A.16) recall that $W^* C^* = S_\mathcal{Y}^* T_\Theta E_\mathcal{E}$. Hence

$$\begin{aligned} DD^* + CPC^* &= \Theta(0)^* \Theta(0) + CWW^* C^* \\ &= E_\mathcal{E}^* T_\Theta^* E_\mathcal{Y} E_\mathcal{Y}^* T_\Theta E_\mathcal{E} + E_\mathcal{E}^* T_\Theta^* S_\mathcal{Y} S_\mathcal{Y}^* T_\Theta E_\mathcal{E} \\ &= E_\mathcal{E}^* T_\Theta^* T_\Theta E_\mathcal{E} = I. \end{aligned}$$

Hence $DD^* + CPC^* = I$. Moreover,

$$\begin{aligned} BD^* + ZPC^* &= [B \quad ZW] \begin{bmatrix} \Theta(0) \\ W^* C^* \end{bmatrix} = [B \quad ZW] \begin{bmatrix} \Theta(0) \\ S_\mathcal{Y}^* T_\Theta E_\mathcal{E} \end{bmatrix} \\ &= WT_\Theta E_\mathcal{E} = 0. \end{aligned}$$

Thus $BD^* + ZPC^* = 0$. This with $P = BB^* + ZPZ^*$, yields (A.16).

To obtain (A.17), notice that T_Θ admits a decomposition of the form

$$T_\Theta = \begin{bmatrix} D^* & 0 \\ W^* C^* & T_\Theta \end{bmatrix} : \begin{bmatrix} \mathcal{E} \\ \ell_+^2(\mathcal{E}) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{Y} \\ \ell_+^2(\mathcal{Y}) \end{bmatrix}.$$

Because $\text{Ker } W = \text{Im } T_\Theta$ and $W^*P^{-1}W$ is the orthogonal projection onto the range of W^* , we have $T_\Theta T_\Theta^* = I - W^*P^{-1}W$. Using $W = \begin{bmatrix} B & ZW \end{bmatrix}$, we obtain

$$\begin{aligned} \begin{bmatrix} I - B^*P^{-1}B & -B^*P^{-1}ZW \\ -W^*Z^*P^{-1}B & I - W^*Z^*P^{-1}ZW \end{bmatrix} &= I - \begin{bmatrix} B^* \\ W^*Z^* \end{bmatrix} P^{-1} \begin{bmatrix} B & ZW \end{bmatrix} \\ &= I - W^*P^{-1}W = T_\Theta T_\Theta^* = \begin{bmatrix} D^*D & D^*CW \\ W^*C^*D & W^*C^*CW + T_\Theta T_\Theta^* \end{bmatrix} \\ &= \begin{bmatrix} D^*D & D^*CW \\ W^*C^*D & W^*C^*CW + I - W^*P^{-1}W \end{bmatrix}. \end{aligned}$$

By comparing the upper left hand corner of the first and last matrices, we have $D^*D + B^*P^{-1}B = I$. Because W is onto, comparing the upper right hand corner shows that $D^*C + B^*P^{-1}Z = 0$. Since W^* is one to one, comparing the lower right hand corner shows that $P^{-1} = ZP^{-1}Z^* + C^*C$. This yields (A.17). Therefore $\{C, D\}$ is an admissible pair of complementary operators. \square

Alternative proof of Proposition A.9. To gain some further insight, let us derive Proposition A.9 as a corollary of Proposition A.8 using the uniqueness part of the Beurling-Lax-Halmos theorem; see [21, Theorem 3.1.1].

Let $\tilde{C} : \mathcal{Z} \rightarrow \tilde{\mathcal{E}}$ and $\tilde{D} : \mathcal{Y} \rightarrow \tilde{\mathcal{E}}$ be Hilbert space operators such that (A.16) and (A.17) are satisfied with \tilde{C} and \tilde{D} in place of C and D , respectively. Set

$$\tilde{\Theta}(\lambda) = \tilde{D}^* + \lambda B^*(I - \lambda Z^*)^{-1}\tilde{C}^*.$$

Then, by Proposition A.8, the function $\tilde{\Theta}$ is inner and $\text{Ker } W = \text{Im } T_{\tilde{\Theta}}$. Thus $\text{Im } T_{\tilde{\Theta}} = \text{Im } T_\Theta$, and hence using the uniqueness part of the Beurling-Lax-Halmos theorem there exists a unitary operator U from $\tilde{\mathcal{E}}$ onto \mathcal{E} such that

$$\Theta(\lambda)U = \tilde{\Theta}(\lambda) \quad (\lambda \in \mathbb{D}).$$

Now put $C = U\tilde{C}$ and $D = U\tilde{D}$. From the final part of Proposition A.7 we know $\{C, D\}$ form an admissible pair of complementary operators determined by $\{B, Z\}$.

It remains to show that C and D are given by (A.28). From the second and third identity in (A.24) we know that

$$(A.29) \quad \tilde{C} = E_{\tilde{\mathcal{E}}}^* T_{\tilde{\Theta}}^* S_{\mathcal{Y}} W^* P^{-1} \quad \text{and} \quad \tilde{D} = \tilde{\Theta}(0)^*.$$

Since $U : \tilde{\mathcal{E}} \rightarrow \mathcal{E}$ is unitary we have $UE_{\tilde{\mathcal{E}}}^* T_{\tilde{\Theta}}^* = E_{\mathcal{E}}^* T_\Theta^*$. Thus the first identity in (A.29) shows that $C = U\tilde{C}$ is given by the first identity in (A.28). Similarly, we have

$$D = U\tilde{D} = U\tilde{\Theta}(0)^* = U(\Theta(0)U)^* = \Theta(0)^*,$$

which proves the second identity in (A.28). \square

An example. Let \mathcal{M} be a subspace of $\ell_+^2(\mathcal{Y})$ invariant under the block forward shift $S_{\mathcal{Y}}$. The Beurling-Lax-Halmos theorem [21, Theorem 3.1.1] tells us that there exist a Hilbert space \mathcal{E} and an inner function $\Theta \in \mathcal{S}(\mathcal{E}, \mathcal{Y})$ such that $\mathcal{M} = \text{Im } T_\Theta$. Moreover, if Ψ is an inner function in $\mathcal{S}(\mathcal{E}_\circ, \mathcal{Y})$ satisfying $\mathcal{M} = \text{Im } T_\Psi$, then $\Theta(\lambda)U = \Psi(\lambda)$ where U is a constant unitary operator mapping \mathcal{E}_\circ into \mathcal{E} .

We shall derive this result as a special case of Proposition A.8. Put $\mathcal{Z} = \ell_+^2(\mathcal{Y}) \ominus \mathcal{M}$, and define

$$(A.30) \quad Z = \Pi_{\mathcal{Z}} S_{\mathcal{Y}} \Pi_{\mathcal{Z}}^* : \mathcal{Z} \rightarrow \mathcal{Z} \quad \text{and} \quad B = \Pi_{\mathcal{Z}} E_{\mathcal{Y}} : \mathcal{Y} \rightarrow \mathcal{Z}.$$

Note that Z is the compression of $S_{\mathcal{Y}}$ to \mathcal{Z} , and \mathcal{Z} is an invariant subspace for $S_{\mathcal{Y}}^*$. Let W be the operator mapping $\ell_+^2(\mathcal{Y})$ onto \mathcal{Z} defined by $W = \Pi_{\mathcal{Z}}$. Since \mathcal{M} is an invariant subspace for $S_{\mathcal{Y}}$, we have

$$S_{\mathcal{Y}} = \begin{bmatrix} Z & 0 \\ \star & \star \end{bmatrix} : \begin{bmatrix} \mathcal{Z} \\ \mathcal{M} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{Z} \\ \mathcal{M} \end{bmatrix}$$

where \star represents an unspecified entry. In particular, this implies that

$$WS_{\mathcal{Y}} = [I \ 0] \begin{bmatrix} Z & 0 \\ \star & \star \end{bmatrix} = [Z \ 0] = Z [I \ 0] = ZW.$$

Hence $ZW = WS_{\mathcal{Y}}$. By construction $B = WE_{\mathcal{Y}}$. Thus $I = WW^*$ is the unique solution to the Stein equation $P = ZPZ^* + BB^*$.

The fact that $W = \Pi_{\mathcal{Z}}$ implies that $\text{Ker } W = \ell_+^2(\mathcal{Y}) \ominus \mathcal{Z} = \mathcal{M}$. But then Proposition A.8 tells us that there exist a Hilbert space \mathcal{E} and an inner function $\Theta \in \mathcal{S}(\mathcal{E}, \mathcal{Y})$ such $\mathcal{M} = \text{Im } T_{\Theta}$ which is the Beurling-Lax-Halmos result. Moreover, Propositions A.7 and A.8 together provide a procedure to construct Θ .

To prove the uniqueness, assume that Ψ is another inner function in $\mathcal{S}(\mathcal{E}_{\circ}, \mathcal{Y})$ satisfying $\mathcal{M} = \text{Im } T_{\Psi}$. Because T_{Θ} and T_{Ψ} are two isometries whose range equals \mathcal{M} , it follows that $T_{\Theta}T_{\Theta}^* = T_{\Psi}T_{\Psi}^* = P_{\mathcal{M}}$, the orthogonal projection onto \mathcal{M} . According to the variant of the Douglas factorization lemma discussed in the preceding subsection (see Lemma A.3) we have $T_{\Theta}V = T_{\Psi}$ where V is a unitary operator from $\ell_+^2(\mathcal{E}_{\circ})$ onto $\ell_+^2(\mathcal{E})$. Because $S_{\mathcal{Y}}T_{\Theta} = T_{\Theta}S_{\mathcal{E}}$ and $S_{\mathcal{Y}}T_{\Psi} = T_{\Psi}S_{\mathcal{E}_{\circ}}$, we see that $S_{\mathcal{E}}V = VS_{\mathcal{E}_{\circ}}$. So V is a lower triangular unitary Toeplitz operator. Hence $V = T_U$ where U is a constant function on \mathbb{D} whose value is a unitary operator, also denoted by U , mapping \mathcal{E}_{\circ} into \mathcal{E} . Therefore $\Theta(\lambda)U = \Psi(\lambda)$.

A.4. Construction of a co-isometric realization. In Section 2 an important role is played by the classical fact that an operator-valued function F is a Schur class function if and only if F admits an observable co-isometric realization (see Theorem 2.5). The ‘‘if part’’ in this theorem is straightforward and holds true for any contraction. Indeed, assume that

$$(A.31) \quad M = \begin{bmatrix} \delta & \gamma \\ \beta & \alpha \end{bmatrix} : \begin{bmatrix} \mathcal{U} \\ \mathcal{X} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{Y} \\ \mathcal{X} \end{bmatrix}$$

is a contraction. Then α is a contraction, and thus $(I - \lambda\alpha)^{-1}$ is well defined for all λ in the open unit disc \mathbb{D} . Hence $F(\lambda) = \delta + \lambda\gamma(I - \lambda\alpha)^{-1}\beta$ is analytic in \mathbb{D} . Now observe that for u in \mathcal{U} , we have

$$\begin{bmatrix} F(\lambda)u \\ (I - \lambda\alpha)^{-1}\beta u \end{bmatrix} = \begin{bmatrix} \delta & \gamma \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} u \\ \lambda(I - \lambda\alpha)^{-1}\beta u \end{bmatrix}.$$

Using the fact that M is contraction, we see that

$$\|F(\lambda)u\|^2 \leq \|F(\lambda)u\|^2 + \|(I - \lambda\alpha)^{-1}\beta u\|^2(1 - |\lambda|^2) \leq \|u\|^2.$$

Hence $\|F(\lambda)\| \leq 1$ for each $\lambda \in \mathbb{D}$. Therefore F is in the Schur class $\mathcal{S}(\mathcal{U}, \mathcal{Y})$.

The only ‘‘only if part’’ is much less trivial and has a long and interesting history (see the paragraph directly after Theorem 2.5). Here we present an alternative proof of the ‘‘only if part’’ inspired by the proof of Theorem 2.1; see the end of this section for more details.

Proof of the “only if” part of Theorem 2.5. Let $F \in S(\mathcal{U}, \mathcal{Y})$, and let $T = T_F$ be the block Toeplitz operator mapping $\ell_+^2(\mathcal{U})$ into $\ell_+^2(\mathcal{Y})$ defined by F . The fact that F is a Schur class function implies that T is a contraction, and hence the defect operator $D_{T^*} = (I - TT^*)^{\frac{1}{2}}$ is well defined. With T we associate the following two auxiliary operators:

$$K = \begin{bmatrix} E_{\mathcal{Y}} & S_{\mathcal{Y}}D_{T^*} \end{bmatrix} : \begin{bmatrix} \mathcal{Y} \\ \ell_+^2(\mathcal{Y}) \end{bmatrix} \rightarrow \ell_+^2(\mathcal{Y}),$$

$$L = \begin{bmatrix} TE_{\mathcal{U}} & D_{T^*} \end{bmatrix} : \begin{bmatrix} \mathcal{U} \\ \ell_+^2(\mathcal{Y}) \end{bmatrix} \rightarrow \ell_+^2(\mathcal{Y}).$$

Here D_{T^*} is the positive square root of $I - TT^*$.

PART 1. We first show that there exists a co-isometry M mapping $\mathcal{U} \oplus \ell_+^2(\mathcal{Y})$ into $\mathcal{Y} \oplus \ell_+^2(\mathcal{Y})$ such that $KM = L$. To see this, note that

$$\begin{aligned} KK^* &= E_{\mathcal{Y}}E_{\mathcal{Y}}^* + S_{\mathcal{Y}}(I - TT^*)S_{\mathcal{Y}}^* = E_{\mathcal{Y}}E_{\mathcal{Y}}^* + S_{\mathcal{Y}}S_{\mathcal{Y}}^* - TS_{\mathcal{U}}S_{\mathcal{U}}^*T^* \\ &= I_{\ell_+^2(\mathcal{Y})} - TS_{\mathcal{U}}S_{\mathcal{U}}^*T^*; \\ LL^* &= TE_{\mathcal{U}}E_{\mathcal{U}}^*T^* + (I - TT^*) = I_{\ell_+^2(\mathcal{Y})} - T(I - E_{\mathcal{U}}E_{\mathcal{U}}^*)T^* \\ &= I_{\ell_+^2(\mathcal{Y})} - TS_{\mathcal{U}}S_{\mathcal{U}}^*T^*. \end{aligned}$$

Thus $KK^* = LL^*$. It follows (apply Lemma A.3 with $K_1 = K$ and $K_2 = L$) that there exists a unique unitary operator τ_1 mapping $\overline{\text{Im } K^*}$ onto $\overline{\text{Im } L^*}$ such that $\tau_1 K^* f = L^* f$ for each $f \in \ell_+^2(\mathcal{Y})$. Furthermore, $\begin{bmatrix} y & x \end{bmatrix}^\top \in \text{Ker } K$ if and only if $y = 0$ and $x \in \text{Ker } D_{T^*}$. The latter implies that the operator τ_2 from $\text{Ker } K$ to $\text{Ker } L$ defined by

$$\tau_2 \begin{bmatrix} 0 \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ x \end{bmatrix}, \quad x \in \text{Ker } D_{T^*}$$

is a well defined isometry from $\text{Ker } K$ to $\text{Ker } L$. Since

$$\overline{\text{Im } K^*} \oplus \text{Ker } K = \mathcal{Y} \oplus \ell_+^2(\mathcal{Y}) \quad \text{and} \quad \overline{\text{Im } L^*} \oplus \text{Ker } L = \mathcal{U} \oplus \ell_+^2(\mathcal{Y}).$$

It follows that $N = \tau_1 \oplus \tau_2$ is an isometry from $\mathcal{Y} \oplus \ell_+^2(\mathcal{Y})$ into $\mathcal{U} \oplus \ell_+^2(\mathcal{Y})$ such that $NK^* = L^*$. But then $M = N^*$ is a co-isometry from $\mathcal{U} \oplus \ell_+^2(\mathcal{Y})$ into $\mathcal{Y} \oplus \ell_+^2(\mathcal{Y})$ such that $KM = L$.

We partition M as a 2×2 operator matrix using the Hilbert space direct sums $\mathcal{U} \oplus \ell_+^2(\mathcal{Y})$ and $\mathcal{Y} \oplus \ell_+^2(\mathcal{Y})$, as follows:

$$M = \begin{bmatrix} \delta & \gamma \\ \beta & \alpha \end{bmatrix} : \begin{bmatrix} \mathcal{U} \\ \ell_+^2(\mathcal{Y}) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{Y} \\ \ell_+^2(\mathcal{Y}) \end{bmatrix}.$$

Finally, using this decomposition with $KM = L$, we obtain

$$\begin{bmatrix} E_{\mathcal{Y}} & S_{\mathcal{Y}}D_{T^*} \end{bmatrix} \begin{bmatrix} \delta & \gamma \\ \beta & \alpha \end{bmatrix} = \begin{bmatrix} TE_{\mathcal{U}} & D_{T^*} \end{bmatrix}.$$

PART 2. We show that F is given by the state space realization

$$(A.32) \quad F(\lambda) = \delta + \lambda\gamma(I - \lambda\alpha)^{-1}\beta \quad (\lambda \in \mathbb{D}).$$

Since M is a co-isometry, M is a contraction, and hence the operator $\begin{bmatrix} \gamma & \alpha \end{bmatrix}^\top$ is also a contraction. But then we can apply Lemma 3.1 in [20] (see Lemma A.10

below) to show that the observability operator

$$(A.33) \quad \Gamma := \begin{bmatrix} \gamma \\ \gamma\alpha \\ \gamma\alpha^2 \\ \vdots \end{bmatrix} : \mathcal{Z} \rightarrow \ell_+^2(\mathcal{Y})$$

is well defined and a contraction. Note that

$$(A.34) \quad \Gamma - S_{\mathcal{Y}}\Gamma\alpha = E_{\mathcal{Y}}\gamma.$$

Furthermore, the identity $KM = L$ is equivalent to

$$(A.35) \quad TE_{\mathcal{U}} = E_{\mathcal{Y}}\delta + S_{\mathcal{Y}}D_{T^*}\beta \quad \text{and} \quad D_{T^*} = E_{\mathcal{Y}}\gamma + S_{\mathcal{Y}}D_{T^*}\alpha.$$

Using the second identity in (A.35) along with (A.34) we see that

$$(A.36) \quad \begin{aligned} D_{T^*} - \Gamma &= (E_{\mathcal{Y}}\gamma + S_{\mathcal{Y}}D_{T^*}\alpha) - (E_{\mathcal{Y}}\gamma + S_{\mathcal{Y}}\Gamma\alpha) = S_{\mathcal{Y}}(D_{T^*} - \Gamma)\alpha \\ &= S_{\mathcal{Y}}^n(D_{T^*} - \Gamma)\alpha^n, \quad n = 0, 1, 2, \dots \end{aligned}$$

Since α is a contraction and $S_{\mathcal{Y}}^*$ is pointwise stable, it follows that for each $f \in \ell_+^2(\mathcal{Y})$ we have

$$(D_{T^*} - \Gamma^*)f = (\alpha^*)^n (D_{T^*} - \Gamma^*) (S_{\mathcal{Y}}^*)^n f \rightarrow 0 \quad (n \rightarrow \infty).$$

But then we have $\Gamma = D_{T^*}$. Thus, by the first identity in (A.35), we obtain

$$\begin{aligned} TE_{\mathcal{U}} &= E_{\mathcal{Y}}\delta + S_{\mathcal{Y}}D_{T^*}\beta = E_{\mathcal{Y}}\delta + S_{\mathcal{Y}}\Gamma\beta \\ &= \begin{bmatrix} \delta \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} + \begin{bmatrix} 0 \\ \gamma \\ \gamma\alpha \\ \gamma\alpha^2 \\ \vdots \end{bmatrix} \beta. \end{aligned}$$

Since the first column of T is given by the Fourier coefficients F_0, F_1, F_2, \dots of the Schur class function F , we conclude that

$$F_0 = \delta \quad \text{and} \quad F_n = \gamma\alpha^{n-1}\beta, \quad n = 1, 2, \dots$$

This proves (A.32). \square

Lemma A.10. ([20, Lemma 3.1]) *Assume that $[\gamma \ \alpha]^\top$ is a contraction mapping \mathcal{Z} into $\mathcal{Y} \oplus \mathcal{Z}$. Then the observability operator $\Gamma = \text{col} [\gamma\alpha^j]_{j=0}^\infty$ is also a contraction mapping \mathcal{Z} into $\ell_+^2(\mathcal{Y})$.*

Proof. Because $[\gamma \ \alpha]^\top$ is a contraction, $I \geq \gamma^*\gamma + \alpha^*\alpha$. By recursively using this fact, we obtain

$$\begin{aligned} I &\geq \gamma^*\gamma + \alpha^*\alpha \geq \gamma^*\gamma + \alpha^*(\gamma^*\gamma + \alpha^*\alpha)\alpha \\ &\geq \gamma^*\gamma + \alpha^*\gamma^*\gamma\alpha + \alpha^{*2}(\gamma^*\gamma + \alpha^*\alpha)\alpha^2 \dots \\ &\geq \sum_{j=0}^n \alpha^{*j}\gamma^*\gamma\alpha^j + \alpha^{*n+1}\alpha^{n+1}, \quad n = 0, 1, 2, \dots \end{aligned}$$

In particular, $I \geq \sum_0^n \alpha^{*j}\gamma^*\gamma\alpha^j$ for any integer $n \geq 0$. Therefore $I \geq \Gamma^*\Gamma$ and Γ is a contraction. \square

A.5. Outer functions. The first lemma presented in this section plays an important role in the proof of Proposition 6.1. Recall that an operator-valued function Φ whose values are operators mapping \mathcal{U} into \mathcal{Y} is called *outer* if Φ is analytic on \mathbb{D} , for each $u \in \mathcal{U}$ the function $\Phi(\cdot)u$ is in $H^2(\mathcal{Y})$, and $\Phi(\cdot)\mathcal{U}$ is cyclic with respect to the forward shift on $H^2(\mathcal{Y})$. The latter is equivalent to the following condition:

$$(A.37) \quad \bigvee_{n \geq 0} S_{\mathcal{Y}}^n \begin{bmatrix} \Phi_0 \\ \Phi_1 \\ \Phi_2 \\ \vdots \end{bmatrix} \mathcal{U} = \ell_+^2(\mathcal{Y}) \quad \text{where } \Phi(\lambda) = \sum_{j=0}^{\infty} \lambda^j \Phi_j.$$

The following result has its roots in [19] and its proof is presented for the sake of completeness.

Lemma A.11. *Let A be a strict contraction mapping $\ell_+^2(\mathcal{U})$ into an auxiliary Hilbert space \mathcal{H}' satisfying the inequality $S_{\mathcal{U}}^* A^* A S_{\mathcal{U}} \leq A^* A$. Then*

$$\Phi(\lambda) = E_{\mathcal{U}}^*(I - \lambda S_{\mathcal{U}}^*)^{-1}(I - A^* A)^{-1} E_{\mathcal{U}}, \quad \lambda \in \mathbb{D},$$

is an outer function. Furthermore, there exists a function $\Psi \in H^\infty(\mathcal{U}, \mathcal{U})$ such that $\Psi(\lambda)\Phi(\lambda)u = u$ for each $u \in \mathcal{U}$ and $\lambda \in \mathbb{D}$. In particular, if $\Phi(\lambda)$ is invertible for each $\lambda \in \mathbb{D}$, then $\Phi(\lambda)^{-1}$ is in $H^\infty(\mathcal{U}, \mathcal{U})$.

We shall derive the above lemma as a corollary of the following somewhat more general lemma.

Lemma A.12. *Let Ω be a strictly positive operator on $\ell_+^2(\mathcal{U})$, and assume that $\Omega \leq S_{\mathcal{U}}^* \Omega S_{\mathcal{U}}$. Then the function $\Phi(\lambda) = E_{\mathcal{U}}^*(I - \lambda S_{\mathcal{U}}^*)^{-1} \Omega^{-1} E_{\mathcal{U}}$ is outer. Furthermore, there exists a function $\Psi \in H^\infty(\mathcal{U}, \mathcal{U})$ such that $\Psi(\lambda)\Phi(\lambda)u = u$ for each $u \in \mathcal{U}$ and $\lambda \in \mathbb{D}$. In particular, if $\Phi(\lambda)$ is invertible for each $\lambda \in \mathbb{D}$, then $\Phi(\lambda)^{-1}$ is in $H^\infty(\mathcal{U}, \mathcal{U})$.*

The additional invertibility condition appearing in the final sentences of the above two lemmas is always fulfilled if \mathcal{U} is finite dimensional; see Remark 3.2.3 in [21]. Moreover, this invertibility condition is also satisfied if $\Phi = \Upsilon_{22}$, where Υ_{22} is given by (1.19).

Proof of Lemma A.11. Put $\Omega = I - A^* A$. Since $S_{\mathcal{U}}^* A^* A S_{\mathcal{U}} \leq A^* A$, we have

$$\Omega = I - A^* A \leq I - S_{\mathcal{U}}^* A^* A S_{\mathcal{U}} = S_{\mathcal{U}}^*(I - A^* A) S_{\mathcal{U}} = S_{\mathcal{U}}^* \Omega S_{\mathcal{U}}.$$

Applying the Lemma A.12 with $\Omega = I - A^* A$ yields the desired result. \square

Proof of Lemma A.12. Notice that

$$\Omega^{\frac{1}{2}} \Omega^{\frac{1}{2}} = \Omega \leq S_{\mathcal{U}}^* \Omega S_{\mathcal{U}} = \left(\Omega^{\frac{1}{2}} S_{\mathcal{U}} \right)^* \Omega^{\frac{1}{2}} S_{\mathcal{U}}.$$

According to the Douglas factorization lemma there exists a contraction C mapping the subspace $\mathcal{M} = \overline{\Omega^{\frac{1}{2}} S_{\mathcal{U}} \ell_+^2(\mathcal{U})}$ into $\ell_+^2(\mathcal{U})$ satisfying $C \Omega^{\frac{1}{2}} S_{\mathcal{U}} = \Omega^{\frac{1}{2}}$. We extend C to the whole space $\ell_+^2(\mathcal{U})$ by setting $C|_{\mathcal{M}^\perp} = 0$. So C is a well defined contraction on $\ell_+^2(\mathcal{U})$. The remaining part of the proof is split into two parts.

PART 1. In this part we show that the function $\Phi(\lambda)$ is outer. Assume that h is a vector in $\ell_+^2(\mathcal{U})$ which is orthogonal to $S_{\mathcal{U}}^n \Omega^{-1} E_{\mathcal{U}} \mathcal{U}$ for all integer $n \geq 0$. We have to show that $h = 0$. Since h is orthogonal to $S_{\mathcal{U}}^n \Omega^{-1} E_{\mathcal{U}} \mathcal{U}$ for all $n \geq 0$, we obtain $\Omega^{-1} S_{\mathcal{U}}^{*n} h$ is orthogonal to $E_{\mathcal{U}} \mathcal{U}$ for all $n \geq 0$. So there exists a vector h_n

in $\ell_+^2(\mathcal{U})$ such that $\Omega^{-1}S_{\mathcal{U}}^{*n}h = S_{\mathcal{U}}h_n$. Multiplying on the left by $\Omega^{\frac{1}{2}}$ shows that $\Omega^{-\frac{1}{2}}S_{\mathcal{U}}^{*n}h = \Omega^{\frac{1}{2}}S_{\mathcal{U}}h_n$ is a vector in \mathcal{M} for all $n \geq 0$. We claim that

$$(A.38) \quad C^*\Omega^{-\frac{1}{2}}S_{\mathcal{U}}^{*n+1}h = \Omega^{-\frac{1}{2}}S_{\mathcal{U}}^{*n}h \quad (\text{for all integers } n \geq 0).$$

To see this notice that for g in $\ell_+^2(\mathcal{U})$, we have

$$\begin{aligned} \langle C^*\Omega^{-\frac{1}{2}}S_{\mathcal{U}}^{*n+1}h, \Omega^{\frac{1}{2}}S_{\mathcal{U}}g \rangle &= \langle \Omega^{-\frac{1}{2}}S_{\mathcal{U}}^{*n+1}h, C\Omega^{\frac{1}{2}}S_{\mathcal{U}}g \rangle \\ &= \langle \Omega^{-\frac{1}{2}}S_{\mathcal{U}}^{*n+1}h, \Omega^{\frac{1}{2}}g \rangle = (S_{\mathcal{U}}^{*n+1}h, g) \\ &= \langle S_{\mathcal{U}}^{*n}h, S_{\mathcal{U}}g \rangle = \langle \Omega^{-\frac{1}{2}}S_{\mathcal{U}}^{*n}h, \Omega^{\frac{1}{2}}S_{\mathcal{U}}g \rangle. \end{aligned}$$

Since $\Omega^{\frac{1}{2}}S_{\mathcal{U}}\ell_+^2(\mathcal{U})$ is dense in \mathcal{M} and $\Omega^{-\frac{1}{2}}S_{\mathcal{U}}^{*n}h \in \mathcal{M}$, we obtain (A.38). The recursion relation in (A.38) implies that

$$\Omega^{-\frac{1}{2}}h = C^*\Omega^{-\frac{1}{2}}S_{\mathcal{U}}^*h = C^{*2}\Omega^{-\frac{1}{2}}S_{\mathcal{U}}^{*2}h = \dots = C^{*n}\Omega^{-\frac{1}{2}}S_{\mathcal{U}}^{*n}h.$$

In other words, $\Omega^{-\frac{1}{2}}h = C^{*n}\Omega^{-\frac{1}{2}}S_{\mathcal{U}}^{*n}h$ for all integers $n \geq 0$. Because C is a contraction, we have

$$\|\Omega^{-\frac{1}{2}}h\| = \|C^{*n}\Omega^{-\frac{1}{2}}S_{\mathcal{U}}^{*n}h\| \leq \|\Omega^{-\frac{1}{2}}S_{\mathcal{U}}^{*n}h\| \rightarrow 0 \quad (n \rightarrow \infty).$$

Since $\Omega^{-\frac{1}{2}}$ is invertible, $h = 0$. So the closed linear span of $\{S_{\mathcal{U}}^n\Omega^{-1}E_{\mathcal{U}}\mathcal{U}\}_0^\infty$ equals $\ell_+^2(\mathcal{U})$ and the function Φ is outer.

PART 2. In this part we prove the remaining claims. In order to do this, let \mathcal{L} be the linear space of all sequences $u = \{u_j\}_{j=0}^\infty$, $u_j \in \mathcal{U}$ for $j = 0, 1, 2, \dots$, with compact support. The latter means that $u_j \neq 0$ for a finite number of indices j only. Note that $\mathcal{L} \subset \ell_+^2(\mathcal{U})$ and that \mathcal{L} is invariant under the forward shift $S_{\mathcal{U}}$. Given \mathcal{L} we consider the linear map M from \mathcal{L} into $\ell_+^2(\mathcal{U})$ defined by

$$Mu = \begin{bmatrix} \Omega^{-1}E_{\mathcal{U}} & S_{\mathcal{U}}\Omega^{-1}E_{\mathcal{U}} & S_{\mathcal{U}}^2\Omega^{-1}E_{\mathcal{U}} & \dots \end{bmatrix} u = \sum_{j=0}^{\infty} S_{\mathcal{U}}^j\Omega^{-1}E_{\mathcal{U}}u_j.$$

If we identify $\ell_+^2(\mathcal{U})$ with the Hardy space $H^2(\mathcal{U})$ using the Fourier transform, then \mathcal{L} is just the space of all \mathcal{U} -valued polynomials, and M is the operator of multiplication by Φ acting on the \mathcal{U} -valued polynomials.

We shall show that there exists $\epsilon > 0$ such that $\|Mu\| \geq \epsilon\|u\|$ for each $u = \{u_j\}_{j=0}^\infty$. Note that

$$\begin{aligned} \|\Omega^{\frac{1}{2}}Mu\|^2 &= \left\| \Omega^{\frac{1}{2}} \sum_{j=0}^{\infty} S_{\mathcal{U}}^j\Omega^{-1}E_{\mathcal{U}}u_j \right\|^2 = \left\langle \Omega \sum_{j=0}^{\infty} S_{\mathcal{U}}^j\Omega^{-1}E_{\mathcal{U}}u_j, \sum_{k=0}^{\infty} S_{\mathcal{U}}^k\Omega^{-1}E_{\mathcal{U}}u_k \right\rangle \\ &= \left\langle \Omega \left(\Omega^{-1}E_{\mathcal{U}}u_0 + S_{\mathcal{U}} \sum_{j=0}^{\infty} S_{\mathcal{U}}^j\Omega^{-1}E_{\mathcal{U}}u_{j+1} \right), \Omega^{-1}E_{\mathcal{U}}u_0 + S_{\mathcal{U}} \sum_{k=0}^{\infty} S_{\mathcal{U}}^k\Omega^{-1}E_{\mathcal{U}}u_{k+1} \right\rangle. \end{aligned}$$

Set $\Delta = E_{\mathcal{U}}^*\Omega^{-1}E_{\mathcal{U}}$. Using the fact that $E_{\mathcal{U}}^*S_{\mathcal{U}} = 0$ and $S_{\mathcal{U}}^*\Omega S_{\mathcal{U}} \geq \Omega$ we obtain that

$$\begin{aligned}
 \|\Omega^{\frac{1}{2}}Mu\|^2 &= \langle \Delta u_0, u_0 \rangle + \left\langle S_{\mathcal{U}}^* \Omega S_{\mathcal{U}} \sum_{j=0}^{\infty} S_{\mathcal{U}}^j \Omega^{-1} E_{\mathcal{U}} u_{j+1}, \sum_{k=0}^{\infty} S_{\mathcal{U}}^k \Omega^{-1} E_{\mathcal{U}} u_{k+1} \right\rangle \\
 &\geq \langle \Delta u_0, u_0 \rangle + \left\langle \Omega \sum_{j=0}^{\infty} S_{\mathcal{U}}^j \Omega^{-1} E_{\mathcal{U}} u_{j+1}, \sum_{k=0}^{\infty} S_{\mathcal{U}}^k \Omega^{-1} E_{\mathcal{U}} u_{k+1} \right\rangle \\
 &= \|\Delta^{\frac{1}{2}}u_0\|^2 + \|\Omega^{\frac{1}{2}}MS_{\mathcal{U}}^*u\|^2.
 \end{aligned}$$

Applying the above computation to $S_{\mathcal{U}}^*u$ instead of u , and continuing recursively we obtain that

$$(A.39) \quad \|\Omega^{\frac{1}{2}}Mu\|^2 \geq \sum_{j=0}^{\infty} \|\Delta^{\frac{1}{2}}u_j\|^2.$$

Since Δ is strictly positive, there exists a $\epsilon_1 > 0$ such that $\|\Delta^{\frac{1}{2}}u_j\| \geq \epsilon_1 \|u_j\|$ for all $j = 0, 1, 2, \dots$. But then the inequality (A.39) shows that

$$\begin{aligned}
 \|Mu\|^2 &\geq \|\Omega^{\frac{1}{2}}\|^{-1} \|\Omega^{\frac{1}{2}}Mu\|^2 \geq \|\Omega^{\frac{1}{2}}\|^{-1} \sum_{j=0}^{\infty} \|\Delta^{\frac{1}{2}}u_j\|^2 \\
 &\geq \epsilon_1^2 \|\Omega^{\frac{1}{2}}\|^{-1} \sum_{j=0}^{\infty} \|u_j\|^2 \geq \epsilon_1^2 \|\Omega^{\frac{1}{2}}\|^{-1} \|u\|^2 \\
 (A.40) \quad &= \epsilon^2 \|u\|^2, \quad \text{where } \epsilon = \epsilon_1 \|\Omega^{\frac{1}{2}}\|^{-\frac{1}{2}}.
 \end{aligned}$$

We conclude that M is bounded from below.

Next, put $\mathcal{R} = M\mathcal{L} \subset \ell_+^2(\mathcal{U})$. Then M maps \mathcal{L} in a one-to-one way onto \mathcal{R} . By T we denote the corresponding inverse operator. Then the result of the previous paragraph tells us that $\|Tf\| \leq \epsilon^{-1} \|f\|$ for each $f \in \mathcal{R}$. The fact that Φ is outer implies that \mathcal{R} is dense in $\ell_+^2(\mathcal{U})$. It follows that T extends to a bounded linear operator from $\ell_+^2(\mathcal{U})$ into $\ell_+^2(\mathcal{U})$ which we also denote by T . Recall that \mathcal{L} is invariant under the forward shift $S_{\mathcal{U}}$. Since $S_{\mathcal{U}}Mu = MS_{\mathcal{U}}u$ for each $u \in \mathcal{L}$, we also have $S_{\mathcal{U}}Tf = TS_{\mathcal{U}}f$ for each $f \in \mathcal{R}$. But then the fact that T is a bounded linear operator on $\ell_+^2(\mathcal{U})$ implies by continuity that $S_{\mathcal{U}}Tg = TS_{\mathcal{U}}g$ for each $g \in \ell_+^2(\mathcal{U})$. It follows that T is a (block) lower triangular Toeplitz operator. Let $\Psi \in H^\infty(\mathcal{U}, \mathcal{U})$ be its defining function, i.e., $T = T_\Psi$. Since $TMu = u$ for each $u \in \mathcal{L}$, we have

$$(A.41) \quad \Psi(\lambda)\Phi(\lambda)u = u, \quad u \in \mathcal{U}, \lambda \in \mathbb{D}.$$

Now if $\Phi(\lambda)$ is invertible for each $\lambda \in \mathbb{D}$, then it is clear that $\Phi(\lambda)^{-1} = \Psi(\lambda)$ is in $H^\infty(\mathcal{U}, \mathcal{U})$. \square

Observe that for the case when $\dim \mathcal{U} < \infty$ the identity (A.41) implies that $\Phi(\lambda)$ is invertible for each $\lambda \in \mathbb{D}$ without using Remark 3.2.3. in [21].

Remark A.13. It is interesting to consider the special case when Ω is a strictly positive Toeplitz operator on $\ell_+^2(\mathcal{U})$. In this case $\Omega = S_{\mathcal{U}}^* \Omega S_{\mathcal{U}}$, and the proof of Lemma A.12 yields a classical result on spectral factorization; see, e.g., Proposition 10.2.1 in [21]. Indeed, put $\Psi(\lambda) = (E_{\mathcal{U}}^* \Omega^{-1} E_{\mathcal{U}})^{\frac{1}{2}} \Phi(\lambda)^{-1}$ where, as before, $\Phi(\lambda) = E_{\mathcal{U}}^*(I - \lambda S_{\mathcal{U}}^*)^{-1} \Omega^{-1} E_{\mathcal{U}}$. The fact that Ω is a strictly positive Toeplitz operator then implies that $\Phi(\lambda)$ is invertible for each $\lambda \in \mathbb{D}$, and $\Psi(\lambda)$ and $\Psi(\lambda)^{-1}$ are both functions in $H^\infty(\mathcal{U}, \mathcal{U})$. Moreover, Ψ is the outer spectral factor for Ω , that is,

$\Omega = T_\Psi^* T_\Psi$ and Ψ is an outer function. To prove the latter using elements of the proof of Lemma A.12, observe that in this setting, we have equality in (A.39), that is,

$$\|\Omega^{\frac{1}{2}} T_\Phi u\|^2 = \sum_{j=0}^{\infty} \|\Delta^{\frac{1}{2}} u_j\|^2 \text{ for all } u \text{ in } \ell_+^2(\mathcal{U}) \text{ with compact support.}$$

Because T_Φ^{-1} is a bounded operator, we have $\|\Omega^{\frac{1}{2}} u\|^2 = \|\Delta^{\frac{1}{2}} T_\Phi^{-1} u\|^2$ for all u in $\ell_+^2(\mathcal{U})$. In other words, $\Omega = T_\Psi^* T_\Psi$. Since Ω is strictly positive and Φ is outer, T_Ψ is well defined bounded invertible operator. Hence Ψ and Ψ^{-1} are both functions in $H^\infty(\mathcal{U}, \mathcal{U})$, and Ψ is the outer spectral factor for Ω . See Section 10.2 in [21] for further details.

A.6. An operator optimization problem. The results in this subsection provide background material for Section 7. We begin with an elementary optimization problem. Let $A_1 : \mathcal{H} \rightarrow \mathcal{U}$ and $A_2 : \mathcal{H} \rightarrow \mathcal{R}$ be Hilbert space operators, where $\overline{\text{Im } A_2} = \mathcal{R}$ and $\mathcal{R} \subset \mathcal{H}$. With these two operators we associate a *cost function* $\sigma(u)$ on \mathcal{U} , namely

$$(A.42) \quad \sigma(u) = \inf\{\|u - A_1 h\|^2 + \|A_2 h\|^2 \mid h \in \mathcal{H}\}, \quad u \in \mathcal{U}.$$

To understand the problem better let A be the operator given by:

$$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} : \mathcal{H} \rightarrow \begin{bmatrix} \mathcal{U} \\ \mathcal{R} \end{bmatrix} \quad \text{and} \quad \text{put } \mathcal{A} = \overline{\text{Im } A}.$$

Then by the projection theorem

$$\sigma(u) = \inf \left\{ \left\| \begin{bmatrix} u \\ 0 \end{bmatrix} - Ah \right\|^2 \mid h \in \mathcal{H} \right\} = \left\| (I - P_{\mathcal{A}}) \begin{bmatrix} u \\ 0 \end{bmatrix} \right\|^2.$$

Here $P_{\mathcal{A}}$ is the orthogonal projection on $\mathcal{U} \oplus \mathcal{R}$ with range $\mathcal{A} = \overline{\text{Im } A}$. Next, let $\Pi_{\mathcal{U}}$ be the orthogonal projection of $\mathcal{U} \oplus \mathcal{R}$ onto \mathcal{U} , and thus $\Pi_{\mathcal{U}}^*$ is the canonical embedding of \mathcal{U} into $\mathcal{U} \oplus \mathcal{R}$. Using this notation we see that

$$(A.43) \quad \begin{aligned} \sigma(u) &= \left\| (I - P_{\mathcal{A}}) \begin{bmatrix} u \\ 0 \end{bmatrix} \right\|^2 = \|(I - P_{\mathcal{A}}) \Pi_{\mathcal{U}}^* u\|^2 \\ &= \langle \Pi_{\mathcal{U}} P_{\mathcal{A}^\perp} \Pi_{\mathcal{U}}^* u, u \rangle, \quad u \in \mathcal{U}. \end{aligned}$$

In particular, $\sigma(u) = \langle \Pi_{\mathcal{U}} P_{\mathcal{A}^\perp} \Pi_{\mathcal{U}}^* u, u \rangle$ is quadratic function in u . Here \mathcal{A}^\perp is the orthogonal complement of \mathcal{A} in $\mathcal{U} \oplus \mathcal{R}$.

The case when $A_2^* A_2$ is strictly positive is of particular interest. In case $A_2^* A_2$ is strictly positive, $A^* A = A_1^* A_1 + A_2^* A_2$ is also strictly positive. It follows that $P_{\mathcal{A}} = A(A^* A)^{-1} A^*$. Moreover, we have

$$\begin{aligned} \Pi_{\mathcal{U}} P_{\mathcal{A}^\perp} \Pi_{\mathcal{U}}^* &= I_{\mathcal{U}} - \Pi_{\mathcal{U}} P_{\mathcal{A}} \Pi_{\mathcal{U}}^* = I_{\mathcal{U}} - \Pi_{\mathcal{U}} A (A^* A)^{-1} A^* \Pi_{\mathcal{U}}^* \\ &= I_{\mathcal{U}} - A_1 (A_1^* A_1 + A_2^* A_2)^{-1} A_1^* \\ &= I_{\mathcal{U}} - A_1 \left(I_{\mathcal{H}} + (A_2^* A_2)^{-1} A_1^* A_1 \right)^{-1} (A_2^* A_2)^{-1} A_1^* \\ &= I_{\mathcal{U}} - \left(I_{\mathcal{U}} + A_1 (A_2^* A_2)^{-1} A_1^* \right)^{-1} A_1 (A_2^* A_2)^{-1} A_1^* \\ &= \left(I_{\mathcal{U}} + A_1 (A_2^* A_2)^{-1} A_1^* \right)^{-1}. \end{aligned}$$

Thus when A_2 is strictly positive, then the cost function is given by

$$(A.44) \quad \sigma(u) = \langle (I_{\mathcal{U}} + A_1(A_2^*A_2)^{-1}A_1^*)^{-1}u, u \rangle, \quad u \in \mathcal{U}.$$

A special choice of A_1 and A_2 . Let C be a contraction from the Hilbert space \mathcal{E} into the Hilbert space \mathcal{H} , let \mathcal{U} be a subspace of \mathcal{E} , and let $\mathcal{R} = \mathcal{D}_{C^*}$ where \mathcal{D}_{C^*} is the closure of the range of the defect operator $D_{C^*} = (I_{\mathcal{H}} - CC^*)^{\frac{1}{2}}$. Put

$$A_1 = \tau_{\mathcal{U}}^* C^* : \mathcal{H} \rightarrow \mathcal{U} \quad \text{and} \quad A_2 = D_{C^*} : \mathcal{H} \rightarrow \mathcal{R}.$$

Here $\tau_{\mathcal{U}}$ is the canonical embedding of \mathcal{U} into \mathcal{E} . Thus $C\tau_{\mathcal{U}}$ maps \mathcal{U} into \mathcal{H} . In this case the cost function σ is given by

$$(A.45) \quad \sigma(u) = \inf\{\|u - \tau_{\mathcal{U}}^* C^* h\|^2 + \langle (I - CC^*)h, h \rangle \mid h \in \mathcal{H}\}, \quad u \in \mathcal{U}.$$

Furthermore, the operator A is given by

$$(A.46) \quad A = \begin{bmatrix} \tau_{\mathcal{U}}^* C^* \\ D_{C^*} \end{bmatrix} \Big|_{\mathcal{H}} \rightarrow \begin{bmatrix} \mathcal{U} \\ \mathcal{R} \end{bmatrix}, \quad \text{where } \mathcal{R} = \mathcal{D}_{C^*}.$$

Finally, if C is a strict contraction, then D_{C^*} is invertible and $\mathcal{R} = \mathcal{H}$. Using (A.44) it follows that

$$\begin{aligned} \sigma(u) &= \langle (I_{\mathcal{U}} + \tau_{\mathcal{U}}^* C^* (I_{\mathcal{H}} - CC^*)^{-1} C \tau_{\mathcal{U}}) u, u \rangle \\ &= \langle \tau_{\mathcal{U}}^* (I_{\mathcal{E}} + C^* (I_{\mathcal{H}} - CC^*)^{-1} C) \tau_{\mathcal{U}} u, u \rangle \\ &= \langle \tau_{\mathcal{U}}^* (I_{\mathcal{E}} + (I_{\mathcal{E}} - C^* C)^{-1} C^* C) \tau_{\mathcal{U}} u, u \rangle \\ &= \langle \tau_{\mathcal{U}}^* (I_{\mathcal{E}} + (I_{\mathcal{E}} - C^* C)^{-1} (C^* C - I_{\mathcal{E}} + I_{\mathcal{E}})) \tau_{\mathcal{U}} u, u \rangle \\ &= \langle \tau_{\mathcal{U}}^* (I_{\mathcal{E}} - C^* C)^{-1} \tau_{\mathcal{U}} u, u \rangle, \quad u \in \mathcal{U}. \end{aligned}$$

Thus in this case the cost function is given by

$$(A.47) \quad \sigma(u) = \langle (I_{\mathcal{E}} - C^* C)^{-1} u, u \rangle, \quad u \in \mathcal{U} \subset \mathcal{E}.$$

The next lemma shows that additional information on $\mathcal{E} \ominus \mathcal{U}$ yields alternative formulas for the cost function.

Lemma A.14. *Let V be an isometry on \mathcal{E} such that $\text{Im } V = \mathcal{E} \ominus \mathcal{U}$. Then the cost function σ defined by (A.45) is also given by*

$$(A.48) \quad \sigma(u) = \inf\{\|D_C(\tau_{\mathcal{U}} u - Ve)\|^2 \mid u \in \mathcal{E}\}, \quad u \in \mathcal{U}.$$

Proof. To prove the lemma we shall use the so-called *rotation matrix* R associated with the contraction C . Recall (see, e.g., the paragraph after Proposition 1.2 in [26, Section XXVII.1]) that

$$(A.49) \quad R = \begin{bmatrix} C^* & D_C \\ D_{C^*} & -C \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{D}_C \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{E} \\ \mathcal{D}_{C^*} \end{bmatrix}$$

is a unitary operator. As before, let A be the operator given in (A.46). Using (A.46) one sees that $f \oplus g$ is a vector in \mathcal{A}^{\perp} if and only if $f \oplus g \in \mathcal{U} \oplus \mathcal{D}_{C^*}$ and $f \oplus g$ is orthogonal to \mathcal{A} , that is,

$$0 = \left\langle \begin{bmatrix} f \\ g \end{bmatrix}, \begin{bmatrix} \tau_{\mathcal{U}}^* C^* \\ D_{C^*} \end{bmatrix} h \right\rangle = \left\langle \begin{bmatrix} \tau_{\mathcal{U}} f \\ g \end{bmatrix}, \begin{bmatrix} C^* \\ D_{C^*} \end{bmatrix} h \right\rangle = \left\langle \begin{bmatrix} f \\ g \end{bmatrix}, \begin{bmatrix} C^* \\ D_{C^*} \end{bmatrix} h \right\rangle, \quad h \in \mathcal{H}.$$

Thus $f \oplus g$ is a vector in \mathcal{A}^{\perp} if and only if $f \oplus g \in \mathcal{U} \oplus \mathcal{D}_{C^*}$ and $f \oplus g$ is orthogonal to the range of the first column of the operator matrix R . Since R is unitary, we conclude that $f \oplus g \in \mathcal{A}^{\perp}$ if and only if $f \oplus g \in \mathcal{U} \oplus \mathcal{D}_{C^*}$ and is contained in the range of the second column of R . In other words, $f \oplus g \in \mathcal{A}^{\perp}$ if and only if

$f \oplus g \in \mathcal{U} \oplus \mathcal{D}_C^*$ and $f = D_C v$ and $g = -Cv$ for some $v \in \mathcal{D}_C$. Clearly, $D_C v \in \mathcal{U}$ if and only if $D_C v$ is orthogonal to $\mathcal{U}^\perp = \mathcal{E} \ominus \mathcal{U}$. However, $D_C v$ is orthogonal to \mathcal{U}^\perp if and only if v is orthogonal to $D_C \mathcal{U}^\perp$, or equivalently, v is in $\mathcal{D}_C \ominus D_C \mathcal{U}^\perp$. Since $D_C \mathcal{U}^\perp = D_C V \mathcal{E}$, we have

$$(\text{Im } A)^\perp = \begin{bmatrix} D_C \\ -C \end{bmatrix} (\mathcal{D}_C \ominus D_C V \mathcal{E}) = \begin{bmatrix} D_C \\ -C \end{bmatrix} \mathfrak{D},$$

where $\mathfrak{D} = \mathcal{D}_C \ominus D_C V \mathcal{E}$. Therefore the orthogonal projection $P_{\mathcal{A}^\perp}$ is given by

$$P_{\mathcal{A}^\perp} \begin{bmatrix} u \\ 0 \end{bmatrix} = \begin{bmatrix} D_C \\ -C \end{bmatrix} P_{\mathfrak{D}} \begin{bmatrix} D_C & -C^* \end{bmatrix} \begin{bmatrix} \tau_{\mathcal{U}} u \\ 0 \end{bmatrix}, \quad u \in \mathcal{U}.$$

Notice that $\mathfrak{D}^\perp = \overline{D_C V \mathcal{E}}$. Using (A.43) it follows that

$$\begin{aligned} \sigma(u) &= \left\langle P_{\mathcal{A}^\perp} \begin{bmatrix} u \\ 0 \end{bmatrix}, \begin{bmatrix} u \\ 0 \end{bmatrix} \right\rangle = \langle P_{\mathfrak{D}} D_C \tau_{\mathcal{U}} u, D_C \tau_{\mathcal{U}} u \rangle \\ &= \|P_{\mathfrak{D}} D_C \tau_{\mathcal{U}} u\|^2 = \inf\{\|D_C \tau_{\mathcal{U}} u - d\|^2 \mid d \in \mathfrak{D}^\perp\} \\ &= \inf\{\|D_C \tau_{\mathcal{U}} u - D_C V e\|^2 \mid e \in \mathcal{E}\}. \end{aligned}$$

Therefore the cost function σ in the two optimization problems (A.45) and (A.48) are the same. \square

A.7. A connection to prediction theory and multiplicative diagonals. Let T_R be a non-negative Toeplitz operator on $\ell_+^2(\mathcal{U})$ with symbol R in $L^\infty(\mathcal{U}, \mathcal{U})$. Then a classical prediction problem is solve the following optimization problem:

$$(A.50) \quad \sigma(u) = \inf\{\langle T_R(E_{\mathcal{U}} u - S_{\mathcal{U}} h), E_{\mathcal{U}} u - S_{\mathcal{U}} h \rangle : h \in \ell_+^2(\mathcal{U})\}$$

where u is a specified vector in \mathcal{U} ; see Helson-Lowdenslager [31, 32].

Recall that a non-negative Toeplitz operator T_R on $\ell_+^2(\mathcal{U})$ with defining function R in $L^\infty(\mathcal{U}, \mathcal{U})$ admits an *outer spectral factor* if there exists an outer function Ψ in $H^\infty(\mathcal{U}, \mathcal{E})$ such that $T_R = T_\Psi^* T_\Psi$, or equivalently, $R(e^{i\theta}) = \Psi(e^{i\theta})^* \Psi(e^{i\theta})$ almost everywhere. In this case, the outer spectral factor Ψ for R is unique up to a unitary constant on the left. In other words, if Φ in $H^\infty(\mathcal{U}, \mathcal{V})$ is another outer function satisfying $T_R = T_\Phi^* T_\Phi$, then $\Psi(\lambda) = U \Phi(\lambda)$ where U is a constant unitary operator mapping \mathcal{V} onto \mathcal{E} . Finally, it is noted that not all non-negative Toeplitz operators admit an outer spectral factor. For example, if $R(e^{i\theta}) = 1$ for $0 \leq \theta \leq \pi$ and zero otherwise, then T_R is a non-negative Toeplitz operator on ℓ_+^2 and does not admit an outer spectral factor. For further results concerning outer spectral factorization see [43, 21]. Following some ideas in Sz.-Nagy-Foias [43], we obtain the following result.

Proposition A.15. *Assume that T_R admits an outer spectral factorization $T_R = T_\Psi^* T_\Psi$ where Ψ is an outer function in $H^\infty(\mathcal{U}, \mathcal{E})$. Then the function σ in (A.50) is also given by $\sigma(u) = \|\Psi(0)u\|^2$ for each $u \in \mathcal{U}$. Moreover, the cost function σ is independent of the outer spectral factor Ψ chosen for R .*

Proof. Observe that in this case

$$\begin{aligned}
 \sigma(u) &= \inf\{\langle T_{\Psi}^* T_{\Psi}(E_{\mathcal{U}}u - S_{\mathcal{U}}h), E_{\mathcal{U}}u - S_{\mathcal{U}}h \rangle : h \in \ell_+^2(\mathcal{U})\} \\
 &= \inf\{\|T_{\Psi}(E_{\mathcal{U}}u - S_{\mathcal{U}}h)\|^2 : h \in \ell_+^2(\mathcal{U})\} \\
 &= \inf\{\|E_{\mathcal{E}}E_{\mathcal{E}}^*T_{\Psi}E_{\mathcal{U}}u + S_{\mathcal{E}}S_{\mathcal{E}}^*T_{\Psi}E_{\mathcal{U}}u - T_{\Psi}S_{\mathcal{U}}h\|^2 : h \in \ell_+^2(\mathcal{U})\} \\
 &= \inf\{\|E_{\mathcal{E}}\Psi(0)u + S_{\mathcal{E}}S_{\mathcal{E}}^*T_{\Psi}E_{\mathcal{U}}u - S_{\mathcal{E}}T_{\Psi}h\|^2 : h \in \ell_+^2(\mathcal{U})\} \\
 &= \inf\{\|\Psi(0)u\|^2 + \|S_{\mathcal{E}}S_{\mathcal{E}}^*T_{\Psi}E_{\mathcal{U}}u - S_{\mathcal{E}}T_{\Psi}h\|^2 : h \in \ell_+^2(\mathcal{U})\} \\
 &= \|\Psi(0)u\|^2 + \inf\{\|S_{\mathcal{E}}^*T_{\Psi}E_{\mathcal{U}}u - T_{\Psi}h\|^2 : h \in \ell_+^2(\mathcal{U})\} = \|\Psi(0)u\|^2.
 \end{aligned}$$

The last equality follows from the fact that Ψ is outer, that is, the range of T_{Ψ} is dense in $\ell_+^2(\mathcal{E})$. Therefore

$$(A.51) \quad \sigma(u) = \|\Psi(0)u\|^2 = \langle \Psi(0)^* \Psi(0)u, u \rangle, \quad u \in \mathcal{U}.$$

The final statement follows from the fact that the outer spectral factor Ψ for R is unique up to a unitary constant on the left. \square

If \mathcal{U} is finite dimensional, then R admits an outer spectral factor Ψ in $H^\infty(\mathcal{U}, \mathcal{U})$ if and only if

$$(A.52) \quad \frac{1}{2\pi} \int_0^{2\pi} \ln \det[R(e^{i\theta})] d\theta > -\infty.$$

In this case, the classical Szegő formula tells us that

$$(A.53) \quad \det[\Psi(0)^* \Psi(0)] = \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \ln \det[R(e^{i\theta})] d\theta\right)$$

where $\det[T]$ is the determinant of a finite dimensional operator with respect to any basis.

The following proposition is well known. The equality in (A.54) follows by a standard Schur complement computation.

Proposition A.16. *If T_R is a strictly positive operator on $\ell_+^2(\mathcal{U})$, then T_R admits an outer spectral factor Ψ in $H^\infty(\mathcal{U}, \mathcal{U})$ and*

$$(A.54) \quad \sigma(u) = \|\Psi(0)u\|^2 = \langle (E_{\mathcal{U}}^* T_R^{-1} E_{\mathcal{U}})^{-1} u, u \rangle, \quad u \in \mathcal{U}.$$

Moreover, $\Psi(\lambda)^{-1}$ is also a function in $H^\infty(\mathcal{U}, \mathcal{U})$.

When T_R is strictly positive, then R also admits a factorization of the form:

$$R(e^{i\theta}) = \Psi(e^{i\theta})^* \Psi(e^{i\theta}) = \Psi_{\circ}(e^{i\theta})^* \Delta \Psi_{\circ}(e^{i\theta})$$

where Ψ_{\circ} is an outer function in $H^\infty(\mathcal{U}, \mathcal{U})$ satisfying $\Psi_{\circ}(0) = I$ and Δ is a strictly positive operator on \mathcal{U} . In fact, $\Delta = \Psi(0)^* \Psi(0)$ and $\Psi_{\circ}(\lambda) = \Psi(0)^{-1} \Psi(\lambda)$. The factorization $R(e^{i\theta}) = \Psi_{\circ}(e^{i\theta})^* \Delta \Psi_{\circ}(e^{i\theta})$ where Ψ_{\circ} is an outer function in $H^\infty(\mathcal{U}, \mathcal{U})$ satisfying $\Psi_{\circ}(0) = I$ is unique. Moreover, Δ is called the (*right*) *multiplicative diagonal* of R . In this setting, $\sigma(u) = \langle \Delta u, u \rangle$. Finally, it is noted that the multiplicative diagonal is usually mentioned in the framework of the Wiener algebra (see Remark A.17 below).

Now assume that F is a Schur function in $\mathcal{S}(\mathcal{U}, \mathcal{Y})$. Then $I - T_F^* T_F$ is a non-negative Toeplitz operator on $\ell_+^2(\mathcal{U})$. In this case, the optimization problem in (A.48) with $V = S_{\mathcal{U}}$ is equivalent to

$$(A.55) \quad \sigma(u) = \inf\{\langle (I - T_F^* T_F)(E_{\mathcal{U}}u - S_{\mathcal{U}}h), E_{\mathcal{U}}u - S_{\mathcal{U}}h \rangle : h \in \ell_+^2(\mathcal{U})\}$$

where u is a specified vector in \mathcal{U} . Assume that $I - F^*F$ admits an outer spectral factor, that is, $I - T_F^*T_F = T_\Psi^*T_\Psi$ for some outer function Ψ in $H^\infty(\mathcal{U}, \mathcal{E})$. Then the corresponding cost function $\sigma(u) = \|\Psi(0)u\|^2$.

If T_F is a strict contraction, or equivalently, $\|F\|_\infty < 1$, then $I - T_F^*T_F$ is a strictly positive operator on $\ell_+^2(\mathcal{U})$. Hence $I - T_F^*T_F$ admits an outer spectral Ψ factor in $H^\infty(\mathcal{U}, \mathcal{U})$ and $\Psi(\lambda)^{-1}$ is also in $H^\infty(\mathcal{U}, \mathcal{U})$. Choosing $R = I - F^*F$ in (A.54), yields

$$(A.56) \quad \sigma(u) = \|\Psi(0)u\|^2 = \langle (E_{\mathcal{U}}^*(I - T_F^*T_F)^{-1}E_{\mathcal{U}})^{-1}u, u \rangle.$$

Finally, if \mathcal{U} is finite dimensional, then

$$(A.57) \quad \det[\Psi(0)^*\Psi(0)] = \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \ln \det[I - F(e^{i\theta})^*F(e^{i\theta})]d\theta\right).$$

Remark A.17. Let \mathcal{H} be a Hilbert spaces, and $W_{\mathcal{H}}(\mathbb{T})$ we denote the operator Wiener algebra on the unite circle which consists of all $\mathcal{L}(\mathcal{H}, \mathcal{H})$ -valued functions on \mathbb{T} of the form

$$F(\lambda) = \sum_{j=-\infty}^{\infty} \lambda^j F_j, \quad \lambda \in \mathbb{T},$$

where $F_j \in \mathcal{L}(\mathcal{H}, \mathcal{H})$ for each j and $\sum_{j=-\infty}^{\infty} \|F_j\| < \infty$. By $W_{\mathcal{H},+}(\mathbb{T})$ we denote the subalgebra of $W_{\mathcal{H}}(\mathbb{T})$ consisting of all F in $W_{\mathcal{H}}(\mathbb{T})$ with $F_j = 0$ for each $j \leq -1$. Now assume that $F(\lambda)$ is strictly positive for each $\lambda \in \mathbb{D}$. Then there exists a unique function Ψ in $W_{\mathcal{H},+}(\mathbb{T})$ and a unique strictly positive operator $\Delta(F)$ on \mathcal{H} such that Ψ is invertible in $W_{\mathcal{H},+}(\mathbb{T})$, its index zero Fourier coefficient $\Psi_0 = I_{\mathcal{H}}$, and

$$F(\lambda) = \Psi(\lambda)^* \Delta(F) \Psi(\lambda), \quad \lambda \in \mathbb{T}.$$

The operator $\Delta(F)$ is called the (*right*) *multiplicative diagonal* of F . It is known that $\Delta(F)$ is also given by

$$\Delta(F) = (E_{\mathcal{H}}^* T_F^{-1} E_{\mathcal{H}})^{-1}.$$

See [29] where the notion of multiplicative diagonal is introduced in a \star -algebra setting, and Sections XXXIV.4 and XXXV.1 in [26] for further information.

Acknowledgement. We thank Joseph A. Ball for his valuable comments on an earlier version of the present paper. His observations varied from remarks on the used terminology to relating some of our results and proofs to those in multivariable interpolation theory, in particular, in his work with Vladimir Bolotnikov [8].

NRF statement disclaimer. The present work is based on the research supported in part by the National Research Foundation of South Africa. Any opinion, finding and conclusion or recommendation expressed in this material is that of the authors and the NRF does not accept any liability in this regard.

REFERENCES

- [1] V.M. Adamjan, D.Z. Arov, and M.G. Krein, Infinite Hankel block matrices and related problems of extension (Russian), *Izv. Akad. Nauk Armjan. SSR Ser. Mat.* **6** (1971), no. 2–3, 87–112.
- [2] J. Agler and J.E. McCarthy, Nevanlinna-Pick interpolation on the bidisk, *J. reine angew. Math.* **506** (1999), 191–204.
- [3] J. Agler and J.E. McCarthy, *Pick interpolation and Hilbert function spaces*, American Mathematical Society, 2002.

- [4] T. Ando, *De Branges spaces and analytic operator functions*, Sapporo, Japan, 1990.
- [5] D.Z. Arov and H. Dym, *J-contractive matrix-valued functions and related topics*, Encyclopedia of Mathematics and its Applications **116**, Cambridge University Press, 2008.
- [6] D.Z. Arov and L.Z. Grossman, Scattering matrices in the theory of unitary extensions of isometric operators, *Math. Nachr.* **157** (1992), 105123.
- [7] J.A. Ball, Linear systems, operator model theory and scattering: multivariable generalizations, in: *Operator theory and its applications (Winnipeg, MB, 1998)*, pp. 151–178, Fields Inst. Commun. **25**, Amer. Math. Soc., Providence, RI, 2000.
- [8] J.A. Ball and V. Bolotnikov, Interpolation problems for Schur multipliers on the Drury-Arveson space: from Nevanlinna-Pick to Abstract Interpolation Problem, *Integr. Equ. Oper. Theory* **62** (2008), 301–349.
- [9] J.A. Ball and V. Bolotnikov, Weighted Bergman Spaces: Shift-Invariant Subspaces and Input/State/Output Linear Systems, *Integr. Equ. Oper. Theory* **76** (2013), 301–356.
- [10] J.A. Ball, V. Bolotnikov, and Q. Fang, Multivariable backward-shift-invariant subspaces and observability operators, *Multidim Syst Sign Process* **18** (2007), 191–248
- [11] J.A. Ball and A. Kheifets, The inverse commutant lifting problem. I.: Coordinate-free formalism, *Integral Equ. Oper. Theory* **70** (2011), 17–62.
- [12] J.A. Ball and T.T. Trent, Unitary colligations, reproducing kernel Hilbert spaces, and Nevanlinna-Pick interpolation in several variables, *J. Funct. Anal.* **157** (1998), 1–61.
- [13] J.A. Ball, T.T. Trent and V. Vinnikov, Interpolation and commutant lifting for multipliers on reproducing kernel Hilbert spaces, *Oper. Theory Adv. Appl.* **122** (2002), 89–138.
- [14] L. de Branges and J. Rovnyak, *Square summable power series*, Holt, Rinehart and Winston, New York-Toronto, Ont.-London, 1966.
- [15] L. de Branges and J. Rovnyak, Canonical models in quantum scattering theory, in: *Perturbation Theory and its Applications in Quantum Mechanics (Proc. Adv. Sem. Math. Res. Center, U.S. Army, Theoret. Chem. Inst., Univ. of Wisconsin, Madison, Wis., 1965)*, pp. 295392, Wiley, New York, 1966.
- [16] M. Cotlar and C. Sadosky, On the Helson-Szegö theorem and a related class of modified Toeplitz kernels, *Proc. Symp. Pure Math. AMS*, **35**(1) (1979), 383–407.
- [17] V. Derkach, Abstract Interpolation Problem in Nevanlinna Classes, *Oper. Theory Adv. Appl.* **190** (2009), 283–298.
- [18] C. Foias and A. Frazho, *The commutant lifting approach to interpolation problems*, *Oper. Theory Adv. Appl.* **100**, Birkhäuser Verlag, Basel, 1990.
- [19] C. Foias, A. Frazho, I. Gohberg, and M.A. Kaashoek, *Metric constrained interpolation, commutant lifting and systems*, *Oper. Theory Adv. Appl.* **100**, Birkhäuser Verlag, Basel, 1998.
- [20] C. Foias, A.E. Frazho, and M.A. Kaashoek, Contractive liftings and the commutator, *C.R. Acad. Sci. Paris Ser. I* **335**, (2002), 1–6.
- [21] A. Frazho and W. Boshri, *An operator perspective on signals and systems*, *Oper. Theory Adv. Appl.* **204**, Birkhäuser Verlag, Basel, 2010.
- [22] A.E. Frazho, S. ter Horst, and M.A. Kaashoek, State space formulas for stable rational matrix solutions of a Leech problem, *Indagationes Math.* **25** (2014), 250–274.
- [23] A.E. Frazho, S. ter Horst, and M.A. Kaashoek, State space formulas for a suboptimal rational Leech problem I: Maximum entropy solution, *Integral Equ. Oper. Theory* **79** (2014), 533–553.
- [24] A.E. Frazho, S. ter Horst, and M.A. Kaashoek, State space formulas for a suboptimal rational Leech problem II: Parametrization of all solutions, *Oper. Theory Adv. Appl.* **244** (2015), 149–179.
- [25] I. Gohberg, S. Goldberg, and M.A. Kaashoek, *Classes of Linear Operators, Volume I*, *Oper. Theory Adv. Appl.* **49** Birkhäuser Verlag, Basel, 1990.
- [26] I. Gohberg, S. Goldberg, and M.A. Kaashoek, *Classes of Linear Operators, Volume II*, *Oper. Theory Adv. Appl.* **63** Birkhäuser Verlag, Basel, 1993.
- [27] I. Gohberg, M.A. Kaashoek, and H.J. Woerdeman, The band method for positive and contractive extension problems, *J. Operator Theory* **22** (1989), 109–155.
- [28] I. Gohberg, M.A. Kaashoek, and H.J. Woerdeman, The band method for positive and contractive extension problems: An alternative version and new applications, *Integr. Equ. Oper. Theory* **12** (1989), 343–382.
- [29] I. Gohberg, M.A. Kaashoek, and H.J. Woerdeman, A maximum entropy principle in the general frame work of the band method' *J. Funct. Anal.* **95** (1991), 231–254
- [30] H. Helson, *Lectures on Invariant Subspaces*, Academic Press, New York, 1964.

- [31] H. Helson and D. Lowdenslager, Prediction theory and Fourier series in several variables, *Acta Math.*, **99** (1958), 165–202.
- [32] H. Helson and D. Lowdenslager, Prediction theory and Fourier series in several variables II, *Acta Math.*, **106** (1961), 175–213.
- [33] M.A. Kaashoek, *The band method and tangential Nevanlinna-Pick interpolation for operator-valued functions at operator points*, unpublished manuscript.
- [34] M.A. Kaashoek and C.G. Zeinstra, The band method and generalized Carathéodory-Toeplitz interpolation at operator points, *Integr. Equ. Oper. Theory* **33** (1999), 175–210.
- [35] M.A. Kaashoek and J. Rovnyak, On the preceding paper by R. B. Leech, *Integr. Equ. Oper. Theory* **78** (2014), 75–77.
- [36] V.E. Katsnelson, A.Ya. Kheifets, P.M. Yuditskii, An abstract interpolation problem and the extension theory of isometric operators, *Oper. Theory Adv. Appl.* **95** (1994), 283–298.
- [37] A.Ya. Kheifets, The abstract interpolation problem and applications, in: *Holomorphic spaces (Berkeley, CA, 1995)*, pp. 351–379, *Math. Sci. Res. Inst. Publ.* **33**, Cambridge Univ. Press, Cambridge, 1998.
- [38] J. Kos, *Time-dependent problems in linear operator theory*, Ph. D. thesis, Department of Mathematics, Vrije Universiteit, Amsterdam, 1995.
- [39] A.E. Taylor and D.C. Lay, *Introduction to Functional Analysis*, 2nd edition, John Wiley and Sons, Inc, 1980.
- [40] R.B. Leech, Factorization of analytic functions and operator inequalities, *Integr. Equ. Oper. Theory* **78** (2014), 71–73.
- [41] R. Nevanlinna, Über beschränkte Funktionen, die in gegebene Punkten vorgeschriebene Werte annehmen, *Ann. Acad. Sci. Fenn. Ser. A* **13** (1919), no. 1.
- [42] G. Pick, Über die Beschränkungen analytischer Funktionen, welche durch vorgegebene Funktionswerte bewirkt werden, *Math. Ann.* **77** (1916), 7–23.
- [43] B. Sz.-Nagy and C. Foias, *Harmonic Analysis of Operators on Hilbert Space*, North Holland Publishing Co., Amsterdam-Budapest, 1970.
- [44] B. Sz.-Nagy, C. Foias, H. Bercovici, and L. Kérchy, *Harmonic Analysis of Operators on Hilbert Space*, Springer Verlag, New York, 2010.
- [45] B. Sz.-Nagy and A. Koranyi, Relations d’un problème de Nevanlinna et Pick avec la théorie des opérateurs de l’espace Hilbertien, *Acta Sci Math.* **7** (1956), 295–302.

DEPARTMENT OF AERONAUTICS AND ASTRONAUTICS, PURDUE UNIVERSITY, WEST LAFAYETTE, IN 47907, USA

E-mail address: frazho@ecn.purdue.edu

DEPARTMENT OF MATHEMATICS, UNIT FOR BMI, NORTH-WEST UNIVERSITY, PRIVATE BAG X6001-209, POTCHEFSTROOM 2520, SOUTH AFRICA

E-mail address: sanne.terhorst@nwu.ac.za

DEPARTMENT OF MATHEMATICS, VU UNIVERSITY AMSTERDAM, DE BOELELAAN 1081A, 1081 HV AMSTERDAM, THE NETHERLANDS

E-mail address: m.a.kaashoek@vu.nl