Orbital integrals on Lorentzian symmetric spaces

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Abstract

In this paper, we address the problem of determining a function in terms of its orbital integrals on Lorentzian symmetric spaces. It has been solved by S. Helgason [13] for even-dimensional isotropic Lorentzian symmetric spaces via a limit formula involving the Laplace-Beltrami operator. The result has been extended by J. Orloff [21] for rank-one semisimple pseudo-Riemannian symmetric spaces giving the keys to treat the odd-dimensional isotropic Lorentzian symmetric spaces. Indecomposable Lorentzian symmetric spaces are either isotropic or have solvable transvection group. We study orbital integrals including an inversion formula on the solvable ones which have been explicitly described by M. Cahen and N. Wallach [5].

Keywords : *symmetric spaces; Lorentzian metrics; orbital integrals; Riesz potentials.*
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Introduction

On a pseudo-Riemannian space $(M, \hat{g})$, one defines *pseudo-spherical integrals* which are parametrized by a pseudo-radial coordinate r and a sign. They are denoted $M^r_\pm$ and associate to any compactly supported continuous function f on M the functions $(M^r_\pm f)$ on M whose value at a point x is given by the integrals of f over pseudo-spheres centered at x, namely

$$\mathrm{Exp}_x(\Sigma_{\pm r^2}(x)), \quad \text{where } \Sigma_{\pm r^2}(x) := \{X \in T_xM \mid \hat{g}_x(X,X) = \pm r^2\}$$

and Exp_x is the exponential mapping at x associated to the Levi-Civita connection. If the metric $\hat{g}$ is Lorentzian, we integrate f over the connected components of the pseudo-spheres at x to define its pseudo-spherical integrals at x. When the space M is G-homogeneous for a Lie subgroup G of the isometry group, one also defines the *orbital integrals* of f at any point x as the integrals

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of f over orbits of the isotropy group K_x of x in G, with respect to an invariant measure, provided it exists. If the pseudo-spheres centered at any point x as well as the light cone $\{\mathrm{Exp}_x(X) \mid X \in T_xM,\ X \neq 0,\ \hat{g}_x(X,X) = 0\}$ are orbits of K_x, then M is said to be *isotropic* ; in that case, the pseudo-spherical integrals of f are equal to its orbital integrals.

If the metric $\hat{g}$ is Riemannian, only the integral operators M_+^r make sense and they are called *spherical integrals*. Assuming that the space M is two-point homogeneous, they appear when the totally geodesic Radon transforms are composed with dual transforms. An inversion formula for these Radon transforms is obtained by applying the Laplace-Beltrami operator to the result of this composition. The proof uses the following Darboux equation

$$L_x(M_+^r f)(x) = L_r(M_+^r f)(x)$$

where L_r is the radial part of L (see details in [13] for spaces of constant curvature and [17] for general two-point homogeneous spaces : the compact ones have been uniformly treated in [14] and the non-compact ones in [23]). Spherical integrals also allow to inverse the Fourier transform defined on Riemannian symmetric spaces of the non-compact type by S. Helgason in [15]. Concretely, they are used to reduce the problem to Harish-Chandra's inversion formula for the spherical Fourier transform on semisimple Lie groups (see details in [16], chapter III).

Orbital integrals were first defined on Lie groups : they are integrals over the conjugacy classes and they play a role in Harmonic Analysis. Indeed, in [11], Harish-Chandra gave the Plancherel formula for a Fourier transform he defined on any complex semisimple Lie group extending the result of Gelfand and Naimark [9] on $\mathrm{SL}(n, \mathbb{C})$. Later, Gelfand and Graev noticed [8] that the Plancherel formula for this Fourier transform on classical complex Lie groups is obtained from a limit formula which expresses the value of a function at the neutral element in terms of its orbital integrals. In [12], Harish-Chandra gave a limit formula for the orbital integrals on any real semisimple Lie group. Even in this case, it turns out to be useful to get a Plancherel formula. A. Bouaziz did the same for the orbital integrals on any real reductive Lie group G [2] and P. Harinck on the quotient $G_{\mathbb{C}}/G$ where $G_{\mathbb{C}}$ is the complexified group of G [10].

We are interested in the general problem of determining a function in terms of its orbital integrals on symmetric spaces. Like in the group case, we expect to solve it with a limit formula. On isotropic Riemannian symmetric spaces, the orbital integral problem is trivially solved since the orbits of the isotropy group of any point x are the spheres centered at x and they shrink to the point as their radius r goes to zero. In other words,

$$f(x) = \lim_{r \to 0^+} (M_+^r f)(x).$$

In contrast, on isotropic Lorentzian symmetric spaces, the "limit" of the pseudo-spheres as their pseudo-radius r goes to zero is the light cone. However, there still exists a solution to the orbital integral problem. Indeed, if the dimension of the space $n > 2$ is even, the expression of a function f in terms of its orbital integrals is given as the following limit formula due to S. Helgason [13]

$$f(x) = c \lim_{r \to 0^+} r^{n-2} P(L)(M_+^r f)(x),$$

where P is a polynomial, L is the Laplace-Beltrami operator associated to the metric and c is a real constant. J. Orloff generalized this limit formula to semisimple pseudo-Riemannian symmetric spaces of rank one [21]. In this paper, we address the orbital integral problem on all other indecomposable Lorentzian symmetric spaces using the known classification of them. More precisely, when the indecomposable Lorentzian symmetric space is not isotropic, its transvection group is solvable and the orbits of the isotropy group are parametrized by two variables. A limit formula is then obtained for the associated orbital integrals. In summary, we get the following result.

Theorem 1. *On any connected simply connected indecomposable Lorentzian symmetric space, any compactly supported smooth function is determined in terms of its orbital integrals via a limit formula involving invariant differential operators.*

Section 1 introduces the necessary background about symmetric spaces and gives the classification of indecomposable Lorentzian symmetric spaces due to M. Cahen and N. Wallach [5]. Orbital integrals are defined in section 2 for semisimple pseudo-Riemannian symmetric spaces. We lay out S. Helgason's determination of a function in terms of its orbital integrals via a limit formula for even-dimensional isotropic Lorentzian symmetric spaces and J. Orloff's generalization which gives the keys to treat the odd-dimensional isotropic Lorentzian symmetric spaces. The proof of these limit formulas uses some integral operators called Riesz potentials and the Laplace-Beltrami operator associated to the metric. In section 3, we give an explicit description of the model spaces for solvable Lorentzian symmetric spaces. We compute the exponential mapping at any point and determine the orbits of the isotropy group. Section 4 presents the steps and arguments leading to a limit formula to determine a function in terms of its orbital integrals on solvable Lorentzian symmetric spaces.

1 Framework of symmetric spaces

1.1 General features of symmetric spaces

We follow the description of symmetric spaces by O. Loos [20].

Definition 1. A *symmetric space* is a smooth manifold M endowed with a smooth map

$$s : M \times M \to M : (x, y) \mapsto s_x(y)$$

such that

1. $\forall x \in M$, s_x is an involutive diffeomorphism,
2. $\forall x \in M$, x is an isolated fixed point of s_x,
3. $\forall x, y$, $s_x \circ s_y \circ s_x = s_{s_x(y)}$.

The diffeomorphism s_x is called a *symmetry* at x.

For any connected symmetric space (M, s), we define the *transvection group* $G(M, s)$ as the group generated by the automorphisms $s_x \circ s_y$ for any $x, y \in M$. There exists a Lie group structure on $G(M, s)$ such that it acts transitively on M. Therefore, M is a homogeneous space.

Let us fix a base-point x_0 of M. We then consider the involutive automorphism of $G(M, s)$ defined as the conjugation by the symmetry at x_0. Then its differential at the neutral element of $G(M, s)$, denoted by σ is an involutive automorphism of the Lie algebra $\mathfrak{g}$ of $G(M, s)$. It induces a decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

into the eigenspaces $\mathfrak{k}$, $\mathfrak{p}$ of σ with respect to the eigenvalues 1, -1 respectively. In particular,

$$[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}, \quad [\mathfrak{p}, \mathfrak{k}] \subset \mathfrak{p}, \quad [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}.$$

Definition 2. A *symmetric Lie algebra* is a finite dimensional real Lie algebra $\mathfrak{g}$ endowed with an involutive automorphisme σ of $\mathfrak{g}$. This pair $(\mathfrak{g}, \sigma)$ is a *transvection* symmetric Lie algebra if, in addition, $\mathfrak{k} = [\mathfrak{p}, \mathfrak{p}]$ where $\mathfrak{k}$, $\mathfrak{p}$ are the eigenspaces for σ with respect to the eigenvalues 1, -1 respectively.

Definition 3. A transvection symmetric Lie algebra $(\mathfrak{g}, \sigma)$ is said to be *effective* if one of the following equivalent conditions is satisfied

- $\mathfrak{k}$ contains no nonzero ideal of $\mathfrak{g}$,
- $\mathfrak{k} \cap \mathfrak{z}(\mathfrak{g}) = \{0\}$,
- the map $\mathrm{ad}_{\mathfrak{g}}(\cdot)|_{\mathfrak{p}} : \mathrm{k} \to \mathfrak{gl}(\mathfrak{p})$ is injective.

The transvection symmetric Lie algebra $(\mathfrak{g}, \sigma)$ defined above for a connected symmetric space (M, s), where $\mathfrak{g}$ is the Lie algebra of the transvection group $G(M)$ and σ the differential of the map $(g \mapsto s_{x_0} \circ g \circ s_{x_0})$, is effective. Conversely, for any symmetric Lie algebra $(\mathfrak{g}, \sigma)$, there exists a connected, simply connected symmetric space (M, s) which is G-homogeneous for a Lie subgroup G of the automorphisms of (M, s) whose Lie algebra is $\mathfrak{g}$ and such that the differential of the induced involution is σ.

On the other hand, there exists a unique affine connection on M such that every symmetry is an affine transformation. This connection is complete, that is every geodesic is defined on $\mathbb{R}$. Therefore, at any $x \in M$, the exponential mapping, denoted by Exp_x, is defined on the whole T_xM. The well-known property about the exponential mapping of a symmetric space (M, s) is

$$\mathrm{Exp}_{x_0}(X) = \exp(\bar{X}).x_0, \quad \text{for } X \in T_{x_0}M$$

where $\exp : \mathfrak{g} \to G$ is the exponential mapping of the Lie group G and $\bar{X}$ is the unique element in $\mathfrak{p}$ such that $\phi(\bar{X}) = X$ using the isomorphism

$$\phi : \begin{cases} \mathfrak{p} & \to T_{x_0}M \\ \bar{X} & \mapsto \frac{d}{dt}\big|_0 \exp(t\bar{X}).x_0 \end{cases}.$$

In 1959 [13], Sigurdur Helgason gave an explicit expression of the differential of the exponential mapping which is very useful for the computations.

Lemma 2. *Let (M, s) be a symmetric space and x_0 be a base-point of M. If $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the decomposition of the transvection Lie algebra with respect to the natural involution, then the exponential mapping $\mathrm{Exp}_{x_0} : T_{x_0}M \to M$ associated to the canonical affine connection has differential*

$$\left(\mathrm{Exp}_{x_0}\right)_{*X} = \tau(\exp(X))_{*x_0} \circ \sum_{k=0}^{\infty} \frac{\mathrm{ad}(X)\big|_{\mathfrak{p}}^{2k}}{(2k+1)!}, \quad \textit{for } X \in T_{x_0}M \simeq \mathfrak{p},$$

where $\tau(\exp(X))$ is the diffeomorphism defined by the action of the group element $\exp(X)$ on M and ad the adjoint action on the Lie algebra $\mathfrak{g}$.

1.2 Rank of symmetric Lie algebras

In [19], Lepowsky and McCollum defined the rank of any symmetric Lie algebra even when it is not necessarily semisimple. This requires introducing Cartan subspaces in a very general way.

Definition 4. Let $(\mathfrak{g}, \sigma)$ be a symmetric Lie algebra and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the usual decomposition of $\mathfrak{g}$ into the (± 1)-eigenspaces of σ. A *Cartan subspace* of the symmetric Lie algebra is a subspace $\mathfrak{a}$ of $\mathfrak{p}$ such that

$$\mathfrak{a} = \mathfrak{p}_{\mathfrak{a}}^0 := \{X \in \mathfrak{p} \mid \forall Y \in \mathfrak{a}, \exists n \geq 0, \mathrm{ad}(Y)^n(X) = 0\}.$$

Definition 5. Let $(\mathfrak{g}, \sigma)$ be a symmetric Lie algebra and $\mathfrak{a}$ a subspace of $\mathfrak{p}$. Then $\mathfrak{a}$ is said to be a $\mathfrak{p}$-*subalgebra* if

$$\forall X \in \mathfrak{a}, \quad \mathrm{ad}^2(X)\mathfrak{a} \subset \mathfrak{a}.$$

Furthermore, it is said to be *natural* if there exists $X_0 \in \mathfrak{a}$ such that $\mathrm{ad}^2(X_0)$ induces a nonsingular endomorphism of $\mathfrak{p}/\mathfrak{a}$.

Proposition 3 (Lepowsky - McCollum, 1976 [19]). *Let $(\mathfrak{g}, \sigma)$ be a symmetric Lie algebra and $\mathfrak{a}$ be a subspace of $\mathfrak{p}$. Then $\mathfrak{a}$ is a Cartan subspace if and only if $\mathfrak{a}$ is a minimal natural $\mathfrak{p}$-subalgebra of $\mathfrak{p}$.*

Theorem 4 (Lepowsky - McCollum, 1976 [19]). *In any symmetric Lie algebra, there exist Cartan subspaces and they all have the same dimension. We call* rank *of the symmetric Lie algebra this common dimension.*

1.3 Pseudo-Riemannian symmetric spaces

We need to add a compatible metric on the symmetric spaces in order to define generalized spheres on them.

Definition 6. A *pseudo-Riemannian symmetric space* is a symmetric space (M, s) endowed with a pseudo-Riemannian metric $\hat{g}$ such that, for all $x \in M$, s_x is an isometry of $(M, \hat{g})$.

On a pseudo-Riemannian symmetric space, the unique affine connection such that every symmetry is an affine transformation is the Levi-Civita connection associated to the metric.

Definition 7. A *pseudo-Riemannian symmetric Lie algebra* is a symmetric Lie algebra $(\mathfrak{g}, \sigma)$ endowed with a non-degenerate symmetric bilinear form B on $\mathfrak{g}$ such that

1. $\forall X, Y \in \mathfrak{g}, \quad B(\sigma X, \sigma Y) = B(X, Y)$,
2. $\forall X, Y, Z \in \mathfrak{g}, \quad B(\mathrm{ad}(Z)X, Y) + B(X, \mathrm{ad}(Z)Y) = 0$.

The *signature* of the triple $(\mathfrak{g}, \sigma, B)$ is the signature of $B|_{\mathfrak{p}\times\mathfrak{p}}$ and its *dimension* is the dimension of $\mathfrak{p}$.

Note that the transvection pseudo-Riemannian symmetric Lie algebras are automatically effective.

Definition 8. An isomorphism of pseudo-Riemannian symmetric Lie algebras $(\mathfrak{g}_1, \sigma_1, B_1)$ and $(\mathfrak{g}_2, \sigma_2, B_2)$ is a Lie algebra isomorphism $\alpha : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that $\alpha \circ \sigma_1 = \sigma_2 \circ \alpha$ and $\alpha^* B_2 = B_1$.

Theorem 5. *There is one-to-one correspondence between isometry classes of connected, simply connected pseudo-Riemannian symmetric spaces and isomorphism classes of transvection pseudo-Riemannian symmetric Lie algebras.*

Definition 9. A connected pseudo-Riemannian manifold $(M, \hat{g})$ is said to be *decomposable* if there exists a proper subspace V of $T_{x_0}M$ which is invariant under the holonomy group $\mathrm{Hol}(M, x_0)$ at x_0 and such that $\hat{g}_{x_0}|_{V\times V}$ is non-degenerate. Otherwise $(M, \hat{g})$ is said to be *indecomposable*.

Definition 10. A pseudo-Riemannian symmetric Lie algebra $(\mathfrak{g}, \sigma, B)$ is said to be *decomposable* if, when denoting by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the usual decomposition of $\mathfrak{g}$ with respect to σ, there exists a proper subspace $\mathfrak{q}$ of $\mathfrak{p}$ which is invariant under $\mathrm{ad}(\mathfrak{k})$ and such that $B|_{\mathfrak{q}\times\mathfrak{q}}$ is non-degenerate. Otherwise $(\mathfrak{g}, \sigma, B)$ is said to be *indecomposable*.

The de Rham-Wu theorem [26] asserts that any decomposable connected pseudo-Riemannian manifold which is simply connected and complete is isometric to a product of indecomposable ones. In addition, a pseudo-Riemannian product manifold is a symmetric space if and only if every factor of the product is a symmetric space. Finally, a connected, simply connected pseudo-Riemnnian symmetric space $(M, s, \hat{g})$ is indecomposable if and only if its associated transvection pseudo-Riemannian symmetric Lie algebra $(\mathfrak{g}, \sigma, B)$ is indecomposable.

1.4 Classification of indecomposable Lorentzian symmetric spaces

Indecomposable Lorentzian symmetric spaces fall into two categories : those whose transvection Lie algebra is semisimple and those whose transvection Lie algebra is solvable.

Proposition 6 (M. Cahen and N. Wallach, 1970 [5])**.** *Let $(\mathfrak{g}, \sigma, B)$ be an indecomposable transvection Lorentzian symmetric Lie algebra. Then $\mathfrak{g}$ is either semisimple or solvable.*

In the semisimple case, the associated Lorentzian symmetric spaces are automatically of constant sectional curvature as ensures the following theorem whose proof is based on Berger's list [1].

Theorem 7 (M. Cahen, J. Leroy, M. Parker, F. Tricerri and L. Vanhecke, 1990 [4])**.** *Let $(\mathfrak{g}, \sigma, B)$ be a semisimple indecomposable Lorentzian symmetric Lie algebra of dimension ≥ 3. Then the sectional curvature of the associated Lorentzian symmetric space is constant and non-zero.*

It is possible to prove this result without using Berger's list. It involves a result of A. J. Di Scala and C. Olmos [6] about connected Lie subgroups of the Lorentzian group acting irreducibly on the flat Lorentzian vector space.

Proof. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the usual decomposition of $\mathfrak{g}$ with respect to σ. Then the representation

$$\mathrm{ad}_{\mathfrak{g}}(\cdot)|_{\mathfrak{p}} : \mathfrak{k} \to \mathfrak{gl}(\mathfrak{p})$$

is faithful and irreducible. If $\dim(\mathfrak{p}) = n$, the image of $\mathfrak{k}$ by ad is contained in $\mathfrak{o}(\mathfrak{p}, B|_{\mathfrak{p}}) \simeq \mathfrak{o}(1, n-1)$. By integrating, we get a connected Lie subgroup K of $SO(1, n-1)$ which acts irreducibly on $(\mathbb{R}^{1,n-1}, \mathbb{I}_{1,n-1})$. By theorem 1.1 in [6], $K = SO_0(1, n-1)$. Therefore, the sectional curvature of the associated Lorentzian symmetric space is constant and non-zero. □

In dimension 2, the Lorentzian symmetric spaces are automatically of constant sectional curvature. The classification in the semisimple case is given by the following theorem.

Theorem 8 (S. Helgason, 1959 [13]). *Any Lorentzian symmetric space with non-zero constant sectional curvature is locally isometric to one of the two following model spaces, up to a positive constant factor on the metric,*

1. $Q_{(+1)} := \{(x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 - x_2^2 - ... - x_n^2 + x_{n+1}^2 = 1\}$,
2. $Q_{(-1)} := \{(x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 - x_2^2 - ... - x_n^2 - x_{n+1}^2 = -1\}$,

endowed with the Lorentzian metric induced by the flat metric on $\mathbb{R}^{1,n}$.

The proof of this theorem, found in [13], relies on the fact that a pseudo-Riemannian symmetric space is determined on a neighborhood of any point by its metric and curvature tensor at this point. Finally, it is clear that any Lorentzian symmetric space whose sectional curvature is constant and non-zero is of rank one.

In the solvable case, the classification is given at the algebraic level. Any solvable indecomposable Lorentzian symmetric Lie algebra is completely described in a particular basis by $n-2$ nonzero real numbers.

Theorem 9 (M. Cahen and N. Wallach, 1970 [5]). *Let* $(\mathfrak{g}, \sigma, B)$ *be a solvable indecomposable Lorentzian symmetric Lie algebra and* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ *be the decomposition associated to the involution* σ. *There exist* $\lambda_1, ..., \lambda_{n-2} \in \mathbb{R}_0$ *and a basis*

$$\{Z, U, W_1, ..., W_{n-2}, K_1, ..., K_{n-2}\}$$

of $\mathfrak{g}$ *such that*

- $\mathfrak{k} = \mathbb{R}K_1 \oplus ... \oplus \mathbb{R}K_{n-2}$, $\mathfrak{p} = \mathbb{R}Z \oplus \mathbb{R}U \oplus \mathbb{R}W_1 \oplus ... \oplus \mathbb{R}W_{n-2}$,
- $\mathfrak{z}(\mathfrak{g}) = \mathbb{R}Z$, $[U, K_i] = \lambda_i W_i$, $[U, W_i] = -K_i$, $[W_j, K_i] = \lambda_i \delta_{ij} Z$, $[W_i, W_j] = 0 = [K_i, K_j]$,
- $B(Z, Z) = 0 = B(U, U)$, $B(Z, U) = 1$, $B(W_i, W_j) = -\delta_{ij}$ *and* $B(K_i, K_j) = -\lambda_i \delta_{ij}$.

Proposition 10. *Let $(\mathfrak{g}, \sigma, B)$ be a solvable indecomposable Lorentzian symmetric Lie algebra, explicitly described in theorem 9 for a certain choice of parameters $\lambda_1, ..., \lambda_{n-2} \in \mathbb{R}_0$. The rank of this symmetric Lie algebra is equal to 2 and any Cartan subspace is of the form*

$$\mathbb{R}Z \oplus \mathbb{R}\Big(U - \sum_{i=1}^{n-2} \lambda_i y_i W_i\Big)$$

for some fixed $y_1, ..., y_{n-2} \in \mathbb{R}$.

Proof. If $\mathfrak{a}$ is a Cartan subspace of $\mathfrak{p}$, then $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{p} \subset \mathfrak{a}$. Otherwise $\mathfrak{a}$ wouldn't be natural. Therefore, $Z \in \mathfrak{a}$. Nonetheless, $\mathbb{R}Z \neq \mathfrak{a}$ because $\mathrm{ad}^2(Z)$ doesn't induce a nonsingular endomorphism of $\mathfrak{p}/\mathfrak{a}$. Thus M is not of rank one.

Let us consider $\mathfrak{a}_0 = \mathbb{R}Z \oplus \mathbb{R}U$. It is clearly a $\mathfrak{p}$-subalgebra. Moreover, it is natural since $\mathrm{ad}^2(U)$ induces a nonsingular endomorphism of $\mathfrak{p}/\mathfrak{a}$. At last, the minimality is obvious. As a conclusion, $\mathfrak{a}_0$ is a Cartan subspace of $\mathfrak{p}$ and $(\mathfrak{g}, \sigma, B)$ is of rank 2.

Back to the beginning of the proof, if $\mathfrak{a}$ is any Cartan subspace of $\mathfrak{p}$, we know that it is of dimension 2 and it contains Z. Then it is of the form

$$\mathfrak{a} = \mathbb{R}Z \oplus \mathbb{R}\Big(\nu U + \sum_{i=1}^{n-2} \mu_i W_i\Big)$$

for some $\nu, \mu_1, ..., \mu_{n-2} \in \mathbb{R}$ not all zero. Since $\mathrm{ad}(\sum_{i=1}^{n-2} \mu_i W_i)^2$ doesn't induce a nonsingular endomorphism of $\mathfrak{p}/\mathfrak{a}$, then $\nu \neq 0$ hence the result. □

1.5 Isotropic pseudo-Riemannian symmetric spaces

On any connected pseudo-Riemannian symmetric space $(M, s, \hat{g})$, the group of isometries, denoted by $I(M, \hat{g})$, acts transitively. Then the isotropy group K_x of any point $x \in M$ acts on the tangent space T_xM in the following way

$$K_x \times T_xM \to T_xM : (k, X) \mapsto \tau(k)_{*x}(X)$$

where $\tau(k)$ is the action of k on M. The second definition below comes from A. J. Wolf's paper [25] and the first one is a weaker version of it.

Definition 11. Let $(M, s, \hat{g})$ be a connected pseudo-Riemannian symmetric space. Then M is said to be *quasi-isotropic* if, for any $x \in M$, the pseudo-spheres in T_xM, namely

$$\Sigma_\alpha(x) := \{X \in T_xM \mid \hat{g}_x(X, X) = \alpha\}$$

for any $\alpha \in \mathbb{R}_0$, are orbits under the action of the isotropy group K_x of x in $I(M, \hat{g})$, the group of isometries.

Furthermore, M is said to be *isotropic* if it is quasi-isotropic and, for any $x \in M$, the light cone in T_xM, namely

$$\Sigma_0(x) := \{X \in T_xM \mid \hat{g}_x(X, X) = 0, \quad X \neq 0\},$$

is also an orbit of the isotropy group K_x of x in $I(M, \hat{g})$.

Proposition 11 (J. Orloff, 1987 [21])**.** *Let $(M, s, \hat{g})$ be a connected pseudo-Riemannian symmetric space whose group of isometries is semisimple. Then M is quasi-isotropic if and only if it is of rank one.*

Note that there exist rank-one pseudo-Riemannian symmetric spaces whose group of isometries is semisimple which are not isotropic like the space $X = \mathrm{SL}(n, \mathbb{R})/\mathrm{GL}(n-1, \mathbb{R})$ studied by M. T. Kosters and G. Van Dijk in [18].

2 Limit formulas on Lorentzian symmetric spaces of constant sectional curvature

We address the problem of determining a function in terms of its orbital integrals on Lorentzian symmetric spaces of constant sectional curvature. These are either the flat Lorentzian vector space or, up to a positive constant factor on the metric, one of the two model spaces in theorem 8. S. Helgason solved the problem when their dimension is even via a limit formula [13]. J. Orloff extended it to rank-one semisimple pseudo-Riemannian symmetric spaces [21] and gave the keys to treat the odd-dimensional Lorentzian symmetric spaces of constant sectional curvature.

Definition 12. Let $(M, s, \hat{g})$ be a pseudo-Riemannian symmetric space. Then M is said to be *semisimple* if there exists a Lie subgroup G of $I(M, \hat{g})$ which is semisimple, acts transitively on M and is invariant under the conjugation by s_{x_0} where x_0 is a base-point of M.

2.1 Definition of orbital integrals

On the pseudo-Euclidean vector space, that is the space $\mathbb{R}^n$ endowed with the flat metric of signature (p, q), the orbital integrals are defined as pseudo-spherical integrals.

Definition 13. Let $\langle ., . \rangle$ be the standard inner product of signature (p, q) on $\mathbb{R}^n$. For any function $f \in \mathcal{C}_c(\mathbb{R}^n)$, the *orbital integrals* of f are the pseudo-radial functions denoted by $(M_+ f)$ and $(M_- f)$ and defined by

$$(M_\pm f)(r) := \frac{1}{r^{n-1}} \int_{\Sigma_{\pm r^2}} f(x)\, d\eta(x), \quad \text{for } r > 0,$$

where $d\eta$ is the measure induced by the metric on the pseudo-spheres centered at 0 in $\mathbb{R}^n$, namely

$$\Sigma_{r^2} := \{x \in \mathbb{R}^n \mid \langle x, x\rangle = r^2\}, \quad \Sigma_{-r^2} := \{x \in \mathbb{R}^n \mid \langle x, x\rangle = -r^2\}.$$

Let $(M, s, \hat{g})$ be a semisimple pseudo-Riemannian symmetric space whose metric is of signature (p, q) and x_0 a base-point of M. Let G be a semisimple Lie subgroup of $I(M, \hat{g})$ which acts transitively on M and is invariant under the conjugation by s_{x_0}. Then the Lie algebra $\mathfrak{g}$ of G decomposes into the (± 1)-eigenspaces of the involution given by the conjugation by s_{x_0}

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

and we get the usual isomorphism $\phi : \mathfrak{p} \to T_{x_0}M$ defined in section 1. Following J. Orloff's paper [21], we assume that the isotropy group K of x_0 is connected and we define the orbital integrals as follows.

Definition 14. Let $\mathfrak{a}$ be a Cartan subspace of $\mathfrak{g}$. For any function $f \in \mathcal{C}_c(M)$ and any point $x \in M$, the *orbital integrals* of f at x are given by

$$(M^X f)(x) := \int_{K/H_X} f(gk.\mathrm{Exp}_{x_0}(X))\, d\mu(kH_X),$$

where $g \in G$ such that $x = g.x_0$, H_X is the stabilizer of X in K and $d\mu$ is the K-invariant measure on K/H_X induced by the metric on the K-orbit of $X \in T_{x_0}M$ providing that

$$[\bar{X}, \mathfrak{k}] = \mathfrak{a}^{\perp}$$

where $\bar{X} \in \mathfrak{p}$ such that $\phi(\bar{X}) = X$ and $\mathfrak{a}^{\perp}$ is the orthogonal complement of $\mathfrak{a}$ in $\mathfrak{p}$ with respect to the metric $\phi^* \hat{g}_{x_0}$.

2.2 Limit formulas in the even-dimensional Lorentzian case

Let $(M, s, \hat{g})$ be a Lorentzian symmetric space of non-zero constant sectional curvature. On the model spaces introduced in theorem 8, the group of isometries is respectively $O(2, n-1)$ if $M = Q_{(+1)}$ and $O(1, n)$ if $M = Q_{(-1)}$. Therefore,

$$Q_{(+1)} \simeq SO_0(2, n-1)/SO_0(1, n-1), \quad Q_{(-1)} \simeq SO_0(1, n)/SO_0(1, n-1),$$

with the base-point $x_0 = (0, ..., 0, 1)$, and the orbit of any vector $X \in T_{x_0}M$ such that $\hat{g}_{x_0}(X, X) \neq 0$ under the action of the isotropy group $K = SO_0(1, n-1)$ is the connected component of a pseudo-sphere. We then use the following notation for the orbital integrals

$$(M^r_+ f)(x) := (M^X f)(x), \quad \text{where } X = (r, 0, ..., 0) \in \mathbb{R}^{1,n-1} \simeq T_{x_0}M,$$
$$(\tilde{M}^r_+ f)(x) := (M^X f)(x), \quad \text{where } X = (-r, 0, ..., 0) \in \mathbb{R}^{1,n-1} \simeq T_{x_0}M,$$
$$(M^r_- f)(x) := (M^X f)(x), \quad \text{where } X = (0, ..., 0, r) \in \mathbb{R}^{1,n-1} \simeq T_{x_0}M,$$
$$(\tilde{M}^r_- f)(x) := (M^X f)(x), \quad \text{where } X = (0, ..., 0, -r) \in \mathbb{R}^{1,n-1} \simeq T_{x_0}M,$$

for any $r > 0$. Note that $M^r_- f = \tilde{M}^r_- f$ except when $n = \dim(M) = 2$.

As in S. Helgason's book [17], we focus on the model space $M = Q_{(-1)}$ and the orbital integrals $M^r_+ f$ to exhibit results and arguments leading to the limit formula. Everything works in the same manner with the other model space and the other series of orbital integrals.

Proposition 12 (S. Helgason, 1959 [13])**.** *Let $\Box$ be the Laplace-Beltrami operator associated to the Lorentzian metric $\hat{g}$ on $Q_{(-1)}$. For any function $f \in \mathcal{C}^\infty_c(Q_{(-1)})$, any point $x \in M$ and any pseudo-radius $0 < r < r_0$,*

$$\begin{aligned}\Box(M^r_+ f)(x) &= M^r_+(\Box f)(x)\\ &= \frac{1}{A(r)}\frac{\partial}{\partial r}\Big(A(r)\frac{\partial}{\partial r}(M^r_+ f)(x)\Big).\end{aligned}$$

where $A(r) := \sinh^{n-1}(r)$.

Definition 15. For any function $f \in \mathcal{C}_c(Q_{(-1)})$, its *Riesz potentials* at any point $x \in Q_{(-1)}$ and for any parameter $\lambda \in \mathbb{C}$ such that $\mathrm{Re}(\lambda) > n$, are given by

$$(I^\lambda_+ f)(x) := \frac{1}{H_n(\lambda)}\int_{D^+_x} f(y)\sinh^{\lambda-n}\left(\sqrt{\hat{g}_x(Y,Y)}\right)dm(y), \quad y = \mathrm{Exp}_x(Y),$$

where dm is the measure induced by the metric $\hat{g}$ on $Q_{(-1)}$, D^+_x is the connected component of $\{\mathrm{Exp}_x(Y) \mid Y \in T_x Q_{(-1)},\ \hat{g}_x(Y,Y) > 0\}$ containing the vector $g.\mathrm{Exp}_{x_0}(1,0,...,0)$ if $x = g.x_0$ and

$$H_n(\lambda) := 2^{\lambda-1}\pi^{\frac{n-2}{2}}\Gamma\Big(\frac{\lambda}{2}\Big)\Gamma\Big(\frac{\lambda+2-n}{2}\Big).$$

The Riesz potentials associate to a function f a one-parameter family of integrals $(I^\lambda f)$ defined for $\lambda \in \mathbb{C}$ such that $\mathrm{Re}(\lambda) > n$.

Proposition 13 (S. Helgason, 1959 [13])**.** *For any function $f \in \mathcal{C}^\infty_c(Q_{(-1)})$, any point $x \in Q_{(-1)}$ and any parameter $\lambda \in \mathbb{C}$ such that $Re(\lambda) > n$,*

(i) $\Box(I^\lambda_+ f)(x) = I^\lambda_+(\Box f)(x)$,

(ii) *if $Re(\lambda) > n+2$,* $\Box(I^\lambda_+ f)(x) = (\lambda - n)(\lambda - 1)(I^\lambda_+ f)(x) + (I^{\lambda-2}_+ f)(x)$,

(iii) $(I^\lambda_+ f)(x)$ *extends holomorphically to $\mathbb{C}$ in the λ-variable so that the value at $\lambda = 0$ is*

$$(I^0_+ f)(x) = f(x).$$

Finally, the Riesz potentials express in terms of the orbital integrals in the following way

$$(I^\lambda_+ f)(x) = \frac{1}{H_n(\lambda)}\int_0^\infty (M^r_+ f)(x)\sinh^{\lambda-1}(r)dr. \tag{1}$$

This yields the desired limit formula whose proof is sketched below.

Theorem 14 (S. Helgason, 1959 [13]). *We assume $n := dim(Q_{(-1)}) > 2$ to be even. Then there exists a polynomial P and a real number c such that, for any $f \in \mathcal{C}_c^\infty(Q_{(-1)})$,*

$$f(x) = c. \lim_{r \to 0^+} r^{n-2} P(\Box)(M_+^r f)(x),$$

where $\Box$ is the associated Laplace-Beltrami operator on $Q_{(-1)}$.

Proof. First, when the dimension n is strictly greater than 2, the limit

$$\lim_{r \to 0^+} r^{n-2}(M_+^r f)(x) = \lim_{r \to 0^+} \sinh^{n-2}(r)(M_+^r f)(x)$$

exists. Moreover, thanks to formula 1, the Riesz potential $(I_+^\lambda f)(x)$ is equal to the following Riemann-Liouville integral

$$\frac{1}{\Gamma(\mu)} \int_0^\infty F(r) \sinh^{\mu-1}(r) dr,$$

where $F(r) := \sinh^{n-2}(r)(M_+^r f)(x)$ and $\mu := \lambda - n + 2$. We thus get

$$(I_+^{n-2} f)(x) = \frac{(4\pi)^{(2-n)/2}}{\Gamma((n-2)/2)} \lim_{r \to 0^+} F(r) = \frac{(4\pi)^{(2-n)/2}}{\Gamma((n-2)/2)} \lim_{r \to 0^+} r^{n-2}(M_+^r f)(x).$$

Since n is also even, we deduce from proposition 13 the existence of a polynomial P such that

$$P(\Box)(I_+^{n-2} f)(x) = I_+^{n-2}(P(\Box) f)(x) = f(x).$$

Therefore,

$$f(x) = \frac{(4\pi)^{(n-2)/2}}{\Gamma((n-2)/2)} \lim_{r \to 0^+} r^{n-2} M_+^r(P(\Box) f))(x)$$

and proposition 12 leads to the limit formula. $\Box$

2.3 Generalization to rank-one semisimple symmetric spaces

Let $(M, s, \hat{g})$ be a semisimple pseudo-Riemannian symmetric space whose metric is of signature (p, q) and x_0 a base-point of M. We assume that M is of rank one. Then by proposition 11, M is quasi-isotropic. Like in the Lorentzian case in subsection 2.2, the orbital integrals of a function $f \in \mathcal{C}_c(M)$, namely $M^X f$, for vectors $X \in T_{x_0}M$ such that $\hat{g}_{x_0}(X, X)$ are then integrals over connected components of the pseudo-spheres in M. We also adopt the following alternative notation

$$(M_+^r f)(x) := (M^X f)(x), \quad \text{where } X = (r, 0, ..., 0) \in \mathbb{R}^{p,q} \simeq T_{x_0}M,$$
$$(M_-^r f)(x) := (M^X f)(x), \quad \text{where } X = (0, ..., 0, r) \in \mathbb{R}^{p,q} \simeq T_{x_0}M,$$

for any $r > 0$.

Remark 1. There exists $r_0 > 0$ such that the exponential mapping of M at x_0 is a diffeomorphism from the open subset

$$\{X \in T_{x_0}M \mid -r_0^2 < \hat{g}_{x_0}(X, X) < r_0^2\}$$

of $T_{x_0}M$ to its image. Therefore, the orbital integrals on M at the base-point x_0 are the pseudo-spherical integrals on the pseudo-Euclidean vector space through the exponential mapping except in the Lorentzian case where Σ_{r^2} is not connected. More precisely, whenever $p > 1$ and $q > 1$, for any function $f \in \mathcal{C}_c(M)$ and any $0 < r < r_0$,

$$(M_{\pm}^r f)(x_0) = M_{\pm}(f \circ \mathrm{Exp}_{x_0})(r),$$

where we identify $(T_{x_0}M, \hat{g}_{x_0})$ with $(\mathbb{R}^n, \langle ., . \rangle_{p,q})$.

In order to get a limit formula on M, J. Orloff solved the problem of determining a function in terms of its orbital integrals on the flat space $\mathbb{R}^n$ endowed with the standard inner product of signature (p, q). He studied the generalized Riesz potentials defined below.

Theorem 15 (J. Orloff, 1987 [21])**.** *Let $D_+ := \{x \in \mathbb{R}^n \mid \langle x, x \rangle > 0\}$ and $D_- := \{x \in \mathbb{R}^n \mid \langle x, x \rangle < 0\}$. For any function $f \in \mathcal{S}(\mathbb{R}^n)$, we define its* Riesz potentials *for any parameter $\lambda \in \mathbb{C}$ such that* $\mathrm{Re}(\lambda) > n$

$$I_+^\lambda f := \frac{1}{H_n(\lambda)} \int_{D_+} f(x) |\langle x, x \rangle|^{\frac{\lambda - n}{2}} dx,$$
$$I_-^\lambda f := \frac{1}{H_n(\lambda)} \int_{D_-} f(x) |\langle x, x \rangle|^{\frac{\lambda - n}{2}} dx,$$
$$I_0^\lambda f := \Gamma\Big(\frac{\lambda - n + 2}{2}\Big)\Big(I_+^\lambda f - \cos\Big(\frac{\lambda - n + 2}{2}\pi\Big) I_-^\lambda f\Big),$$

where $H_n(\lambda) := 2^{\lambda - 1} \pi^{\frac{n-2}{2}} \Gamma\big(\frac{\lambda}{2}\big)\Gamma\big(\frac{\lambda - n + 2}{2}\big)$. Then

1. *$I_+^\lambda f$, $I_-^\lambda f$ and $I_0^\lambda f$ extend to entire functions in λ,*
2. *$I_+^{\lambda+2}(Lf) = I_+^\lambda f$ and $I_-^{\lambda+2}(Lf) = -I_-^\lambda f$ where L is the Laplace-Beltrami operator associated to $\langle ., . \rangle$,*
3. *$I_+^0 f = 2\sin(p\frac{\pi}{2}) f(0)$ and $I_-^0 f = 2\sin(q\frac{\pi}{2}) f(0)$,*
4. *The map $(f \mapsto I_\pm^\lambda f)$ is a tempered distribution for all λ,*
5. *For p and q both even, $I_0^0 f = \frac{(-1)^{q/2} 2\pi}{(\frac{n-2}{2})!} f(0)$.*

Theorem 16 (J. Orloff, 1987 [21])**.** *Whenever $p > 1$ and $q > 1$, for any function $f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, we have the following limit formulas.*

1. If p and q are both odd then

$$\lim_{r\to 0^+} r^{n-2}\Big(\frac{d^2}{dr^2}+\frac{n-1}{r}\frac{d}{dr}\Big)^{\frac{n-2}{2}}(M_+f)(r)=cf(0)$$

and

$$\lim_{r\to 0^+} r^{n-2}\Big(-\frac{d^2}{dr^2}-\frac{n-1}{r}\frac{d}{dr}\Big)^{\frac{n-2}{2}}(M_-f)(r)=cf(0),$$

2. If p is odd and q is even then

$$\lim_{r\to 0^+}\Big(\frac{d}{dr}\Big)^{n-2} r^{n-2}(M_+f)(r)=cf(0),$$

3. If p is even and q is odd then

$$\lim_{r\to 0^+}\Big(\frac{d}{dr}\Big)^{n-2} r^{n-2}(M_-f)(r)=cf(0),$$

4. If p and q are both even then

$$\lim_{r\to 0^+}\Big(\frac{d}{dr}\Big)^{n-2} r^{n-2}(M_+f)(r)+(-1)^{\frac{n}{2}}\lim_{r\to 0^+}\Big(\frac{d}{dr}\Big)^{n-2} r^{n-2}(M_-f)(r)=cf(0),$$

where, in each equation, c is a nonzero constant independent of f.

Then the Riesz potentials and the limit formulas are lifted to the symmetric space M via its exponential mapping.

Theorem 17 (J. Orloff, 1987 [21])**.** *For $p>1$ and $q>1$, and $f\in\mathcal{C}_c^\infty(M)$,*

1. If p and q are both odd,

$$\lim_{r\to 0^+}\sinh^{n-2}(r)P_+(L_r^+)(M_+^r f)(x_0)=cf(x_0),$$

and

$$\lim_{r\to 0^+}\sin^{n-2}(r)P_-(L_r^-)(M_-^r f)(x_0)=cf(x_0),$$

2. If p is odd and q is even,

$$\lim_{r\to 0^+}\Big(\frac{d}{dr}\Big)^{n-2} r^{n-2}(M_+^r f)(x_0)=cf(x_0),$$

3. If p is even and q is odd,

$$\lim_{r\to 0^+}\Big(\frac{d}{dr}\Big)^{n-2} r^{n-2}(M_-^r f)(x_0)=cf(x_0),$$

4. If p and q are both even,

$$\lim_{r\to 0^+}\Big(\frac{d}{dr}\Big)^{n-2} r^{n-2}(M_+^r f)(x_0)+(-1)^{\frac{n}{2}}\lim_{r\to 0^+}\Big(\frac{d}{dr}\Big)^{n-2} r^{n-2}(M_+^r f)(x_0)=cf(x_0),$$

where L_r^+ (respectively L_r^-) is the radial part of the Laplace-Beltrami operator in polar geodesic coordinates on D_+ (respectively D_-), P_+ and P_- are polynomials and, in each equation, c is a non-zero constant independent of f.

2.4 Summarizing result in the Lorentzian case

For Lorentzian symmetric spaces of constant sectional curvature, we collect the limit formulas given by S. Helgason in the even-dimensional case and we use the arguments of J. Orloff to get the limit formulas in odd-dimensional case. This gives the following summarizing result.

Theorem 18 (S. Helgason, 1959 [13] - J. Orloff, 1987 [21]). *Let $(M, s, \hat{g})$ be either the flat Lorentzian vector space $\mathbb{R}^{1,n-1}$ or one of the two model spaces $Q_{(+1)}$ and Q_{-1} introduced in theorem 8. Let κ be the constant sectional curvature of M. If $n = \dim(M)$, for any $f \in \mathcal{C}_c^\infty(M)$,*

1. *If $n > 2$ is even,*
$$f(x) = c \lim_{r\to 0^+} r^{n-2} P_+^\kappa(\Box)(M_+^r f)(x)$$
and
$$f(x) = c \lim_{r\to 0^+} r^{n-2} P_-^\kappa(\Box)(M_-^r f)(x),$$
2. *If $n = 2$,*
$$f(x) = -\frac{1}{2} \lim_{r\to 0^+} r \frac{d}{dr}(M_+^r f)(x)$$
and
$$f(x) = -\frac{1}{2} \lim_{r\to 0^+} r \frac{d}{dr}(M_-^r f)(x),$$
3. *If n is odd,*
$$f(x) = c \lim_{r\to 0^+} \left(\frac{d}{dr}\right)^{n-2} \big(r^{n-2}(M_+^r f)(x)\big),$$

where P_+^κ and P_-^κ are polynomials and, in each equation, c is non-zero constant independent of f.

In order to prove theorem 1, taking theorem 6 into account, it remains to look at orbital integrals on solvable indecomposable Lorentzian symmetric spaces.

3 Solvable indecomposable Lorentzian symmetric spaces

Let $(\mathfrak{g}, \sigma, B)$ be one of the triples described in theorem 9 for a particular choice of parameters $\lambda_1, ..., \lambda_{n-2} \in \mathbb{R}_0$. Up to isomorphism, the connected, simply connected Lie group associated to $\mathfrak{g}$ is the smooth manifold

$$G := \mathbb{R}^{2n-2}$$

endowed with the group law

$$(t,p,q,r)*(t',p',q',r') := (t+t', (p,q,r)\cdot\varphi_t(p',q',r'))$$

for $t,t',r,r' \in \mathbb{R}$, $p,p',q,q' \in \mathbb{R}^{n-2}$, where $\cdot$ is the *weighted* Heisenberg product given by

$$(p,q,r)\cdot(p',q',r') := (p+p', q+q', r+r'+\frac{1}{2}\sum_{i=1}^{n-2}\lambda_i(q_ip'_i - q'_ip_i))$$

and

$$\varphi_t(p',q',r') := \Big(\text{cos.h}(t\sqrt{|\lambda_i|})p'_i - \frac{1}{\sqrt{|\lambda_i|}}\text{sin.h}(t\sqrt{|\lambda_i|})q'_i, \\ \frac{\lambda_i}{\sqrt{|\lambda_i|}}\text{sin.h}(t\sqrt{|\lambda_i|})p'_i + \text{cos.h}(t\sqrt{|\lambda_i|})q'_i{}_{\ (1\le i\le n-2)}, r'\Big).$$

In the above formula and in the following, the notation $\text{sin.h}(t\sqrt{|\lambda_i|})$ (respectively $\text{cos.h}(t\sqrt{|\lambda_i|})$) means $\sin(t\sqrt{\lambda_i})$ (respectively $\cos(t\sqrt{\lambda_i})$) when $\lambda_i > 0$ and $\sinh(t\sqrt{-\lambda_i})$ (respectively $\cosh(t\sqrt{-\lambda_i})$) when $\lambda_i < 0$.

Therefore, G is a semi-direct product of $\mathbb{R}$ and a *weighted* Heisenberg group. The connected Lie subgroup of G whose Lie algebra is $\mathfrak{k}$, namely the isotropy group, is given by

$$K := \{(t,p,q,r) \in G \mid t = q_1 = ... = q_{n-2} = r = 0\}.$$

Let us define $\pi : G \to \mathbb{R}^n$ such that

$$\pi(t,p,q,r) := \Big(t, \frac{\lambda_i}{\sqrt{|\lambda_i|}}\text{sin.h}(t\sqrt{|\lambda_i|})p_i - \text{cos.h}(t\sqrt{|\lambda_i|})q_i{}_{\ (1\le i\le n-2)}, \\ r + \sum_{i=1}^{n-2}\frac{\lambda_i}{2}\Big(\frac{\lambda_i}{\sqrt{|\lambda_i|}}\text{sin.h}(t\sqrt{|\lambda_i|})p_i - \text{cos.h}(t\sqrt{|\lambda_i|})q_i\Big) \\ \Big(p_i\text{cos.h}(t\sqrt{|\lambda_i|}) + \frac{q_i}{\sqrt{|\lambda_i|}}\text{sin.h}(t\sqrt{|\lambda_i|})\Big)\Big)$$

which factors the natural projection from G onto G/K and yields an isomorphism between $M := G/K$ and $\mathbb{R}^n$. We use this identification in the sequel.

The unique involutive automorphism of G whose differential is σ writes

$$\tilde{\sigma} : G \to G : (t,p,q,r) \mapsto (-t,p,-q,-r).$$

The associated symmetric structure on M is given by

$$s_{\pi(g)}(\pi(g')) := \pi(g\sigma(g^{-1}g')), \quad \text{for } g,g' \in G.$$

If we denote by $(t, x_1, ..., x_{n-2}, v)$ the global coordinates on M identified with $\mathbb{R}^n$ as above, the metric on M associated to the bilinear form B is given by

$$\hat{g} := \Big(\sum_{i=1}^{n-2} \lambda_i x_i^2\Big) dt \otimes dt + dt \otimes dv + dv \otimes dt - \sum_{i=1}^{n-2} dx_i \otimes dx_i.$$

As a conclusion, up to isometry, the connected, simply connected, indecomposable solvable Lorentzian symmetric space associated to the triple $(\mathfrak{g}, \sigma, B)$ is $(M, s, \hat{g})$. It is called a *Cahen-Wallach space*. In dimension 4, it is an example of pp-waves in the Brinkmann class which are idealized gravitational wave models in general relativity[3, 7, 24].

3.1 The exponential mapping

Proposition 19. *Let $(M, s, \hat{g})$ be the Cahen-Wallach space corresponding to parameters $\lambda_1, ..., \lambda_{n-2} \in \mathbb{R}_0$. The exponential mapping at any point $y = (t_0, x_0, v_0) \in M$ is given by*

$$\mathrm{Exp}_y(\bar{b}) = \Big(b_0 + t_0, \frac{\sin.\mathrm{h}(b_0\sqrt{|\lambda_i|})}{b_0\sqrt{|\lambda_i|}} b_i + \cos.\mathrm{h}(b_0\sqrt{|\lambda_i|}) x_{0,i} \; {}_{(1 \le i \le n-2)},$$
$$\frac{1}{2b_0}\Big(2b_0 b' - \sum_{i=1}^{n-2} b_i^2 + b_0^2 \sum_{i=1}^{n-2} \lambda_i x_{0,i}^2\Big) + \sum_{i=1}^{n-2} \frac{b_i^2 - b_0^2 \lambda_i x_{0,i}^2}{4b_0^2\sqrt{|\lambda_i|}} \sin.\mathrm{h}(2b_0\sqrt{|\lambda_i|})$$
$$+ \sum_{i=1}^{n-2} \frac{b_i x_{0,i}}{2b_0} \cos.\mathrm{h}(2b_0\sqrt{|\lambda_i|}) + \Big(v_0 - \frac{1}{2b_0}\sum_{i=1}^{n-2} b_i x_{0,i}\Big)\Big)$$

for any $\bar{b} = (b_0, b_1, ..., b_{n-2}, b') \in T_y M \simeq \mathbb{R}^n$ such that $b_0 \neq 0$ and

$$\mathrm{Exp}_y(0, b_1, ..., b_{n-2}, b') = (t_0, b_1 + x_{0,1}, ..., b_{n-2} + x_{0,n-2}, b' + v_0).$$

Proof. The geodesic equations corresponding to the Levi-Civita connection on M are given by

$$\begin{cases} \frac{d^2 t}{ds^2}(s) = 0, \\ \frac{d^2 v}{ds^2}(s) + 2\sum_{i=1}^{n-2} \lambda_i x_i(s) \frac{dt}{ds}(s) \frac{dx_i}{ds}(s) = 0, \\ \frac{d^2 x_i}{ds^2}(s) + \lambda_i x_i(s) \big(\frac{dt}{ds}(s)\big)^2 = 0 \end{cases} .$$

The solutions $\gamma(s) := (t(s), x_1(s), ..., x_{n-2}(s), v(s))$ of these equations with initial data

$$\gamma(0) = (t(0), x_1(0), ..., x_{n-2}(0), v(0)) = (t_0, x_{0,1}, ..., x_{0,n-2}, v_0),$$
$$\dot{\gamma}(0) = \Big(\frac{dt}{ds}(0), \frac{dx_1}{ds}(0), ..., \frac{dx_{n-2}}{ds}(0), \frac{dv}{ds}(0)\Big) = (b_0, b_1, ..., b_{n-2}, b')$$

are given by $\mathrm{Exp}_{\gamma(0)}(s\dot{\gamma}(0))$ as above. □

Corollary 20. *Let* $(M, s, \hat{g})$ *be the Cahen-Wallach space corresponding to parameters* $\lambda_1, ..., \lambda_{n-2} \in \mathbb{R}_0$. *For any point* $y = (t_0, x_0, v_0) \in M$, *the exponential mapping at* y *is a diffeomorphism from the open subset*

$$\{(b_0, b_1, ..., b_{n-2}, b') \in T_yM \mid \forall i \in \{1, ..., n-2\},\ \lambda_i > 0 \Rightarrow b_0\sqrt{\lambda_i} \notin \pi\mathbb{Z}_0\}$$

of T_yM *to its image given by*

$$\{(t, x, v) \in M \mid \forall i \in \{1, ..., n-2\},\ \lambda_i > 0 \Rightarrow (t - t_0)\sqrt{\lambda_i} \notin \pi\mathbb{Z}_0\}.$$

The determinant of the differential of Exp_y *at any vector* $(b_0, b_1, ..., b_{n-2}, b') \in T_yM$ *is equal to*

$$\prod_{i=1}^{n-2} \left| \frac{\mathrm{sin.h}(b_0\sqrt{|\lambda_i|})}{b_0\sqrt{|\lambda_i|}} \right| \text{ if } b_0 \neq 0, \text{ and } 1 \text{ otherwise.}$$

3.2 The orbits of the isotropy group and the orbital integrals

Let us fix $(0, ..., 0)$ as base-point of M. Then we get the following identification $T_0M \simeq \mathfrak{p}$

$$\begin{cases} T_0M \simeq \mathbb{R}^n & \to \mathfrak{p} \\ (b_0, b_1, ..., b_{n-2}, b') & \mapsto b_0U + \sum_{i=1}^{n-2} b_iW_i + b'Z \end{cases}.$$

Proposition 21. *The orbits under the adjoint action of the isotropy group* K *in* $\mathfrak{p} \simeq T_0M$ *are*

$$\{(b_0, b_1, ..., b_{n-2}, b') \in T_0M \mid 2b_0b' - \sum_{i=1}^{n-2} b_i^2 = \alpha,\ b_0 = \beta\} =: \Sigma_{\alpha,\beta},$$
$$\{(0, \gamma_1, ..., \gamma_{n-2}, b') \mid b' \in \mathbb{R}\},$$
$$\{(0, 0, ..., 0, \beta')\}$$

for $\beta \in \mathbb{R}_0, \alpha \in \mathbb{R}, (\gamma_1, ..., \gamma_{n-2}) \in \mathbb{R}^{n-2} \setminus \{0\}$ *and* $\beta' \in \mathbb{R}$.

Proof. The adjoint action of K on $\mathfrak{p} \simeq \mathbb{R}^n$ is given by

$$\begin{aligned} &\mathrm{Ad}(\exp(p_1K_1 + ... + p_{n-2}K_{n-2}))(b_0, b_1, ..., b_{n-2}, b') \\ &= \Big(b_0, b_1 - \lambda_1p_1b_0, ..., b_{n-2} - \lambda_{n-2}p_{n-2}b_0, b' + \frac{b_0}{2}\sum_{i=1}^{n-2}\lambda_i^2p_i^2 - \sum_{i=1}^{n-2}\lambda_ip_ib_i\Big) \end{aligned}$$

for $(p_1, ..., p_{n-2}) \in \mathbb{R}^{n-2}$ and $(b_0, b_1, ..., b_{n-2}, b') \in T_0M \simeq \mathfrak{p}$. Once $(b_0, b_1, ..., b_{n-2}, b')$ is fixed, we must distinguish the cases when $b_0 \neq 0$ and $b_0 = 0$ to identify the different isotropy orbits. ∎

Note that the stabilizer of any point $X = (b_0, b_1, ..., b_{n-2}, b') \in T_0M \simeq \mathbb{R}^n$ such that $b_0 \neq 0$ under the action of the isotropy group K is just the neutral element. Therefore, the K-orbit of X is diffeomorphic to K.

Definition 16. For any $f \in \mathcal{C}_c(M)$ and any point $y = (t_0, x_0, v_0) \in M$, the *orbital integrals* of f at y are given by

$$(M^X f)(y) := \int_K f(gk.\mathrm{Exp}_0(X))\, dk$$

where $g \in G$ such that $y = g.0$, $X = (b_0, b_1, ..., b_{n-2}, b') \in T_0M \simeq \mathbb{R}^n$ such that $b_0 \neq 0$ and if $\lambda_i > 0$, $b_0\sqrt{\lambda_i} \notin \pi\mathbb{Z}$, and dk is the invariant measure $(\prod_{i=1}^{n-2} |\lambda_i|)\, dp_1...dp_{n-2}$ on $K \simeq \mathbb{R}^{n-2}$.

The orbital integrals of a function $f \in \mathcal{C}_c(M)$ at $y \in M$ are integrals of f over the orbits $\mathrm{Exp}_y(\Sigma_{\pm r^2, b_0}(y))$ where

$$\Sigma_{\pm r^2, b_0}(y) := \Big\{(\tilde{b}_0, b_1, ..., b_{n-2}, b') \in T_yM \mid \tilde{b}_0 = b_0, \\ \sum_{i=1}^{n-2} \lambda_i x_{0,i}^2 b_0^2 + 2b_0 b' - \sum_{i=1}^{n-2} b_i^2 = \pm r^2\Big\}$$

for $r > 0$ and $b_0 \in \mathbb{R}_0$ such that if $\lambda_i > 0$, $b_0\sqrt{\lambda_i} \notin \pi\mathbb{Z}_0$. We use the following notation for the orbital integrals

$$(M_+^{r,b_0} f)(y) := (M^X f)(y), \quad \text{where } X = (b_0, 0, ..., 0, r^2/2b_0) \in T_0M \simeq \mathbb{R}^n,$$
$$(M_-^{r,b_0} f)(y) := (M^X f)(y), \quad \text{where } X = (b_0, 0, ..., 0, -r^2/2b_0) \in T_0M \simeq \mathbb{R}^n,$$

for $r > 0$ and $b_0 \in \mathbb{R}_0$ such that if $\lambda_i > 0$, $b_0\sqrt{\lambda_i} \notin \pi\mathbb{Z}_0$.

Lemma 22. *For any $f \in \mathcal{C}_c(M)$, $y = g.0 \in M$, $r > 0$ and $b_0 \in \mathbb{R}_0$ such that if $\lambda_i > 0$, $b_0\sqrt{\lambda_i} \notin \pi\mathbb{Z}_0$,*

$$(M_\pm^{r,b_0} f)(y) = \frac{1}{|b_0|^{n-2}} \int_{\mathbb{R}^{n-2}} f\Big(g.\mathrm{Exp}_0\Big(b_0, b_1, ..., b_{n-2}, \frac{1}{2b_0}\big(\pm r^2 + \sum_{i=1}^{n-2} b_i^2\big)\Big)\Big) \\ db_1...db_{n-2}.$$

Proof. Using the expression of the action of K on T_0M given in the proof of proposition 21, we get

$$(M_\pm^{r,b_0} f)(y) = \Big(\prod_{i=1}^{n-2} |\lambda_i|\Big) \int_{\mathbb{R}^{n-2}} f\Big(g.\mathrm{Exp}_0\Big(b_0, -\lambda_1 p_1 b_0, ..., -\lambda_{n-2} p_{n-2} b_0, \\ \frac{1}{2b_0}\big(\pm r^2 + \sum_{i=1}^{n-2} \lambda_i^2 p_i^2 b_0^2\big)\Big)\Big)\, dp_1...dp_{n-2}.$$

Then a simple change of variables in the integral yields the result. □

4 Limit formulas on solvable Lorentzian symmetric spaces

Let $(M, s, \hat{g})$ be the Cahen-Wallach space corresponding to parameters $\lambda_1, ..., \lambda_{n-2} \in \mathbb{R}_0$. In order to determine a function in terms of its orbital integrals on M, we consider invariant differential operators.

4.1 Invariant differential operators

Let us recall that M is G-homogeneous space for the connected, simply connected Lie group G presented in section 3.

Proposition 23. *The Laplace-Beltrami operator on M associated to the Lorentzian $\hat{g} = \sum_{i=1}^{n-2} \lambda_i x_i^2 dt \otimes dt + dt \otimes dv + dv \otimes dt - \sum_{i=1}^{n-2} dx_i \otimes dx_i$*

$$L := \frac{\partial^2}{\partial t \partial v} + \frac{\partial^2}{\partial v \partial t} - \sum_{i=1}^{n-2} \lambda_i x_i^2 \frac{\partial^2}{\partial v^2} - \sum_{i=1}^{n-2} \frac{\partial^2}{\partial x_i^2}$$

as well as the order-one differential operator

$$\frac{\partial}{\partial v}$$

are G-invariant.

At any point $y = (t_0, x_0, v_0) = g.0 \in M$, we consider coordinates $(r, b_0, b_1, ..., b_{n-2})$ on either $\{\mathrm{Exp}_y(\bar{b}) \mid \bar{b} \in T_y M,\ \hat{g}_y(\bar{b}, \bar{b}) > 0\}$ or $\{\mathrm{Exp}_y(\bar{b}) \mid \bar{b} \in T_y M,\ \hat{g}_y(\bar{b}, \bar{b}) < 0\}$ which are defined by

$$(r, b_0, b_1, ..., b_{n-2}) \mapsto g.\mathrm{Exp}_0\Big(b_0, b_1, ..., b_{n-2}, \frac{1}{2b_0}\big(\varepsilon\, r^2 + \sum_{i=1}^{n-2} b_i^2\big)\Big),$$

where $\epsilon = 1$ or -1 respectively. In this coordinates, the expression of the two invariant differential operator in proposition 23 is

$$\begin{aligned} L = \varepsilon \Big\{ &\frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial}{\partial r}\Big) + \frac{1}{r}\frac{\partial}{\partial r}\Big(b_0 \frac{\partial}{\partial b_0}\Big) + \frac{1}{\frac{r}{|b_0|}\prod_{i=1}^{n-2}\frac{\sin.\mathrm{h}(b_0\sqrt{|\lambda_i|})}{\sqrt{|\lambda_i|}}} \\ &\frac{\partial}{\partial b_0}\Big(\frac{b_0}{|b_0|}\prod_{i=1}^{n-2}\frac{\sin.\mathrm{h}(b_0\sqrt{|\lambda_i|})}{\sqrt{|\lambda_i|}}\frac{\partial}{\partial r}\Big)\Big\} + L_{\mathrm{Exp}_y(\Sigma_{\varepsilon\, r^2, b_0}(y))}, \\ \frac{\partial}{\partial v} &= \varepsilon\, \frac{b_0}{r}\frac{\partial}{\partial r}, \end{aligned}$$

where $L_{\mathrm{Exp}_y(\Sigma_{\varepsilon\, r^2, b_0}(y))}$ is the Laplace-Beltrami operator on the isotropy orbit associated to the induced metric and ϵ is either 1 or -1 depending on whether the domain is $\{\mathrm{Exp}_y(\bar{b}) \mid \bar{b} \in T_y M,\ \hat{g}_y(\bar{b}, \bar{b}) > 0\}$ or $\{\mathrm{Exp}_y(\bar{b}) \mid \bar{b} \in T_y M,\ \hat{g}_y(\bar{b}, \bar{b}) < 0\}$.

Proposition 24. *For any $f \in \mathcal{C}_c^\infty(M)$, $y \in M$, $r > 0$ and $b_0 \in \mathbb{R}_0$ such that if $\lambda_i > 0$, $b_0\sqrt{\lambda_i} \notin \pi\mathbb{Z}_0$,*

1. $L(M_\pm^{r,b_0} f)(y) = M_\pm^{r,b_0}(Lf)(y)$,
2. $\frac{\partial}{\partial v}(M_\pm^{r,b_0} f)(y) = M_\pm^{r,b_0}\left(\frac{\partial f}{\partial v}\right)(y)$,
3. $M_\pm^{r,b_0}\left(\frac{\partial f}{\partial v}\right)(y) = \pm\frac{b_0}{r}\frac{\partial}{\partial r}(M_\pm^{r,b_0} f)(y)$.

Proof. By a theorem stated in S. Helgason's book [13], for every G-invariant differential operator D on M, there exists a bi-invariant differential operator $\tilde{D}$ on G such that

$$\forall f \in \mathcal{C}^\infty(M),\ \tilde{D}(f \circ \pi) = (Df) \circ \pi,$$

where $\pi : G \to M$ is the canonical projection. Therefore, when we apply D to the orbital integrals of a function $f \in \mathcal{C}_c^\infty(M)$, we get

$$\begin{aligned} D(M_\pm^{r,b_0} f)(g.0) &= \tilde{D}\big((M_\pm^{r,b_0} f) \circ \pi\big)(g) \\ &= \tilde{D}\Big\{\frac{1}{|b_0|^{n-2}} \int_{\mathbb{R}^{n-2}} f\Big(g.\mathrm{Exp}_0\Big(b_0, b_1, ..., b_{n-2}, \frac{1}{2b_0}\big(\pm r^2 + \sum_{i=1}^{n-2} b_i^2\big)\Big)\Big) \\ &\quad db_1 ... db_{n-2}\Big\} \\ &= \frac{1}{|b_0|^{n-2}} \int_{\mathbb{R}^{n-2}} (Df)\Big(g.\mathrm{Exp}_0\Big(b_0, b_1, ..., b_{n-2}, \frac{1}{2b_0}\big(\pm r^2 + \sum_{i=1}^{n-2} b_i^2\big)\Big)\Big) \\ &\quad db_1 ... db_{n-2} \end{aligned}$$

because $\tilde{D}$ is bi-invariant. This implies the points 1 and 2. For point 3, we use the local expression of $\frac{\partial}{\partial v}$. □

4.2 One-parameter generalized Riesz potentials

Let us focus on the first set of orbital integrals, namely $(M_+^{r,b_0} f)$. The corresponding one-parameter generalized Riesz potentials are then given below.

Definition 17. For any $f \in \mathcal{C}_c(M)$, $y \in M$, $b_0 \in \mathbb{R}_0$ such that if $\lambda_i > 0$, $b_0\sqrt{\lambda_i} \notin \pi\mathbb{Z}_0$ and $\mu \in \mathbb{C}$ such that $\mathrm{Re}(\mu) > n - 1$,

$$(J_+^{\mu,b_0} f)(y) := \frac{1}{H_{n-1}(\mu)} \int_0^\infty (M_+^{r,b_0} f)(y) r^{\mu-n+2} dr$$

where $H_{n-1}(\mu) := \pi^{(n-3)/2} 2^{\mu-1} \Gamma(\mu/2)\Gamma((\mu + 3 - n)/2)$.

Remark 2. For any $y = g.0 \in M$ and fixed $b_0 \in \mathbb{R}_0$ such that if $\lambda_i > 0$, $b_0\sqrt{\lambda_i} \notin \pi\mathbb{Z}_0$,

$$(J_+^{\mu,b_0} f)(y) = \frac{1}{|b_0|^{n-2}} \frac{1}{H_{n-1}(\mu)} \int_{(D_0^+)_{cc}} F_{g,b_0}(u_1, ..., u_{n-1}) \\ u_{n-1}\Big(\sqrt{u_{n-1}^2 - u_{n-2}^2 - ... - u_1^2}\Big)^{\mu-n+1} du_1...du_{n-1}$$

where $F_{g,b_0}(u_1, ..., u_{n-1}) := f\big(g.\mathrm{Exp}_0\big(b_0, u_1, ..., u_{n-2}, \frac{u_{n-1}^2}{2b_0}\big)\big)$ and

$$(D_0^+)_{cc} := \{(u_1, ..., u_{n-1}) \in \mathbb{R}^{n-1} \mid u_{n-1}^2 - u_{n-2}^2 - ... - u_1^2 > 0,\ u_{n-1} > 0\}.$$

This means that the one-parameter generalized Riesz potentials on M turn out to be some flat Lorentzian Riesz potentials on $(\mathbb{R}^{n-1}, \mathbb{I}_{1,n-2})$.

Proposition 25. *For any $f \in \mathcal{C}_c^\infty(M)$, $y = g.0 \in M$ and fixed $b_0 \in \mathbb{R}_0$ such that if $\lambda_i > 0$, $b_0\sqrt{\lambda_i} \notin \pi\mathbb{Z}_0$,*

1. *$(J_+^{\mu,b_0} f)(y)$ holomorphically extends to the whole complex plane $\mathbb{C}$ with respect to μ,*
2. *$\lim_{\mu\to 0} (J_+^{\mu,b_0} f)(y) = \frac{1}{|b_0|^{n-2}} F_{g,b_0}(0, ..., 0).0 = 0$,*
3. *$\frac{\partial}{\partial v}(J_+^{\mu,b_0} f)(y) = J_+^{\mu,b_0}\Big(\frac{\partial f}{\partial v}\Big)(y) = -\frac{b_0}{\mu-2}(J_+^{\mu-2,b_0} f)(y)$.*

Proof. Property 1 is true thanks to remark 2 and theorem 15 which states that flat Lorentzian Riesz potentials can be holomorphically extended on the whole complex plane with respect to the parameter. Property 2 also follows from theorem 15 thanks to remark 2. Finally, we use the local expression of the invariant differential operator

$$\frac{\partial}{\partial v} = \frac{b_0}{r}\frac{\partial}{\partial r}$$

on a dense open subset of $\{\mathrm{Exp}_y(\bar{b}) \mid \bar{b} \in T_yM,\ \hat{g}_y(\bar{b}, \bar{b}) > 0\}$ to get property 3 whenever $\mathrm{Re}(\mu) > n-1$. Furthermore, it remains true for every $\mu \in \mathbb{C}$ thanks to the uniqueness of the holomorphic extensions. □

Corollary 26. *For any $f \in \mathcal{C}_c^\infty(M)$, $y \in M$, $b_0 \in \mathbb{R}_0$ such that if $\lambda_i > 0$, $b_0\sqrt{\lambda_i} \notin \pi\mathbb{Z}_0$ and any $k \in \mathbb{N}_0$,*

$$\frac{\partial^k}{\partial v^k}(J_+^{\mu,b_0} f)(y) = J_+^{\mu,b_0}\Big(\frac{\partial^k f}{\partial v^k}\Big)(y) = \frac{(-b_0)^k}{(\mu-2)(\mu-4)...(\mu-2k)}(J_+^{\mu-2k,b_0} f)(y).$$

Proposition 27. *For any $f \in \mathcal{C}_c^\infty(M)$, $y \in M$ and $b_0 \in \mathbb{R}_0$ such that if $\lambda_i > 0$, $b_0\sqrt{\lambda_i} \notin \pi\mathbb{Z}_0$,*

$$J_+^{1,b_0}\Big(\frac{\partial f}{\partial v}\Big)(y) = \lim_{\mu\to -1} \frac{-b_0}{\mu}(J_+^{\mu,b_0} f)(y) = \frac{-b_0\Gamma\big(\frac{n-1}{2}\big)}{|b_0|^{n-2}\sqrt{\pi}\Gamma\big(\frac{n}{2}\big)} f(g.\mathrm{Exp}_0(b_0, 0, ..., 0)).$$

Proof. We use property 3 in proposition 25 and the following change of variables in the expression in remark 2

$$(u_1, ..., u_{n-1}) = (r\sinh(\zeta)\omega(\theta_1, ..., \theta_{n-3}), r\cosh(\zeta))$$

where $\zeta \in]0, +\infty[$ and $\omega(\bar{\theta}) \in \mathbb{S}^{n-3}$, to get

$$J_+^{1,b_0}\Big(\frac{\partial f}{\partial v}\Big)(y) = \lim_{\mu\to -1} \frac{-b_0}{|b_0|^{n-2}\mu H_{n-1}(\mu)} \int_0^\infty \int_0^\infty \int_{\mathbb{S}^{n-3}} F_{g,b_0}(r\sinh(\zeta)\omega(\bar{\theta}), r\cosh(\zeta))\, r^\mu \cosh(\zeta)\sinh^{n-3}(\zeta) d\omega(\bar{\theta}) d\zeta dr.$$

Then the consecutive changes $T = e^{-2\zeta}$ and $r = \sigma\sqrt{T}$ lead to

$$\lim_{\mu\to -1} \frac{-b_0\Gamma\big(\frac{\mu+1}{2}\big)}{|b_0|^{n-2}\pi^{(n-2)/2}\Gamma(\mu+1)\Gamma\big(\frac{\mu+3-n}{2}\big)} \int_0^1 \int_0^\infty \int_{\mathbb{S}^{n-3}} F_{g,b_0}\Big(\frac{\sigma}{2}(1-T)\omega(\bar{\theta}), \frac{\sigma}{2}(1+T)\Big)\, \sigma^{(\mu+1)-1} 2^{1-n} T^{(\mu+1-n)/2}(1+T)(1-T)^{n-3} d\omega(\bar{\theta}) d\sigma dT$$

which can be seen as a double Riemann-Liouville integral. Thanks to arguments found in Riesz's paper [22], the limit is equal to

$$\lim_{\nu\to\frac{2-n}{2}} \frac{-b_0 2^{1-n}\Omega_{n-2}\Gamma\big(\nu + \frac{n-2}{2}\big)}{|b_0|^{n-2}\pi^{(n-2)/2}\Gamma(\nu)} F_{g,b_0}(0, ..., 0) \int_0^1 T^{\nu-1}(1+T)(1-T)^{n-3} dT$$

where Ω_{n-2} is the area of the $(n-3)$-dimensional standard sphere. Using Euler's beta functions to rewrite the remaining integral and since $F_{g,b_0}(0, ..., 0) = f(g.\mathrm{Exp}_0(b_0, 0, ..., 0))$, we get the expected equality. □

Corollary 28. *For any $f \in \mathcal{C}_c^\infty(M)$, $y = g.0 \in M$, $b_0 \in \mathbb{R}_0$ such that if $\lambda_i > 0$, $b_0\sqrt{\lambda_i} \notin \pi\mathbb{Z}_0$ and any $k \in \mathbb{N}_0$,*

$$J_+^{2k-1,b_0}\Big(\frac{\partial^k f}{\partial v^k}\Big)(y) = \frac{\Gamma\big(\frac{n-1}{2}\big)}{2^{k-1}\Gamma\big(\frac{n}{2}\big)} \frac{(-b_0)^k}{\Gamma\big(\frac{2k-1}{2}\big)|b_0|^{n-2}} f(g.\mathrm{Exp}_0(b_0, 0, ..., 0)).$$

4.3 Limit formulas for the orbital integrals

Lemma 29. *For any function $f \in \mathcal{C}_c^\infty(M)$, any $b_0 \in \mathbb{R}_0$ such that if $\lambda_i > 0$, $b_0\sqrt{\lambda_i} \notin \pi\mathbb{Z}_0$ and any $y \in M$, the limit*

$$\lim_{r \overset{>}{\to} 0} (M_\pm^{r,b_0} f)(y)$$

exists whenever $n = \dim(M) > 2$.

Proof. By lemma 22, if $y = g.0$, the orbital integrals $(M_\pm^{r,b_0} f)(y)$ writes

$$\frac{1}{|b_0|^{n-2}} \int_{\mathbb{R}^{n-2}} f\Big(g.\mathrm{Exp}_0\Big(b_0, b_1, ..., b_{n-2}, \frac{1}{2b_0}\big(\pm r^2 + \sum_{i=1}^{n-2} b_i^2\big)\Big)\Big) db_1 ... db_{n-2}.$$

In the first series of integrals, namely $(M_+^{r,b_0}f)$, we apply the change of coordinates

$$(b_1, ..., b_{n-2}) = r\sinh(\zeta)\omega(\theta_1, ..., \theta_{n-3})$$

where $\zeta \in]0, +\infty[$ and $\omega(\bar{\theta}) \in \mathbb{S}^{n-3}$, to get

$$\frac{1}{|b_0|^{n-2}} \int_0^\infty \int_{\mathbb{S}^{n-3}} f\Big(g.\mathrm{Exp}_0\Big(b_0, r\sinh(\zeta)\omega(\bar{\theta}), \frac{r^2}{2b_0}\cosh^2(\zeta)\Big)\Big) \\ r^{n-2}\sinh^{n-3}(\zeta)\cosh(\zeta)d\omega(\bar{\theta})d\zeta.$$

Next, we define $F_{g,b_0}(u_1, ..., u_{n-1}) := f\big(g.\mathrm{Exp}_0\big(b_0, u_1, ..., u_{n-2}, \frac{u_{n-1}^2}{2b_0}\big)\big)$ and set $t = r\sinh(\zeta)$ in the integral. Thus the orbital integrals write

$$(M_+^{r,b_0}f)(g.0) = \frac{1}{|b_0|^{n-2}} \int_0^\infty \int_{\mathbb{S}^{n-3}} F_{g,b_0}\big(t\omega(\bar{\theta}), \sqrt{r^2+t^2}\big)t^{n-3}d\omega(\bar{\theta})dt$$

and it is clear that the limit when r goes to zero exists. This remains true with the other series of orbital integrals, namely $(M_-^{r,b_0}f)$. □

Corollary 30. *Assume* $n = \dim(M) > 3$. *Then, for any function* $f \in \mathcal{C}_c(M)$, *any* $y \in M$ *and fixed* $b_0 \in \mathbb{R}_0$ *such that if* $\lambda_i > 0$, $b_0\sqrt{\lambda_i} \notin \pi\mathbb{Z}_0$,

$$(J_+^{n-3,b_0}f)(y) = \frac{1}{(4\pi)^{(n-3)/2}\Gamma\big(\frac{n-3}{2}\big)} \lim_{r \xrightarrow{>} 0} (M_+^{r,b_0}f)(y).$$

Proof. The definition of one-parameter generalized Riesz potentials gives us

$$(J_+^{n-3,b_0}f)(y) = \lim_{\mu\to n-3} \frac{\Gamma\big(\frac{\mu+4-n}{2}\big)}{2^{n-3}\pi^{(n-2)/2}\Gamma\big(\frac{\mu}{2}\big)} \frac{1}{\Gamma(\mu+3-n)} \int_0^\infty \tilde{F}_{b_0}(r)r^{(\mu+3-n)-1}dr$$

where $\tilde{F}_{b_0}(r) := (M_+^{r,b_0}f)(y)$. Since $\tilde{F}_{b_0}$ is a compactly supported, continuous function whose limit when r goes to zero exists thanks to lemma 29, the limit of the Riemann-Liouville integral above is equal to

$$\frac{\sqrt{\pi}}{2^{n-3}\pi^{(n-2)/2}\Gamma\big(\frac{n-3}{2}\big)} \lim_{r \xrightarrow{>} 0} \tilde{F}_{b_0}(r).$$

□

Theorem 31. *If* $n = \dim(M) > 2$ *is even, for any* $f \in \mathcal{C}_c^\infty(M)$ *and* $y \in M$,

$$f(y) = \frac{\Gamma\big(\frac{n}{2}\big)}{\pi^{(n-3)/2}\Gamma\big(\frac{n-1}{2}\big)} \lim_{b_0 \xrightarrow{\neq} 0} \lim_{r \xrightarrow{>} 0} \Big\{\frac{-b_0^2}{2r}\frac{\partial}{\partial r}\Big\}^{(n-2)/2} (M_+^{r,b_0}f)(y).$$

Proof. Since $n = \dim(M) > 2$ is even, there exists $k \in \mathbb{N}_0$ such that $n = 2k+2$. Let us first fix $b_0 \in \mathbb{R}_0$ such that if $\lambda_i > 0$, $-\pi/\sqrt{\lambda_i} < b_0 < \pi/\sqrt{\lambda_i}$. By corollary 30,

$$J_+^{n-3,b_0}\Big(\frac{\partial^k f}{\partial v^k}\Big)(y) = \frac{1}{(4\pi)^{(n-3)/2}\Gamma\big(\frac{n-3}{2}\big)} \lim_{r \xrightarrow{>} 0} M_+^{r,b_0}\Big(\frac{\partial^k f}{\partial v^k}\Big)(y).$$

Furthermore, thanks to property 3 in proposition 24, we get

$$J_+^{n-3,b_0}\Big(\frac{\partial^k f}{\partial v^k}\Big)(y) = \frac{1}{(4\pi)^{(n-3)/2}\Gamma\big(\frac{n-3}{2}\big)} \lim_{r \overset{>}{\to} 0} \Big\{\frac{b_0}{r}\frac{\partial}{\partial r}\Big\}^k (M_+^{r,b_0} f)(y).$$

On the other hand, by corollary 28, since $n-3 = 2k-1$,

$$J_+^{n-3,b_0}\Big(\frac{\partial^k f}{\partial v^k}\Big)(y) = \frac{\Gamma\big(\frac{n-1}{2}\big)}{2^{(n-4)/2}\Gamma\big(\frac{n}{2}\big)} \frac{(-b_0)^{(n-2)/2}}{\Gamma\big(\frac{n-3}{2}\big)|b_0|^{n-2}} f(g.\mathrm{Exp}_0(b_0, 0, ..., 0)),$$

hence the result. □

The same kind of limit formula holds for the other series of orbital integrals, namely $(M_-^{r,b_0} f)$. In other words, the function f is also determined in terms of $(M_-^{r,b_0} f)$.

Theorem 32. *If $n = \dim(M) > 2$ is even, for any $f \in \mathcal{C}_c^\infty(M)$ and $y \in M$,*

$$f(y) = \frac{\Gamma\big(\frac{n}{2}\big)}{\pi^{(n-3)/2}\Gamma\big(\frac{n-1}{2}\big)} \lim_{b_0 \overset{\neq}{\to} 0} \lim_{r \overset{>}{\to} 0} \Big\{\frac{-b_0^2}{2r}\frac{\partial}{\partial r}\Big\}^{(n-2)/2} (M_-^{r,b_0} f)(y).$$

This limit formula is obtained in the same way by considering the corresponding one-parameter generalized Riesz potentials

$$(J_-^{\mu,b_0} f)(y) := \frac{1}{H_{n-1}(\mu)} \int_0^\infty (M_-^{r,b_0} f)(y) r^{\mu-n+2} dr.$$

Finally, we deal with the odd-dimensional case.

Theorem 33. *If $n = \dim(M) \geq 3$ is odd, for any $f \in \mathcal{C}_c^\infty(M)$ and $y \in M$,*

$$f(y) = \frac{\Gamma\big(\frac{n}{2}\big)}{\pi^{(n-2)/2}\Gamma\big(\frac{n-1}{2}\big)} \lim_{b_0 \overset{\neq}{\to} 0} \int_0^\infty \Big\{\frac{-b_0}{r}\frac{\partial}{\partial r}\Big\}\Big\{\frac{-b_0^2}{2r}\frac{\partial}{\partial r}\Big\}^{(n-3)/2} (M_+^{r,b_0} f)(y) dr.$$

In the same way,

$$f(y) = \frac{\Gamma\big(\frac{n}{2}\big)}{\pi^{(n-2)/2}\Gamma\big(\frac{n-1}{2}\big)} \lim_{b_0 \overset{\neq}{\to} 0} \int_0^\infty \Big\{\frac{-b_0}{r}\frac{\partial}{\partial r}\Big\}\Big\{\frac{-b_0^2}{2r}\frac{\partial}{\partial r}\Big\}^{(n-3)/2} (M_-^{r,b_0} f)(y) dr.$$

Proof. Since $n = \dim(M) \geq 3$ is odd, there exists $k \in \mathbb{N}_0$ such that $n = 2k+1$. Let us first fix $b_0 \in \mathbb{R}_0$ such that if $\lambda_i > 0$, $-\pi/\sqrt{\lambda_i} < b_0 < \pi/\sqrt{\lambda_i}$. By corollary 28, since $n-2 = 2k-1$,

$$J_+^{n-2,b_0}\Big(\frac{\partial^k f}{\partial v^k}\Big)(y) = \frac{\Gamma\big(\frac{n-1}{2}\big)}{2^{(n-3)/2}\Gamma\big(\frac{n}{2}\big)} \frac{(-b_0)^{(n-1)/2}}{\Gamma\big(\frac{n-2}{2}\big)|b_0|^{n-2}} f(g.\mathrm{Exp}_0(b_0, 0, ..., 0)).$$

Furthermore, by definition of the one-parameter generalized Riesz potentials,

$$J_+^{n-2,b_0}\Big(\frac{\partial^k f}{\partial v^k}\Big)(y) = \frac{1}{H_{n-1}(n-2)} \int_0^\infty M_+^{r,b_0}\Big(\frac{\partial^k f}{\partial v^k}\Big)(y) dr$$

where the integral converges due to lemma 29 and because f is compactly supported. Thanks to property 3 in proposition 24,

$$J_+^{n-2,b_0}\Big(\frac{\partial^k f}{\partial v^k}\Big)(y) = \frac{1}{2^{n-3}\pi^{(n-2)/2}\Gamma\big(\frac{n-2}{2}\big)}\int_0^\infty \Big\{\frac{b_0}{r}\frac{\partial}{\partial r}\Big\}^k (M_+^{r,b_0}f)(y)dr,$$

hence the result. It works the same way for the other series of orbital integrals, namely $(M_-^{r,b_0}f)$. □

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