arXiv:1804.08510v1 [math.FA] 23 Apr 2018

STANDARD VERSUS STRICT BOUNDED REAL LEMMA WITH INFINITE-DIMENSIONAL STATE SPACE III: THE DICHOTOMOUS AND BICAUSAL CASES

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Abstract. This is the third installment in a series of papers concerning the Bounded Real Lemma for infinite-dimensional discrete-time linear input/state/output systems. In this setting, under appropriate conditions, the lemma characterizes when the transfer function associated with the system has contractive values on the unit circle, expressed in terms of a Linear Matrix Inequality, often referred to as the Kalman-Yakubovich-Popov (KYP) inequality. Whereas the first two installments focussed on causal systems with the transfer functions extending to an analytic function on the disk, in the present paper the system is still causal but the state operator is allowed to have nontrivial dichotomy (the unit circle is not contained in its spectrum), implying that the transfer function is analytic in a neighborhood of zero and on a neighborhood of the unit circle rather than on the unit disk. More generally, we consider bicausal systems, for which the transfer function need not be analytic in a neighborhood of zero. For both types of systems, by a variation on Willems' storage-function approach, we prove variations on the standard and strict Bounded Real Lemma. We also specialize the results to nonstationary discrete-time systems with a dichotomy, thereby recovering a Bounded Real Lemma due to Ben-Artzi–Gohberg-Kaashoek for such systems.

1. Introduction

This is the third installment in a series of papers on the bounded real lemma for infinite-dimensional discrete-time linear systems and the related Kalman-Yakubovich-Popov (KYP) inequality. We consider discrete-time input-state-output linear systems determined by the following equations

$$\Sigma := \left\{ \begin{array}{ccc} \mathbf{x}(n+1) & = & A\mathbf{x}(n) + B\mathbf{u}(n), \\ \mathbf{y}(n) & = & C\mathbf{x}(n) + D\mathbf{u}(n), \end{array} \right. \qquad (n \in \mathbb{Z}) \tag{1.1}$$

where $A : \mathcal{X} \to \mathcal{X}$, $B : \mathcal{U} \to \mathcal{X}$, $C : \mathcal{X} \to \mathcal{Y}$ and $D : \mathcal{U} \to \mathcal{Y}$ are bounded linear Hilbert space operators i.e., $\mathcal{X}$, $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces and the *system matrix* associated with Σ takes the form

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}. \tag{1.2}$$

2010 *Mathematics Subject Classification.* Primary 47A63; Secondary 47A48, 93B20, 93C55, 47A56.

Key words and phrases. KYP inequality, storage function, bounded real lemma, infinite dimensional linear systems, dichotomous systems, bicausal systems.

This work is based on the research supported in part by the National Research Foundation of South Africa (Grant Numbers 93039, 90670, and 93406).

Associated with the system Σ is the transfer function

$$F_\Sigma(z) = D + zC(I - zA)^{-1}B \tag{1.3}$$

which necessarily defines an analytic function on some neighborhood of the origin in the complex plane with values in the space $\mathcal{L}(\mathcal{U},\mathcal{Y})$ of bounded linear operators from $\mathcal{U}$ to $\mathcal{Y}$. The Bounded Real Lemma is concerned with the question of characterizing (in terms of A, B, C, D) when F_Σ has analytic continuation to the whole unit disk $\mathbb{D}$ such that the supremum norm of F over the unit disk $\|F\|_{\infty,\mathbb{D}} := \sup\{\|F(z)\|\colon z \in \mathbb{D}\}$ satisfies either (i) $\|F_\Sigma\|_{\infty,\mathbb{D}} \le 1$ (*standard version*), or (ii) $\|F_\Sigma\|_{\infty,\mathbb{D}} < 1$ (*strict version*).

We first note the following terminology which we shall use. Given a selfadjoint operator H on a Hilbert space $\mathcal{X}$, we say that

(i) H is *strictly positive-definite* ($H \succ 0$) if there is a $\delta > 0$ so that $\langle Hx, x\rangle \ge \delta\|x\|^2$ for all $x \in \mathcal{X}$.
(ii) H is *positive-definite* if $\langle Hx, x\rangle > 0$ for all $0 \ne x \in \mathcal{X}$.
(iii) H is *positive-semidefinite* ($H \succeq 0$) if $\langle Hx, x\rangle \ge 0$ for all $x \in \mathcal{X}$.

Given two selfadjoint operators H, K on $\mathcal{X}$, we write $H \succ K$ or $K \prec H$ if $K - H \succ 0$ and similarly for $H \succeq K$ or $K \preceq H$. Note that if $\mathcal{X}$ is finite-dimensional, then *strictly positive-definite* and *positive-definite* are equivalent. Then the standard and strict Bounded Real Lemmas for the finite-dimensional setting (where $\mathcal{X}$, $\mathcal{U}$, $\mathcal{Y}$ are all finite-dimensional and one can view A, B, C, D as finite matrices) are as follows.

Theorem 1.1. *Suppose that we are given $\mathcal{X}$, $\mathcal{U}$, $\mathcal{Y}$, $M = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$, and F_Σ as in* (1.1), (1.2), (1.3)*, with $\mathcal{X}$, $\mathcal{U}$, $\mathcal{Y}$ all finite-dimensional Hilbert spaces. Then:*

(1) **Standard Bounded Real Lemma** (see [1]): *Assume that (A, B) is* controllable *(i.e.* $\operatorname{span}_{k\ge 0}\{\operatorname{Im} A^k B\} = \mathcal{X}$*) and (C, A) is* observable *(i.e.* $\bigcap_{k\ge 0} \ker CA^k = \{0\}$*). Then $\|F_\Sigma\|_{\infty,\mathbb{D}} \le 1$ if and only if there exists a positive-definite matrix H satisfying the Kalman-Yakubovich-Popov (KYP) inequality:*

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^* \begin{bmatrix} H & 0 \\ 0 & I_\mathcal{Y} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \preceq \begin{bmatrix} H & 0 \\ 0 & I_\mathcal{U} \end{bmatrix}. \tag{1.4}$$

(2) **Strict Bounded Real Lemma** (see [18]): Assume that all eigenvalues of A are in the unit disk $\mathbb{D}$. Then $\|F_\Sigma\|_{\infty,\mathbb{D}} < 1$ if and only if there exists a positive-definite matrix H so that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^* \begin{bmatrix} H & 0 \\ 0 & I_\mathcal{Y} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \prec \begin{bmatrix} H & 0 \\ 0 & I_\mathcal{U} \end{bmatrix}. \tag{1.5}$$

Infinite-dimensional versions of the standard Bounded Real Lemma have been studied by Arov-Kaashoek-Pik [2] and the authors [6, 7], while infinite-dimensional versions of the strict Bounded Real Lemma have been analyzed by Yakubovich [23, 24], Opmeer-Staffans [17] and the authors [6, 7].

In this paper we wish to study the following variation of the Bounded Real Lemma, which we shall call the *dichotomous Bounded Real Lemma*. Given the system with system matrix $M = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ and associated transfer function F_Σ as in (1.1), (1.2), (1.3), we now assume that the operator A *admits dichotomy*, i.e., we assume that A *has no spectrum on the unit circle* $\mathbb{T}$. Under this assumption it follows that the transfer function F_Σ in (1.3) can be viewed as an analytic

$\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued function on a neighborhood of the unit circle $\mathbb{T}$. The **dichotomous Bounded Real Lemma** is concerned with the question of characterizing in terms of A, B, C, D when it is the case that $\|F_\Sigma\|_{\infty,\mathbb{T}} := \sup\{\|F(z)\| \colon z \in \mathbb{T}\}$ satisfies either $\|F_\Sigma\|_{\infty,\mathbb{T}} \le 1$ (standard version) or (ii) $\|F_\Sigma\|_{\infty,\mathbb{T}} < 1$ (strict version). For the finite-dimensional case we have the following result.

Theorem 1.2. *Suppose that we are given $\mathcal{X}$, $\mathcal{U}$, $\mathcal{Y}$ and $M = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ and F_Σ as in* (1.1), (1.2), (1.3)*, with $\mathcal{X}$, $\mathcal{U}$, $\mathcal{Y}$ all finite-dimensional Hilbert spaces and with A having no eigenvalues on the unit circle $\mathbb{T}$. Then:*

1. **Finite-dimensional standard dichotomous Bounded Real Lemma:** *Assume that Σ is minimal ((A, B) is controllable and (C, A) is observable). Then the inequality $\|F_\Sigma\|_{\infty,\mathbb{T}} \le 1$ holds if and only if there exists an invertible selfadjoint matrix H which satisfies the KYP-inequality* (1.4)*. Moreover, the dimension of the spectral subspace of A over the unit disk is equal to the number of positive eigenvalues (counting multiplicities) of H and the dimension of the spectral subspace of A over the exterior of the closed unit disk is equal to the number of negative eigenvalues (counting multiplicities) of H.*
2. **Finite-dimensional strict dichotomous Bounded Real Lemma:** *The strict inequality $\|F_\Sigma\|_{\infty,\mathbb{T}} < 1$ holds if and only if there exists an invertible selfadjoint matrix H which satisfies the strict KYP-inequality* (1.5)*. Moreover, the inertia of A (the dimensions of the spectral subspace of A for the disk and for the exterior of the closed unit disk) is related to the inertia of H (dimension of negative and positive eigenspaces) as in the standard dichotomous Bounded Real Lemma (item (1) above).*

We note that Theorem 1.2 (2) appears as Corollary 1.2 in [10] as a corollary of more general considerations concerning input-output operators for nonstationary linear systems with an indefinite metric; to make the connection between the result there and the strict KYP-inequality (1.5), one should observe that a standard Schur-complement computation converts the strict inequality (1.5) to the pair of strict inequalities

$$\begin{aligned} & I - B^*HB - D^*D \succ 0, \\ & H - A^*HA - C^*C - (A^*HB + C^*D)Z^{-1}(B^*HA + D^*C) \succ 0 \\ & \qquad \text{where } Z = I - B^*HB - D^*D. \end{aligned}$$

We have not located an explicit statement of Theorem 1.2 (1) in the literature; this will be a corollary of the infinite-dimensional standard dichotomous Bounded Real Lemma which we present in this paper (Theorem 7.1 below).

Note that if $F = F_\Sigma$ is a transfer function of the form (1.3), then necessarily F is analytic at the origin. One approach to remove this restriction is to designate some other point z_0 where F is to be analytic and adapt the formula (1.3) to a realization "centered at z_0" (see [5, page 141] for details): e.g. for the case $z_0 = \infty$, one can use $F(z) = D + C(zI - A)^{-1}B$. To get a single chart to handle an arbitrary location of poles, one can use the bicausal realizations used in [4] (see [8] for the nonrational operator-valued case); for the setting here, where we are interested in a rational matrix functions analytic on a neighborhood of the unit circle $\mathbb{T}$, we suppose that

$$M_+ = \begin{bmatrix} \widetilde{A}_+ & \widetilde{B}_+ \\ \widetilde{C}_+ & \widetilde{D} \end{bmatrix} : \begin{bmatrix} \mathcal{X}_+ \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X}_+ \\ \mathcal{Y} \end{bmatrix}, \ M_- = \begin{bmatrix} \widetilde{A}_- & \widetilde{B}_- \\ \widetilde{C}_- & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{X}_- \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X}_- \\ \mathcal{Y} \end{bmatrix} \tag{1.6}$$

are two system matrices with spectrum of A_+, $\sigma(A_+)$, and of A_-, $\sigma(\widetilde{A}_-)$, contained in the unit disk $\mathbb{D}$ and that $F(z)$ is given by

$$F(z) = \widetilde{D} + z\widetilde{C}_+(I - z\widetilde{A}_+)^{-1}\widetilde{B}_+ + \widetilde{C}_-(I - z^{-1}\widetilde{A}_-)^{-1}\widetilde{B}_-. \tag{1.7}$$

We shall give an interpretation of (1.7) as the transfer function of a bicausal exponentially stable system in Section 3 below. In any case we can now pose the question for the rational case where all spaces $\mathcal{X}_\pm, \mathcal{U}, \mathcal{Y}$ in (1.6) are finite-dimensional: *characterize in terms of* M_+ *and* M_- *when it is the case that* $\|F\|_{\infty,\mathbb{T}} \le 1$ *(standard case) or* $\|F\|_{\infty,\mathbb{T}} < 1$ *(strict case).* To describe the result we need to introduce the *bicausal KYP-inequality* to be satisfied by a selfadjoint operator

$$H = \begin{bmatrix} H_- & H_0 \\ H_0^* & H_+ \end{bmatrix} \text{ on } \mathcal{X} = \begin{bmatrix} \mathcal{X}_- \\ \mathcal{X}_+ \end{bmatrix} \tag{1.8}$$

given by

$$\begin{aligned} &\begin{bmatrix} I & 0 & \widetilde{A}_-^*\widetilde{C}_-^* \\ 0 & \widetilde{A}_+^* & \widetilde{C}_+^* \\ 0 & \widetilde{B}_+^* & \widetilde{B}_-^*\widetilde{C}_-^* + \widetilde{D}^* \end{bmatrix} \begin{bmatrix} H_- & H_0 & 0 \\ H_0^* & H_+ & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & \widetilde{A}_+ & \widetilde{B}_+ \\ \widetilde{C}_-\widetilde{A}_- & \widetilde{C}_+ & \widetilde{C}_-\widetilde{B}_- + \widetilde{D} \end{bmatrix} \\ &\qquad \preceq \begin{bmatrix} \widetilde{A}_-^* & 0 & 0 \\ 0 & I & 0 \\ \widetilde{B}_-^* & 0 & I \end{bmatrix} \begin{bmatrix} H_- & H_0 & 0 \\ H_0^* & H_+ & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \widetilde{A}_- & 0 & \widetilde{B}_- \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \end{aligned} \tag{1.9}$$

as well as the *strict bicausal KYP-inequality*: for some $\epsilon > 0$ we have

$$\begin{aligned} &\begin{bmatrix} I & 0 & \widetilde{A}_-^*\widetilde{C}_-^* \\ 0 & \widetilde{A}_+^* & \widetilde{C}_+^* \\ 0 & \widetilde{B}_+^* & \widetilde{B}_-^*\widetilde{C}_-^* + \widehat{D}^* \end{bmatrix} \begin{bmatrix} H_- & H_0 & 0 \\ H_0^* & H_+ & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & \widetilde{A}_+ & \widetilde{B}_+ \\ \widetilde{C}_-\widetilde{A}_- & \widetilde{C}_- & \widetilde{C}_-\widetilde{B}_- + \widetilde{D} \end{bmatrix} + \epsilon^2 \begin{bmatrix} \widetilde{A}_-^*\widetilde{A}_- & 0 & \widetilde{A}_-^*\widetilde{B}_- \\ 0 & I & 0 \\ \widetilde{B}_-^*\widetilde{A}_- & 0 & \widetilde{B}_-^*\widetilde{B}_- + I \end{bmatrix} \\ &\qquad \preceq \begin{bmatrix} \widetilde{A}_-^* & 0 & 0 \\ 0 & I & 0 \\ \widetilde{B}_-^* & 0 & I \end{bmatrix} \begin{bmatrix} H_- & H_0 & 0 \\ H_0^* & H_+ & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \widetilde{A}_- & 0 & \widetilde{B}_- \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}. \end{aligned} \tag{1.10}$$

We say that the system matrix-pair (M_+, M_-) is *controllable* if both $(\widetilde{A}_+, \widetilde{B}_+)$ and $(\widetilde{A}_-, \widetilde{A}_-\widetilde{B}_-)$ are controllable, and that (M_+, M_-) is *observable* if both $(\widetilde{C}_+, \widetilde{A}_+)$ and $(\widetilde{C}_-\widetilde{A}_-, \widetilde{A}_-)$ are observable. We then have the following result.

Theorem 1.3. *Suppose that we are given* $\mathcal{X}_+$, $\mathcal{X}_-$, $\mathcal{U}$, $\mathcal{Y}$, M_+, M_- *as in* (1.6) *with* $\mathcal{X}_+$, $\mathcal{X}_-$, $\mathcal{U}$, $\mathcal{Y}$ *finite-dimensional Hilbert spaces and both* A_+ *and* $\widetilde{A}_-$ *having spectrum inside the unit disk* $\mathbb{D}$. *Further, suppose that* F *is the rational matrix function with no poles on the unit circle* $\mathbb{T}$ *given by* (1.7). *Then:*

(1) **Finite-dimensional standard bicausal Bounded Real Lemma:** *Assume that* (M_+, M_-) *is controllable and observable. Then we have* $\|F\|_{\infty,\mathbb{T}} \le 1$ *if and only there exists an invertible selfadjoint solution* H *as in* (1.8) *of the bicausal KYP-inequality* (1.9). *Moreover* $H_+ \succ 0$ *and* $H_- \prec 0$.

(2) **Finite-dimensional strict bicausal Bounded Real Lemma:** *The strict inequality* $\|F\|_{\infty,\mathbb{T}} < 1$ *holds if and only if there exists an invertible selfadjoint solution* H *as in* (1.8) *of the strict bicausal KYP-inequality* (1.10). *Moreover, in this case* $H_+ \succ 0$ *and* $H_- \prec 0$.

We have not located an explicit statement of these results in the literature; they also are corollaries of the infinite-dimensional results which we develop in this paper (Theorem 7.3 below).

The goal of this paper is to explore infinite-dimensional analogues of Theorems 1.2 and 1.3 (both standard and strict versions). For the case of trivial dichotomy (the stable case where $\sigma(A) \subset \mathbb{D}$ in Theorem 1.2), we have recently obtained such results via two distinct approaches: (i) the State-Space-Similarity theorem approach (see [6]), and (ii) the storage-function approach (see [7]) based on the work of Willems [21, 22]. Both approaches in general involve additional complications in the infinite-dimensional setting. In the first approach (i), one must deal with possibly unbounded pseudo-similarity rather than true similarity transformations, as explained in the penetrating paper of Arov-Kaashoek-Pik [3]; one of the contributions of [6] was to identify additional hypotheses (exact or ℓ^2-exact controllability and observability) which guarantee that the pseudo-similarities guaranteed by the Arov-Kaashoek-Pik theory can in fact be taken to be bounded and boundedly invertible. In the second approach (ii), no continuity properties of a storage function are guaranteed a priori and in general one must allow a storage function to take the value $+\infty$; nevertheless, as shown in [7], it is possible to show that the Willems available storage function S_a and a regularized version of the Willems required supply $\underline{S}_r$ (at least when suitably restricted) have a quadratic form coming from a possibly unbounded positive-definite operator (H_a and H_r respectively) which leads to a solution (in an adjusted generalized sense required for the formulation of what it should mean for an unbounded operator to be a solution) of the KYP-inequality. Again, if the system satisfies an exact or ℓ^2-exact controllability/observability hypothesis, then we get finite-valued quadratic storage functions and associated bounded and boundedly invertible solutions of the KYP-inequality.

It seems that the first approach (i) (involving the State-Space-Similarity theorem with pseudo-similarities) does not adapt well in the dichotomous setting, so we here focus on the second approach (ii) (computation of extremal storage functions). For the dichotomous setting, there again is a notion of storage function but now the storage functions S can take values on the whole real line rather than just positive values, and quadratic storage functions should have the form $S(x) = \langle Hx, x \rangle$ (at least for x restricted to some appropriate domain) with H (possibly unbounded) selfadjoint rather than just positive-definite. Due to the less than satisfactory connection between closed forms and closed operators for forms not defined on the whole space and not necessarily semi-bounded (see e.g. [20, 15]), it is difficult to make sense of quadratic storage functions in the infinite-dimensional setting unless the storage function is finite-valued and the associated self-adjoint operator is bounded. Therefore, for the dichotomous setting here we deal only with the case where ℓ^2-exact controllability/observability assumptions are imposed at the start, and we are able to consider only storage functions S which are finite real-valued with the associated selfadjoint operators in an quadratic representation equal to bounded operators. Consequently our results require either the strict inequality condition $\|F\|_{\infty,\mathbb{T}} < 1$ on the transfer function F, or an ℓ^2-exact or exact controllability/observability assumption on the operators in the system matrices. Consequently, unlike what is done in [6, 7] for the causal trivial-dichotomy setting, the present paper has nothing in the direction of a Bounded Real Lemma for a dichotomous or exponentially dichotomous system under only (approximate) controllability and observability assumptions for the case where $\|F\|_{\infty,\mathbb{T}} = 1$.

The paper is organized as follows. Apart from the current introduction, the paper consists of seven sections. In Sections 2 and 3 we introduce the dichotomous systems and bicausal systems, respectively, studied in this paper and derive various basic results used in the sequel. Next, in Section 4 we introduce the notion of a storage function for discrete-time dichotomous linear systems as well as the available storage S_a and required supply S_r storage functions in this context and show that they indeed are storage functions (pending the proof of a continuity condition which is obtained later from Theorem 5.2). In Section 5 we show, under certain conditions, that S_a and S_r are quadratic storage functions by explicit computation of the corresponding invertible selfadjoint operators H_a and H_r. The results of Sections 4 and 5 are extended to bicausal systems in Section 6. The main results of the present paper, i.e., the infinite-dimensional versions of Theorems 1.2 and 1.3, are proven in Section 7. In the final section, Section 8, we apply our Dichotomous Bounded Real Lemma to discrete-time, nonstationary, dichotomous l inear systems and recover a result of Ben-Artzi–Gohberg–Kaashoek [10].

2. Dichotomous system theory

We assume that we are given a system Σ as in (1.1) with system matrix $M = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ and associated transfer function F_Σ as in (1.3) with A having dichotomy. As a neighborhood of the unit circle $\mathbb{T}$ is in the resolvent set of A, by definition of A having a dichotomy, we see that $F_\Sigma(z)$ is analytic and uniformly bounded in z on a neighborhood of $\mathbb{T}$. One way to make this explicit is to decompose F_Σ in the form $F_\Sigma = F_{\Sigma,+} + F_{\Sigma,-}$ where $F_{\Sigma,+}(z)$ is analytic and uniformly bounded on a neighborhood of the closed unit disk $\overline{\mathbb{D}}$ and where $F_{\Sigma,-}(z)$ is analytic and uniformly bounded on a neighborhood of the closed exterior unit disk $\overline{\mathbb{D}_e}$ as follows.

The fact that A admits a dichotomy implies there is a direct (not necessarily orthogonal) decomposition of the state space $\mathcal{X} = \mathcal{X}_+ \dot{+} \mathcal{X}_-$ so that with respect to this decomposition A has a block diagonal matrix decomposition of the form

$$A = \begin{bmatrix} A_- & 0 \\ 0 & A_+ \end{bmatrix} : \begin{bmatrix} \mathcal{X}_- \\ \mathcal{X}_+ \end{bmatrix} \to \begin{bmatrix} \mathcal{X}_- \\ \mathcal{X}_+ \end{bmatrix} \tag{2.1}$$

where $A_+ := A|_{\mathcal{X}_+} \in \mathcal{L}(\mathcal{X}_+)$ has spectrum inside the unit disk $\mathbb{D}$ and $A_- := A|_{\mathcal{X}_-} \in \mathcal{L}(\mathcal{X}_-)$ has spectrum in the exterior of the closed unit disk $\mathbb{D}_e = \mathbb{C} \backslash \overline{\mathbb{D}}$. It follows that A_+ is *exponentially stable*, $r_{\rm spec}(A_+) < 1$, and A_- is invertible with inverse A_-^{-1} exponentially stable. Occasionally we will view A_+ and A_- as operators acting on $\mathcal{X}$ and, with some abuse of notation, write A_-^{-1} for what is really a generalized inverse of A_-:

$$A_-^{-1} \cong \begin{bmatrix} A_-^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{X}_- \\ \mathcal{X}_+ \end{bmatrix} \to \begin{bmatrix} \mathcal{X}_- \\ \mathcal{X}_+ \end{bmatrix}$$

i.e., the Moore-Penrose generalized inverse of A_- in case the decomposition $\mathcal{X}_- \dot{+} \mathcal{X}_+$ is orthogonal—the meaning will be clear from the context. Now decompose B and C accordingly:

$$B = \begin{bmatrix} B_- \\ B_+ \end{bmatrix} : \mathcal{U} \to \begin{bmatrix} \mathcal{X}_- \\ \mathcal{X}_+ \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} C_- & C_+ \end{bmatrix} : \begin{bmatrix} \mathcal{X}_- \\ \mathcal{X}_+ \end{bmatrix} \to \mathcal{Y}. \tag{2.2}$$

We may then write

$$
\begin{aligned}
F_\Sigma(z) &= D + zC(I - zA)^{-1}B \\
&= D + z\begin{bmatrix} C_- & C_+ \end{bmatrix} \begin{bmatrix} I - zA_- & 0 \\ 0 & I - zA_+ \end{bmatrix}^{-1} \begin{bmatrix} B_- \\ B_+ \end{bmatrix} \\
&= D + zC_-(I - zA_-)^{-1}B_- + zC_+(I - zA_+)^{-1}B_+ \\
&= -C_-A_-^{-1}(z^{-1}I - A_-^{-1})^{-1}B_- + D + zC_+(I - zA_+)^{-1}B_+ \\
&= F_{\Sigma,-}(z) + F_{\Sigma,+}(z)
\end{aligned}
$$

where

$$
F_{\Sigma,-}(z) = -C_-A_-^{-1}(I - z^{-1}A_-^{-1})^{-1}B_- = -\sum_{n=0}^{\infty} C_-(A_-^{-1})^{n+1}B_-z^{-n} \tag{2.3}
$$

is analytic on a neighborhood of $\overline{\mathbb{D}_e}$, with the series converging in operator norm on $\mathbb{D}_e$ due to the exponential stability of A_-^{-1}, and where

$$
F_{\Sigma,+}(z) = D + zC_+(I - zA_+)^{-1}B_+ = D + \sum_{n=1}^{\infty} C_+A_+^{n-1}B_+z^n \tag{2.4}
$$

is analytic on a neighborhood of $\overline{\mathbb{D}}$, with the series converging in operator norm on $\mathbb{D}$ due to the exponential stability of A_+. Furthermore, from the convergent-series expansions for $F_{\Sigma,+}$ in (2.4) and for $F_{\Sigma,-}$ in (2.3) we read off that F_Σ has the convergent Laurent expansion on the unit circle $\mathbb{T}$

$$
F_\Sigma(z) = \sum_{n=-\infty}^{\infty} F_n z^n
$$

with Laurent coefficients F_n given by

$$
F_n = \begin{cases} D - C_-A_-^{-1}B_- & \text{if } n = 0, \\ C_+A_+^{n-1}B_+ & \text{if } n > 0, \\ -C_-A_-^{n-1}B_- & \text{if } n < 0. \end{cases} \tag{2.5}
$$

As $\|F_\Sigma\|_{\infty,\mathbb{T}} := \sup\{\|F_\Sigma(z)\| \colon z \in \mathbb{T}\} < \infty$, it follows that F_Σ defines a bounded multiplication operator:

$$
M_{F_\Sigma} : L^2_{\mathcal{U}}(\mathbb{T}) \to L^2_{\mathcal{Y}}(\mathbb{T}), \quad M_{F_\Sigma} \colon f(z) \mapsto F_\Sigma(z)f(z)
$$

with $\|M_{F_\Sigma}\| = \|F_\Sigma\|_{\infty,\mathbb{T}}$. If we write this operator as a block matrix $M_{F_\Sigma} = [M_{F_\Sigma}]_{ij}$ $(-\infty < i, j < \infty)$ with respect to the orthogonal decompositions

$$
L^2_{\mathcal{U}}(\mathbb{T}) = \bigoplus_{n=-\infty}^{\infty} z^n\mathcal{U}, \quad L^2_{\mathcal{Y}}(\mathbb{T}) = \bigoplus_{n=-\infty}^{\infty} z^n\mathcal{Y}
$$

for the input and output spaces for M_{F_Σ}, it is a standard calculation to verify that $[M_{F_\Sigma}]_{ij} = F_{i-j}$, i.e., the resulting bi-infinite matrix $[M_{F_\Sigma}]_{ij}$ is the *Laurent matrix* $\mathfrak{L}_{F_\Sigma}$ associated with F_Σ given by

$$
\mathfrak{L}_{F_\Sigma} = [F_{i-j}]_{i,j=-\infty}^{\infty} \tag{2.6}
$$

where F_n is as in (2.5). Another expression of this identity is the fact that $M_{F_\Sigma} \colon L^2_{\mathcal{U}}(\mathbb{T}) \to L^2_{\mathcal{Y}}(\mathbb{T})$ is just the frequency-domain expression of the time-domain operator $\mathfrak{L}_{F_\Sigma} \colon \ell^2_{\mathcal{U}}(\mathbb{Z}) \to \ell^2_{\mathcal{Y}}(\mathbb{Z})$, i.e., if we let $\widehat{\mathbf{u}}(z) = \sum_{n=-\infty}^{\infty} \mathbf{u}(n)z^n$ in $L^2_{\mathcal{U}}(\mathbb{T})$ be the bilateral Z-transform of $\mathbf{u}$ in $\ell^2_{\mathcal{U}}(\mathbb{Z})$ and similarly let $\widehat{\mathbf{y}}(z) = \sum_{n=-\infty}^{\infty} \mathbf{y}(n) z^n$ in $L^2_{\mathcal{Y}}(\mathbb{T})$ be the bilateral Z-transform of $\mathbf{y}$ in $\ell^2_{\mathcal{Y}}(\mathbb{Z})$, then we have the relationship

$$\mathbf{y} = \mathfrak{L}_{F_\Sigma} \mathbf{u} \quad \Longleftrightarrow \quad \widehat{\mathbf{y}}(z) = F_\Sigma(z) \cdot \widehat{\mathbf{u}}(z) \text{ for almost all } z \in \mathbb{T}. \tag{2.7}$$

We now return to analyzing the system-theoretic properties of the dichotomous system (1.1). Associated with the system operators A, B, C, D are the diagonal operators $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ acting between the appropriate ℓ^2-spaces indexed by $\mathbb{Z}$:

$$\begin{aligned} \mathcal{A} &= \operatorname{diag}_{k\in\mathbb{Z}}[A] \colon \ell^2_{\mathcal{X}}(\mathbb{Z}) \to \ell^2_{\mathcal{X}}(\mathbb{Z}), \quad \mathcal{B} = \operatorname{diag}_{k\in\mathbb{Z}}[B] \colon \ell^2_{\mathcal{U}}(\mathbb{Z}) \to \ell^2_{\mathcal{X}}(\mathbb{Z}), \\ \mathcal{C} &= \operatorname{diag}_{k\in\mathbb{Z}}[C] \colon \ell^2_{\mathcal{X}}(\mathbb{Z}) \to \ell^2_{\mathcal{Y}}(\mathbb{Z}), \quad \mathcal{D} = \operatorname{diag}_{k\in\mathbb{Z}}[D] \colon \ell^2_{\mathcal{Y}}(\mathbb{Z}) \to \ell^2_{\mathcal{U}}(\mathbb{Z}). \end{aligned} \tag{2.8}$$

We also introduce the bilateral shift operator

$$\mathcal{S} : \ell^2_{\mathcal{X}}(\mathbb{Z}) \to \ell^2_{\mathcal{X}}(\mathbb{Z}), \quad \mathcal{S} \colon \{\mathbf{x}(k)\}_{k\in\mathbb{Z}} \mapsto \{\mathbf{x}(k-1)\}_{k\in\mathbb{Z}}$$

and its inverse

$$\mathcal{S}^{-1} = \mathcal{S}^* \colon \{\mathbf{x}(k)\}_{k\in\mathbb{Z}} \mapsto \{\mathbf{x}(k+1)\}_{k\in\mathbb{Z}}.$$

We can then rewrite the system equations (1.1) in aggregate form

$$\Sigma := \left\{ \begin{array}{ccc} \mathcal{S}^{-1}\mathbf{x} & = & \mathcal{A}\mathbf{x} + \mathcal{B}\mathbf{u}, \\ \mathbf{y} & = & \mathcal{C}\mathbf{x} + \mathcal{D}\mathbf{u}, \end{array} \right. \tag{2.9}$$

We shall say that a system trajectory $(\mathbf{u}, \mathbf{x}, \mathbf{y}) = \{(\mathbf{u}(n), \mathbf{x}(n), \mathbf{y}(n)\}_{n\in\mathbb{Z}}$ is ℓ^2-*admissible* if all of $\mathbf{u}$, $\mathbf{x}$, and $\mathbf{y}$ are in ℓ^2: $\mathbf{u} = \{\mathbf{u}(n)\}_{n\in\mathbb{Z}} \in \ell^2_{\mathcal{U}}(\mathbb{Z})$, $\mathbf{x} = \{\mathbf{x}(n)\}_{n\in\mathbb{Z}} \in \ell^2_{\mathcal{X}}(\mathbb{Z})$, $\mathbf{y} = \{\mathbf{y}(n)\}_{n\in\mathbb{Z}} \in \ell^2_{\mathcal{Y}}(\mathbb{Z})$. Note that the constant-diagonal structure of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ implies that each of these operators intertwines the bilateral shift operator on the appropriate $\ell^2(\mathbb{Z})$-space:

$$\mathcal{A}\mathcal{S} = \mathcal{S}\mathcal{A}, \quad \mathcal{B}\mathcal{S} = \mathcal{S}\mathcal{B}, \quad \mathcal{C}\mathcal{S} = \mathcal{S}\mathcal{C}, \quad \mathcal{D}\mathcal{S} = \mathcal{S}\mathcal{D} \tag{2.10}$$

where $\mathcal{S}$ is the bilateral shift operator on $\ell^2_{\mathcal{W}}$ with $\mathcal{W}$ is any one of $\mathcal{U}$, $\mathcal{X}$, $\mathcal{Y}$ depending on the context.

It is well known (see e.g. [9, Theorem 2]) that the operator A admitting a dichotomy is equivalent to $\mathcal{S}^{-1} - \mathcal{A}$ being invertible as an operator on $\ell^2_{\mathcal{X}}(\mathbb{Z})$. Hence the dichotomy hypothesis enables us to solve uniquely for $\mathbf{x} \in \ell^2_{\mathcal{X}}(\mathbb{Z})$ and $\mathbf{y} \in \ell^2_{\mathcal{Y}}(\mathbb{Z})$ for any given $\mathbf{u} \in \ell^2_{\mathcal{U}}(\mathbb{Z})$:

$$\begin{aligned} \mathbf{x} &= (\mathcal{S}^{-1} - \mathcal{A})^{-1}\mathcal{B}\mathbf{u} = (I - \mathcal{S}\mathcal{A})^{-1}\mathcal{S}\mathcal{B}\mathbf{u} =: T_{\Sigma,\text{is}}\mathbf{u}, \\ \mathbf{y} &= (\mathcal{D} + \mathcal{C}(\mathcal{S}^{-1} - \mathcal{A})^{-1}\mathcal{B})\mathbf{u} = (\mathcal{D} + \mathcal{C}(I - \mathcal{S}\mathcal{A})^{-1}\mathcal{S}\mathcal{B})\mathbf{u} =: T_\Sigma \mathbf{u}. \end{aligned} \tag{2.11}$$

where

$$T_{\Sigma,\text{is}} = (\mathcal{S}^{-1} - \mathcal{A})^{-1}\mathcal{B} : \ell^2_{\mathcal{U}}(\mathbb{Z}) \to \ell^2_{\mathcal{X}}(\mathbb{Z}) \tag{2.12}$$

$$T_\Sigma = \mathcal{D} + \mathcal{C}(\mathcal{S}^{-1} - \mathcal{A})^{-1}\mathcal{B} : \ell^2_{\mathcal{U}}(\mathbb{Z}) \to \ell^2_{\mathcal{Y}}(\mathbb{Z}) \tag{2.13}$$

are the respective *input-state* and *input-output* maps. In general the input-output map T_Σ in (2.13) is not causal. Given an $\ell^2_{\mathcal{U}}(\mathbb{Z})$-input signal $\mathbf{u}$, rather than specification of an initialization condition on the state $\mathbf{x}(0)$, as in standard linear system theory for systems running on $\mathbb{Z}_+$, in order to specify a uniquely determined state trajectory $\mathbf{x}$ for a given input trajectory $\mathbf{u}$, the extra information required to solve uniquely for the state trajectory $\mathbf{x}$ in the dichotomous system (1.1) or (2.9) is the specification that $\mathbf{x} \in \ell^2_{\mathcal{X}}(\mathbb{Z})$, i.e., that the resulting trajectory $(\mathbf{u}, \mathbf{x}, \mathbf{y})$ be ℓ^2-admissible.

Next we express various operators explicitly in terms of $A_\pm$, $B_\pm$, $C_\pm$ and D. The following lemma provides the basis for the formulas derived in the remainder of the section. In fact this lemma amounts to the easy direction of the result of Ben-Artzi–Gohberg–Kaashoek [9, Theorem 2] mentioned above.

Lemma 2.1. *Let Σ be the dichotomous system* (1.1) *with A decomposing as in* (2.1)*. Then $(\mathcal{S}^{-1}-\mathcal{A})^{-1}=(I-\mathcal{S}\mathcal{A})^{-1}\mathcal{S}$ acting on $\ell^2_{\mathcal{X}}(\mathbb{Z})$ is given explicitly as the following block matrix with rows and columns indexed by $\mathbb{Z}$*

$$[(\mathcal{S}^{-1}-\mathcal{A})^{-1}]_{ij}=\begin{cases} A_+^{i-j-1} & \text{for } i>j,\\ -A_-^{i-j-1} & \text{for } i\leq j\end{cases}, \quad \text{with } A_+^0=P_{\mathcal{X}_+}. \tag{2.14}$$

Proof. Via the decomposition $\mathcal{X}=\mathcal{X}_-\dot{+}\mathcal{X}_+$, we can identify $\ell^2_{\mathcal{X}}(\mathbb{Z})$ with $\ell^2_{\mathcal{X}_-}(\mathbb{Z})\dot{+}\ell^2_{\mathcal{X}_+}(\mathbb{Z})$. Write $\mathcal{S}_+$ for the bilateral shift operator and $\mathcal{A}_+$ for the block diagonal operator with A_+ diagonal entries, both acting on $\ell^2_{\mathcal{X}_+}(\mathbb{Z})$, and write $\mathcal{S}_-$ for the bilateral shift operator and $\mathcal{A}_-$ for the block diagonal operator with A_- diagonal entries, both acting on $\ell^2_{\mathcal{X}_-}(\mathbb{Z})$. Then with respect to the above decomposition of $\ell^2_{\mathcal{X}}(\mathbb{Z})$ we have

$$\begin{aligned}(I-\mathcal{S}\mathcal{A})^{-1}&=\begin{bmatrix} I-\mathcal{S}_-\mathcal{A}_- & 0\\ 0 & I-\mathcal{S}_+\mathcal{A}_+\end{bmatrix}^{-1}\\ &=\begin{bmatrix} (I-\mathcal{S}_-\mathcal{A}_-)^{-1} & 0\\ 0 & (I-\mathcal{S}_+\mathcal{A}_+)^{-1}\end{bmatrix}.\end{aligned}$$

Since A_+ has its spectrum in $\mathbb{D}$, so do $\mathcal{A}_+$ and $\mathcal{S}_+\mathcal{A}_+$, and thus

$$(I-\mathcal{S}_+\mathcal{A}_+)^{-1}=\sum_{k=0}^{\infty}(\mathcal{S}_+\mathcal{A}_+)^k=\sum_{k=0}^{\infty}\mathcal{S}_+^k\mathcal{A}_+^k$$

where we make use of observation (2.10) to arrive at the final infinite-series expression. Similarly, A_-^{-1} has spectrum in $\mathbb{D}$ implies that $\mathcal{A}_-^{-1}$ and $\mathcal{A}_-^{-1}\mathcal{S}_-^{-1}$ have spectrum in $\mathbb{D}$, and hence

$$\begin{aligned}(I-\mathcal{S}_-\mathcal{A}_-)^{-1}&=(\mathcal{S}_-\mathcal{A}_-)^{-1}((\mathcal{S}_-\mathcal{A}_-)^{-1}-I)^{-1}\\ &=-\mathcal{A}_-^{-1}\mathcal{S}_-^{-1}(I-(\mathcal{S}_-\mathcal{A}_-)^{-1})^{-1}=-\mathcal{A}_-^{-1}\mathcal{S}_-^{-1}\sum_{k=0}^{\infty}(\mathcal{S}_-\mathcal{A}_-)^{-k}\\ &=-\mathcal{A}_-^{-1}\mathcal{S}_-^{-1}\sum_{k=0}^{\infty}\mathcal{A}_-^{-k}\mathcal{S}_-^{-k}=-\sum_{k=1}^{\infty}\mathcal{A}_-^{-k}\mathcal{S}_-^{-k}.\end{aligned}$$

Inserting the formulas for $(I-\mathcal{S}_+\mathcal{A}_+)^{-1}$ and $(I-\mathcal{S}_-\mathcal{A}_-)^{-1}$ in the formula for $(I-\mathcal{S}\mathcal{A})^{-1}$, multiplying with $\mathcal{S}$ from the left and writing out in block matrix form we obtain the desired formula for $(I-\mathcal{S}\mathcal{A})^{-1}\mathcal{S}=(\mathcal{S}^{-1}-A)^{-1}$. □

We now compute the input-output map T_Σ and input-state map $T_{\Sigma,\,\mathrm{is}}$ explicitly.

Proposition 2.2. *Let Σ be the dichotomous system* (1.1) *with A decomposing as in* (2.1) *and B and C as in* (2.2)*. The input-output map $T_\Sigma:\ell^2_{\mathcal{U}}(\mathbb{Z})\to\ell^2_{\mathcal{Y}}(\mathbb{Z})$ and input-state map $T_{\Sigma,\,\mathrm{is}}:\ell^2_{\mathcal{U}}(\mathbb{Z})\to\ell^2_{\mathcal{X}}(\mathbb{Z})$ of Σ are then given by the following block matrix, with row and columns indexed over $\mathbb{Z}$:*

$$[T_\Sigma]_{ij}=\begin{cases} C_+A_+^{i-j-1}B_+ & \text{if } i>j,\\ D-C_-A_-^{-1}B_- & \text{if } i=j,\\ -C_-A_-^{i-j-1}B_- & \text{if } i<j,\end{cases}\quad [T_{\Sigma,\,\mathrm{is}}]_{ij}=\begin{cases} A_+^{i-j-1}B_+ & \text{if } i>j,\\ -A_-^{i-j-1}B_- & \text{if } i\leq j.\end{cases}$$

In particular, T_Σ is equal to the Laurent operator $\mathfrak{L}_{F_\Sigma}$ of the transfer function F_Σ given in (2.6)*, and for $\mathbf{u} \in \ell^2_{\mathcal{U}}(\mathbb{Z})$ and $\mathbf{y} \in \ell^2_{\mathcal{Y}}(\mathbb{Z})$ with bilateral Z-transform notation as in* (2.7)*,*

$$\mathbf{y} = T_\Sigma \mathbf{u} \iff \widehat{\mathbf{y}}(z) = F_\Sigma(z) \cdot \widehat{\mathbf{u}}(z) \text{ for almost all } z \in \mathbb{T}. \tag{2.15}$$

Proof. Recall that T_Σ can be written as $T_\Sigma = \mathcal{D} + \mathcal{C}(I - \mathcal{S}\mathcal{A})^{-1}\mathcal{S}\mathcal{B} = \mathcal{D} + \mathcal{C}(\mathcal{S}^{-1} - \mathcal{A})^{-1}\mathcal{B}$. The block matrix formula for T_Σ now follows directly from the block matrix formula for $(\mathcal{S}^{-1} - \mathcal{A})^{-1}$ obtained in Lemma 2.1. Comparison of the formula for $[T_\Sigma]_{ij}$ with the formula (2.5) for the Laurent coefficients $\{F_n\}_{n\in\mathbb{Z}}$ of F_Σ shows that $T_\Sigma = \mathfrak{L}_{F_\Sigma}$ as operators from $L^2_{\mathcal{U}}(\mathbb{T})$ to $L^2_{\mathcal{Y}}(\mathbb{T})$. Finally the identity (2.15) follows upon combining the identity $\mathfrak{L}_{F_\Sigma} = T_\Sigma$ with the general identity (2.7). □

It is convenient to also view $T_\Sigma = \mathfrak{L}_{F_\Sigma}$ as a block 2×2 matrix with respect to the decomposition $\ell^2_{\mathcal{U}}(\mathbb{Z}) = \ell^2_{\mathcal{U}}(\mathbb{Z}_-) \oplus \ell^2_{\mathcal{U}}(\mathbb{Z}_+)$ for the input-signal space and $\ell^2_{\mathcal{Y}}(\mathbb{Z}) = \ell^2_{\mathcal{Y}}(\mathbb{Z}_-) \oplus \ell^2_{\mathcal{Y}}(\mathbb{Z}_+)$ for the output-signal space. We can then write

$$\mathfrak{L}_{F_\Sigma} = \begin{bmatrix} \widetilde{\mathfrak{T}}_{F_\Sigma} & \widetilde{\mathfrak{H}}_{F_\Sigma} \\ \mathfrak{H}_{F_\Sigma} & \mathfrak{T}_{F_\Sigma} \end{bmatrix} : \begin{bmatrix} \ell^2_{\mathcal{U}}(\mathbb{Z}_-) \\ \ell^2_{\mathcal{U}}(\mathbb{Z}_+) \end{bmatrix} \to \begin{bmatrix} \ell^2_{\mathcal{Y}}(\mathbb{Z}_-) \\ \ell^2_{\mathcal{Y}}(\mathbb{Z}_+) \end{bmatrix} \tag{2.16}$$

where

$$\begin{aligned} [\widetilde{\mathfrak{T}}_{F_\Sigma}]_{ij\,:\,i<0, j<0} &= \begin{cases} C_+ A_+^{i-j-1} B_+ & \text{for } 0 > i > j, \\ D - C_- A_-^{-1} B_- & \text{for } i = j < 0, \\ -C_- A_-^{i-j-1} B_- & \text{for } i < j < 0, \end{cases} \\ [\mathfrak{T}_{F_\Sigma}]_{ij\,:\,i\geq 0, j\geq 0} &= \begin{cases} C_+ A_+^{i-j-1} B_+ & \text{for } i > j \geq 0, \\ D - C_- A_-^{-1} B_- & \text{for } i = j \geq 0, \\ -C_- A_-^{i-j-1} B_- & \text{for } 0 \leq i < j \end{cases} \end{aligned} \tag{2.17}$$

are noncausal Toeplitz operators, and

$$\begin{aligned} [\widetilde{\mathfrak{H}}_{F_\Sigma}]_{ij\,:\,i<0, j\geq 0} &= -C_- A_-^{i-j-1} B_- \text{ for } i < 0,\, j \geq 0, \\ [\mathfrak{H}_{F_\Sigma}]_{ij\,:\,i\geq 0, j<0} &= C_+ A_+^{i-j-1} B_+ \text{ for } i \geq 0,\, j < 0 \end{aligned} \tag{2.18}$$

are Hankel operators.

Next we consider the observability and controllability operators of Σ. For any integer n, let $\Pi_n : \ell^2_{\mathcal{X}}(\mathbb{Z}) \to \mathcal{X}$ be the projection onto the n^{th} component of $\ell^2_{\mathcal{X}}(\mathbb{Z})$. We then define the controllability operator $\mathbf{W}_c$ and observability operator $\mathbf{W}_o$ associated with the system Σ as

$$\begin{aligned} &\mathbf{W}_c : \ell^2_{\mathcal{U}}(\mathbb{Z}) \to \mathcal{X}, \quad \mathbf{W}_c \mathbf{u} = \Pi_0 \mathbf{x} = \Pi_0 T_{\Sigma,\,\text{is}} \mathbf{u} = \Pi_0 (\mathcal{S}^{-1} - \mathcal{A})^{-1} \mathcal{B} \mathbf{u}, \\ &\mathbf{W}_o : \mathcal{X} \to \ell^2_{\mathcal{Y}}(\mathbb{Z}), \quad \mathbf{W}_o x = \mathcal{C}(I - \mathcal{S}\mathcal{A})^{-1} \Pi_0^* x. \end{aligned}$$

Lemma 2.3. *Let Σ be the dichotomous system* (1.1) *with A decomposing as in* (2.1) *and B and C as in* (2.2)*. Let the observability operator $\mathbf{W}_o$ and controllability operator $\mathbf{W}_c$ decompose as*

$$\begin{aligned} \mathbf{W}_c &= \begin{bmatrix} \mathbf{W}_c^+ & \mathbf{W}_c^- \end{bmatrix} : \ell^2_{\mathcal{U}}(\mathbb{Z}) = \begin{bmatrix} \ell^2_{\mathcal{U}}(\mathbb{Z}_-) \\ \ell^2_{\mathcal{U}}(\mathbb{Z}_+) \end{bmatrix} \to \mathcal{X}, \\ \mathbf{W}_o &= \begin{bmatrix} \mathbf{W}_o^- \\ \mathbf{W}_o^+ \end{bmatrix} : \mathcal{X} \to \begin{bmatrix} \ell^2_{\mathcal{Y}}(\mathbb{Z}_-) \\ \ell^2_{\mathcal{Y}}(\mathbb{Z}_+) \end{bmatrix}. \end{aligned}$$

Then $\mathbf{W}_c^+$ *and* $\mathbf{W}_c^-$ *are given by*

$$\mathbf{W}_c^+ = \mathrm{row}_{j<0}[A_+^{-j-1}B_+], \qquad \mathbf{W}_c^- = \mathrm{row}_{j\geq 0}[-A_-^{-j-1}B_-], \tag{2.19}$$

and $\mathbf{W}_c^+$ *maps into* $\mathcal{X}_+$ *and* $\mathbf{W}_c^-$ *into* $\mathcal{X}_-$. *Furthermore,* $\mathbf{W}_o^+$ *and* $\mathbf{W}_o^-$ *are given by*

$$\mathbf{W}_o^+ = \mathrm{col}_{i:\, i\geq 0}[C_+A_+^i], \qquad \mathbf{W}_o^- = \mathrm{col}_{i:\, i<0}[-C_-A_-^i], \tag{2.20}$$

and $(\mathbf{W}_o^+)^*$ *maps into* $\mathcal{X}_+$ *and* $(\mathbf{W}_o^-)^*$ *maps into* $\mathcal{X}_-$. *Finally, the Hankel operators* $\widetilde{\mathfrak{H}}_{F_-}$ *and* $\mathfrak{H}_{F_+}$ *in* (2.18) *have the following factorizations:*

$$\widetilde{\mathfrak{H}}_{F_\Sigma} = \mathbf{W}_o^-\mathbf{W}_c^-, \quad \mathfrak{H}_{F_\Sigma} = \mathbf{W}_o^+\mathbf{W}_c^+. \tag{2.21}$$

Proof. The formulas in (2.19) follow directly by restricting the matrix representation of $T_{\Sigma,\,\mathrm{is}}$ obtained in Proposition 2.2 to the zero-indexed row. Since $A_\pm$ and $B_\pm$ map into $\mathcal{X}_\pm$, it follows directly that $\mathbf{W}_c^\pm$ maps into $\mathcal{X}_\pm$. The analogous statements for $\mathbf{W}_o$ follow by similar arguments, now using (2.14) to compute the zero-indexed column of $\mathcal{C}(I-\mathcal{S}\mathcal{A})^{-1}$ explicitly. The factorization formulas for $\widetilde{\mathfrak{H}}_{F_\Sigma}$ and $\mathfrak{H}_{F_\Sigma}$ follow directly from an inspection of the entries in the block matrix decompositions. □

Remark 2.4. Let $(\mathbf{u},\mathbf{x},\mathbf{y})$ be an ℓ^2-admissible trajectory for the system Σ. Then $\mathbf{x}(0) = \Pi_0\mathbf{x} = \mathbf{W}_c\mathbf{u} \in \mathrm{Im}\,\mathbf{W}_o$. In fact, by shift invariance of the system Σ we have $\mathbf{x}(n) \in \mathrm{Im}\,\mathbf{W}_c$ for each $n \in \mathbb{Z}$, because

$$\begin{aligned}\mathbf{x}(n) &= \Pi_n\mathbf{x} = \Pi_0\mathcal{S}^{-n}\mathbf{x} = \Pi_0\mathcal{S}^{-n}(I-\mathcal{S}\mathcal{A})^{-1}\mathcal{S}\mathcal{B}\mathbf{u}\\ &= \Pi_0(I-\mathcal{S}\mathcal{A})^{-1}\mathcal{S}\mathcal{B}\mathcal{S}^{-n}\mathbf{u} = \mathbf{W}_c\mathcal{S}^{-n}\mathbf{u},\end{aligned}$$

which holds since $\mathcal{S}^{-n}$ commutes with $\mathcal{S}$, $\mathcal{A}$ and $\mathcal{B}$.

Another topic playing a prominent role in the theory of causal linear systems (see [12]) is that of controllability and observability. For a causal system Σ of the form (1.1) we say that Σ (or the input pair (A,B)) is *controllable* if $\overline{\mathrm{span}}_{k\geq 0}\mathrm{Im}\,A^kB = \mathcal{X}$, which in case $\mathbf{W}_c$ is bounded is equivalent to $\mathbf{W}_c$ having dense range, while Σ (or the output pair (C,A)) is said to be *observable* if $\bigcap_{k\geq 0}\ker CA^k = \{0\}$, which in turn is equivalent to $\mathrm{Ker}\,\mathbf{W}_o = \{0\}$ in case $\mathbf{W}_o$ is bounded.

Note that if Σ in (1.1) is a system having a dichotomy with associated decomposition (2.1) and (2.2), then the assumed exponential stability of the operators A_+ and A_-^{-1} implies that the associated controllability operators $\mathbf{W}_c^-\colon \ell^2_{\mathcal{U}}(\mathbb{Z}_+) \to \mathcal{X}_-$ and $\mathbf{W}_c^+\colon \ell^2_{\mathcal{U}}(\mathbb{Z}_-) \to \mathcal{X}_+$ as well as the observability operators $\mathbf{W}_o^+\colon \mathcal{X}_+ \to \ell^2_{\mathcal{Y}}(\mathbb{Z}_+)$ and $\mathbf{W}_o^-\colon \mathcal{X}_- \to \ell^2_{\mathcal{Y}}(\mathbb{Z}_-)$ are all bounded. We then say that Σ (or the pair (A,B)) is *controllable* if $\mathbf{W}_c$ has dense range and that Σ (or the pair (C,A)) is *observable* if $\mathrm{Ker}\,\mathbf{W}_o = \{0\}$. With (A,B,C) decomposed as in (2.1) and (2.2), we see from Lemma 2.3 that controllability of Σ is equivalent to

$$(A_+,B_+) \text{ controllable} \quad \text{and} \quad (A_-^{-1}, A_-^{-1}B_-) \text{ controllable}, \tag{2.22}$$

hence to $\overline{\mathrm{span}}_{k\geq 0}\mathrm{Im}\,A_+^kB_+ = \mathcal{X}_+$ and $\overline{\mathrm{span}}_{k\geq 1}\mathrm{Im}\,A_-^{-k}B_- = \mathcal{X}_-$, while Σ being controllable is equivalent to

$$(C_+,A_+) \text{ observable} \quad \text{and} \quad (C_-A_-^{-1}, A_-^{-1}) \text{ observable}, \tag{2.23}$$

hence to $\bigcap_{k\geq 0}\ker C_+A_+^k = \{0\}$ and $\bigcap_{k\geq 1}\ker C_-A_-^{-k} = \{0\}$.

We shall have need for stronger controllability/observability notions for a dichotomous system defined as follows. We shall say that Σ (or (A, B)) is *dichotomously ℓ^2-exactly controllable* if

$$\operatorname{Im} \mathbf{W}_c^+ = \mathcal{X}_+ \quad \text{and} \quad \operatorname{Im} \mathbf{W}_c^- = \mathcal{X}_-, \text{ or equivalently } \quad \operatorname{Im} \mathbf{W}_c = \mathcal{X}. \tag{2.24}$$

Similarly, we say that Σ (or (C, A)) is *dichotomously ℓ^2-exactly observable* if

$$\operatorname{Im} (\mathbf{W}_o^+)^* = \mathcal{X}_+ \text{ and } \operatorname{Im} (\mathbf{W}_o^-)^* = \mathcal{X}_-, \text{ or equivalently } \operatorname{Im} \mathbf{W}_o^* = \mathcal{X}. \tag{2.25}$$

In case Σ is both dichotomously ℓ^2-exactly controllable and dichotomously ℓ^2-exactly observable, we shall say simply that Σ is *dichotomously ℓ^2-exactly minimal.* We note that these notions for the stable (non-dichotomous) case played a key role in the results of [6, 7].

Remark 2.5. In that case that $\mathcal{X}$ is finite dimensional, the notion of controllability (respectively, observability) for dichotomous systems introduced here coincides with the more standard notion, namely, that $\overline{\operatorname{span}}_{k\geq 0} \operatorname{Im} A^k B = \mathcal{X}$(respectively, $\bigcap_{k\geq 0} \ker CA^k = \{0\}$). Indeed, to see that this is the case, note that it suffices to show that $(A_-^{-1}, A_-^{-1}B_-)$ being a controllable pair is equivalent to (A_-, B_-) being a controllable pair. Since the two statements are symmetric, it suffices to prove only one direction. Hence, assume the pair (A_-, B_-) is controllable. Since $\mathcal{X}$ is finite dimensional, this implies there is a positive integer n such that

$$\begin{aligned} \mathcal{X} &= \operatorname{Im} \begin{bmatrix} B_- & A_-B_- & \cdots & A_-^{n-1}B_- \end{bmatrix} \\ &= \operatorname{Im} A_-^n \begin{bmatrix} A_-^{-n}B_- & A_-^{-n+1}B_- & \cdots & A_-^{-1}B_- \end{bmatrix} \\ &= A_-^n \operatorname{Im} \begin{bmatrix} A_-^{-1}B_- & \cdots & A_-^{-n+1}B_- & A_-^{-n}B_- \end{bmatrix}. \end{aligned}$$

Thus $\mathcal{X} = \operatorname{Im} \begin{bmatrix} A_-^{-1}B_- & \cdots & A_-^{-n+1}B_- & A_-^{-n}B_- \end{bmatrix}$, and we obtain that $(A_-^{-1}, A_-^{-1}B_-)$ is a controllable pair. For the notions of observability the claim follows by a duality argument.

If $\mathcal{X}$ is infinite-dimensional, it is not clear whether the two notions coincide. Let us discuss here only the situation for controllability as that for observability is similar. Let $A \in \mathcal{L}(\mathcal{X})$ and $B \in \mathcal{L}(\mathcal{X}, \mathcal{U})$ where now both $\mathcal{X}$ and $\mathcal{U}$ are allowed to be infinite-dimensional Hilbert spaces. If (A, B) is a controllable pair, then, by definition, for a given $x \in \mathcal{X}$ and $\epsilon > 0$, there is an $N = N(x, \epsilon) \in \mathbb{N}$ and vectors $u_0, u_1, \dots, u_N \in \mathcal{U}$ so that $\| \sum_{k=0}^N A^k B u_k - x \| < \epsilon$. Similarly, given $x \in \mathcal{X}$, $N \in \mathbb{N}$ and $\epsilon > 0$, there is a $\widetilde{N} = \widetilde{N}(x, N, \epsilon) \in \mathbb{N}$ so that there exist vectors $u_0', u_1', \dots, u_{\widetilde{N}}' \in \mathcal{U}$ so that $\| \sum_{k=0}^{\widetilde{N}} A_k B u_k' - A^{N+1} x \| < \epsilon / \| (A^{-1})^{N+1} \|$. Let us say that the pair (A, B) is *uniformly controllable* if it is possible to take $\widetilde{N}(x, N(x, \epsilon), \epsilon) = N(x, \epsilon)$, i.e., if: *given $x \in \mathcal{X}$ and $\epsilon > 0$ there is an $N = N_{x,\epsilon} \in \mathbb{N}$ so that there is a choice of $u_0, u_1, \dots, u_N \in \mathcal{U}$ so that*

$$\left\| \sum_{k=0}^N A^k B u_k - A^{N+1} x \right\| < \frac{\epsilon}{\| (A^{-1})^{N+1} \|}.$$

Note that the notions of *uniform controllability* and *controllability* are equivalent in the finite-dimensional case—take $N_{x,\epsilon} = \dim \mathcal{X}$ and then use the Cayley-Hamilton theorem to approximate $A^{N+1}x$ exactly by a vector of the form $\sum_{k=0}^N A^k B u_k$ ($N = \dim \mathcal{X}$). In the infinite-dimensional case arguably the condition appears to be somewhat contrived and is difficult to check; nevertheless it is what is needed for the following result.

Proposition. *Assume that A is invertible and that the input pair (A, B) is uniformly controllable. Then $(A^{-1}, A^{-1}B)$ is controllable.*

Proof. Let $x \in \mathcal{X}$ and $\epsilon > 0$. Let $N = N_{x,\epsilon}$ as in the uniformly-controllable condition: thus there exist vectors $u_0, u_1, \dots, u_N \in \mathcal{U}$ so that

$$\left\| \sum_{k=0}^{N} A^k B u_k - A^{N+1} x \right\| < \frac{\epsilon}{\|(A^{-1})^{N+1}\|}.$$

Rewrite this as

$$\left\| A^{N+1} \left(\sum_{k=0}^{N} (A^{-1})^k A^{-1} B u_{N-k} - x \right) \right\| < \frac{\epsilon}{\|(A^{-1})^{N+1}\|}$$

from which we get

$$\begin{aligned} &\left\| \sum_{k=0}^{N} (A^{-1})^k A^{-1} B u_{N-k} - x \right\| \\ &\qquad = \left\| (A^{-1})^{N+1} \cdot A^{N+1} \left(\sum_{k=0}^{N} (A^{-1})^k A^{-1} B u_{N-k} - x \right) \right\| \\ &\qquad < \|(A^{-1})^{N+1}\| \cdot \frac{\epsilon}{\|(A^{-1})^{N+1}\|} = \epsilon. \end{aligned}$$

As $x \in \mathcal{X}$ and $\epsilon > 0$ are arbitrary, we conclude that $(A^{-1}, A^{-1}B)$ is controllable. □

The following ℓ^2-admissible-trajectory interpolation result will be useful in the sequel.

Proposition 2.6. *Suppose that Σ is a dichotomous linear system as in* (1.1), (2.1), (2.2), *and that we are given a vector $u \in \mathcal{U}$ and $x \in \mathcal{X}$. Assume that Σ is dichotomously ℓ^2-exactly controllable. Then there exists an ℓ^2-admissible system trajectory $(\mathbf{u}, \mathbf{x}, \mathbf{y})$ for Σ such that*

$$\mathbf{u}(0) = u, \quad \mathbf{x}(0) = x.$$

Proof. As Σ is ℓ^2-exactly controllable, we know that $\mathbf{W}_c^-$ and $\mathbf{W}_c^+$ are surjective. Write $x = x_+ + x_-$ with $x_\pm \in \mathcal{X}_\pm$. $\mathbf{u}_- \in \ell^2_{\mathcal{U}}(\mathbb{Z}_-)$ so that $\mathbf{W}_c^+ \mathbf{u}_- = x_+$. Choose $\mathbf{u}_- \in \ell^2_{\mathcal{U}}(\mathbb{Z}_-)$ so that $\mathbf{W}_c^+ \mathbf{u}_- = x_+$. Next solve for x_-' so that $x_- = A_-^{-1} x_-' - A_-^{-1} B_- u$, i.e., set

$$x_-' := A_- x_- + A_- B_- u. \tag{2.26}$$

Use the surjectivity of the controllability operator $\mathbf{W}_c^-$ to find $\mathbf{u}_+ \in \ell^2_{\mathcal{U}}(\mathbb{Z}_+)$ so that $\mathbf{W}_c^- \mathbf{u}_+ = x_-'$. We now define a new input signal $\mathbf{u}$ by

$$\mathbf{u}(n) = \begin{cases} \mathbf{u}_-(n) & \text{if } n < 0, \\ u & \text{if } n = 0, \\ \mathbf{u}_+(n-1) & \text{if } n \geq 1. \end{cases}$$

Since $\mathbf{u}_+$ and $\mathbf{u}_-$ are ℓ^2-sequences, we obtain that $\mathbf{u} \in \ell^2_{\mathcal{U}}(\mathbb{Z})$. Now let $(\mathbf{u}, \mathbf{x}, \mathbf{y})$ be the ℓ^2-admissible system trajectory determined by the input sequence $\mathbf{u}$. Clearly $\mathbf{u}(0) = u$. So it remains to show that $\mathbf{x}(0) = x$. To see this, note that

$$\begin{aligned}\mathbf{x}(0) &= \mathbf{W}_c\mathbf{u} = \mathbf{W}_c^+\mathbf{u}_- + \mathbf{W}_c^-(\begin{bmatrix} u & 0 & \cdots \end{bmatrix} + \mathcal{S}\mathbf{u}_+)\\ &= x_+ - A_-^{-1}B_-u + A_-^{-1}\mathbf{W}_c^-\mathbf{u}_+ = x_+ - A_-^{-1}B_-u + A_-^{-1}x'_-\\ &= x_+ + x_- = x.\end{aligned}$$

□

3. Bicausal systems

Even for the setting of rational matrix functions, it is not the case that a rational matrix function F which is analytic on a neighborhood of the unit circle $\mathbb{T}$ necessarily has a realization of the form (1.3), as such a realization for F implies that F must be analytic at the origin. What is required instead is a slightly more general notion of a system, which we will refer to as a *bicausal system*, defined as follows.

A *bicausal system* Σ consists of a pair of input-state-output linear systems Σ_+ and Σ_- with Σ_+ running in forward time and Σ_- running in backward time

$$\Sigma_- : \begin{cases} \mathbf{x}_-(n) &= \widetilde{A}_-\mathbf{x}_-(n+1) + \widetilde{B}_-\mathbf{u}(n), \\ \mathbf{y}_-(n) &= \widetilde{C}_-\mathbf{x}_-(n) \end{cases} \quad (n \in \mathbb{Z}) \tag{3.1}$$

$$\Sigma_+ : \begin{cases} \mathbf{x}_+(n+1) &= \widetilde{A}_+\mathbf{x}_+(n) + \widetilde{B}_+\mathbf{u}(n), \\ \mathbf{y}_+(n) &= \widetilde{C}_+\mathbf{x}_+(n) + \widetilde{D}\mathbf{u}(n) \end{cases} \quad (n \in \mathbb{Z}) \tag{3.2}$$

with Σ_- having state space $\mathcal{X}_-$ and state operator $\widetilde{A}_-$ on $\mathcal{X}_-$ exponentially stable (i.e., $\sigma(\widetilde{A}_-) \subset \mathbb{D}$) and Σ_+ having state space $\mathcal{X}_+$ and $\widetilde{A}_+$ on $\mathcal{X}_+$ exponentially stable ($\sigma(\widetilde{A}_+) \subset \mathbb{D}$). A system trajectory consists of a triple $\{\mathbf{u}(n), \mathbf{x}(n), \mathbf{y}(n)\}_{n\in\mathbb{Z}}$ such that

$$\mathbf{u}(n) \in \mathcal{U}, \quad \mathbf{x}(n) = \begin{bmatrix} \mathbf{x}_- \\ \mathbf{x}_+ \end{bmatrix} \in \begin{bmatrix} \mathcal{X}_- \\ \mathcal{X}_+ \end{bmatrix}, \quad \mathbf{y}(n) = \mathbf{y}_-(n) + \mathbf{y}_+(n) \text{ with } \mathbf{y}_\pm(n) \in \mathcal{Y}$$

such that $(\mathbf{u}, \mathbf{x}_-, \mathbf{y}_-)$ is a system trajectory of Σ_- and $(\mathbf{u}, \mathbf{x}_+, \mathbf{y}_+)$ is a system trajectory of Σ_+. We say that the system trajectory $(\mathbf{u}, \mathbf{x}, \mathbf{y}) = (\mathbf{u}, \begin{bmatrix} \mathbf{x}_- \\ \mathbf{x}_+ \end{bmatrix}, \mathbf{y}_- + \mathbf{y}_+)$ is ℓ^2-*admissible* if all system signals are in ℓ^2:

$$\mathbf{u} \in \ell^2_{\mathcal{U}}(\mathbb{Z}), \quad \mathbf{x}_+ \in \ell^2_{\mathcal{X}_+}(\mathbb{Z}), \quad \mathbf{x}_- \in \ell^2_{\mathcal{X}_-}(\mathbb{Z}), \quad \mathbf{y}_\pm \in \ell^2_{\mathcal{Y}}(\mathbb{Z}).$$

Due to the assumed exponential stability of $\widetilde{A}_+$, given $\mathbf{u} \in \ell^2_{\mathcal{U}}(\mathbb{Z})$, there is a uniquely determined $\mathbf{x}_+ \in \ell^2_{\mathcal{X}_+}(\mathbb{Z})$ and $\mathbf{y}_+ \in \ell^2_{\mathcal{Y}}(\mathbb{Z})$ so that $(\mathbf{u}, \mathbf{x}_+, \mathbf{y}_+)$ is an ℓ^2-admissible system trajectory for Σ_+ and similarly for $\widetilde{A}_-$ due to the assumed exponential stability of $\widetilde{A}_-$. The result is as follows.

Proposition 3.1. *Suppose that $\Sigma = (\Sigma_+, \Sigma_-)$ is a bicausal system, with $\widetilde{A}_+$ exponentially stable as an operator on $\mathcal{X}_+$ and $\widetilde{A}_-$ exponentially stable as an operator on $\mathcal{X}_-$. Then:*

1. *Given any $\mathbf{u} \in \ell^2_{\mathcal{U}}(\mathbb{Z})$, there is a unique $\mathbf{x}_+ \in \ell^2_{\mathcal{X}_+}(\mathbb{Z})$ satisfying the first system equation in* (3.2)*, with the resulting input-state map $T_{\Sigma_+,is}$ mapping $\ell^2_{\mathcal{U}}(\mathbb{Z})$ to $\ell^2_{\mathcal{X}_+}(\mathbb{Z})$ given by the block matrix*

$$[T_{\Sigma_+,is}]_{ij} = \begin{cases} \widetilde{A}_+^{i-j-1}\widetilde{B}_+ & \text{for } i > j, \\ 0 & \text{for } i \leq j. \end{cases} \tag{3.3}$$

The unique output signal $\mathbf{y}_+ \in \ell^2_{\mathcal{Y}}(\mathbb{Z})$ *resulting from the system equations* (3.2) *with given input* $\mathbf{u} \in \ell^2_{\mathcal{U}}(\mathbb{Z})$ *and resulting uniquely determined state trajectory* $\mathbf{x}_+$ *in* $\ell^2_{\mathcal{X}_+}(\mathbb{Z})$ *is then given by* $\mathbf{y}_+ = T_{\Sigma_+}\mathbf{u}$ *with* $T_{\Sigma_+}\colon \ell^2_{\mathcal{U}}(\mathbb{Z}) \to \ell^2_{\mathcal{Y}}(\mathbb{Z})$ *having block matrix representation given by*

$$[T_{\Sigma_+}]_{ij} = \begin{cases} \widetilde{C}_+\widetilde{A}_+^{i-j-1}\widetilde{B}_+ & \text{for } i > j, \\ \widetilde{D} & \text{for } i = j, \\ 0 & \text{for } i < j. \end{cases} \tag{3.4}$$

Thus $T_{\Sigma_+,is}$ *and* T_{Σ_+} *are block lower-triangular (causal) Toeplitz operators.*

(2) *Given any* $\mathbf{u} \in \ell^2_{\mathcal{U}}(\mathbb{Z})$, *there is a unique* $\mathbf{x}_- \in \ell^2_{\mathcal{X}_-}(\mathbb{Z})$ *satisfying the first system equation in* (3.1), *with resulting input-state map* $T_{\Sigma_-,is}\colon \ell^2_{\mathcal{U}}(\mathbb{Z}) \to \ell^2_{\mathcal{X}_+}(\mathbb{Z})$ *having block matrix representation given by*

$$[T_{\Sigma_-,is}]_{ij} = \begin{cases} 0 & \text{for } i > j, \\ \widetilde{A}_-^{j-i}\widetilde{B}_- & \text{for } i \le j. \end{cases} \tag{3.5}$$

The unique output signal $\mathbf{y}_- \in \ell^2_{\mathcal{Y}}(\mathbb{Z})$ *resulting from the system equations* (3.1) *with given input* $\mathbf{u} \in \ell^2_{\mathcal{U}}(\mathbb{Z})$ *and resulting uniquely determined state trajectory* $\mathbf{x}_-$ *in* $\ell^2_{\mathcal{X}_+}(\mathbb{Z})$ *is then given by* $\mathbf{y} = T_{\Sigma_-}\mathbf{u}$ *with* $T_{\Sigma_-}\colon \ell^2_{\mathcal{U}}(\mathbb{Z}) \to \ell^2_{\mathcal{Y}}(\mathbb{Z})$ *having block matrix representation given by*

$$[T_{\Sigma_-}]_{ij} = \begin{cases} 0 & \text{for } i > j, \\ \widetilde{C}_-\widetilde{A}_-^{j-i}\widetilde{B}_- & \text{for } i \le j. \end{cases} \tag{3.6}$$

Thus $T_{\Sigma_-,is}$ *and* T_{Σ_-} *are upper-triangular (anticausal) Toeplitz operators.*

(3) *The input-state map for the combined bicausal system* $\Sigma = (\Sigma_+, \Sigma_-)$ *is then given by*

$$T_{\Sigma,is} = \begin{bmatrix} T_{\Sigma_-,is} \\ T_{\Sigma_+,is} \end{bmatrix} : \ell^2_{\mathcal{U}}(\mathbb{Z}) \to \ell^2_{\mathcal{X}}(\mathbb{Z}) = \begin{bmatrix} \ell^2_{\mathcal{X}_-}(\mathbb{Z}) \\ \ell^2_{\mathcal{X}_+}(\mathbb{Z}) \end{bmatrix}$$

with block matrix entries (with notation using the natural identifications $\mathcal{X}_+ \cong \left[\begin{smallmatrix} 0 \\ \mathcal{X}_+ \end{smallmatrix}\right]$ *and* $\mathcal{X}_- \cong \left[\begin{smallmatrix} \mathcal{X}_- \\ 0 \end{smallmatrix}\right]$*)*

$$[T_{\Sigma,is}]_{ij} = \begin{cases} \widetilde{A}_+^{i-j-1}\widetilde{B}_+ & \text{for } i > j, \\ \widetilde{A}_-^{j-i}\widetilde{B}_- & \text{for } i \le j. \end{cases} \tag{3.7}$$

Moreover, the input-output map $T_\Sigma\colon \ell^2_{\mathcal{U}}(\mathbb{Z}) \to \ell^2_{\mathcal{Y}}(\mathbb{Z})$ *of* Σ *is given by*

$$T_\Sigma = T_{\Sigma_+} + T_{\Sigma_-} : \ell^2_{\mathcal{U}}(\mathbb{Z}) \to \ell^2_{\mathcal{Y}}(\mathbb{Z}),$$

having block matrix decomposition given by

$$[T_\Sigma]_{ij} = \begin{cases} \widetilde{C}_+\widetilde{A}_+^{i-j-1}\widetilde{B}_+ & \text{for } i > j, \\ \widetilde{D} + \widetilde{C}_-\widetilde{B}_- & \text{for } i = j, \\ \widetilde{C}_-\widetilde{A}_-^{j-i}\widetilde{B}_- & \text{for } i < j. \end{cases} \tag{3.8}$$

(4) *For* $\mathbf{u} \in \ell^2_{\mathcal{U}}(\mathbb{Z})$ *and* $\mathbf{y} \in \ell^2_{\mathcal{Y}}(\mathbb{Z})$, $\widehat{\mathbf{u}}$ *and* $\widehat{\mathbf{y}}$ *be the respective bilateral Z-transforms*

$$\mathbf{u}(z) = \sum_{n=-\infty}^{\infty} \mathbf{u}(n)z^n \in L^2_{\mathcal{U}}(\mathbb{T}), \quad \mathbf{y}(z) = \sum_{n=-\infty}^{\infty} \mathbf{y}(n)z^n \in L^2_{\mathcal{U}}(\mathbb{T}).$$

Then

$$\mathbf{y} = T_\Sigma \mathbf{u} \iff \widehat{\mathbf{y}}(z) = F_\Sigma(z) \cdot \widehat{\mathbf{u}}(z) \text{ for almost all } z \in \mathbb{T}$$

where $F_\Sigma(z)$ is the transfer function *of the bicausal system Σ given by*

$$\begin{aligned} F_\Sigma(z) &= \widetilde{C}_-(I - z^{-1}\widetilde{A}_-)^{-1}\widetilde{B}_- + \widetilde{D} + z\widetilde{C}_+(I - z\widetilde{A}_+)^{-1}\widetilde{B}_+ \\ &= \sum_{n=1}^{\infty} \widetilde{C}_-\widetilde{A}_-^n\widetilde{B}_- z^{-n} + (\widetilde{D} + \widetilde{C}_-\widetilde{B}_-) + \sum_{n=1}^{\infty} \widetilde{C}_+\widetilde{A}_+^n\widetilde{B}_+ z^n. \end{aligned} \tag{3.9}$$

Furthermore, the Laurent operator $\mathfrak{L}_{F_\Sigma} \colon \ell^2_{\mathcal{U}}(\mathbb{Z}) \to \ell^2_{\mathcal{Y}}(\mathbb{Z})$ associated with the function $F_\Sigma \in L^\infty_{\mathcal{L}(\mathcal{U},\mathcal{Y})}(\mathbb{T})$ as in (2.6) is identical to the input-output operator T_Σ for the bicausal system Σ

$$\mathfrak{L}_{F_\Sigma} = T_\Sigma, \tag{3.10}$$

and hence also, for $\mathbf{u} \in \ell^2_{\mathcal{U}}(\mathbb{Z})$ and $\mathbf{y} \in \ell^2_{\mathcal{Y}}(\mathbb{Z})$ and notation as in (2.7),

$$\mathbf{y} = T_\Sigma \mathbf{u} \iff \widehat{\mathbf{y}}(z) = F_\Sigma(z) \cdot \widehat{\mathbf{u}}(z) \text{ for almost all } z \in \mathbb{T}. \tag{3.11}$$

Proof. We first consider item (1). Let us rewrite the system equations (3.2) in aggregate form

$$\Sigma_+ \colon \left\{ \begin{array}{rcl} \mathcal{S}^{-1}\mathbf{x}_+ &=& \widetilde{\mathcal{A}}_+\mathbf{x}_+ + \widetilde{\mathcal{B}}_+\mathbf{u} \\ \mathbf{y}_+ &=& \widetilde{\mathcal{C}}_+\mathbf{x}_+ + \widetilde{\mathcal{D}}\mathbf{u} \end{array} \right. \tag{3.12}$$

where

$$\begin{aligned} \widetilde{\mathcal{A}}_+ &= \mathrm{diag}_{k\in\mathbb{Z}}[\widetilde{A}_+] \in \mathcal{L}(\ell^2_{\mathcal{X}_+}), \\ \widetilde{\mathcal{B}}_+ &= \mathrm{diag}_{k\in\mathbb{Z}}[\widetilde{B}_+] \in \mathcal{L}(\ell^2_{\mathcal{U}}(\mathbb{Z}), \ell^2_{\mathcal{X}_+}(\mathbb{Z})), \\ \widetilde{\mathcal{C}}_+ &= \mathrm{diag}_{k\in\mathbb{Z}}[\widetilde{C}_+] \in \mathcal{L}(\ell^2_{\mathcal{X}_+}(\mathbb{Z}), \ell^2_{\mathcal{Y}}(\mathbb{Z})), \\ \widetilde{\mathcal{D}} &= \mathrm{diag}_{k\in\mathbb{Z}}[D] \in \mathcal{L}(\ell^2_{\mathcal{U}}(\mathbb{Z}), \ell^2_{\mathcal{Y}}(\mathbb{Z})). \end{aligned} \tag{3.13}$$

The exponential stability assumption on $\widetilde{A}_+$ implies that $\widetilde{A}_+$ has trivial exponential dichotomy (with state-space $\mathcal{X}_- = \{0\}$). As previously observed (see [9]), the exponential dichotomy of $\widetilde{A}_+$ implies that we can solve the first system equation (3.12) of Σ_+ uniquely for $\mathbf{x}_+ \in \ell^2_{\mathcal{X}_+}(\mathbb{Z})$:

$$\mathbf{x}_+ = (\mathcal{S}^{-1} - \widetilde{\mathcal{A}}_+)^{-1}\widetilde{\mathcal{B}}_+\mathbf{u} =: T_{\Sigma_+,is}\mathbf{u} \tag{3.14}$$

and item (1) follows. From the general formula (2.14) for $(\mathcal{S}^{-1} - \mathcal{A})^{-1}$ in (2.14), we see that for our case here the formula for the input-state map $T_{\Sigma_+,is}$ for the system Σ_+ is given by (3.3). From the aggregate form of the system equations (3.12) we see that the resulting input-output map $T_{\Sigma_+} \colon \ell^2_{\mathcal{U}}(\mathbb{Z}) \to \ell^2_{\mathcal{Y}}(\mathbb{Z})$ is then given by

$$T_{\Sigma_+} = \widetilde{\mathcal{D}} + \widetilde{\mathcal{C}}(\mathcal{S}^{-1} - \widetilde{\mathcal{A}}_+)^{-1}\widetilde{\mathcal{B}}_+ = \widetilde{\mathcal{D}} + \widetilde{\mathcal{C}}T_{\Sigma_+,is}.$$

The block matrix decomposition (3.4) for the input-output map T_{Σ_+} now follows directly from plugging in the matrix decomposition (3.3) for $T_{\Sigma_+,is}$ into this last formula.

The analysis for item (2) proceeds in a similar way. Introduce operators

$$\begin{aligned} \widetilde{\mathcal{A}}_- &= \mathrm{diag}_{k\in\mathbb{Z}}[\widetilde{A}_-] \in \mathcal{L}(\ell^2_{\mathcal{X}_-}), \\ \widetilde{\mathcal{B}}_- &= \mathrm{diag}_{k\in\mathbb{Z}}[\widetilde{B}_-] \in \mathcal{L}(\ell^2_{\mathcal{U}}(\mathbb{Z}), \ell^2_{\mathcal{X}_-}(\mathbb{Z})), \\ \widetilde{\mathcal{C}}_- &= \mathrm{diag}_{k\in\mathbb{Z}}[\widetilde{C}_-] \in \mathcal{L}(\ell^2_{\mathcal{X}_-}(\mathbb{Z}), \ell^2_{\mathcal{Y}}(\mathbb{Z})), \end{aligned} \tag{3.15}$$

Write the system (3.1) in the aggregate form

$$\Sigma_- : \begin{cases} \mathbf{x}_- &= \widetilde{\mathcal{A}}_- \mathcal{S}^{-1}\mathbf{x}_- + \widetilde{\mathcal{B}}_- \mathbf{u} \\ \mathbf{y}_- &= \widetilde{\mathcal{C}}_- \mathbf{x}_-. \end{cases} \tag{3.16}$$

As $\widetilde{A}_-$ is exponentially stable, so also is $\widetilde{\mathcal{A}}_-$ and we may compute $(I - \widetilde{A}_- \mathcal{S}^{-1})^{-1}$ via the geometric series, using also that $\widetilde{A}_-$ and $\mathcal{S}^{-1}$ commute as observed in (2.10),

$$(I - \widetilde{A}_- \mathcal{S}^{-1})^{-1} = \sum_{k=0}^{\infty} \widetilde{A}_-^k \mathcal{S}^{-k}$$

from which we deduce the block matrix representation

$$[(I - \widetilde{\mathcal{A}}_- \mathcal{S}^{-1})^{-1}]_{ij} = \begin{cases} 0 & \text{for } i > j, \\ \widetilde{A}_-^{j-i} & \text{for } i \leq j. \end{cases}$$

We next note that we can solve the first system equation in (3.15) for $\mathbf{x}_-$ in terms of $\mathbf{u}$:

$$\mathbf{x}_- = (I - \mathcal{A}_- \mathcal{S}^{-1})^{-1} \widetilde{B}\mathbf{u} =: T_{\Sigma_-, is}\mathbf{u}.$$

Combining this with the previous formula for the block-matrix entries for $(I - \widetilde{\mathcal{A}}_- \mathcal{S}^{-1})^{-1}$ leads to the formula (3.5) for the matrix entries of $T_{\Sigma_-, is}$. From the second equation for the system (3.16) we see that then $\mathbf{y}_-$ is uniquely determined via the formula

$$\mathbf{y}_- = \widetilde{\mathcal{C}}_- \mathbf{x}_- = \widetilde{\mathcal{C}}_- T_{\Sigma_-, is}\mathbf{u}.$$

Plugging in the formula (3.5) for the block matrix entries of $T_{\Sigma_-, is}$ then leads to the formula (3.6) for the block matrix entries of the input-output map T_{Σ_-} for the system Σ_-.

Item (3) now follows by definition of the input-output map T_Σ of the bicausal system Σ as the sum $T_\Sigma = T_{\Sigma,-} + T_{\Sigma,+}$ of the input-output maps for the anticausal system Σ_- and the causal system Σ_+ along with the formulas for $T_{\Sigma,\pm}$ obtained in items (1) and (2).

We now analyze item (4). Define F_Σ by either of the equivalent formulas in (3.9). Due to the exponential stability of $\widetilde{A}_+$ and $\widetilde{A}_-$, we see that F_Σ is a continuous $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued function on the unit circle $\mathbb{T}$, and hence the multiplication operator $M_{F_\Sigma} \colon f(z) \mapsto F_\Sigma(z) \cdot f(z)$ is a bounded operator from $L^2_{\mathcal{U}}(\mathbb{T})$ into $L^2_{\mathcal{Y}}(\mathbb{T})$. From the second formula for $F_\Sigma(z)$ in (3.9) combined with the formula (3.8) for the block matrix entries of T_Σ, we see that the Laurent expansion for $F(z) = \sum_{n=-\infty}^{\infty} F_n z^n$ on $\mathbb{T}$ is given by $F_n = [T_\Sigma]_{n,0}$ and that the Laurent matrix $[\mathfrak{L}_{F_\Sigma}]_{ij} = F_{i-j}$ is the same as the matrix for the input-output operator $[T_\Sigma]_{ij}$. We now see the identity (3.10) as an immediate consequence of the general identity (2.7). Finally, the transfer-function property (3.11) follows immediately from (3.10) combined with the general identity (2.7). □

Remark 3.2. From the form of the input-output operator T_Σ and transfer function F_Σ of the dichotomous system Σ in (1.1)–(1.2) that were obtained in Proposition 2.2 with respect to the decompositions of A in (2.1) and of B and C in (2.2) it follows that a dichotomous system can be represented as a bicausal system (3.2)–(3.1) with

$$\begin{aligned} (\widetilde{C}_+, \widetilde{A}_+, \widetilde{B}_+, \widetilde{D}) &= (C_+, A_+, B_+, D), \\ (\widetilde{C}_-, \widetilde{A}_-, \widetilde{B}_-) &= (C_-, A_-^{-1}, -A_-^{-1}B_-). \end{aligned} \tag{3.17}$$

The extra feature that a bicausal system coming from a dichotomous system has is that $\widetilde{A}_-$ is invertible. In fact, if the operator $\widetilde{A}_-$ in a bicausal system (3.2)-(3.1) is invertible, it can be represented as a dichotomous system (1.1) as well, by reversing the above transformation. Indeed, one easily verifies that if $\Sigma = (\Sigma_+, \Sigma_-)$ is a bicausal system given by (3.2)-(3.1) with $\widetilde{A}_-$ invertible, then the system (1.1) with

$$A = \begin{bmatrix} \widetilde{A}_+ & 0 \\ 0 & \widetilde{A}_-^{-1} \end{bmatrix}, \quad B = \begin{bmatrix} \widetilde{B}_+ \\ -\widetilde{A}_-^{-1}\widetilde{B}_- \end{bmatrix}, \quad C = \begin{bmatrix} \widetilde{C}_+ & \widetilde{C}_- \end{bmatrix}, \quad D = \widetilde{D}$$

is a dichotomous system whose input-output operator and transfer function are equal the input-output operator and transfer function from the original bicausal system.

To a large extent, the theory of dichotomous system presented in Section 2 carries over to bicausal systems, with proofs that can be directly obtained from the translation between the two systems given above. We describe here the main features.

The Laurent operator $\mathfrak{L}_{F_\Sigma} = T_\Sigma$ can again be decomposed as in (2.16) where now the Toeplitz operators $\widetilde{\mathfrak{T}}_{F_\Sigma}$ and $\mathfrak{T}_{F_\Sigma}$ are given by

$$\text{(3.18)} \quad \begin{aligned} [\widetilde{\mathfrak{T}}_{F_\Sigma}]_{ij\colon i<0, j<0} &= \begin{cases} \widetilde{C}_-\widetilde{A}_-^{j-i}\widetilde{B}_- & \text{for } i<j<0, \\ \widetilde{D} + \widetilde{C}_-\widetilde{B}_- & \text{for } i=j<0, \\ \widetilde{C}_+\widetilde{A}_+^{i-j-1}\widetilde{B}_+ & \text{for } j<i<0, \end{cases} \\ [\mathfrak{T}_{F_\Sigma}]_{ij\colon i\geq 0, j\geq 0} &= \begin{cases} \widetilde{C}_-\widetilde{A}_-^{j-i}\widetilde{B}_- & \text{for } 0\leq i<j, \\ \widetilde{D} + \widetilde{C}_-\widetilde{B}_- & \text{for } 0\leq i=j, \\ \widetilde{C}_+\widetilde{A}_+^{i-j-1}\widetilde{B}_+ & \text{for } 0\leq j<i, \end{cases} \end{aligned}$$

while the Hankel operators $\widetilde{\mathfrak{H}}_{F_\Sigma}$ and $\mathfrak{H}_{F_\Sigma}$ are given by

$$\text{(3.19)} \quad [\widetilde{\mathfrak{H}}_{F_\Sigma}]_{ij: i<0, j\geq 0} = \widetilde{C}_-\widetilde{A}_-^{j-i}\widetilde{B}_-, \quad [\mathfrak{H}_{F_\Sigma}]_{ij\colon i\geq 0, j<0} = \widetilde{C}_+\widetilde{A}_+^{i-j-1}\widetilde{B}_+.$$

For the subsystems Σ_+ and Σ_- we define controllability operators $\mathbf{W}_c^+$ and $\mathbf{W}_c^-$, respectively, as well as observability operators $\mathbf{W}_o^+$ and $\mathbf{W}_o^-$, respectively, just as in the case of regular forward-time and backward-time systems:

$$\text{(3.20)} \quad \begin{aligned} \mathbf{W}_c^+ &= \text{row}_{j\in\mathbb{Z}_-}[\widetilde{A}_+^{-j-1}\widetilde{B}_+]\colon \ell^2_{\mathcal{U}}(\mathbb{Z}_-) \to \mathcal{X}_+, \\ \mathbf{W}_c^- &= \text{row}_{j\in\mathbb{Z}_+}[\widetilde{A}_-^{j}\widetilde{B}_-]\colon \ell^2_{\mathcal{U}}(\mathbb{Z}_+) \to \mathcal{X}_-, \\ \mathbf{W}_o^+ &= \text{col}_{i\in\mathbb{Z}_+}[\widetilde{C}_+\widetilde{A}_+^{i}]\colon \mathcal{X}_+ \to \ell^2_{\mathcal{Y}}(\mathbb{Z}_+), \\ \mathbf{W}_o^- &= \text{col}_{i\in\mathbb{Z}_-}[\widetilde{C}_-\widetilde{A}_-^{-i}]\colon \mathcal{X}_- \to \ell^2_{\mathcal{Y}}(\mathbb{Z}_-). \end{aligned}$$

Setting $\mathcal{X} = \mathcal{X}_+\dot{+}\mathcal{X}_-$, we put these operators together to define the controllability operator $\mathbf{W}_c$ and observability operator $\mathbf{W}_o$ of the bicausal system Σ via

$$\text{(3.21)} \quad \begin{aligned} \mathbf{W}_c &= \begin{bmatrix} \mathbf{W}_c^+ & \mathbf{W}_c^- \end{bmatrix} : \begin{bmatrix} \ell^2_{\mathcal{U}}(\mathbb{Z}_-) \\ \ell^2_{\mathcal{U}}(\mathbb{Z}_+) \end{bmatrix} \to \mathcal{X}, \\ \mathbf{W}_o &= \begin{bmatrix} \mathbf{W}_o^- \\ \mathbf{W}_o^+ \end{bmatrix} : \mathcal{X} \to \begin{bmatrix} \ell^2_{\mathcal{Y}}(\mathbb{Z}_-) \\ \ell^2_{\mathcal{Y}}(\mathbb{Z}_+) \end{bmatrix}. \end{aligned}$$

The fact that $\widetilde{A}_+$ and $\widetilde{A}_-$ are both stable implies that all the operators $\mathbf{W}_c^\pm$, $\mathbf{W}_c$, $\mathbf{W}_o^\pm$ and $\mathbf{W}_o$ are bounded. Then it is now easily checked that we still recover factorizations of the Hankel operators as before:

$$\widetilde{\mathfrak{H}}_{F_\Sigma} = \mathbf{W}_o^- \mathbf{W}_c^-, \quad \mathfrak{H}_{F_\Sigma} = \mathbf{W}_o^+ \mathbf{W}_c^+. \tag{3.22}$$

For the bicausal system $\Sigma = (\Sigma_+, \Sigma_-)$ given by (3.2)–(3.1) we say that Σ is *controllable* (respectively, *observable*) whenever the two systems Σ_+ and Σ_- are both controllable (respectively, observable), i.e., $\operatorname{Im} \mathbf{W}_c^+$ dense in $\mathcal{X}_+$ and $\operatorname{Im} \mathbf{W}_c^-$ dense in $\mathcal{X}_-$, or equivalently, $\operatorname{Im} \mathbf{W}_c$ dense in $\mathcal{X}$ (respectively, $\operatorname{Ker} \mathbf{W}_o^+ = \{0\}$ and $\operatorname{Ker} \mathbf{W}_o^- = \{0\}$, or equivalently, $\operatorname{Ker} \mathbf{W}_o = \{0\}$). Analogously, we say that Σ is *ℓ^2-exactly controllable* (respectively, *ℓ^2-exactly observable*) whenever $\operatorname{Im} \mathbf{W}_c = \mathcal{X}$ (respectively, $\operatorname{Im} \mathbf{W}_o^* = \mathcal{X}$).

An exception to the general rubric that the theory of dichotomous systems carries over directly to the theory of bicausal systems is the following analogue of the ℓ^2-admissible-trajectory interpolation result (Proposition 2.6), which has a somewhat different form for the bicausal setting.

Proposition 3.3. *Let $\Sigma = (\Sigma_-, \Sigma_+)$ be a bicausal linear system as in* (3.1) *and* (3.2) *with $\widetilde{A}_-$ exponentially stable on $\mathcal{X}_-$ and $\widetilde{A}_+$ exponentially stable on $\mathcal{X}_+$ and suppose that Σ is ℓ^2-exactly controllable in the bicausal sense. Then given any vectors $x_- \in \mathcal{X}_-$, $x_+ \in \mathcal{X}_+$, $u \in \mathcal{U}$, there is an ℓ^2-admissible system trajectory $(\mathbf{u}, \mathbf{x}_- \oplus \mathbf{x}_+, \mathbf{y})$ for Σ satisfying the interpolation conditions*

$$\mathbf{x}_-(1) = x_-, \quad \mathbf{x}_+(0) = x_+, \quad \mathbf{u}(0) = u. \tag{3.23}$$

Proof. By the ℓ^2-exact controllability assumption, we can find $\mathbf{u}_- \in \ell^2_{\mathcal{U}}(\mathbb{Z}_-)$ so that $\mathbf{W}_c^+ \mathbf{u} = x_+$. Similarly, we can find $\mathbf{u}_+ \in \ell^2_{\mathcal{U}}(\mathbb{Z}_+)$ so that $\mathbf{W}_c^- \mathbf{u}_+ = x_-$. Define a input signal $\mathbf{u} \in \ell^2_{\mathcal{U}}(\mathbb{Z})$ by

$$\mathbf{u}(n) = \begin{cases} \mathbf{u}_-(n) & \text{if } n < 0, \\ u & \text{if } n = 0, \\ \mathbf{u}_+(n-1) & \text{if } n > 0. \end{cases}$$

Now it is simple direct check that the ℓ^2-admissible trajectory $(\mathbf{u}, \mathbf{x}, \mathbf{y})$ determined by the input $\mathbf{u}$ has the desired interpolation properties (3.23).

The proof is close to that of the the corresponding result for the dichotomous setting, Proposition 2.6. The key difference is that we must use $\mathbf{x}_-(1)$ rather than $\mathbf{x}_-(0)$ as a free parameter since in general we are not able to solve the equation $x_- = \widetilde{A}_- \mathbf{x}_-(1) - \widetilde{A}_- \widetilde{B}_- u$ for $\mathbf{x}_-(1)$ (the analogue of equation (2.26)) since $\widetilde{A}_-$ need not be invertible in the bicausal setting. □

4. Storage functions

Let Σ be the dichotomous system given by (1.1). A *storage function* for the system Σ is a function $S \colon \mathcal{X} \to \mathbb{R}$ so that

(1) S is continuous at 0:

$$\{x_n\}_{n \in \mathbb{N}} \subset \mathcal{X}, \ \lim_{n \to \infty} x_n = 0 \text{ in } \mathcal{X} \implies \lim_{n \to \infty} S(x_n) = S(0) \text{ in } \mathbb{R},$$

(2) S satisfies the energy-balance relation:

$$S(\mathbf{x}(n+1)) - S(\mathbf{x}(n)) \le \|\mathbf{u}(n)\|^2_{\mathcal{U}} - \|\mathbf{y}(n)\|^2_{\mathcal{Y}} \qquad (n \in \mathbb{Z}) \tag{4.1}$$

along all ℓ^2-admissible system trajectories $(\mathbf{u}, \mathbf{x}, \mathbf{y})$ of Σ, and

(3) S satisfies the normalization condition $S(0) = 0$.

We further say that S is a *strict storage function* for Σ if S is a storage function for Σ with condition (4.1) replaced by the stronger condition: *there is a* $\epsilon > 0$ *so that*

$$S(\mathbf{x}(n+1)) - S(\mathbf{x}(n)) + \epsilon^2 \|\mathbf{x}(n)\|^2 \le (1-\epsilon^2)\|\mathbf{u}(n)\|^2_{\mathcal{U}} - \|\mathbf{y}(n)\|^2_{\mathcal{Y}} \quad (n \in \mathbb{Z}) \tag{4.2}$$

along all ℓ^2*-admissible system trajectories* $(\mathbf{u}, \mathbf{x}, \mathbf{y})$ *of* Σ.

Then we have the following result.

Proposition 4.1. *Suppose that the dichotomous system* (1.1) *has a storage function* S*. Then the input-output map* T_Σ *is contractive, i.e.,* $\|T_\Sigma\| \le 1$*. In case* Σ *has a strict storage function* S*, the input-output map is a strict contraction, i.e.,* $\|T_\Sigma\| < 1$.

Proof. Let S be a storage function for Σ. Take $\mathbf{u} \in \ell^2_{\mathcal{U}}(\mathbb{Z})$. Define $\mathbf{x}$ by $\mathbf{x} = T_{\Sigma,is}\mathbf{u}$ and $\mathbf{y}$ by $\mathbf{y} = T_\Sigma \mathbf{u}$, where $T_{\Sigma,is}$ and T_Σ are as in (2.12)–(2.13), so $(\mathbf{u}, \mathbf{x}, \mathbf{y})$ is an ℓ^2-admissible system trajectory. If we sum (4.1) from $n = -N$ to $n = N$ we get

$$S(\mathbf{x}(N+1)) - S(\mathbf{x}(-N)) \le \sum_{n=-N}^{N} \|\mathbf{u}(n)\|^2_{\mathcal{U}} - \sum_{n=-N}^{N} \|\mathbf{y}(n)\|^2_{\mathcal{Y}}.$$

Taking the limit as $N \to \infty$ and using the fact that both $\mathbf{x}(-N) \to 0$ and $\mathbf{x}(N) \to 0$ as $N \to \infty$, since $\mathbf{x} \in \ell^2_{\mathcal{X}}(\mathbb{Z})$, we obtain from the continuity of S at 0 and the normalization condition $S(0) = 0$ that both $S(\mathbf{x}_{N+1}) \to 0$ and $S(\mathbf{x}(-N)) \to 0$ as $N \to \infty$. Hence taking the limit as $N \to \infty$ in the preceding estimate gives

$$0 \le \|\mathbf{u}\|^2_{\ell^2_{\mathcal{U}}(\mathbb{Z})} - \|\mathbf{y}\|^2_{\ell^2_{\mathcal{Y}}(\mathbb{Z})} = \|\mathbf{u}\|^2_{\ell^2_{\mathcal{U}}(\mathbb{Z})} - \|T_\Sigma \mathbf{u}\|^2_{\ell^2_{\mathcal{Y}}(\mathbb{Z})}.$$

Since $\mathbf{u}$ was chosen arbitrarily in $\ell^2_{\mathcal{U}}(\mathbb{Z})$, it follows that $\|T_\Sigma\| \le 1$.

If Σ has a strict storage function we see that there is an $\epsilon > 0$ so that

$$\begin{aligned} S(\mathbf{x}(n+1)) - S(\mathbf{x}(n)) &\le S(\mathbf{x}(n+1)) - S(\mathbf{x}(n)) + \epsilon^2\|\mathbf{x}(n)\|^2 \\ &\le (1-\epsilon^2)\|\mathbf{u}(n)\|^2 - \|\mathbf{y}(n)\|^2 \end{aligned}$$

so in particular we have

$$S(\mathbf{x}(n+1) - S(\mathbf{x}(n)) \le (1-\epsilon^2)\|\mathbf{u}(n)\|^2 - \|\mathbf{y}(n)\|^2.$$

Summing this last inequality from $n = -N$ to $n = N$ leaves us with

$$S(\mathbf{x}(N+1)) - S(\mathbf{x}(-N)) \le (1-\epsilon^2) \sum_{n=-N}^{N} \|\mathbf{u}(n)\|^2_{\mathcal{U}} - \sum_{n=-N}^{N} \|\mathbf{y}(n)\|^2_{\mathcal{Y}}.$$

Taking the limit as $N \to \infty$ and using again the fact that both $\mathbf{x}(-N) \to 0$ and $\mathbf{x}(N) \to 0$ as $N \to \infty$ along ℓ^2-admissible system trajectories then gives us

$$0 \le (1-\epsilon^2)\|\mathbf{u}\|^2_{\ell^2_{\mathcal{U}}(\mathbb{Z})} - \|\mathbf{y}\|^2_{\ell^2_{\mathcal{Y}}(\mathbb{Z})} = (1-\epsilon^2)\|\mathbf{u}\|^2_{\ell^2_{\mathcal{U}}(\mathbb{Z})} - \|T_\Sigma \mathbf{u}\|^2_{\ell^2_{\mathcal{Y}}(\mathbb{Z})}$$

and we are able to conclude that $\|T_\Sigma\|^2 \le 1 - \epsilon^2 < 1$. □

To get further results on storage functions for dichotomous systems, we shall assume from now on that the transfer function F_Σ is contractive on the unit circle ($\|F_\Sigma\|_{\infty,\mathbb{T}} \le 1$) as well as that Σ is dichotomously ℓ^2-exactly minimal (see (2.24)–(2.25)), i.e.,

$$\|F_\Sigma\|_{\infty,\mathbb{T}} \le 1, \quad \operatorname{Im} \mathbf{W}_c = \mathcal{X}, \quad \operatorname{Im} \mathbf{W}_o^* = \mathcal{X}. \tag{4.3}$$

Remark 4.2. A particular consequence of assumption (4.3) is that Σ is ℓ^2-exactly controllable. As a consequence of Proposition 2.6 we then see that the second condition (4.1) in the definition of storage function can be replaced by the localized version: *given $u \in \mathcal{U}$ and $x \in \mathcal{X}$ we have the inequality*

$$S(Ax+Bu) - S(x) \le \|u\|^2 - \|Cx+Du\|^2 \tag{4.4}$$

for the standard case, and

$$S(Ax+Bu) - S(x) + \epsilon^2\|x\|^2 \le (1-\epsilon^2)\|u\|^2 - \|Cx+Du\|^2 \tag{4.5}$$

for the strict case. Once we have this formulation, we also see that we could equally well replace the phrase *ℓ^2-admissible system trajectories* simply with *system trajectories* in (4.1) and (4.2).

With all the assumptions (4.3) in force, we now define two candidate storage functions, referred to as the available storage and required supply functions for the dichotomous linear system (1.1), namely

$$S_a(x_0) = \sup_{\mathbf{u}\in\ell^2_{\mathcal{U}}(\mathbb{Z}):\, \mathbf{W}_c\mathbf{u}=x_0} \sum_{n=0}^{\infty} \left(\|\mathbf{y}(n)\|^2 - \|\mathbf{u}(n)\|^2\right) \qquad (x_o \in \operatorname{Im}\mathbf{W}_c) \tag{4.6}$$

$$S_r(x_0) = \inf_{\mathbf{u}\in\ell^2_{\mathcal{U}}(\mathbb{Z}):\, \mathbf{W}_c\mathbf{u}=x_0} \sum_{n=-\infty}^{-1} \left(\|\mathbf{u}(n)\|^2 - \|\mathbf{y}(n)\|^2\right) \quad (x_o \in \operatorname{Im}\mathbf{W}_c) \tag{4.7}$$

where $\mathbf{y}$ is the output signal determined by (2.13).

In order to show that S_a and S_r are storage functions, we shall need multiple applications of the following elementary patching lemma.

Lemma 4.3. *Suppose that $(\mathbf{u}',\mathbf{x}',\mathbf{y}')$ and $(\mathbf{u}'',\mathbf{x}'',\mathbf{y}'')$ are two ℓ^2-admissible system trajectories of the system Σ such that $\mathbf{x}'(0)=\mathbf{x}''(0)=: x_0$. Define a new triple of signals $(\mathbf{u},\mathbf{x},\mathbf{y})$ by*

$$\mathbf{u}(n) = \begin{cases} \mathbf{u}'(n) & \text{if } n<0,\\ \mathbf{u}''(n) & \text{if } n\ge 0,\end{cases} \qquad \mathbf{x}(n) = \begin{cases} \mathbf{x}'(n) & \text{if } n\le 0,\\ \mathbf{x}''(n) & \text{if } n> 0,\end{cases}$$
$$\mathbf{y}(n) = \begin{cases} \mathbf{y}'(n) & \text{if } n<0,\\ \mathbf{y}''(n) & \text{if } n\ge 0.\end{cases} \tag{4.8}$$

Then $(\mathbf{u},\mathbf{x},\mathbf{y})$ is again an ℓ^2-admissible system trajectory.

Proof. We must verify that $(\mathbf{u},\mathbf{x},\mathbf{y})$ satisfy the system equations (1.1) for all $n\in\mathbb{Z}$. For $n<0$ this is clear since $(\mathbf{u}',\mathbf{x}',\mathbf{y}')$ is a system trajectory. Since $\mathbf{x}'(0)=\mathbf{x}''(0)$, we see that this holds for $n=0$. That it holds for $n>0$ follows easily from the fact that $(\mathbf{u}'',\mathbf{x}'',\mathbf{y}'')$ is a system trajectory. Finally note that $(\mathbf{u}',\mathbf{x}',\mathbf{y}')$ and $(\mathbf{u}'',\mathbf{x}'',\mathbf{y}'')$ both being ℓ^2-admissible implies that $(\mathbf{u},\mathbf{x},\mathbf{y})$ is ℓ^2-admissible. $\square$

Our next goal is to show that S_a and S_r are storage functions for Σ, and among all storage functions they are the minimum and maximum ones. We postpone the proof of Step 4 in the proof of items (1) and (2) in the following proposition to Section 5 below.

Proposition 4.4. *Let Σ be a dichotomous linear system as in (1.1) such that (4.3) holds. Then:*

(1) *S_a is a storage function for Σ.*

(2) *S_r is a storage function for Σ.*
(3) *If $\widetilde{S}$ is any other storage function for Σ, then*

$$S_a(x_0) \le \widetilde{S}(x_0) \le S_r(x_0) \text{ for all } x_0 \in \mathcal{X}.$$

Proof of (1) and (2). The proof proceeds in several steps.
Step 1: S_a and S_r are finite-valued on $\mathcal{X}$. Let $x_0 \in \mathcal{X} = \operatorname{Im} \mathbf{W}_c$. By the ℓ^2-exact controllability assumption, there is a $\mathbf{u}_0 \in \ell^2_{\mathcal{U}}(\mathbb{Z})$ so that $x_0 = \mathbf{W}_c \mathbf{u}_0$. Let $(\mathbf{u}_0, \mathbf{x}_0, \mathbf{y}_0)$ be the unique ℓ^2-admissible system trajectory of Σ defined by the input $\mathbf{u}_0$. Then

$$S_a(x_0) \ge \sum_{n=0}^{\infty} \|\mathbf{y}_0(n)\|^2 - \|\mathbf{u}_0(n)\|^2 \ge -\|\mathbf{u}_0\|^2 > -\infty$$

and similarly

$$S_r(x_0) \le \|\mathbf{u}_0\|^2 < \infty.$$

It remains to show $S_a(x_0) < \infty$ and $S_r(x_0) > -\infty$.

Let $(\mathbf{u}, \mathbf{x}, \mathbf{y})$ be any ℓ^2-admissible system trajectory of Σ with $\mathbf{x}(0) = x_0$. Let $(\mathbf{u}_0, \mathbf{x}_0, \mathbf{y}_0)$ be the particular ℓ^2-admissible system trajectory with $\mathbf{x}(0) = x_0$ as chosen above. Then by Lemma 4.3 we may piece together these two trajectories to form a new ℓ^2-admissible system trajectory $(\mathbf{u}', \mathbf{x}', \mathbf{y}')$ of Σ defined as follows:

$$\mathbf{u}'(n) = \begin{cases} \mathbf{u}_0(n) & \text{if } n < 0 \\ \mathbf{u}(n) & \text{if } n \ge 0 \end{cases}, \quad \mathbf{x}'(n) = \begin{cases} \mathbf{x}_0(n) & \text{if } n \le 0 \\ \mathbf{x}(n) & \text{if } n > 0 \end{cases},$$
$$\mathbf{y}'(n) = \begin{cases} \mathbf{y}_0(n) & \text{if } n < 0, \\ \mathbf{y}(n) & \text{if } n \ge 0 \end{cases}.$$

Since $\|T_\Sigma\| \le 1$ and $(\mathbf{u}', \mathbf{x}', \mathbf{y}')$ is a system trajectory, we know that

$$\sum_{n=-\infty}^{+\infty} \|\mathbf{y}'(n)\|^2 = \|\mathbf{y}'\|^2 = \|T_\Sigma \mathbf{u}'\|^2 \le \|\mathbf{u}'\|^2 = \sum_{n=-\infty}^{+\infty} \|\mathbf{u}'(n)\|^2.$$

Let us rewrite this last inequality in the form

$$\sum_{n=0}^{\infty} \|\mathbf{y}(n)\|^2 - \sum_{n=0}^{\infty} \|\mathbf{u}(n)\|^2 \le \sum_{n=-\infty}^{-1} \|\mathbf{u}_0(n)\|^2 - \sum_{n=-\infty}^{-1} \|\mathbf{y}_0(n)\|^2 < \infty.$$

It follows that the supremum of the left hand side over all ℓ^2-admissible trajectories $(\mathbf{u}, \mathbf{x}, \mathbf{y})$ of Σ with $\mathbf{x}(0) = x_0$ is finite, i.e., $S_a(x_0) < \infty$.

A similar argument shows that $S_r(x_0) > -\infty$ as follows. Given an arbitrary ℓ^2-admissible system trajectory $(\mathbf{u}, \mathbf{x}, \mathbf{y})$ with $\mathbf{x}(0) = x_0$, Lemma 4.3 enables us to form the composite ℓ^2-admissible system trajectory $(\mathbf{u}'', \mathbf{x}'', \mathbf{y}'')$ of Σ defined by

$$\mathbf{u}''(n) = \begin{cases} \mathbf{u}(n) & \text{if } n < 0 \\ \mathbf{u}_0(n) & \text{if } n \ge 0 \end{cases}, \quad \mathbf{x}''(n) = \begin{cases} \mathbf{x}(n) & \text{if } n \le 0 \\ \mathbf{x}_0(n) & \text{if } n > 0 \end{cases},$$
$$\mathbf{y}''(n) = \begin{cases} \mathbf{y}(n) & \text{if } n < 0, \\ \mathbf{y}_0(n) & \text{if } n \ge 0 \end{cases}.$$

Then the fact that $\sum_{n=-\infty}^{+\infty} \|\mathbf{y}''(n)\|^2 \le \sum_{n=-\infty}^{\infty} \|\mathbf{u}''(n)\|^2$ gives us that

$$\sum_{n=-\infty}^{-1} \|\mathbf{u}(n)\|^2 - \sum_{n=-\infty}^{-1} \|\mathbf{y}(n)\|^2 \ge \sum_{n=0}^{\infty} \|\mathbf{y}_0(n)\|^2 - \sum_{n=0}^{\infty} \|\mathbf{u}_0(n)\|^2 > -\infty$$

and it follows from the definition (4.7) that $S_r(x_0) > -\infty$. By putting all these pieces together we see that both S_a and S_r are finite-valued on $\mathcal{X} = \operatorname{Im} \mathbf{W}_c$.

Step 2: $S_a(0) = S_r(0) = 0$**.** This fact follows from the explicit quadratic form for S_a and S_r obtained in Theorem 5.2 below, but we include here an alternative more conceptual proof to illustrate the ideas. By noting that $(0, 0, 0)$ is an ℓ^2-admissible system trajectory, we see from the definitions of S_a in (4.6) and S_r in (4.7) that $S_a(0) \geq 0$ and $S_r(0) \leq 0$. Now let $(\mathbf{u}, \mathbf{x}, \mathbf{y})$ be any ℓ^2-admissible system trajectory such that $\mathbf{x}(0) = 0$. Another application of Lemma 4.3 then implies that $(\mathbf{u}', \mathbf{x}', \mathbf{y}')$ given by

$$(\mathbf{u}'(n), \mathbf{x}'(n), \mathbf{y}'(n)) = \begin{cases} (0, 0, 0) & \text{if } n < 0, \\ (\mathbf{u}(n), \mathbf{x}(n), \mathbf{y}(n)) & \text{if } n \geq 0 \end{cases}$$

is also an ℓ^2-admissible system trajectory. From the assumption that $\|T_\Sigma\| \leq 1$ we get that

$$0 \leq \sum_{n=-\infty}^{\infty} (\|\mathbf{u}'(n)\|^2_{\mathcal{U}} - \|\mathbf{y}'(n)\|^2_{\mathcal{Y}}) = \sum_{n=0}^{\infty} (\|\mathbf{u}(n)\|^2_{\mathcal{U}} - \|\mathbf{y}(n)\|^2_{\mathcal{Y}}),$$

so

$$\sum_{n=0}^{\infty} (\|\mathbf{y}(n)\|^2_{\mathcal{Y}} - \|\mathbf{u}(n)\|^2_{\mathcal{U}}) \leq 0 \tag{4.9}$$

whenever $(\mathbf{u}, \mathbf{x}, \mathbf{y})$ is an ℓ^2-admissible system trajectory with $\mathbf{x}(0) = 0$. From the definition in (4.6) we see that $S_a(0)$ is the supremum over all such expressions on the left hand side of (4.9), and we conclude that $S_a(0) \leq 0$. Putting this together with the first piece above gives $S_a(0) = 0$.

Similarly, note that if $(\mathbf{u}, \mathbf{x}, \mathbf{y})$ is an ℓ^2-admissible trajectory with $\mathbf{x}(0) = \mathbf{W}_c \mathbf{u} = 0$, then again by Lemma 4.3

$$(\mathbf{u}''(n), \mathbf{x}''(n), \mathbf{y}''(n)) = \begin{cases} (\mathbf{u}(n), \mathbf{x}(n), \mathbf{y}(n)) & \text{if } n < 0, \\ (0, 0, 0) & \text{if } n \geq 0 \end{cases}$$

is also an ℓ^2-admissible system trajectory. Since $\|T_\Sigma\| \leq 1$ we get

$$0 \leq \sum_{n=-\infty}^{\infty} (\|\mathbf{u}''(n)\|^2_{\mathcal{U}} - \|\mathbf{y}''(n)\|^2_{\mathcal{Y}}) = \sum_{n=-\infty}^{-1} (\|\mathbf{u}(n)\|^2_{\mathcal{U}} - \|\mathbf{y}(n)\|^2_{\mathcal{Y}})$$

From the definition (4.7) of $S_r(0)$ it follows that $S_r(0) \geq 0$. Putting all these pieces together, we arrive at $S_a(0) = S_r(0) = 0$.

Step 3: Both S_a **and** S_r **satisfy the energy balance inequality** (4.1)**.** For $x_0 \in \mathcal{X}$, set

$$\overrightarrow{\mathcal{U}}_{x_0} = \{\widetilde{\mathbf{u}} \in \ell^2_{\mathcal{U}}(\mathbb{Z}) \colon \widetilde{\mathbf{x}}(0) = W_c \widetilde{\mathbf{u}} = x_0\},$$

Also, for any Hilbert space $\mathcal{W}$ let P_+ on $\ell^2_{\mathcal{W}}(\mathbb{Z})$ be the orthogonal projection on $\ell^2_{\mathcal{W}}(\mathbb{Z}_+)$ and $P_- = I - P_+$. Then we can write $S_a(x_0)$ and $S_r(x_0)$ as

$$S_a(x_0) = \sup_{\widetilde{\mathbf{u}} \in \overrightarrow{\mathcal{U}}_{x_0}} \|P_+ \widetilde{\mathbf{y}}\|^2 - \|P_+ \widetilde{\mathbf{u}}\|^2 \quad \text{and} \quad S_r(x_0) = \inf_{\widetilde{\mathbf{u}} \in \overrightarrow{\mathcal{U}}_{x_0}} \|P_- \widetilde{\mathbf{u}}\|^2 - \|P_- \widetilde{\mathbf{y}}\|^2,$$

where $\widetilde{\mathbf{y}} = T_\Sigma \widetilde{\mathbf{u}}$ is the output of Σ defined by the input $\widetilde{\mathbf{u}} \in \ell^2_{\mathcal{U}}(\mathbb{Z})$. In general, if $\widetilde{\mathbf{u}} \in \ell^2_{\mathcal{U}}(\mathbb{Z})$ is an input trajectory, then the corresponding uniquely determined ℓ^2-admissible state and output trajectories are denoted by $\widetilde{\mathbf{x}} := T_{\Sigma,is} \widetilde{\mathbf{u}}$ and $\widetilde{\mathbf{y}} := T_\Sigma \widetilde{\mathbf{u}}$.

Now let $(\mathbf{u}, \mathbf{x}, \mathbf{y})$ be an arbitrary fixed system trajectory for the dichotomous system Σ and fix $n \in \mathbb{Z}$. Set

$$\overrightarrow{\mathcal{U}}_* = \{\widetilde{\mathbf{u}} \in \ell^2_{\mathcal{U}}(\mathbb{Z})\colon \widetilde{\mathbf{x}}(0) = \mathbf{x}(n),\ \widetilde{\mathbf{u}}(0) = \mathbf{u}(n)\}.$$

Note that $\overrightarrow{\mathcal{U}}_*$ is nonempty by simply quoting Proposition 2.6. Observe that $\overrightarrow{\mathcal{U}}_* \subset \overrightarrow{\mathcal{U}}_{\mathbf{x}(n)}$. For an ℓ^2-admissible system trajectory $(\widetilde{\mathbf{u}}, \widetilde{\mathbf{x}}, \widetilde{\mathbf{y}})$ with $\widetilde{\mathbf{u}} \in \overrightarrow{\mathcal{U}}_*$ we have $\widetilde{\mathbf{y}}(0) = \mathbf{y}(n)$ and $\widetilde{\mathbf{x}}(1) = \mathbf{x}(n+1)$. Furthermore, $(\mathcal{S}^*\widetilde{\mathbf{u}}, \mathcal{S}^*\widetilde{\mathbf{x}}, \mathcal{S}^*\widetilde{\mathbf{y}})$ is also an ℓ^2-admissible system trajectory of Σ and

$$(\mathcal{S}^*\widetilde{\mathbf{x}})(0) = \widetilde{\mathbf{x}}(1) = \mathbf{x}(n+1).$$

Hence

$$\mathcal{S}^*\overrightarrow{\mathcal{U}}_* = \{\mathcal{S}^*\widetilde{\mathbf{u}}\colon \widetilde{\mathbf{u}} \in \overrightarrow{\mathcal{U}}_*\} \subset \overrightarrow{\mathcal{U}}_{\mathbf{x}(n+1)}.$$

Next, since $\widetilde{\mathbf{y}}(0) = \mathbf{y}(n)$ and $\widetilde{\mathbf{u}}(0) = \mathbf{u}(n)$ we have

$$\|P_+\widetilde{\mathbf{y}}\|^2 - \|P_+\widetilde{\mathbf{u}}\|^2 = \|P_+\mathcal{S}^*\widetilde{\mathbf{y}}\|^2 - \|P_+\mathcal{S}^*\widetilde{\mathbf{u}}\|^2 + \|\mathbf{y}(n)\|^2 - \|\mathbf{u}(n)\|^2$$

and

$$\|P_-\mathcal{S}^*\widetilde{\mathbf{u}}\|^2 - \|P_-\mathcal{S}^*\widetilde{\mathbf{y}}\|^2 = \|P_-\widetilde{\mathbf{u}}\|^2 - \|P_-\widetilde{\mathbf{y}}\|^2 + \|\mathbf{u}(n)\|^2 - \|\mathbf{y}(n)\|^2.$$

We thus obtain that

$$\begin{aligned} S_a(\mathbf{x}(n)) &= \sup_{\widetilde{\mathbf{u}} \in \overrightarrow{\mathcal{U}}_{\mathbf{x}(n)}} \|P_+\widetilde{\mathbf{y}}\|^2 - \|P_+\widetilde{\mathbf{u}}\| \geq \sup_{\widetilde{\mathbf{u}} \in \overrightarrow{\mathcal{U}}_*} \|P_+\widetilde{\mathbf{y}}\|^2 - \|P_+\widetilde{\mathbf{u}}\|^2 \\ &= \|\mathbf{y}(n)\|^2 - \|\mathbf{u}(n)\|^2 + \sup_{\widetilde{\mathbf{u}} \in \overrightarrow{\mathcal{U}}_*} \|P_+\mathcal{S}^*\widetilde{\mathbf{y}}\|^2 - \|P_+\mathcal{S}^*\widetilde{\mathbf{u}}\|^2 \\ &= \|\mathbf{y}(n)\|^2 - \|\mathbf{u}(n)\|^2 + \sup_{\widetilde{\mathbf{u}} \in \mathcal{S}^*\overrightarrow{\mathcal{U}}_*} \|P_+\widetilde{\mathbf{y}}\|^2 - \|P_+\widetilde{\mathbf{u}}\|^2, \end{aligned}$$

and similarly for S_r we have

$$\begin{aligned} S_r(\mathbf{x}(n+1)) &= \inf_{\widetilde{\mathbf{u}} \in \overrightarrow{\mathcal{U}}_{\mathbf{x}(n+1)}} \|P_-\widetilde{\mathbf{u}}\|^2 - \|P_-\widetilde{\mathbf{y}}\|^2 \\ &= \inf_{\mathcal{S}^*\widetilde{\mathbf{u}} \in \overrightarrow{\mathcal{U}}_{\mathbf{x}(n+1)}} \|P_-\mathcal{S}^*\widetilde{\mathbf{u}}\|^2 - \|P_-\mathcal{S}^*\widetilde{\mathbf{y}}\|^2 \\ &\leq \inf_{\mathcal{S}^*\widetilde{\mathbf{u}} \in \mathcal{S}^*\overrightarrow{\mathcal{U}}_*} \|P_-\mathcal{S}^*\widetilde{\mathbf{u}}\|^2 - \|P_-\mathcal{S}^*\widetilde{\mathbf{y}}\|^2 \\ &= \inf_{\widetilde{\mathbf{u}} \in \overrightarrow{\mathcal{U}}_*} \|P_-\mathcal{S}^*\widetilde{\mathbf{u}}\|^2 - \|P_-\mathcal{S}^*\widetilde{\mathbf{y}}\|^2 \\ &= \|\mathbf{u}(n)\|^2 - \|\mathbf{y}(n)\|^2 + \inf_{\widetilde{\mathbf{u}} \in \overrightarrow{\mathcal{U}}_*} \|P_-\widetilde{\mathbf{u}}\|^2 - \|P_-\widetilde{\mathbf{y}}\|^2. \end{aligned}$$

To complete the proof of this step it remains to show that

$$\begin{aligned} S_a(\mathbf{x}(n+1)) &= \sup_{\widetilde{\mathbf{u}} \in \mathcal{S}^*\overrightarrow{\mathcal{U}}_*} \|P_+\widetilde{\mathbf{y}}\|^2 - \|P_+\widetilde{\mathbf{u}}\|^2 =: s_a, \\ S_r(\mathbf{x}(n)) &= \inf_{\widetilde{\mathbf{u}} \in \overrightarrow{\mathcal{U}}_*} \|P_-\widetilde{\mathbf{u}}\|^2 - \|P_-\widetilde{\mathbf{y}}\|^2 =: s_r. \end{aligned} \tag{4.10}$$

We start with S_a. Since $\mathcal{S}^*\overrightarrow{\mathcal{U}}_* \subset \overrightarrow{\mathcal{U}}_{\mathbf{x}(n+1)}$ we see that

$$s_a \leq S_a(\mathbf{x}(n+1)).$$

To show that also $s_a \geq S_a(\mathbf{x}(n+1))$, let $(\widetilde{\mathbf{u}}, \widetilde{\mathbf{x}}, \widetilde{\mathbf{y}})$ be an ℓ^2-admissible system trajectory with $\widetilde{\mathbf{u}} \in \overrightarrow{\mathcal{U}}_{\mathbf{x}(n+1)}$. The problem is to show

$$\sum_{n=0}^{\infty} (\|\widetilde{\mathbf{y}}(n)\|^2 - \|\widetilde{\mathbf{u}}(n)\|^2) \leq s_a. \tag{4.11}$$

Toward this goal, let $(\widehat{\mathbf{u}}, \widehat{\mathbf{x}}, \widehat{\mathbf{y}})$ be any ℓ^2-admissible trajectory with $\widehat{\mathbf{u}} \in \mathcal{S}^* \overrightarrow{\mathcal{U}}_*$. We then patch the two system trajectories together by setting

$$\widetilde{\mathbf{u}}'(k) = \begin{cases} \widehat{\mathbf{u}}(k) & \text{if } k < 0 \\ \widetilde{\mathbf{u}}(k) & \text{if } k \geq 0 \end{cases}, \quad \widetilde{\mathbf{x}}'(k) = \begin{cases} \widehat{\mathbf{x}}(k) & \text{if } k \leq 0 \\ \widetilde{\mathbf{x}}(k) & \text{if } k > 0 \end{cases},$$
$$\widetilde{\mathbf{y}}'(k) = \begin{cases} \widehat{\mathbf{y}}(k) & \text{if } k < 0, \\ \widetilde{\mathbf{y}}(k) & \text{if } k \geq 0 \end{cases}.$$

Clearly the input, state and output trajectories are all ℓ^2-sequences. Note that $(\widehat{\mathbf{u}}, \widehat{\mathbf{x}}, \widehat{\mathbf{y}})$ and $(\widetilde{\mathbf{u}}, \widetilde{\mathbf{x}}, \widetilde{\mathbf{y}})$ are both ℓ^2-admissible system trajectories. Note that $\widehat{\mathbf{u}}(-1) = \mathbf{u}(n)$, $\widehat{\mathbf{x}}(-1) = \mathbf{x}(n)$, $\widehat{\mathbf{y}}(-1) = \mathbf{y}(n)$ and $\widehat{\mathbf{x}}(0) = \mathbf{x}(n+1)$, we see that

$$\widehat{\mathbf{x}}(0) = A\widehat{\mathbf{x}}(-1) + B\widehat{\mathbf{u}}(-1) = A\mathbf{x}(n) + B\mathbf{u}(n) = \mathbf{x}(n+1) = \widetilde{\mathbf{x}}(0).$$

We can now apply once again Lemma 4.3 to conclude that $(\widetilde{\mathbf{u}}', \widetilde{\mathbf{x}}', \widetilde{\mathbf{y}}')$ is also an ℓ^2-admissible trajectory for Σ. Furthermore, we have $\widetilde{\mathbf{u}}' \in \mathcal{S}^* \overrightarrow{\mathcal{U}}_*$, $P_+\widetilde{\mathbf{y}} = P_+\widetilde{\mathbf{y}}'$ and $P_+\widetilde{\mathbf{u}} = P_+\widetilde{\mathbf{u}}'$. Thus

$$\|P_+\widetilde{\mathbf{y}}\|^2 - \|P_+\widetilde{\mathbf{u}}\|^2 = \|P_+\widetilde{\mathbf{y}}'\|^2 - \|P_+\widetilde{\mathbf{u}}'\|^2 \leq \sup_{\widetilde{\mathbf{u}} \in \mathcal{S}^* \overrightarrow{\mathcal{U}}_*} \|P_+\widetilde{\mathbf{y}}\|^2 - \|P_+\widetilde{\mathbf{u}}\|^2 =: s_a.$$

Taking the supremum on the left-hand side over all ℓ^2-admissible system trajectories with $\widetilde{\mathbf{u}} \in \overrightarrow{\mathcal{U}}_{\mathbf{x}(n+1)}$ then yields $S_a(x(n+1)) \leq s_a$, and the first equality in (4.10) holds as required.

To prove the second equality in (4.10) we follow a similar strategy, which we will only sketch here. The inclusion $\overrightarrow{\mathcal{U}}_* \subset \overrightarrow{\mathcal{U}}_{\mathbf{x}(n)}$ shows s_r is an upper bound. Any $(\widetilde{\mathbf{u}}, \widetilde{\mathbf{x}}, \widetilde{\mathbf{y}})$ be ℓ^2-admissible system trajectory with $\widetilde{\mathbf{u}} \in \overrightarrow{\mathcal{U}}_{\mathbf{x}(n)}$ can be patched together with an ℓ^2-admissible system trajectory with input sequence from $\overrightarrow{\mathcal{U}}_*$ to form a new ℓ^2-admissible system trajectory $(\widetilde{\mathbf{u}}', \widetilde{\mathbf{x}}', \widetilde{\mathbf{y}}')$ with $\widetilde{\mathbf{u}}'$ in $\overrightarrow{\mathcal{U}}_*$, $P_-\widetilde{\mathbf{u}} = P_-\widetilde{\mathbf{u}}'$ and $P_-\widetilde{\mathbf{y}} = P_-\widetilde{\mathbf{y}}'$, so that

$$\|P_-\widetilde{\mathbf{u}}\|^2 - \|P_-\widetilde{\mathbf{y}}\|^2 = \|P_-\widetilde{\mathbf{u}}'\|^2 - \|P_-\widetilde{\mathbf{y}}'\|^2 \geq \inf_{\widetilde{\mathbf{u}} \in \overrightarrow{\mathcal{U}}_*} \|P_-\widetilde{\mathbf{u}}\|^2 - \|P_-\widetilde{\mathbf{y}}\|^2 =: s_r$$

which then yields that s_r is also a lower bound for $S_r(\mathbf{x}(n))$ as required.

Step 4: Both S_a and S_r are continuous at 0. This is a consequence of the explicit quadratic form obtained for S_a and S_r in Theorem 5.2 below.

Proof of (3). Let $\widetilde{S}$ be any storage function for Σ. Let $x_0 \in \operatorname{Im} \mathbf{W}_c$ and let $(\mathbf{u}, \mathbf{x}, \mathbf{y})$ be any ℓ^2-admissible dichotomous system trajectory for Σ with $\mathbf{x}(0) = x_0$. Then $\widetilde{S}$ satisfies the energy balance relation

$$\widetilde{S}(\mathbf{x}(n+1)) - \widetilde{S}(\mathbf{x}(n)) \leq \|\mathbf{u}(n)\|^2_{\mathcal{U}} - \|\mathbf{y}(n)\|^2_{\mathcal{Y}}. \tag{4.12}$$

Summing from $n = 0$ to $n = N$ then gives

$$\widetilde{S}(\mathbf{x}(N+1)) - \widetilde{S}(x_0) = \widetilde{S}(\mathbf{x}(N+1)) - \widetilde{S}(\mathbf{x}(0)) \leq \sum_{n=0}^{N} \left(\|\mathbf{u}(n)\|^2_{\mathcal{U}} - \|\mathbf{y}(n)\|^2_{\mathcal{Y}} \right). \tag{4.13}$$

As $\mathbf{x} \in \ell^2_{\mathcal{X}}(\mathbb{Z})$ and $\widetilde{S}$ as part of being a storage function is continuous at 0 with $\widetilde{S}(0) = 0$, we see from $\mathbf{x}(N+1) \to 0$ that $\widetilde{S}(\mathbf{x}(N+1)) \to \widetilde{S}(0) = 0$ as $N \to \infty$. Hence letting $N \to \infty$ in (4.13) gives

$$-\widetilde{S}(x_0) \leq \sum_{n=0}^{\infty} \left(\|\mathbf{u}(n)\|^2_{\mathcal{U}} - \|\mathbf{y}(n)\|^2_{\mathcal{Y}} \right).$$

But by definition, the infimum of the right-hand side of this last expression over all system trajectories $(\mathbf{u}, \mathbf{x}, \mathbf{y})$ of Σ such that $\mathbf{x}(0) = x_0$ is exactly $-S_a(x_0)$. We conclude that $-\widetilde{S}(x_0) \leq -S_a(x_0)$, and thus $S_a(x_0) \leq \widetilde{S}(x_0)$ for any $x_0 \in \operatorname{Im} \mathbf{W}_c$.

Similarly, if we sum up (4.12) from $n = -N$ to $n = -1$ we get

$$\widetilde{S}(x_0) - \widetilde{S}(\mathbf{x}(-N)) = \widetilde{S}(\mathbf{x}(0)) - \widetilde{S}(\mathbf{x}(-N)) \leq \sum_{n=-N}^{-1} (\|\mathbf{u}(n)\|^2_{\mathcal{U}} - \|\mathbf{y}(n)\|^2_{\mathcal{Y}}).$$

Letting $N \to \infty$ in this expression then gives

$$\widetilde{S}(x_0) \leq \sum_{n=-\infty}^{-1} (\|\mathbf{u}(n)\|^2_{\mathcal{U}} - \|\mathbf{y}(n)\|^2_{\mathcal{Y}}).$$

But by definition the infimum of the right-hand side of this last inequality over all ℓ^2-admissible system trajectories $(\mathbf{u}, \mathbf{x}, \mathbf{y})$ with $\mathbf{x}(0) = x_0$ is exactly equal to $S_r(x_0)$. We conclude that $\widetilde{S}(x_0) \leq S_r(x_0)$. This completes the proof of part (3) of Proposition 4.4. □

Quadratic storage functions and spatial KYP-inequalities: the dichotomous setting. Let us say that a function $S : \mathcal{X} \to \mathbb{R}$ is *quadratic* if there exists a bounded selfadjoint operator H on $\mathcal{X}$ such that $S(x) = S_H(x) := \langle Hx, x \rangle$ for all $x \in \mathcal{X}$. Trivially any function $S = S_H$ of this form satisfies conditions (1) and (3) in the definition of storage function (see the discussion around (4.1)). To characterize which bounded selfadjoint operators H give rise to $S = S_H$ being a storage function, as we are assuming that our blanket assumption (4.3) is in force, we may quote the result of Remark 4.2 to substitute the local version (4.4) ((4.5)) of the energy-balance condition in place of the original version (4.1) (respectively (4.2) for the strict case). Condition (4.4) applied to S_H leads us to the condition

$$\langle H(Ax + Bu), Ax + Bu \rangle - \langle Hx, x \rangle \leq \|u\|^2 - \|Cx + Du\|^2,$$

or equivalently

$$\langle H(Ax{+}Bu), Ax{+}Bu \rangle + \langle Cx{+}Du, Cx{+}Du \rangle \leq \langle Hx, x \rangle + \langle u, u \rangle.$$

holding for all $x \in \mathcal{X}$ and $u \in \mathcal{U}$. In a more matricial form, we may write instead

$$\left\langle \begin{bmatrix} H & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}, \begin{bmatrix} x \\ u \end{bmatrix} \right\rangle - \left\langle \begin{bmatrix} H & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}, \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \right\rangle \geq 0 \tag{4.14}$$

for all $x \in \mathcal{X}$ and $u \in \mathcal{U}$. Hence H satisfies the spatial version (4.14) of the KYP-inequality (1.4). By elementary Hilbert-space theory, namely, that a selfadjoint operator X on a complex Hilbert space is uniquely determined by its associated quadratic form $x \mapsto \langle Xx, x\rangle$, it follows that H solves the KYP-inequality (1.4), but now for an infinite-dimensional setup.

If we start with the strict version (4.5) of the local energy-balance condition, we arrive at the following criterion for the quadratic function S_H to be a strict storage function for the system Σ, namely the spatial version of the strict KYP-inequality:

$$\begin{aligned} &\left\langle \begin{bmatrix} H & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}, \begin{bmatrix} x \\ u \end{bmatrix} \right\rangle \\ &- \left\langle \begin{bmatrix} H & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}, \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \right\rangle \geq \epsilon \left\| \begin{bmatrix} x \\ u \end{bmatrix} \right\|^2, \end{aligned} \tag{4.15}$$

and hence also the strict KYP-inequality in operator form (1.5), again now for the infinite-dimensional setting. Following the above computations in reversed order shows that the spatial KYP-inequality (4.14) and strict spatial KYP-inequality (4.15) imply that S_H is a storage function and strict storage function, respectively.

Proposition 4.5. *Let Σ be a dichotomous linear system as in* (1.1). *Let H be a bounded, self adjoint operator on $\mathcal{X}$. Then S_H is a quadratic storage function for Σ if and only if H is a solution of the KYP-inequality* (1.4). *Moreover, S_H is a strict quadratic storage function if and only if H is a solution of the strict KYP-inequality* (1.5).

5. The available storage and required supply

We assume throughout this section that the dichotomous linear system Σ satisfies the standing assumption (4.3). Under these conditions we shall show that the available storage S_a and required supply S_r are quadratic storage functions and we shall obtain explicit formulas for the associated selfadjoint operators H_a and H_r satisfying the KYP-inequality (1.4).

The assumption that $\|F_\Sigma\|_{\infty,\mathbb{T}} \leq 1$ implies that the associated Laurent operator $\mathfrak{L}_{F_\Sigma}$ in (2.6) is a contraction, so that the Toeplitz operators $\mathfrak{T}_{F_\Sigma}$ and $\widetilde{\mathfrak{T}}_{F_\Sigma}$ (2.17) are also contractions. Thus $I - \mathfrak{L}_{F_\Sigma}\mathfrak{L}_{F_\Sigma}^*$ and $I - \mathfrak{L}_{F_\Sigma}^*\mathfrak{L}_{F_\Sigma}$ are both positive operators. Writing out these operators in terms of the operator matrix decomposition (2.16) we obtain

$$\begin{aligned} I - \mathfrak{L}_{F_\Sigma}\mathfrak{L}_{F_\Sigma}^* &= \begin{bmatrix} D^2_{\widetilde{\mathfrak{T}}^*_{F_\Sigma}} - \widetilde{\mathfrak{H}}_{F_\Sigma}\widetilde{\mathfrak{H}}^*_{F_\Sigma} & -\widetilde{\mathfrak{T}}_{F_\Sigma}\mathfrak{H}^*_{F_\Sigma} - \widetilde{\mathfrak{H}}_{F_\Sigma}\mathfrak{T}^*_{F_\Sigma} \\ -\mathfrak{H}_{F_\Sigma}\widetilde{\mathfrak{T}}^*_{F_\Sigma} - \mathfrak{T}_{F_\Sigma}\widetilde{\mathfrak{H}}^*_{F_\Sigma} & D^2_{\mathfrak{T}^*_{F_\Sigma}} - \mathfrak{H}_{F_\Sigma}\mathfrak{H}^*_{F_\Sigma} \end{bmatrix} \\ I - \mathfrak{L}_{F_\Sigma}^*\mathfrak{L}_{F_\Sigma} &= \begin{bmatrix} D^2_{\widetilde{\mathfrak{T}}_{\Sigma}} - \mathfrak{H}^*_{\Sigma}\mathfrak{H}_{F_\Sigma} & -\mathfrak{H}^*_{F_\Sigma}\mathfrak{T}_{F_\Sigma} - \widetilde{\mathfrak{T}}^*_{F_\Sigma}\widetilde{\mathfrak{H}}_{F_\Sigma} \\ -\mathfrak{T}^*_{F_\Sigma}\mathfrak{H}_{F_\Sigma} - \widetilde{\mathfrak{H}}^*_{F_\Sigma}\widetilde{\mathfrak{T}}_{F_\Sigma} & D^2_{\mathfrak{T}_{\Sigma}} - \widetilde{\mathfrak{H}}^*_{\Sigma}\widetilde{\mathfrak{H}}_{F_\Sigma} \end{bmatrix}. \end{aligned} \tag{5.1}$$

In particular, from $I - \mathfrak{L}_{F_\Sigma}\mathfrak{L}_{F_\Sigma}^*$ and $I - \mathfrak{L}_{F_\Sigma}^*\mathfrak{L}_{F_\Sigma}$ being positive operators we read off that

$$D^2_{\mathfrak{T}^*_F} \succeq \mathfrak{H}_{F_\Sigma}\mathfrak{H}^*_{F_\Sigma}, \quad D^2_{\widetilde{\mathfrak{T}}^*_{F_\Sigma}} \succeq \widetilde{\mathfrak{H}}_{F_\Sigma}\widetilde{\mathfrak{H}}^*_{F_\Sigma}, \quad D^2_{\widetilde{\mathfrak{T}}_\Sigma} \succeq \mathfrak{H}^*_\Sigma\mathfrak{H}_{F_\Sigma}, \quad D^2_{\mathfrak{T}_\Sigma} \succeq \widetilde{\mathfrak{H}}^*_\Sigma\widetilde{\mathfrak{H}}_{F_\Sigma}. \tag{5.2}$$

Applying Douglas' Lemma [11] along with the factorizations in (2.21) enables us to prove the following result.

Lemma 5.1. *Assume the dichotomous system* Σ *in* (1.1) *satisfies* (4.3)*. Then there exist unique injective bounded linear operators* $X_{o,+}$*,* $X_{o,-}$*,* $X_{c,+}$ *and* $X_{c,-}$ *such that*

$$X_{o,+} : \mathcal{X}_+ \to \ell^2_{\mathcal{U}}(\mathbb{Z}_+), \quad \mathbf{W}_o^+ = D_{\mathfrak{T}^*_{F_\Sigma}} X_{o,+}, \quad \operatorname{Im} X_{o,+} \subset \overline{\operatorname{Im}}\, D_{\mathfrak{T}^*_{F_\Sigma}}, \tag{5.3}$$

$$X_{o,-} : \mathcal{X}_- \to \ell^2_{\mathcal{U}}(\mathbb{Z}_-), \quad \mathbf{W}_o^- = D_{\widetilde{\mathfrak{T}}^*_{F_\Sigma}} X_{o,-}, \quad \operatorname{Im} X_{o,-} \subset \overline{\operatorname{Im}}\, D_{\widetilde{\mathfrak{T}}^*_{F_\Sigma}}, \tag{5.4}$$

$$X_{c,+} : \mathcal{X}_+ \to \ell^2_{\mathcal{Y}}(\mathbb{Z}_+), \quad (\mathbf{W}_c^+)^* = D_{\widetilde{\mathfrak{T}}_{F_\Sigma}} X_{c,+}, \quad \operatorname{Im} X_{c,+} \subset \overline{\operatorname{Im}}\, D_{\widetilde{\mathfrak{T}}_{F_\Sigma}}, \tag{5.5}$$

$$X_{c,-} : \mathcal{X}_- \to \ell^2_{\mathcal{Y}}(\mathbb{Z}_-), \quad (\mathbf{W}_c^-)^* = D_{\mathfrak{T}_{F_\Sigma}} X_{c,-}, \quad \operatorname{Im} X_{c,-} \subset \overline{\operatorname{Im}}\, D_{\mathfrak{T}_{F_\Sigma}}. \tag{5.6}$$

Proof. We give the details of the proof only for $X_{o,+}$ as the other cases are similar. The argument is very much like the proof of Lemma 4.8 in [7] where the argument is more complicated due the unbounded-operator setting there.

Since $D^2_{\mathfrak{T}^*_{F_\Sigma}} \succeq \mathfrak{H}_{F_\Sigma}\mathfrak{H}_{F_\Sigma}$, the Douglas factorization lemma [11] implies the existence of a unique contraction operator $Y_{o,+} : \ell^2_{\mathcal{U}}(\mathbb{Z}_-) \to \ell^2_{\mathcal{U}}(\mathbb{Z}_+)$ with

$$\mathbf{W}_o^+\mathbf{W}_c^+ = \mathfrak{H}_{F_\Sigma} = D_{\mathfrak{T}^*_{F_\Sigma}} Y_{o,+} \quad \text{and} \quad \operatorname{Im} Y_{o,+} \subset \overline{\operatorname{Im}}\, D_{\mathfrak{T}^*_{F_\Sigma}}.$$

As $\operatorname{Im} \mathbf{W}_c = \mathcal{X}_+$, the Open Mapping Theorem guarantees that $\mathbf{W}_c^+$ has a bounded right inverse $\mathbf{W}_c^{+\dagger} := \mathbf{W}_c^{+*}(\mathbf{W}_c^+\mathbf{W}_c^{+*})^{-1}$. Moreover, $\mathbf{u} = \mathbf{W}_c^{+\dagger}x$ is the least norm solution of the equation $\mathbf{W}_c^+\mathbf{u} = x$:

$$\mathbf{W}_c^{+\dagger}(x) = \arg\min\, \{\|\mathbf{u}\|^2_{\ell^2_{\mathcal{U}}(\mathbb{Z}_-)} : \mathbf{u} \in \mathcal{D}(\mathbf{W}_c^+),\ x = \mathbf{W}_c^+\mathbf{u}\} \quad (x \in \operatorname{Im} \mathbf{W}_c^+).$$

We now define $X_{o,+}$ by

$$X_{o,+} = Y_{o,+}\mathbf{W}_c^{+\dagger}.$$

We then observe

$$D_{\mathfrak{T}^*_{F_\Sigma}} X_{o,+} = D_{\mathfrak{T}^*_{F_\Sigma}} Y_{o,+}\mathbf{W}_c^{+\dagger} = \mathfrak{H}_{F_\Sigma}\mathbf{W}_c^{+\dagger} = \mathbf{W}_o^+\mathbf{W}_c^+\mathbf{W}_c^{+\dagger} = \mathbf{W}_o^+$$

giving the factorization (5.3) as wanted. Moreover, the factorization $X_{o,+} = Y_{o,+}\mathbf{W}_c^{+\dagger}$ implies that $\operatorname{Im} X_{o,+} \subset \operatorname{Im} Y_{o,+} \subset \overline{\operatorname{Im}}\, D_{\mathfrak{T}^*_{F_\Sigma}}$; this property combined with the factorization (5.3) makes the choice of $X_{o,+}$ unique. Moreover, the containment $\operatorname{Im} X_{o,+} \subset \overline{\operatorname{Im}}\, D_{\mathfrak{T}^*_{F_\Sigma}}$ combined with the injectivity of $\mathbf{W}_o^+$ forces the injectivity of $X_{o,+}$. □

We are now ready to analyze both the available storage function S_a and the required supply function S_r for a system meeting hypotheses (4.3).

Theorem 5.2. *Suppose that* Σ *is a dichotomous discrete-time linear system as in* (1.1) *which satisfies hypotheses* (4.3)*. Then* $S_a = S_{H_a}$ *and* $S_r = S_{H_r}$ *are quadratic storage functions with associated selfadjoint operators* H_a *and* H_r *bounded and boundedly invertible on* $\mathcal{X}$*, and* S_a *and* S_r *are given by*

$$S_a(x_0) = \|X_{o,+}(x_0)_+\|^2 - \left\|P_a\mathfrak{T}_F^* X_{o,+}(x_0)_+ - P_a D_{\mathfrak{T}_F}\mathbf{W}_c^{-\dagger}(x_0)_-\right\|^2 \tag{5.7}$$

$$S_r(x_0) = \left\|P_r\widetilde{\mathfrak{T}}_F^* X_{o,-}(x_0)_- - P_r D_{\widetilde{\mathfrak{T}}_F}\mathbf{W}_c^{+\dagger}(x_0)_+\right\|^2 - \|X_{o,-}(x_0)_-\|^2 \tag{5.8}$$

with $x_0 = (x_0)_+ \oplus (x_0)_-$ *the decomposition of* x_0 *with respect to the direct sum* $\mathcal{X} = \mathcal{X}_+ \dot{+} \mathcal{X}_-$*, the operators* $X_{o,+}$ *and* $X_{o,-}$ *as in Lemma 5.1 and*

$$\mathbf{W}_c^{-\dagger} = \mathbf{W}_c^{-*}(\mathbf{W}_c^-\mathbf{W}_c^{-*})^{-1}, \quad \mathbf{W}_c^{+\dagger} = \mathbf{W}_c^{+*}(\mathbf{W}_c^+\mathbf{W}_c^{+*})^{-1},$$
$$P_a = P_{(D_{\mathfrak{T}_F}\operatorname{Ker}\mathbf{W}_c^-)^\perp}, \quad P_r = P_{(D_{\widetilde{\mathfrak{T}}_F}\operatorname{Ker}\mathbf{W}_c^+)^\perp}.$$

In particular, S_a and S_r are continuous.

If we assume that the decomposition $\mathcal{X} = \mathcal{X}_- \dot{+} \mathcal{X}_+$ inducing the decompositions (2.1) *and* (2.2) *is actually orthogonal, which can always be arranged via an invertible similarity-transformation change of coordinates in $\mathcal{X}$ if necessary, then with respect to the the orthogonal decomposition $\mathcal{X} = \mathcal{X}_- \oplus \mathcal{X}_+$, H_a and H_r are given explicitly by*

$$H_a = \begin{bmatrix} X_{o,+}^*(I - \mathfrak{T}_F P_a \mathfrak{T}_F^*) X_{o,+} & X_{o,+}^* \mathfrak{T}_F P_a D_{\mathfrak{T}_F} \mathbf{W}_c^{-\dagger} \\ \mathbf{W}_c^{-\dagger *} D_{\mathfrak{T}_F} P_a \mathfrak{T}_F^* X_{o,+} & -\mathbf{W}_c^{-\dagger *} D_{\mathfrak{T}_F} P_a D_{\mathfrak{T}_F} \mathbf{W}_c^{-\dagger} \end{bmatrix}, \tag{5.9}$$

$$H_r = \begin{bmatrix} \mathbf{W}_c^{+\dagger *} D_{\widetilde{\mathfrak{T}}_F} P_r D_{\widetilde{\mathfrak{T}}_F} \mathbf{W}_c^{+\dagger} & -\mathbf{W}_c^{+\dagger *} D_{\widetilde{\mathfrak{T}}_F} P_r \widetilde{\mathfrak{T}}_F^* X_{o,-} \\ -X_{o,-}^* \widetilde{\mathfrak{T}}_F P_r D_{\widetilde{\mathfrak{T}}_F} \mathbf{W}_c^{*\dagger} & -X_{o,-}^*(I - \widetilde{\mathfrak{T}}_F P_r \widetilde{\mathfrak{T}}_F^*) X_{o,-} \end{bmatrix}. \tag{5.10}$$

Furthermore, the dimension of the spectral subspace of A over the unit disk agrees with the dimension of the spectral subspace of H_a and H_r over the positive real line ($= \dim \mathcal{X}_+$), and the dimension of the spectral subspace of A over the exterior of the closed unit disk agrees with the dimension of the spectral subspace of H_a and H_r over the negative real line ($= \dim \mathcal{X}_-$).

Proof. To simplify notation, in this proof we write simply F rather than F_Σ.

We start with the formula for S_a. Fix $x_0 \in \operatorname{Im} \mathbf{W}_c$. Let $(\mathbf{u}, \mathbf{x}, \mathbf{y})$ be any system trajectory of Σ such that $x_0 = \mathbf{W}_c \mathbf{u}$.

The first step in the calculation of S_a is to reformulate the formula from the definition (4.6) in operator-theoretic form:

$$S_a(x_0) = \sup_{\mathbf{u} \colon \mathbf{W}_c \mathbf{u} = x_0} \|(\mathfrak{L}_F \mathbf{u})|_{\mathbb{Z}_+}\|^2_{\ell^2_{\mathcal{Y}}(\mathbb{Z}_+)} - \|\mathbf{u}|_{\mathbb{Z}_+}\|^2_{\ell^2_{\mathcal{U}}(\mathbb{Z}_+)}. \tag{5.11}$$

From the formulas (2.16), (2.17), and (2.18) for $\mathfrak{L}_F$, in more detail we have

$$S_a(x_0) = \sup_{\mathbf{u}_+, \mathbf{u}_- \colon \mathbf{W}_c^+ \mathbf{u}_- = (x_0)_+, \mathbf{W}_c^- \mathbf{u}_+ = (x_0)_-} \|\mathfrak{H}_{F_+} \mathbf{u}_- + \mathfrak{T}_F \mathbf{u}_+\|^2 - \|\mathbf{u}_+\|^2.$$

where $\mathbf{u}_- \in \ell^2_{\mathcal{U}}(\mathbb{Z}_-)$ and $\mathbf{u}_+ \in \ell^2_{\mathcal{U}}(\mathbb{Z}_+)$ and where $x_0 = (x_0)_- + (x_0)_+$ is the decomposition of x_0 into $\mathcal{X}_-$ and $\mathcal{X}_+$ components. Recalling the factorization $\mathfrak{H}_F = \mathbf{W}_o^+ \mathbf{W}_c^+$ from (2.18) as well as the constraint on $\mathbf{u}_-$, we rewrite the objective function in the formula for $S_a(x_0)$ as

$$\|\mathfrak{H}_F \mathbf{u}_- + \mathfrak{T}_F \mathbf{u}_+\|^2 - \|\mathbf{u}_+\|^2 = \|\mathbf{W}_o^+ (x_0)_+ + \mathfrak{T}_F \mathbf{u}_+\|^2 - \|\mathbf{u}_+\|^2.$$

Furthermore, by assumption $\mathbf{W}_c^+$ is surjective, so there is always a $\mathbf{u}_- \in \ell^2_{\mathcal{U}}(\mathbb{Z}_-)$ which achieves the constraint $\mathbf{W}_c^+ \mathbf{u}_- = (x_0)_+$. In this way we have eliminated the parameter $\mathbf{u}_-$ and the formula for $S_a(x_0)$ becomes

$$S_a(x_0) = \sup_{\mathbf{u}_+ \in \ell^2_{\mathcal{U}}(\mathbb{Z}_+) \colon \mathbf{W}_c^- \mathbf{u}_+ = (x_0)_-} \|\mathbf{W}_o^+ (x_0)_+ + \mathfrak{T}_F \mathbf{u}_+\|^2 - \|\mathbf{u}_+\|^2. \tag{5.12}$$

By Lemma 5.1 there is a uniquely determined injective linear operator $X_{o,+}$ from $\mathcal{X}_+$ to $\overline{\operatorname{Im}} D_{\mathfrak{T}_F^*}$ so that $W_o^+ = D_{\mathfrak{T}_F^*} X_{o,+}$. Then the objective function in (5.12) becomes

$$\begin{aligned}
&\|\mathbf{W}_o^+(x_0)_+ + \mathfrak{T}_F\mathbf{u}_+\|^2 - \|\mathbf{u}_+\|^2 = \\
&\quad = \|D_{\mathfrak{T}_F^*}X_{o,+} + \mathfrak{T}_F\mathbf{u}_+\|^2 - \|\mathfrak{T}_F\mathbf{u}_+\|^2 - \|D_{\mathfrak{T}_F}\mathbf{u}_+\|^2 \\
&\quad = \|D_{\mathfrak{T}_F^*}X_{o,+}(x_0)_+\|^2 + 2\mathrm{Re}\langle D_{\mathfrak{T}_F^*}X_{o,+}(x_0)_+, \mathfrak{T}_F\mathbf{u}_+\rangle - \|D_{\mathfrak{T}_F}\mathbf{u}_+\|^2 \\
&\quad = \|D_{\mathfrak{T}_F^*}X_{o,+}(x_0)_+\|^2 + 2\mathrm{Re}\langle X_{o,+}(x_0)_+, \mathfrak{T}_F D_{\mathfrak{T}_F}u_+\rangle - \|D_{\mathfrak{T}_F}\mathbf{u}_+\|^2 \\
&\quad = \|D_{\mathfrak{T}_F^*}X_{o,+}(x_0)_+\|^2 + 2\mathrm{Re}\langle \mathfrak{T}_F^*X_{o,+}(x_0)_+, D_{\mathfrak{T}_F}\mathbf{u}_+\rangle - \|D_{\mathfrak{T}_F}\mathbf{u}_+\|^2 \\
&\quad = \|D_{\mathfrak{T}_F^*}X_{o,+}(x_0)_+\|^2 + \|\mathfrak{T}_F^*X_{o,+}(x_0)_+\|^2 - \|\mathfrak{T}_F^*X_{o,+}(x_0)_+ - D_{\mathfrak{T}_F}\mathbf{u}_+\|^2 \\
&\quad = \|X_{o,+}(x_0)_+\|^2 - \|\mathfrak{T}_F^*X_{o,+}(x_0)_+ - D_{\mathfrak{T}_F}\mathbf{u}_+\|^2.
\end{aligned}$$

In this way we arrive at the decoupled formula for $S_a(x_0)$:

$$S_a(x_0) = \|X_{o,+}(x_0)_+\|^2 - \inf_{\mathbf{u}_+\colon \mathbf{W}_c^-\mathbf{u}_+=(x_0)_-} \|\mathfrak{T}_F^*X_{o,+}(x_0)_+ - D_{\mathfrak{T}_F}\mathbf{u}_+\|^2. \tag{5.13}$$

By assumption $\mathbf{W}_c^-$ is surjective and hence $\mathbf{W}_c^-$ is right invertible with right inverse equal to $\mathbf{W}_c^{-*}(\mathbf{W}_c^-\mathbf{W}_c^{-*})^{-1}$. In particular, the minimal-norm solution $\mathbf{u}_+^0$ of $\mathbf{W}_c^-\mathbf{u}_+ = (x_0)_-$ is given by

$$\mathbf{u}_+^0 = \mathbf{W}_c^{-*}(\mathbf{W}_c^-\mathbf{W}_c^{-*})^{-1}(x_0)_- = \mathbf{W}_c^{-\dagger}(x_0)_-$$

and then any other solution has the form

$$\mathbf{u}_+ = \mathbf{u}_+^0 + \mathbf{v}_+ \text{ where } \mathbf{v}_+ \in \operatorname{Ker}\mathbf{W}_c^-.$$

By standard Hilbert space theory, it then follows that

$$\begin{aligned}
&\inf_{\mathbf{u}_+\colon \mathbf{W}_c^-\mathbf{u}_+=(x_0)_-} \|\mathfrak{T}_F^*X_{o,+}(x_0)_+ - D_{\mathfrak{T}_F}\mathbf{u}_+\|^2 \\
&\quad = \left\|P_{(D_{\mathcal{T}_F}\mathrm{Ker}\mathbf{W}_c^-)^\perp}\left(\mathfrak{T}_F^*X_{o,+}(x_0)_+ - D_{\mathfrak{T}_F}\mathbf{u}_+^0\right)\right\|^2 \\
&\quad = \left\|P_a\left(\mathfrak{T}_F^*X_{o,+}(x_0)_+ - D_{\mathfrak{T}_F}\mathbf{W}_c^{-\dagger}(x_0)_-\right)\right\|^2
\end{aligned}$$

and we arrive at the formulas (5.7) for S_a. A few more notational manipulation leads to the explicit formula (5.9) for H_a when $\mathcal{X} = \mathcal{X}_- \dot{+} \mathcal{X}_+$ is an orthogonal decomposition.

In a similar vein, the formula (4.7) for S_r can be written in operator form as

$$S_r(x_0) = \inf_{\mathbf{u}\colon \mathbf{W}_c\mathbf{u}=x_0} \|\mathbf{u}_-\|^2 - \|P_{\ell^2_{\mathcal{Y}}(\mathbb{Z}_-)}\mathfrak{L}_F\mathbf{u}\|^2.$$

Then the objective function can be written as

$$\begin{aligned}
&\|\mathbf{u}_-\|^2 - \|P_{\ell^2_{\mathcal{Y}}(\mathbb{Z}_-)}\mathfrak{L}_F\mathbf{u}\|^2 = \|\mathbf{u}_-\|^2 - \|\widetilde{\mathfrak{H}}_F\mathbf{u}_+ + \widetilde{\mathfrak{T}}_F\mathbf{u}_-\|^2\\
&\quad = \|\mathbf{u}_-\|^2 - \|\mathbf{W}_o^-\mathbf{W}_c^-\mathbf{u}_+ + \widetilde{\mathfrak{T}}_F\mathbf{u}_-\|^2 = \|\mathbf{u}_-\|^2 - \|\mathbf{W}_o^-(x_0)_- + \widetilde{\mathfrak{T}}_F\mathbf{u}_-\|^2\\
&\quad = \|\mathbf{u}_-\|^2 - \|D_{\widetilde{\mathfrak{T}}_F^*}X_{o,-}(x_0)_- + \widetilde{\mathfrak{T}}_F\mathbf{u}_-\|^2\\
&\quad = \|\mathbf{u}_-\|^2 - \|D_{\widetilde{\mathfrak{T}}_F^*}X_{o,-}(x_0)_-\|^2 - 2\mathrm{Re}\langle D_{\widetilde{\mathfrak{T}}_F^*}X_{o,-}(x_0)_-, \widetilde{\mathfrak{T}}_F\mathbf{u}_-\rangle - \|\widetilde{\mathfrak{T}}_F\mathbf{u}_-\|^2\\
&\quad = -\|D_{\widetilde{\mathfrak{T}}_F^*}X_{o,-}(x_0)_-\|^2 - 2\mathrm{Re}\langle X_{o,-}(x_0)_-, D_{\widetilde{\mathfrak{T}}_F^*}\widetilde{\mathfrak{T}}_F\mathbf{u}_-\rangle + \|D_{\widetilde{\mathfrak{T}}_F}\mathbf{u}_-\|^2\\
&\quad = -\|D_{\widetilde{\mathfrak{T}}_F^*}X_{o,-}(x_0)_-\|^2 - 2\mathrm{Re}\langle \widetilde{\mathfrak{T}}_F^*X_{o,-}(x_0)_-, D_{\widetilde{\mathfrak{T}}_F}\mathbf{u}_-\rangle + \|D_{\widetilde{\mathfrak{T}}_F}\mathbf{u}_-\|^2\\
&\quad = -\|D_{\widetilde{\mathfrak{T}}_F^*}X_{o,-}(x_0)_-\|^2 - \|\widetilde{\mathfrak{T}}_F^*X_{o,-}(x_0)^-\|^2 + \|\widetilde{\mathfrak{T}}_F^*X_{o,-}(x_0)_- - D_{\widetilde{\mathfrak{T}}_F}\mathbf{u}_-\|^2\\
&\quad = -\|X_{o,-}(x_0)_-\|^2 + \|\widetilde{\mathfrak{T}}_F^*X_{o,-}(x_0)_- - D_{\widetilde{\mathfrak{T}}_F}\mathbf{u}_-\|^2
\end{aligned}$$

where now $\mathbf{u}_+$ is eliminated and the constraint on the free parameter $\mathbf{u}_-$ is $\mathbf{W}_c^+\mathbf{u}_- = (x_0)_+$. Thus

$$S_r(x_0) = -\|X_{o,-}(x_0)_-\|^2 + \inf_{\mathbf{u}_-\,:\,\mathbf{W}_c^+\mathbf{u}_-=(x_0)_+} \|\widetilde{\mathfrak{T}}_F^*X_{o,-}(x_0)_- - D_{\widetilde{\mathfrak{T}}_F}\mathbf{u}_-\|^2.$$

We note that all possible solutions $\mathbf{u}_-$ of the constraint $\mathbf{W}_c^+\mathbf{u}_- = (x_0)_+$ are given by

$$\mathbf{u}_- = \mathbf{W}_c^{+*}(\mathbf{W}_c^+\mathbf{W}_c^{+*})^{-1}(x_0)_+ + \mathbf{v}_- = \mathbf{W}_c^{+\dagger}(x_0)_+ + \mathbf{v}_- \text{ where } \mathbf{v}_- \in \mathrm{Ker}\mathbf{W}_c^+.$$

Then standard Hilbert-space theory leads to the formulas (5.8) for S_r; a little more careful manipulation leads to the explicit form (5.10) for H_r.

We next wish to verify that H_a and H_r are invertible. This follows as an application of results referred to as inertial theorems; as these results are well known for the finite-dimensional settings (see e.g. [16]) but not so well known for the infinite-dimensional settings, we go through the results in some detail here.

As a consequence of Proposition 4.5, we know that H_a is a solution of the KYP-inequality (1.4). From the (1,1)-entry of (1.4) we see in particular that

$$H_a - A^*H_aA - C^*C \succeq 0.$$

Write H_a as a block operator matrix with respect to the direct sum decomposition $\mathcal{X} = \mathcal{X}_+\dot{+}\mathcal{X}_-$ as

$$H_a = \begin{bmatrix} H_{a-} & H_{a0} \\ H_{a0}^* & H_{a+} \end{bmatrix} \text{ on } \begin{bmatrix} \mathcal{X}_- \\ \mathcal{X}_+ \end{bmatrix}. \tag{5.14}$$

We can then rewrite the above inequality as

$$\begin{bmatrix} H_{a-} & H_{a0} \\ H_{a0}^* & H_{a+} \end{bmatrix} - \begin{bmatrix} A_-^* & 0 \\ 0 & A_+^* \end{bmatrix}\begin{bmatrix} H_{a-} & H_{a0} \\ H_{a0}^* & H_{a+} \end{bmatrix}\begin{bmatrix} A_- & 0 \\ 0 & A_+ \end{bmatrix} \succeq \begin{bmatrix} C_-^* \\ C_+^* \end{bmatrix}\begin{bmatrix} C_- & C_+ \end{bmatrix}.$$

From the diagonal entries of this block-operator inequality we get

$$-H_{a-} + A_-^{*-1}H_{a-}A_-^{-1} \succeq A_-^{*-1}C_-^*C_-A_-^{-1}, \quad H_{a+} - A_+^*H_{a+}H_+A_+ \succeq C_+^*C_+.$$

An inductive argument then gives

$$-H_{a-} \succeq \sum_{n=1}^{N} A^{*-n} C_-^* C_- A_-^{-n} - A_-^{*-N} H_{a-} A_-^{-N},$$

$$H_{a+} \succeq \sum_{n=0}^{N} A_+^{*n} C_+^* C_+ A_+^n + A_+^{*N+1} H_{a+} A_+^{N+1}.$$

As both A_-^{-1} and A_+ are exponentially stable, we may take the limit as $N \to \infty$ in both of the above expressions to get

$$-H_{a-} \succeq (\mathbf{W}_o^-)^* \mathbf{W}_o^-, \quad H_{a+} \succeq (\mathbf{W}_o^+)^* \mathbf{W}_o^+.$$

By the dichotomous ℓ^2-exactly observable assumption, both operators $(\mathbf{W}_o^-)^*$ and $(\mathbf{W}_o^+)^*$ are surjective, and hence $(\mathbf{W}_o^-)^* \mathbf{W}_o^-$ and $(\mathbf{W}_o^+)^* \mathbf{W}_o^+$ are also surjective. Thus we can invoke the Open Mapping Theorem to get that both $(\mathbf{W}_o^-)^* \mathbf{W}_o^-$ and $(\mathbf{W}_o^+)^* \mathbf{W}_o^+$ are bounded below. We conclude that both H_{a+} and $-H_{a-}$ are strictly positive-definite, i.e., there is an $\epsilon > 0$ so that $H_{a+} \succeq \epsilon I$ and $H_{a-} \preceq -\epsilon I$. In particular, both H_{a+} and H_{a-} are invertible.

It remains to put all this together to see that H_a and H_r are invertible. We do the details for H_a as the proof for H_r is exactly the same. By Schur complement theory (see e.g. [12]), applied to the block matrix decomposition of H_a in (5.14), given that the operator H_{a+} is invertible (as we have already verified), then H_a is also invertible if and only if the Schur complement $\mathcal{S}(H_a; H_{a+}) := H_{a-} - H_{a0} H_{a+}^{-1} H_{a0}^*$ is invertible. But we have already verified that both H_{a-} and $-H_{a+}$ are strictly positive-definite. Hence the Schur complement is the sum of a strictly positive-definite operator and an at worst positive-semidefinite operator, and hence is itself strictly positive-definite and therefore also invertible. We next note the block diagonalization of H_a associated with the Schur-complement computation:

$$\begin{bmatrix} H_{a-} & H_{a0} \\ H_{a0}^* & H_{a+} \end{bmatrix} = \begin{bmatrix} I & H_{a0} H_{a+}^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} \mathcal{S}(H_a; H_{a+}) & 0 \\ 0 & H_{a+} \end{bmatrix} \begin{bmatrix} I & 0 \\ H_{a+}^{-1} H_{a0}^* & I \end{bmatrix}.$$

Thus H_a is congruent with $\left[\begin{smallmatrix} \mathcal{S}(H_a; H_{a+}) & 0 \\ 0 & H_{a-} \end{smallmatrix}\right]$ where we have seen that

$$\mathcal{S}(H_a; H_{a+}) \succ 0 \text{ on } \mathcal{X}_+, \quad H_{a-} \prec 0 \text{ on } \mathcal{X}_-.$$

In this way we arrive at the (infinite-dimensional) inertial relations between H and A: the dimension of the spectral subspace of A over the unit disk is the same as the dimension of the spectral subspace of H over the positive real axis, namely $\dim \mathcal{X}_+$, and the dimension of the spectral subspace of A over the exterior of the unit disk is the same as the dimension of the spectral subspace of H over the negative real axis, namely $\dim \mathcal{X}_-$. □

Remark 5.3. Rather than the full force of assumption (4.3), let us now only assume that $\|F_\Sigma\|_{\infty,\mathbb{T}} \leq 1$. A careful analysis of the proof shows that H_a and H_r each being bounded requires only the dichotomous ℓ^2-exact controllability assumption (surjectivity of $\mathbf{W}_c$). The invertibility of each of H_a and H_r requires in addition the dichotomous ℓ^2-exact observability assumption (surjectivity of $\mathbf{W}_o^*$). Moreover, if the ℓ^2-exact observability condition is weakened to observability (i.e., $\bigcap_{n\geq 0} \mathrm{Ker} C_+ A_+^n = \{0\}$ and $\bigcap_{n \geq 0} \mathrm{Ker} C_- A_-^{-n-1} = \{0\}$), then one still gets that H_a and H_r are injective but their respective inverses may not be bounded.

Remark 5.4. If $\|F\|_{\infty,\mathbb{T}} < 1$ (where we are setting $F = F_\Sigma$), then $D_{\mathfrak{T}_F^*}$ and $D_{\widetilde{\mathfrak{T}}_F^*}$ are invertible, and we can solve uniquely for the operators $X_{0,+}$ and $X_{0,-}$ in Lemma 5.1:

$$X_{0,+} = D_{\mathfrak{T}_F^*}^{-1}\mathbf{W}_o^+, \quad X_{0,-} = D_{\widetilde{\mathfrak{T}}_F^*}^{-1}\mathbf{W}_o^-.$$

We may then plug in these expressions for $X_{0,+}$ and $X_{0,-}$ into the formulas (5.7), (5.8), (5.9), (5.10) to get even more explicit formulas for S_a, S_r, H_a and H_r.

6. Storage functions for bicausal systems

We now consider how the analysis in Sections 4 and 5, concerning storage functions $S\colon \mathcal{X} \to \mathbb{R}$, available storage S_a and required supply S_r, quadratic storage function S_H, etc., can be adapted to the setting of a bicausal system $\Sigma = (\Sigma_+, \Sigma_-)$ with subsystems (3.2) and (3.1), where now ℓ^2-admissible trajectories refer to signals of the form $(\mathbf{u}, \mathbf{x}_- \oplus \mathbf{x}_+, \mathbf{y})$ such that $\mathbf{y} = \mathbf{y}_- + \mathbf{y}_+$ with $(\mathbf{u}, \mathbf{x}_-, \mathbf{y}_-)$ an ℓ^2-admissible system trajectory of Σ_- and $(\mathbf{u}, \mathbf{x}_+, \mathbf{y}_+)$ an ℓ^2-admissible system trajectory of Σ_+. We define $S\colon \mathcal{X} := \mathcal{X}_- \oplus \mathcal{X}_+ \to \mathbb{R}$ to be a *storage function* for Σ exactly as was done in Section 4 for the dichotomous case, i.e., we demand that

(1) S is continuous at 0,
(2) S satisfies the energy balance relation (4.1) along all ℓ^2-admissible system trajectories of the bicausal system $\Sigma = (\Sigma_-, \Sigma_+)$, and
(3) $S(0) = 0$.

We again say that S is a *strict storage function* for Σ if the strict energy-balance relation (4.2) holds over all ℓ^2-admissible system trajectories $(\mathbf{u}, \mathbf{x}, \mathbf{y})$ for the bicausal system $\Sigma = (\Sigma_-, \Sigma_+)$. By following the proof of Proposition 4.1 verbatim, but now interpreted for the more general setting of a bicausal system $\Sigma = (\Sigma_-, \Sigma_+)$, we arrive at the following result.

Proposition 6.1. *Suppose that S is a storage function for the bicausal system $\Sigma = (\Sigma_-, \Sigma_+)$ in (3.1)–(3.2), with $\widetilde{A}_\pm$ exponentially stable. Then the input-output map T_Σ is contractive ($\|T_\Sigma\| \le 1$). In case S is a strict storage function for Σ, the input-output map is a strict contraction ($\|T_\Sigma\| < 1$).*

To get further results for bicausal systems, we impose the condition (4.3), interpreted property for the bicausal setting as explained in Section 3. In particular, with the bicausal ℓ^2-exact controllability assumption in place, we get the following analogue of Remark 4.2.

Remark 6.2. We argue that *the second condition* (4.1) *(respectively,* (4.2) *for the strict case) in the definition of a storage function for a bicausal system $\Sigma = (\Sigma_-, \Sigma_+)$ (assumed to be ℓ^2-exactly controllable) can be replaced by the local condition*

$$\begin{aligned} &S(x_- \oplus (\widetilde{A}_+ x_+ + B_+ u)) - S((\widetilde{A}_- x_- + B_- u) \oplus x_+) \\ &\qquad \le \|u\|^2 - \|\widetilde{C}\widetilde{A}_- x_- + \widetilde{C}_+ x_+ + (\widetilde{C}_- \widetilde{B}_- + D)u\|^2. \end{aligned} \tag{6.1}$$

for the standard case, and by its strict version

$$\begin{aligned} &S(x_- \oplus (\widetilde{A}_+ x_+ + B_+ u)) - S((\widetilde{A}_- x_- + B_- u) \oplus x_+) \\ &\qquad + \epsilon^2(\|\widetilde{A}_- x_- + \widetilde{B}_- u\|^2 + \|x_+\|^2 + \|u\|^2) \\ &\qquad \le \|u\|^2 - \|\widetilde{C}\widetilde{A}_- x_- + \widetilde{C}_+ x_+ + (\widetilde{C}_- \widetilde{B}_- + D)u\|^2. \end{aligned} \tag{6.2}$$

for the strict case. Indeed, by translation invariance of the system equations, it suffices to check the bicausal energy-balance condition (4.1) only at $n = 0$ for any ℓ^2-admissible trajectory $(\mathbf{u}, \mathbf{x}, \mathbf{y})$. In terms of

$$x_- := \mathbf{x}_-(n+1), \quad x_+ := \mathbf{x}_+(n), \quad u := \mathbf{u}(n), \tag{6.3}$$

we can solve for the other quantities appearing in (4.1) for the case $n = 0$:

$$\begin{aligned}
\mathbf{x}_+(1) &= \widetilde{A}_+ x_+ + \widetilde{B}_+ u, \\
\mathbf{x}_-(0) &= \widetilde{A}_- x_- + \widetilde{B}_- u, \\
\mathbf{y}(0) &= \widetilde{C}_- \widetilde{A}_- x_- + C_+ x_+ + (\widetilde{C}_- \widetilde{B}_- + \widetilde{D}) u.
\end{aligned}$$

Then the energy-balance condition (4.1) for the bicausal system Σ reduces to (6.1), so (6.1) is a sufficient condition for S to be a storage function (assuming conditions (1) and (3) in the definition of a storage function also hold). Conversely, given any $x_- \in \mathcal{X}_-$, $x_+ \in \mathcal{X}_+$, $u \in \mathcal{U}$, the trajectory-interpolation result Proposition 3.3 assures us that we can always embed the vectors x_-, x_+, u into an ℓ^2-admissible trajectory so that (6.3) holds. We then see that condition (6.1) holding for all x, x', u is also necessary for S to be a storage function. The strict version works out in a similar way, again by making use of the interpolation result Proposition 3.3 .

We next define functions $S_a \colon \mathcal{X} \to \mathbb{R}$ and $S_r \colon \mathcal{X} \to \mathbb{R}$ via the formulas (4.6) and (4.7) but with $\mathbf{W}_c$ taken to be the controllability operator as in (3.21) for a bicausal system. One can also check that the following bicausal version of Proposition 4.4 holds, but again with the verification of the continuity property for S_a and S_r postponed until more detailed information concerning S_a and S_r is developed below.

Proposition 6.3. *Let $\Sigma = (\Sigma_-, \Sigma_+)$ be a bicausal system as in* (3.1)–(3.2)*, with $\widetilde{A}_\pm$ exponentially stable. Assume that* (4.3) *holds. Then:*

(1) *S_a is a storage function for Σ.*
(2) *S_r is a storage function for Σ.*
(3) *If $\widetilde{S}$ is any storage function for Σ, then*

$$S_a(x_0) \le \widetilde{S}(x_0) \le S_r(x_0) \text{ for all } x_0 \in \mathcal{X}.$$

Proof. The proof of Proposition 4.4 for the causal dichotomous setting extends verbatim to the bicausal setting once we verify that the patching technique of Lemma 4.3 holds in exactly the same form for the bicausal setting. We therefore suppose that

$$(\mathbf{u}', \mathbf{x}', \mathbf{y}'), \quad (\mathbf{u}'', \mathbf{x}'', \mathbf{y}'') \tag{6.4}$$

are ℓ^2-admissible trajectories for the bicausal system Σ such that $\mathbf{x}'(0) = \mathbf{x}''(0)$. In more detail, this means that there are two ℓ^2-admissible system trajectories of the form $(\mathbf{u}', \mathbf{x}'_-, \mathbf{y}'_-)$ and $(\mathbf{u}'', \mathbf{x}''_-, \mathbf{y}''_-)$ for the anticausal system Σ_- such that $\mathbf{x}'_-(0) = \mathbf{x}''_-(0)$ and two ℓ^2-admissible system trajectories of the form $(\mathbf{u}', \mathbf{x}'_+, \mathbf{y}'_+)$ and $(\mathbf{u}'', \mathbf{x}''_+, \mathbf{y}''_+)$ for the causal system Σ_+ with $\mathbf{x}'_+(0) = \mathbf{x}''_+(0)$ such that we recover the state and output components of the original trajectories for the bicausal system (6.4) via

$$\begin{aligned}
\mathbf{x}'(n) &= \mathbf{x}'_-(n) \oplus \mathbf{x}'_+(n), \quad & \mathbf{x}''(n) &= \mathbf{x}''_-(n) \oplus \mathbf{x}''_+(n), \\
\mathbf{y}'(n) &= \mathbf{y}'_-(n) + \mathbf{y}'_+(n), \quad & \mathbf{y}''(n) &= \mathbf{y}''_-(n) + \mathbf{y}''_+(n).
\end{aligned}$$

Let us define a composite input trajectory by

$$\mathbf{u}(n) = \begin{cases} \mathbf{u}'(n) & \text{if } n < 0, \\ \mathbf{u}''(n) & \text{if } n \geq 0. \end{cases}$$

We apply the causal patching lemma to the system Σ_+ (having trivial dichotomy) to see that the composite trajectory $(\mathbf{u}, \mathbf{x}_+, \mathbf{y}_+)$ with state and output given by

$$\mathbf{x}_+(n) = \begin{cases} \mathbf{x}'_+(n) & \text{if } n \leq 0, \\ \mathbf{x}''_+(n) & \text{if } n > 0, \end{cases} \quad \mathbf{y}_+(n) = \begin{cases} \mathbf{y}'_+(n) & \text{if } n < 0, \\ \mathbf{y}''_+(n) & \text{if } n \geq 0, \end{cases}$$

is an ℓ^2-admissible trajectory for the causal system Σ_+. Similarly, we apply a reversed-orientation version of the patching given by Lemma 4.3 to see that the trajectory $(\mathbf{u}, \mathbf{x}_-, \mathbf{y}_-)$ with state and output given by

$$\mathbf{x}_-(n) = \begin{cases} \mathbf{x}'_-(n) & \text{if } n < 0, \\ \mathbf{x}'_-(n) & \text{if } n \geq 0, \end{cases} \quad \mathbf{y}_-(n) = \begin{cases} \mathbf{y}'_-(n) & \text{if } n < 0, \\ \mathbf{y}''_-(n) & \text{if } n \geq 0 \end{cases}$$

is an ℓ^2-admissible system trajectory for the anticausal system Σ_-. It then follows from the definitions that the composite trajectory $(\mathbf{u}, \mathbf{x}, \mathbf{y})$ given by (4.8) is an ℓ^2-admissible system trajectory for the bicausal system Σ as wanted. $\square$

Quadratic storage functions and spatial KYP-inequalities: the bicausal setting. We define a quadratic function $S\colon \mathcal{X} \to \mathbb{R}$ as was done at the end of Section 4 above: $S(x) = \langle Hx, x\rangle$ where H is a bounded selfadjoint operator on $\mathcal{X}$. For the bicausal setting, we wish to make explicit that $\mathcal{X}$ has a decomposition as $\mathcal{X} = \mathcal{X}_- \oplus \mathcal{X}_+$ which we now wish to write as a column decomposition $\mathcal{X} = \left[\begin{smallmatrix} \mathcal{X}_- \\ \mathcal{X}_+ \end{smallmatrix}\right]$. After a change of coordinates which we choose not to go through explicitly, we may assume that this decomposition is orthogonal. Then any selfadjoint operator H on $\mathcal{X}$ has a 2×2 matrix representation

$$H = \begin{bmatrix} H_- & H_0 \\ H_0^* & H_+ \end{bmatrix} \text{ on } \mathcal{X} = \begin{bmatrix} \mathcal{X}_- \\ \mathcal{X}_+ \end{bmatrix}. \tag{6.5}$$

with associated quadratic function S_H now given by

$$S_H(x_- \oplus x_+) = \langle H(x_- \oplus x_+), x_- \oplus x_+\rangle = \left\langle \begin{bmatrix} H_- & H_0 \\ H_0^* & H_+ \end{bmatrix} \begin{bmatrix} x_- \\ x_+ \end{bmatrix}, \begin{bmatrix} x_- \\ x_+ \end{bmatrix} \right\rangle.$$

If we apply the local criterion for a given function S to be a storage function in the bicausal setting as given by Remark 6.2, we arrive at the following criterion for S_H to be a storage function for the bicausal system Σ:

$$\begin{aligned} &\left\langle \begin{bmatrix} H_- & H_0 \\ H_0^* & H_+ \end{bmatrix} \begin{bmatrix} x_- \\ \widetilde{A}x_+ + \widetilde{B}_+ u \end{bmatrix}, \begin{bmatrix} x_- \\ \widetilde{A}x_+ + \widetilde{B}_+ u \end{bmatrix} \right\rangle \\ &\qquad - \left\langle \begin{bmatrix} H_- & H_0 \\ H_0^* & H_+ \end{bmatrix} \begin{bmatrix} \widetilde{A}_- x_- + \widetilde{B}_- u \\ x_+ \end{bmatrix}, \begin{bmatrix} \widetilde{A}_- x_- + \widetilde{B}_- u \\ x_+ \end{bmatrix} \right\rangle \\ &\leq \|u\|^2 - \|\widetilde{C}_- \widetilde{A}_- x_- + \widetilde{C}_+ x_+ + (\widetilde{C}_- \widetilde{B}_- + \widetilde{D})u\|^2. \end{aligned}$$

which amounts to the spatial version of the bicausal KYP-inequality (1.9).

Similarly, S_H is a strict storage function exactly when there is an $\epsilon > 0$ so that

$$\begin{aligned} &\left\langle \begin{bmatrix} H_- & H_0 \\ H_0^* & H_+ \end{bmatrix} \begin{bmatrix} x_- \\ \widetilde{A}x_+ + \widetilde{B}_+ u \end{bmatrix}, \begin{bmatrix} x_- \\ \widetilde{A}x_+ + \widetilde{B}_+ u \end{bmatrix} \right\rangle \\ &\qquad - \left\langle \begin{bmatrix} H_- & H_0 \\ H_0^* & H_+ \end{bmatrix} \begin{bmatrix} \widetilde{A}_- x_- + \widetilde{B}_- u \\ x_+ \end{bmatrix}, \begin{bmatrix} \widetilde{A}_- x_- + \widetilde{B}_- u \\ x_+ \end{bmatrix} \right\rangle \\ &\qquad + \epsilon^2(\|\widetilde{A}_- x_- + \widetilde{B}_- u\|^2 + \|x_+\|^2 + \|u\|^2) \\ &\leq \|u\|^2 - \|\widetilde{C}_- \widetilde{A}_- x_- + \widetilde{C}_+ x_+ + (\widetilde{C}_- \widetilde{B}_- + \widetilde{D})u\|^2. \end{aligned}$$

One can check that this is just the spatial version of the strict bicausal KYP-inequality (1.10). One can now check that the assertion of Proposition 4.5 goes through as stated with the *dichotomous linear systems in* (1.1) replaced by a *bicausal system* (3.1)–(3.2) (with $\widetilde{A}_+$ and $\widetilde{A}_-$ both exponentially stable), and with *KYP-inequality* (respectively *strict KYP-inequality* (1.5)) replaced with *bicausal KYP-inequality* (1.9) (respectively *strict bicausal KYP-inequality* (1.10)). We have thus arrived at the following extension of Proposition 4.5 to the bicausal setting.

Proposition 6.4. *Suppose that $\Sigma = (\Sigma_-, \Sigma_+)$ is a bicausal system* (3.1)–(3.2)*, with $\widetilde{A}_\pm$ exponentially stable. Let H be a selfadjoint operator as in* (6.5)*, where we assume that coordinates are chosen so that the decomposition $\mathcal{X} = \mathcal{X}_- \oplus \mathcal{X}_+$ is orthogonal. Then S_H is a quadratic storage function for Σ if and only if H is a solution of the bicausal KYP-inequality* (1.9)*. Moreover, S_H is a strict storage function for Σ if and only if H is a solution of the strict bicausal KYP-inequality* (1.10)*.*

Furthermore, as noted in Section 3, the Hankel factorizations (3.22) also hold in the bicausal setting. Hence Lemma 5.1 goes through as stated, the only modification being the adjustment of the formulas for the operators $\mathbf{W}_o^\pm$, $\mathbf{W}_c^\pm$ to those in (3.20) (rather than (2.19), (2.20)). It then follows that Theorem 5.2 holds with exactly the same formulas (5.7), (5.8), (5.9), (5.10) for S_a, S_r, H_a and H_r, again with the adjusted formulas for the operators $\mathbf{W}_o^\pm$ and $\mathbf{W}_c^\pm$. As $S_a = S_{H_a}$ and $S_r = S_{H_r}$ with H_a and H_r bounded and boundedly invertible selfadjoint operators on $\mathcal{X}_- \oplus \mathcal{X}_+$, it follows that S_a and S_r are continuous, completing the missing piece in the proof of Proposition 6.3 above. We have arrived at the following extension of Theorem 5.2 to the bicausal setting.

Theorem 6.5. *Suppose that $\Sigma = (\Sigma_-, \Sigma_+)$ is a bicausal system* (3.1)–(3.2)*, with $\widetilde{A}_\pm$ exponentially stable, satisfying the standing hypothesis* (4.3)*. Define the available storage S_a and the required supply S_r as in* (4.6)–(4.7) *(properly interpreted for the bicausal rather than dichotomous setting). Then S_a and S_r are continuous. In detail, S_a and S_r are given by the formulas* (5.7)–(5.8)*, or equivalently, $S_a = S_{H_a}$ and $S_r = S_{H_r}$ where H_a and H_r are given explicitly as in* (5.9) *and* (5.10)*.*

Remark 6.6. A nice exercise is to check that the bicausal KYP-inequality (1.9) collapses to the standard KYP-inequality (1.4) in the case that $\widetilde{A}_-$ is invertible so that the the bicausal system $\widetilde{\Sigma}$ can be converted to a dichotomous system as in Remark 3.2. Let us assume that $\widetilde{\Sigma}$ is a bicausal system as in (3.1) and (3.2). We assume that $\widetilde{A}$ is invertible and we make the substitution (3.17) to convert to a dichotomous linear system as in (1.1), (2.1), (2.2). The resulting bicausal

KYP-inequality then becomes

$$\begin{bmatrix} I & 0 & A_-^{-1*}C_-^* \\ 0 & A_+^* & C_+^* \\ 0 & B_+^* & -B_-^*A_-^{*-1}C_-^*+D^* \end{bmatrix} \begin{bmatrix} H_- & H_0 & 0 \\ H_0^* & H_+ & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & A_+ & B_+ \\ C_-A_-^{-1} & C_+ & -C_-A_-^{-1}B_-+D \end{bmatrix}$$

$$\preceq \begin{bmatrix} A_-^{-1*} & 0 & 0 \\ 0 & I & 0 \\ -B_-^*A_-^{*-1} & 0 & I \end{bmatrix} \begin{bmatrix} H_- & H_0 & \\ H_0^* & H_+ & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A_-^{-1} & 0 & -A_-^{-1}B_- \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}. \tag{6.6}$$

However the spatial version of the bicausal KYP-inequality (1.9) corresponds to the quadratic form based at the vector $\begin{bmatrix} \mathbf{x}_-(1) \\ \mathbf{x}_+(0) \\ \mathbf{u}(0) \end{bmatrix}$ while the spatial version of the dichotomous (causal) KYP-inequality (1.4) is the quadratic form based at the vector $\begin{bmatrix} \mathbf{x}_-(0) \\ \mathbf{x}_+(0) \\ \mathbf{u}(0) \end{bmatrix}$, where the conversion from the latter to the former is given by

$$\begin{bmatrix} \mathbf{x}_-(0) \\ \mathbf{x}_+(0) \\ \mathbf{u}(0) \end{bmatrix} = \begin{bmatrix} A_- & 0 & B_- \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \mathbf{x}_-(1) \\ \mathbf{x}_+(0) \\ \mathbf{u}(0) \end{bmatrix}.$$

To recover the dichotomous KYP-inequality (1.4) from (6.6), it therefore still remains to conjugate both sides of (6.6) by $T = \begin{bmatrix} A_- & 0 & B_- \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$ (i.e., multiply on the right by T and on the left by T^*). Note next that

$$\begin{bmatrix} I & 0 & 0 \\ 0 & A_+ & B_+ \\ C_-A_-^{-1} & C_+ & -C_-A_-^{-1}B_-+D \end{bmatrix} \begin{bmatrix} A_- & 0 & B_- \\ 0 & I & 0 \\ 0 & D & 0 \end{bmatrix} = \begin{bmatrix} A_- & 0 & B_- \\ 0 & A_+ & B_+ \\ C_- & C_+ & D \end{bmatrix},$$

$$\begin{bmatrix} A_-^{-1} & 0 & -A_-^{-1}B \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A_- & 0 & B_- \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}.$$

Hence conjugation of both sides of (6.6) by T results in

$$\begin{bmatrix} A_-^* & 0 & C_-^* \\ 0 & A_+^* & C_+^* \\ B_-^* & B_+^* & D^* \end{bmatrix} \begin{bmatrix} H_- & H_0 & 0 \\ H_0^* & H_- & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A_- & 0 & B_- \\ 0 & A_- & B_+ \\ C_- & C_+ & D \end{bmatrix} \preceq \begin{bmatrix} H_- & H_0 & 0 \\ H_0^* & H_- & 0 \\ 0 & 0 & I \end{bmatrix}$$

which is just the dichotomous KYP-inequality (1.4) written out when the matrices are expressed in the decomposed form (2.1), (2.2), (6.5).

The connection between the strict KYP-inequalities for the bicausal setting (1.10) and the dichotomous setting (1.5) works out similarly. In fact all the results presented here for dichotomous systems follow from the corresponding result for the bicausal setting by restricting to the associated bicausal system having $\widetilde{A}_- = A_-^{-1}$ invertible.

7. Dichotomous and bicausal Bounded Real Lemmas

In this section we derive infinite-dimensional versions of the finite-dimensional Bounded Real Lemmas stated in the introduction.

Combining the results of Propositions 4.1, 4.4, 4.5 and Theorem 5.2 leads us to the following infinite-dimensional version of the standard dichotomous Bounded Real Lemma; this result contains Theorem 1.2 (1), as stated in the introduction, as a corollary.

Theorem 7.1. Standard dichotomous Bounded Real Lemma: *Assume that the linear system* Σ *in* (1.1) *has a dichotomy and is dichotomously* ℓ^2*-exactly controllable and observable (both* $\mathbf{W}_c$ *and* $\mathbf{W}_o^*$ *are surjective). Then the following are equivalent:*

(1) $\|F_\Sigma\|_{\infty,\mathbb{T}} := \sup_{z\in\mathbb{T}} \|F_\Sigma(z)\| \leq 1$.
(2) *There is a bounded and boundedly invertible selfadjoint operator H on $\mathcal{X}$ which satisfies the KYP-inequality* (1.4). *Moreover, the dimension of the spectral subspace of A over the unit disk (respectively, exterior of the closed unit disk) agrees with the dimension of the spectral subspace of H over the positive real line (respectively, over the negative real line).*

We shall next show how the infinite-dimensional version of the strict dichotomous Bounded Real Lemma (Theorem 1.2 (2)) can be reduced to the standard version (Theorem 7.1) by the same technique used for the stable (non-dichotomous) case (see [18, 6, 7]). The result is as follows; the reader can check that specializing the result to the case where all signal spaces $\mathcal{U}$, $\mathcal{X}$, $\mathcal{Y}$ are finite-dimensional results in Theorem 1.2 (2) from the introduction as a corollary. Note that, as in the non-dichotomous case (see [6, Theorem 1.6]), there is no controllability or observability condition required here.

Theorem 7.2. Strict dichotomous Bounded Real Lemma: *Assume that the linear system Σ in* (1.1) *has a dichotomy. Then the following are equivalent:*

(1) $\|F_\Sigma\|_{\infty,\mathbb{T}} := \sup_{z\in\mathbb{T}} \|F_\Sigma(z)\| < 1$.
(2) *There is a bounded and boundedly invertible selfadjoint operator H on $\mathcal{X}$ which satisfies the strict KYP-inequality* (1.5). *Moreover the inertia of A partitioned by the unit circle lines up with the inertia of H (partitioned by the point* 0 *on the real line) as in the standard dichotomous Bounded Real Lemma (Theorem 7.1 above).*

Proof. The proof of (2) $\Rightarrow$ (1) is a consequence of Propositions 4.1, 4.4, and 4.5, so it suffices to prove (1) $\Rightarrow$ (2). To simplify the notation, we again write F rather than F_Σ throughout this proof.

We therefore assume that $\|F\|_{\infty,\mathbb{T}} < 1$. For $\epsilon > 0$, we let Σ_ϵ be the discrete-time linear system (1.1) with system matrix M_ϵ given by

$$M_\epsilon = \begin{bmatrix} A & B_\epsilon \\ C_\epsilon & D_\epsilon \end{bmatrix} := \left[\begin{array}{c|cc} A & B & \epsilon I_{\mathcal{X}} \\ \hline C & D & 0 \\ \epsilon I_{\mathcal{X}} & 0 & 0 \\ 0 & \epsilon I_{\mathcal{U}} & 0 \end{array}\right] \tag{7.1}$$

with associated transfer function

$$\begin{aligned} F_\epsilon(z) &= \begin{bmatrix} D & 0 \\ 0 & 0 \\ \epsilon I_{\mathcal{U}} & 0 \end{bmatrix} + z \begin{bmatrix} C \\ \epsilon I_{\mathcal{X}} \\ 0 \end{bmatrix} (I - zA)^{-1} \begin{bmatrix} B & \epsilon I_{\mathcal{X}} \end{bmatrix} \\ &= \begin{bmatrix} F(z) & \epsilon z C (I - zA)^{-1} \\ \epsilon z (I - zA)^{-1} B & \epsilon^2 z (I - zA)^{-1} \\ \epsilon I_{\mathcal{U}} & 0 \end{bmatrix}. \end{aligned} \tag{7.2}$$

As M and M_ϵ have the same state-dynamics operator A, the system Σ_ϵ inherits the dichotomy property from Σ. As the resolvent expression $z(I - zA)^{-1}$ is uniformly bounded in norm on $\mathbb{T}$, the fact that $\|F\|_{\infty,\mathbb{T}} < 1$ implies that $\|F_\epsilon\|_{\infty,\mathbb{T}} < 1$ as long as $\epsilon > 0$ is chosen sufficiently small. Moreover, when we decompose B_ϵ and C_ϵ

according to (2.2), we get

$$B_\epsilon = \begin{bmatrix} B_- & \epsilon I_{\mathcal{X}_-} & 0 \\ B_+ & 0 & \epsilon I_{\mathcal{X}_+} \end{bmatrix} : \begin{bmatrix} \mathcal{U} \\ \mathcal{X}_- \\ \mathcal{X}_+ \end{bmatrix} \to \begin{bmatrix} \mathcal{X}_- \\ \mathcal{X}_+ \end{bmatrix},$$

$$C_\epsilon = \begin{bmatrix} C_- & C_+ \\ \epsilon I_{\mathcal{X}_-} & 0 \\ 0 & \epsilon I_{\mathcal{X}_+} \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{X}_- \\ \mathcal{X}_+ \end{bmatrix} \to \begin{bmatrix} \mathcal{Y} \\ \mathcal{X}_- \\ \mathcal{X}_+ \\ \mathcal{U} \end{bmatrix},$$

or specifically

$$B_{\epsilon -} = \begin{bmatrix} B_- & \epsilon I_{\mathcal{X}_-} & 0 \end{bmatrix}, \quad B_{\epsilon +} = \begin{bmatrix} B_+ & 0 & \epsilon I_{\mathcal{X}_+} \end{bmatrix},$$

$$C_{\epsilon -} = \begin{bmatrix} C \\ \epsilon I_{\mathcal{X}_-} \\ 0 \\ 0 \end{bmatrix}, \quad C_{\epsilon +} = \begin{bmatrix} C_+ \\ 0 \\ \epsilon I_{\mathcal{X}_+} \\ 0 \end{bmatrix}.$$

Hence we see that $(A_+, B_{\epsilon +})$ is exactly controllable in one step and hence is ℓ^2-exactly controllable. Similarly, $(A_-^{-1}, A_-^{-1} B_{\epsilon -})$ is ℓ^2-exactly controllable and both $(C_{\epsilon +}, A_+)$ and $(C_{\epsilon -} A_-^{-1}, A_-^{-1})$ are ℓ^2-exactly observable. As we also have $\|F_\epsilon\|_{\infty,\mathbb{T}} < 1$, in particular $\|F_\epsilon\|_{\infty,\mathbb{T}} \le 1$, so Theorem 7.1 applies to the system Σ_ϵ. We conclude that there is bounded, boundedly invertible, selfadjoint operator H_ϵ on $\mathcal{X}$ so that the KYP-inequality holds with respect to the system Σ_ϵ:

$$\begin{bmatrix} A^* & C_\epsilon^* \\ B_\epsilon^* & D_\epsilon^* \end{bmatrix} \begin{bmatrix} H_\epsilon & 0 \\ 0 & I_{\mathcal{Y} \oplus \mathcal{X} \oplus \mathcal{U}} \end{bmatrix} \begin{bmatrix} A & B_\epsilon \\ C_\epsilon & D_\epsilon \end{bmatrix} \preceq \begin{bmatrix} H_\epsilon & 0 \\ 0 & I_{\mathcal{U} \oplus \mathcal{X}} \end{bmatrix}.$$

Spelling this out gives

$$\begin{bmatrix} A^* H_\epsilon A + C^* C + \epsilon^2 I_{\mathcal{X}} & A^* H_\epsilon B + C^* D & \epsilon A^* H_\epsilon \\ B^* H_\epsilon A + D^* C & B^* H_\epsilon B + D^* D + \epsilon^2 I_{\mathcal{U}} & \epsilon B^* H_\epsilon \\ \epsilon H_\epsilon A & \epsilon H_\epsilon B & \epsilon^2 H_\epsilon \end{bmatrix} \preceq \begin{bmatrix} H_\epsilon & 0 & 0 \\ 0 & I_{\mathcal{U}} & 0 \\ 0 & 0 & I_{\mathcal{X}} \end{bmatrix}.$$

By crossing off the third row and third column, we get the inequality

$$\begin{bmatrix} A^* H_\epsilon A + C^* C + \epsilon^2 I_{\mathcal{X}} & A^* H_\epsilon B + C^* D \\ B^* H_\epsilon A + D^* C & B^* H_\epsilon B + D^* D + \epsilon^2 I_{\mathcal{U}} \end{bmatrix} \preceq \begin{bmatrix} H_\epsilon & 0 \\ 0 & I_{\mathcal{U}} \end{bmatrix}$$

or

$$\begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix} \begin{bmatrix} H_\epsilon & 0 \\ 0 & I_{\mathcal{Y}} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} + \epsilon^2 \begin{bmatrix} I_{\mathcal{X}} & 0 \\ 0 & I_{\mathcal{U}} \end{bmatrix} \preceq \begin{bmatrix} H_\epsilon & 0 \\ 0 & I_{\mathcal{U}} \end{bmatrix}.$$

We conclude that H_ϵ serves as a solution to the strict KYP-inequality (1.5) for the original system Σ as wanted. □

The results in Section 6 for bicausal systems lead to the following extensions of Theorems 7.1 and 7.2 to the bicausal setting; note that Theorem 1.3 in the introduction follows as a corollary of this result.

Theorem 7.3. *Suppose that $\Sigma = (\Sigma_+, \Sigma_-)$ is a bicausal linear system with subsystems Σ_+ and Σ_- as in* (3.1) *and* (3.2)*, respectively, with both A_+ and A_- exponentially stable and with associated transfer function F_Σ as in* (3.9).

(1) *Assume that Σ is ℓ^2-exactly minimal, i.e., the operators $\mathbf{W}_c^+$, $\mathbf{W}_c^-$, $(\mathbf{W}_o^+)^*$, $(\mathbf{W}_o^-)^*$ given by* (3.20) *are all surjective. Then $\|F_\Sigma\|_{\infty,\mathbb{T}} \leq 1$ if and only if there exists a bounded and boundedly invertible selfadjoint solution $H = \left[\begin{smallmatrix} H_- & H_0 \\ H_0^* & H_+ \end{smallmatrix}\right]$ of the bicausal KYP-inequality* (1.9).

(2) *Furthermore, $\|F_\Sigma\|_{\infty,\mathbb{T}} < 1$ holds if and only if there is a bounded and boundedly invertible selfadjoint solution $H = \left[\begin{smallmatrix} H_- & H_0 \\ H_0^* & H_+ \end{smallmatrix}\right]$ of the strict bicausal KYP-inequality* (1.10).

Proof. To verify item (1), simply combine the results of Propositions 6.1, 6.3, 6.4 and Theorem 6.5.

As for item (2), note that sufficiency follows already from the stream of Propositions 6.1, 6.3 and 6.4. As for necessity, let us verify that the same ϵ-augmented-system technique as used in the proof of Theorem 7.2 can be used to reduce the strict case of the result (item (2)) to the standard case (item (1)). Let us rewrite the bicausal KYP-inequality (1.9) as

$$\begin{bmatrix} H_- + \widetilde{A}_-^* \widetilde{C}_-^* \widetilde{C}_- \widetilde{A}_- & H_0 \widetilde{A}_+ + \widetilde{A}_-^* \widetilde{C}_-^* \widetilde{C}_+ & H_0 \widetilde{B}_+ + \widetilde{A}_-^* \widetilde{C}_-^* \widehat{D} \\ \widetilde{A}_+^* H_0^* + \widetilde{C}_+^* \widetilde{C}_- \widetilde{A}_- & \widetilde{A}_+^* H_+ \widetilde{A}_+ + \widetilde{C}_+^* \widetilde{C}_+ & \widetilde{A}_+^* H_+ \widetilde{B}_+ + \widetilde{C}_+^* \widehat{D} \\ \widetilde{B}_+^* H_0^* + \widehat{D}^* \widetilde{C}_- \widetilde{A}_- & \widetilde{B}_+^* H_+ \widetilde{A}_+ + \widehat{D}^* \widetilde{C}_+ & \widetilde{B}_+^* H_+ \widetilde{B}_+ + \widehat{D}^* \widehat{D} \end{bmatrix} \preceq \begin{bmatrix} \widetilde{A}_-^* H_- \widetilde{A}_- & \widetilde{A}_-^* H_0 & \widetilde{A}_-^* H_- \widetilde{B}_- \\ H_0^* \widetilde{A}_- & H_+ & H_0^* \widetilde{B}_- \\ \widetilde{B}_-^* H_- \widetilde{A}_- & \widetilde{B}_-^* H_0 & \widetilde{B}_-^* H_- \widetilde{B}_- + I \end{bmatrix} \tag{7.3}$$

where we set

$$\widehat{D} = \widetilde{C}_- \widetilde{B}_- + \widetilde{D}.$$

Let us now consider the ϵ-augmented system matrices

$$M_{+,\epsilon} = \begin{bmatrix} \widetilde{A}_+ & \widetilde{B}_{+,\epsilon} \\ \widetilde{C}_{+,\epsilon} & \widetilde{D}_\epsilon \end{bmatrix} = \left[\begin{array}{c|cc} \widetilde{A}_+ & \widetilde{B}_+ & \epsilon I_{\mathcal{X}_+} \\ \hline \widetilde{C}_+ & \widetilde{D} & 0 \\ \epsilon I_{\mathcal{X}_+} & 0 & 0 \\ 0 & \epsilon I_{\mathcal{U}} & 0 \end{array}\right] : \begin{bmatrix} \mathcal{X}_+ \\ \hline \mathcal{U} \\ \mathcal{X}_+ \end{bmatrix} \to \begin{bmatrix} \mathcal{X}_+ \\ \hline \mathcal{Y} \\ \mathcal{X}_+ \\ \mathcal{U} \end{bmatrix},$$

$$M_{-,\epsilon} = \begin{bmatrix} \widetilde{A}_- & \widetilde{B}_{-,\epsilon} \\ \widetilde{C}_{-,\epsilon} & 0 \end{bmatrix} = \left[\begin{array}{c|cc} \widetilde{A}_- & \widetilde{B}_- & \epsilon I_{\mathcal{X}_-} \\ \hline \widetilde{C}_- & 0 & 0 \\ \epsilon I_{\mathcal{X}_-} & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] : \begin{bmatrix} \mathcal{X}_- \\ \hline \mathcal{U} \\ \mathcal{X}_- \end{bmatrix} \to \begin{bmatrix} \mathcal{X}_- \\ \hline \mathcal{Y} \\ \mathcal{X}_- \\ \mathcal{U} \end{bmatrix}.$$

Then the system matrix-pair $(M_{\epsilon,+}, M_{\epsilon,-})$ defines a bicausal system $\Sigma_\epsilon = (\Sigma_{\epsilon,+}, \Sigma_{\epsilon,-})$ where the subsystem $\Sigma_{\epsilon,+}$ is associated the system matrix $M_{\epsilon,+}$ and the subsystem $\Sigma_{\epsilon,-}$ is associated the system matrix $M_{\epsilon,-}$. Note that the state-dynamics operators $\widetilde{A}_+$ and $\widetilde{A}_-$ of Σ_ϵ are exponentially stable and has transfer function F_ϵ given by

$$\begin{aligned} F_\epsilon(z) &= \begin{bmatrix} \widetilde{D} & 0 \\ 0 & 0 \\ \epsilon I_{\mathcal{U}} & 0 \end{bmatrix} + z \begin{bmatrix} \widetilde{C}_+ \\ \epsilon I_{\mathcal{X}_+} \\ 0 \end{bmatrix} (I - z\widetilde{A}_+)^{-1} \begin{bmatrix} \widetilde{B}_+ & \epsilon I_{\mathcal{X}_+} \end{bmatrix} \\ &\quad + \begin{bmatrix} \widetilde{C}_- \\ \epsilon I_{\mathcal{X}_-} \\ 0 \end{bmatrix} (I - z^{-1}\widetilde{A}_-)^{-1} \begin{bmatrix} \widetilde{B}_- & \epsilon I_{\mathcal{X}_-} \end{bmatrix} \\ &= \begin{bmatrix} F(z) & \epsilon z \widetilde{C}_+ (I - z\widetilde{A}_+)^{-1} + \epsilon \widetilde{C}_- (I - z\widetilde{A}_-)^{-1} \\ \epsilon z (I - z\widetilde{A}_+)^{-1} \widetilde{B}_+ + \epsilon (I - z^{-1}\widetilde{A}_-)^{-1} \widetilde{B}_- & \epsilon^2 z (I - z\widetilde{A}_+)^{-1} + \epsilon^2 (I - z^{-1}\widetilde{A}_-)^{-1} \\ \epsilon I_{\mathcal{U}} & 0 \end{bmatrix}. \end{aligned}$$

Since by assumption the transfer function F associated with the original bicausal system $\Sigma = (\Sigma_+, \Sigma_-)$ has norm $\|F\|_{\infty,\mathbb{T}} < 1$ and both $\widetilde{A}_+$ and $\widetilde{A}_-$ are exponentially stable, it is clear that we can maintain $\|F_\epsilon\|_{\infty,\mathbb{T}} < 1$ with $\epsilon > 0$ as long as we choose ϵ sufficiently small. Due to the presence of the identity matrices in the input and output operators for $M_{+,\epsilon}$ and $M_{-,\epsilon}$, it is clear that the bicausal system Σ_ϵ is ℓ^2-exactly controllable and ℓ^2-exactly observable in the bicausal sense. Then statement (1) of the present theorem (already verified) applies and we are assured that there is a bounded and boundedly invertible solution $H = \begin{bmatrix} H_+ & H_0 \\ H_0^* & H_- \end{bmatrix}$ of the bicausal KYP-inequality (7.3) associated with $\Sigma_\epsilon = (\Sigma_{+,\epsilon}, \Sigma_{-,\epsilon})$. Replacing $\widetilde{B}_\pm$, $\widetilde{C}_\pm$ and $\widetilde{D}$ in (7.3) by $\widetilde{B}_{\pm,\epsilon}$, $\widetilde{C}_{\pm,\epsilon}$ and $\widetilde{D}_\epsilon$ leads to the ϵ-augmented version of the bicausal KYP-inequality:

$$\begin{bmatrix} H_- + \widetilde{A}_-^* \widetilde{C}_-^* \widetilde{C}_- \widetilde{A}_- + \epsilon^2 \widetilde{A}_-^* \widetilde{A}_- & H_0 \widetilde{A}_+ + \widetilde{A}_-^* \widetilde{C}_-^* \widetilde{C}_+ ! & H_0 \widetilde{B}_+ + \widetilde{A}_-^* \widetilde{C}_-^* \widehat{D} + \epsilon^2 \widetilde{A}_-^* \widetilde{B}_- & X_{14} \\ \widetilde{A}_+^* H_0^* + \widetilde{C}_+^* \widetilde{C}_- \widetilde{A}_- & \widetilde{A}_+^* H_+ \widetilde{A}_+ + \widetilde{C}_+^* \widetilde{C}_+ + \epsilon^2 I & \widetilde{A}_+ H_+ \widetilde{B}_+ + \widetilde{C}_+^* \widehat{D} & X_{24} \\ \widetilde{B}_+^* H_0^* + \widehat{D}^* \widetilde{C}_- \widetilde{A}_- + \epsilon^2 \widetilde{B}_-^* \widetilde{A}_- & \widetilde{B}_+^* H_+ \widetilde{A}_+ + \widehat{D}^* \widetilde{C}_+ & \widetilde{B}_+^* H_+ \widetilde{B}_+ + \widehat{D}^* \widehat{D} + \epsilon^2 \widetilde{B}_-^* \widetilde{B}_- + \epsilon^2 I_{\mathcal{U}} & X_{34} \\ \epsilon H_0^* + \epsilon \widetilde{C}_-^* \widetilde{C}_- \widetilde{A}_- + \epsilon^2 \widetilde{A}_- & \epsilon H_+ \widetilde{A}_+ + \epsilon \widetilde{C}_-^* \widetilde{C}_+ & \epsilon \widetilde{C}_-^* \widehat{D} + \epsilon^2 \widetilde{B}_- & X_{44} \end{bmatrix}$$

(7.4)

$$\preceq \begin{bmatrix} \widetilde{A}_-^* H_- \widetilde{A}_- & \widetilde{A}_-^* H_0 & \widetilde{A}_-^* H_- \widetilde{B}_- & \epsilon \widetilde{A}_-^* H_- \\ H_0^* \widetilde{A}_- & H_+ & H_0^* \widetilde{B}_- & \epsilon H_0^* \\ \widetilde{B}_-^* H_- \widetilde{A}_- & \widetilde{B}_-^* H_0 & \widetilde{B}_-^* H_- \widetilde{B}_- + I & \epsilon \widetilde{B}_-^* H_- \\ \epsilon H_- \widetilde{A}_- & \epsilon H_0 & \epsilon H_- \widetilde{B}_- & \epsilon^2 H_- + I \end{bmatrix}$$

where

$$\begin{bmatrix} X_{14} \\ X_{24} \\ X_{34} \\ X_{44} \end{bmatrix} = \begin{bmatrix} \epsilon H_0 + \epsilon \widetilde{A}_-^* \widetilde{C}_-^* \widetilde{C}_- + \epsilon^2 \widetilde{A}_-^* \\ \epsilon \widetilde{A}_+^* H_+ + \epsilon \widetilde{C}_+^* \widetilde{C}_- \\ \epsilon \widehat{D}^* \widetilde{C}_- + \epsilon^2 \widetilde{B}_-^* \\ \epsilon^2 \widetilde{C}_-^* \widetilde{C}_- + \epsilon^2 I_{\mathcal{X}_-} \end{bmatrix}.$$

The (4×4)-block matrices appearing in (7.4) are to be understood as operators on $\begin{bmatrix} \mathcal{X}_- \\ \mathcal{X}_+ \\ \mathcal{U} \\ \mathcal{X} \end{bmatrix}$ where the last component $\mathcal{X}$ further decomposes as $\mathcal{X} = \begin{bmatrix} \mathcal{X}_- \\ \mathcal{X}_+ \end{bmatrix}$. Note that the operators in the fourth row a priori are operators with range in $\mathcal{X}_-$ or $\mathcal{X}_+$; to get the proper interpretation of these operators as mapping in $\mathcal{X}$, one must compost each of these operators on the left by the canonical injection of $\mathcal{X}_\pm$ into $\mathcal{X} = \begin{bmatrix} \mathcal{X}_- \\ \mathcal{X}_+ \end{bmatrix}$. Similarly the operators in the fourth columns a priori are defined only on $\mathcal{X}_-$ or $\mathcal{X}_+$; each of these should be composed on the right with the canonical projection of $\mathcal{X}$ onto $\mathcal{X}_\pm$. On the other hand the identity operator I appearing in the $(4,4)$-entry of the matrix on the right is the identity on the whole space $\mathcal{X} = \begin{bmatrix} \mathcal{X}_- \\ \mathcal{X}_+ \end{bmatrix}$.

With this understanding of the interpretation for the fourth row and fourth column of the matrices in (7.4) in place, the next step is to simply cross out the last row and last column in (7.4) to arrive at the reduced block-(3×3) inequality

$$\begin{bmatrix} H_- + \widetilde{A}_-^* \widetilde{C}_-^* \widetilde{C}_- \widetilde{A}_- & H_0 \widetilde{A}_+ + \widetilde{A}_-^* \widetilde{C}_-^* \widetilde{C}_+ & H_0 \widetilde{B}_+ + \widetilde{A}_-^* \widetilde{C}_-^* \widehat{D} \\ \widetilde{A}_+^* H_0^* + \widetilde{C}_+^* \widetilde{C}_- \widetilde{A}_- & \widetilde{A}_+^* H_+ \widetilde{A}_+ + \widetilde{C}_+^* \widetilde{C}_+ & \widetilde{A}_+ H_+ \widetilde{B}_+ + \widetilde{C}_+^* \widehat{D} \\ \widetilde{B}_+^* H_0^* + \widehat{D}^* \widetilde{C}_- \widetilde{A}_- & \widetilde{B}_+^* H_+ \widetilde{A}_+ + \widehat{D}^* \widetilde{C}_+ & \widetilde{B}_+^* H_+ \widetilde{B}_+ + \widehat{D}^* \widehat{D} \end{bmatrix}$$

$$+ \epsilon^2 \begin{bmatrix} \widetilde{A}_-^* \widetilde{A}_- & 0 & \widetilde{A}_-^* \widetilde{B}_- \\ 0 & I_{\mathcal{X}_+} & 0 \\ \widetilde{B}_- \widetilde{A}_- & 0 & \widetilde{B}_-^* \widetilde{B}_- + I_{\mathcal{U}} \end{bmatrix} \preceq \begin{bmatrix} \widetilde{A}_-^* H_- \widetilde{A}_- & \widetilde{A}_-^* H_0 & \widetilde{A}_-^* H_- \widetilde{B}_- \\ H_0^* \widetilde{A}_- & H_+ & H_0^* \widetilde{B}_- \\ \widetilde{B}_-^* H_- \widetilde{A}_- & \widetilde{B}_-^* H_0 & \widetilde{B}_-^* H_- \widetilde{B}_- + I_{\mathcal{U}} \end{bmatrix}.$$

This last inequality amounts to the spelling out of the strict version of the bicausal KYP-inequality (1.9), i.e., to (1.10). $\square$

8. Bounded Real Lemma for nonstationary systems with dichotomy

In this section we show how the main result of Ben Artzi-Gohberg-Kaashoek in [10] (see also [14, Chapter 3] for closely related results) follows from Theorem 7.2

by the technique of embedding a nonstationary discrete-time linear system into an infinite-dimensional stationary (time-invariant) linear system (see [13, Chapter X]) and applying the corresponding result for stationary linear systems. We note that Ben Artzi-Gohberg-Kaashoek took the reverse path: they obtain the result for the stationary case as a corollary of the result for the non-stationary case.

In this section we replace the stationary linear system (1.1) with a nonstationary (or time-varying) linear system of the form

$$\Sigma_{\text{non-stat}} \colon = \left\{ \begin{array}{rcl} \mathbf{x}(n+1) & = & A_n \mathbf{x}(n) + B_n \mathbf{u}(n), \\ \mathbf{y}(n) & = & C_n \mathbf{x}(n) + D_n \mathbf{u}(n), \end{array} \right. \quad (n \in \mathbb{Z}) \tag{8.1}$$

where $\{A_n\}_{n\in\mathbb{Z}}$ is a bilateral sequence of state space operators ($A_n \in \mathcal{L}(\mathcal{X})$), $\{B_n\}_{n\in\mathbb{Z}}$ is a bilateral sequence of input operators ($B_n \in \mathcal{L}(\mathcal{U},\mathcal{X})$), $\{C_n\}_{n\in\mathbb{Z}}$ is a bilateral sequence of output operators ($C_n \in \mathcal{L}(\mathcal{X},\mathcal{Y})$), and $\{D_n\}_{n\in\mathbb{Z}}$ is a bilateral sequence of feedthrough operators ($D_n \in \mathcal{L}(\mathcal{U},\mathcal{Y})$). We assume that all the operator sequences $\{A_n\}_{n\in\mathbb{Z}}$, $\{B_n\}_{n\in\mathbb{Z}}$, $\{C_n\}_{n\in\mathbb{Z}}$, $\{D_n\}_{n\in\mathbb{Z}}$ are uniformly bounded in operator norm. The system $\Sigma_{\text{non}-\text{stat}}$ is said to have *dichotomy* if there is a bounded sequence of projection operators $\{R_n\}_{n\in\mathbb{Z}}$ on $\mathcal{X}$ such that

(1) $\operatorname{Rank} R_n$ is constant and the equalities
$$A_n R_n = R_{n+1} A_n$$
hold for all $n \in \mathbb{Z}$,

(2) there are constants a and b with $a < 1$ so that

$$\|A_{n+j-1} \cdots A_n x\| \le b a^j \|x\| \text{ for all } x \in \operatorname{Im} R_n, \tag{8.2}$$

$$\|A_{n+j-1} \cdots A_n y\| \ge \frac{1}{b a^j} \|y\| \text{ for all } y \in \operatorname{Ker} R_n. \tag{8.3}$$

Let us introduce spaces

$$\vec{\mathcal{U}} := \ell^2_{\mathcal{U}}(\mathbb{Z}), \quad \vec{\mathcal{X}} := \ell^2_{\mathcal{X}}(\mathbb{Z}), \quad \vec{\mathcal{Y}} := \ell^2_{\mathcal{Y}}(\mathbb{Z}).$$

and define bounded operators $\mathbf{A} \in \mathcal{L}(\vec{\mathcal{X}})$, $\mathbf{B} \in \mathcal{L}(\vec{\mathcal{U}}, \vec{\mathcal{X}})$, $\mathbf{C} \in \mathcal{L}(\vec{\mathcal{X}}, \vec{\mathcal{Y}})$, $\mathbf{D} \in \mathcal{L}(\vec{\mathcal{U}}, \vec{\mathcal{Y}})$ by

$$\mathbf{A} = \operatorname{diag}_{n\in\mathbb{Z}}[A_n], \quad \mathbf{B} = \operatorname{diag}_{n\in\mathbb{Z}}[B_n], \quad \mathbf{C} = \operatorname{diag}_{n\in\mathbb{Z}}[C_n], \quad \mathbf{D} = \operatorname{diag}_{n\in\mathbb{Z}}[D_n].$$

Define the shift operator $\mathbf{S}$ on $\vec{\mathcal{X}}$ by

$$\mathbf{S} = [\delta_{i,j+1} I_{\mathcal{X}}]_{i,j\in\mathbb{Z}}$$

with inverse $\mathbf{S}^{-1}$ given by

$$\mathbf{S}^{-1} = [\delta_{i+1,j} I_{\mathcal{X}}]_{i,j\in\mathbb{Z}},$$

Then the importance of the dichotomy condition is that $\mathbf{S}^{-1} - \mathbf{A}$ is invertible on $\ell^2_{\mathcal{X}}(\mathbb{Z})$, and conversely, $\mathbf{S}^{-1} - \mathbf{A}$ invertible implies that the system (8.1) has dichotomy (see [10, Theorem 2.2]). In this case, given any ℓ^2-sequence $\mathbf{x}' \in \vec{\mathcal{X}}$, the equation

$$\mathbf{S}^{-1}\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{x}' \tag{8.4}$$

admits a unique solution $\mathbf{x} = (\mathbf{S}^{-1} - \mathbf{A})^{-1}\mathbf{x}' \in \ell^2_{\mathcal{X}}(\mathbb{Z})$. Write out $\mathbf{x}$ as $\mathbf{x} = \{\mathbf{x}(n)\}_{n\in\mathbb{Z}}$. Then the aggregate equation (8.4) amounts to the system of equations

$$\mathbf{x}(n+1) = A_n \mathbf{x}(n) + \mathbf{x}'(n). \tag{8.5}$$

In particular we may take $\mathbf{x}'(n)$ to be of the form $\mathbf{x}'(n) = B_n\mathbf{u}(n)$ where $\mathbf{u} = \{\mathbf{u}(n)\}_{n\in\mathbb{Z}} \in \ell^2_{\mathcal{U}}(\mathbb{Z})$. Then we may uniquely solve for $\mathbf{x} = \{\mathbf{x}(n)\}_{n\in\mathbb{Z}} \in \vec{\mathcal{X}}$ so that

$$\mathbf{x}(n+1) = A_n\mathbf{x}(n) + B\mathbf{u}(n).$$

We may then use the output equation in (8.1) to arrive at an output sequence $\mathbf{y} = \{\mathbf{y}(n)\}_{n\in\mathbb{Z}} \in \vec{\mathcal{Y}}$ by

$$\mathbf{y}(n) = C_n\mathbf{x}(n) + D_n\mathbf{u}(n).$$

Thus there is a well-defined map T_Σ which maps the sequence $\mathbf{u} = \{\mathbf{u}(n)\}_{n\in\mathbb{Z}}$ in $\vec{\mathcal{U}}$ to the sequence $\mathbf{y} = \{\mathbf{y}(n)\}_{n\in\mathbb{Z}}$ in $\vec{\mathcal{Y}}$. Roughly speaking, here the initial condition is replaced by boundary conditions at $\pm\infty$: $R\mathbf{x}(-\infty) = 0$ and $(I - R)\mathbf{x}(+\infty) = 0$; we shall use the more precise albeit more implicit operator-theoretic formulation of the input-output map (compare also to the discussion around (3.2) and (3.1) for this formulation in the stationary setting): $T_\Sigma\colon \vec{\mathcal{U}} \to \vec{\mathcal{Y}}$ is defined as: *given a* $\mathbf{u} = \{\mathbf{u}(n)\}_{n\in\mathbb{Z}} \in \vec{\mathcal{U}}$, $T_\Sigma\mathbf{u}$ *is the unique* $\mathbf{y} = \{\mathbf{y}(n)\}_{n\in\mathbb{Z}} \in \vec{\mathcal{Y}}$ *for which there is a* $\mathbf{x} = \{\mathbf{x}(n)\}_{n\in\mathbb{Z}} \in \vec{\mathcal{X}}$ *so that the system equations* (8.1) *hold for all* $n \in \mathbb{Z}$, or in aggregate operator-form, by

$$T_\Sigma = \mathbf{D} + \mathbf{C}(\mathbf{S}^{-1} - \mathbf{A})^{-1}\mathbf{B} = \mathbf{D} + \mathbf{C}(I - \mathbf{S}\mathbf{A})^{-1}\mathbf{S}\mathbf{B}.$$

The main theorem from [10] can be stated as follows.

Theorem 8.1. *(See* [10, Theorem 1.1]*.) Given a nonstationary input-state-output linear system* (8.1)*, the following conditions are equivalent.*

(1) *The system* (8.1) *is dichotomous and the associated input-output operator* T_Σ *has operator norm strictly less than 1 (*$\|T_\Sigma\| < 1$*).*
(2) *There exists a sequence of constant-inertia invertible selfadjoint operators* $\{H_n\}_{n\in\mathbb{Z}} \in \mathcal{L}(\mathcal{X})$ *with both* $\|H_n\|$ *and* $\|H_n^{-1}\|$ *uniformly bounded in* $n \in \mathbb{Z}$ *such that*

$$\begin{bmatrix} A_n^* & C_n^* \\ B_n^* & D_n^* \end{bmatrix} \begin{bmatrix} H_{n+1} & 0 \\ 0 & I_{\mathcal{Y}} \end{bmatrix} \begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix} \prec \begin{bmatrix} H_n & 0 \\ 0 & I_{\mathcal{U}} \end{bmatrix} \tag{8.6}$$

for all $n \in \mathbb{Z}$.

Proof. One can check that the nonstationary dichotomy condition (8.2)–(8.3) on the operator sequence $\{A_n\}_{n\in\mathbb{Z}}$ translates to the stationary dichotomy condition on $\mathbf{A}$ with $\vec{\mathcal{X}}_+ = \operatorname{Im}\mathbf{R}$, $\vec{\mathcal{X}}_- = \operatorname{Ker}\mathbf{R}$ where $\mathbf{R} = \operatorname{diag}_{n\in\mathbb{Z}}[R_n]$. Then

$$\mathbf{U} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} : \begin{bmatrix} \vec{\mathcal{X}} \\ \vec{\mathcal{U}} \end{bmatrix} \to \begin{bmatrix} \vec{\mathcal{X}} \\ \vec{\mathcal{Y}} \end{bmatrix}$$

is the system matrix for a big stationary dichotomous linear system

$$\boldsymbol{\Sigma} := \left\{ \begin{array}{rcl} \vec{\mathbf{x}}(n+1) & = & \mathbf{A}\vec{\mathbf{x}}(n) + \mathbf{B}\vec{\mathbf{u}}(n) \\ \vec{\mathbf{y}}(n+1) & = & \mathbf{C}\vec{\mathbf{x}}(n) + \mathbf{D}\vec{\mathbf{u}}(n) \end{array} \right. \tag{8.7}$$

where

$$\vec{\mathbf{u}} = \{\vec{\mathbf{u}}(n)\}_{n\in\mathbb{Z}} \in \ell^2_{\vec{\mathcal{U}}}(\mathbb{Z}), \quad \vec{\mathbf{x}} = \{\vec{\mathbf{x}}(n)\}_{n\in\mathbb{Z}} \in \ell^2_{\vec{\mathcal{X}}}(\mathbb{Z}),$$
$$\vec{\mathbf{y}} = \{\vec{\mathbf{y}}(n)\}_{n\in\mathbb{Z}} \in \ell^2_{\vec{\mathcal{Y}}}(\mathbb{Z}).$$

To apply Theorem 7.2 to this enlarged stationary dichotomous system $\mathbf{\Sigma}$, we first need to check that the input-output map $T_{\mathbf{\Sigma}}$ is strictly contractive. Toward this end, for each $k \in \mathbb{Z}$ introduce a linear operator

$$\sigma_{k,\mathcal{U}} \colon \vec{\mathcal{U}} = \ell^2_{\mathcal{U}}(\mathbb{Z}) \to \ell^2_{\vec{\mathcal{U}}}(\mathbb{Z})$$

defined by

$$\mathbf{u} = \{\mathbf{u}(n)\}_{n\in\mathbb{Z}} \to \sigma_k \mathbf{u} = \{\vec{\mathbf{u}}^{(k)}(n)\}_{n\in\mathbb{Z}}$$

where we set

$$\vec{\mathbf{u}}^{(k)}(n) = \{\delta_{m,n+k}\mathbf{u}(n)\}_{m\in\mathbb{Z}} \in \ell^2_{\mathcal{U}}(\mathbb{Z}) = \vec{\mathcal{U}}.$$

In the same way we define $\sigma_{k,\mathcal{X}}$ and $\sigma_{k,\mathcal{Y}}$, changing only $\mathcal{U}$ to $\mathcal{X}$ and $\mathcal{Y}$, respectively, in the definition:

$$\sigma_{k,\mathcal{X}} \colon \vec{\mathcal{X}} = \ell^2_{\mathcal{X}}(\mathbb{Z}) \mapsto \ell^2_{\vec{\mathcal{X}}}(\mathbb{Z}), \quad \sigma_{k,\mathcal{Y}} \colon \vec{\mathcal{Y}} = \ell^2_{\mathcal{Y}}(\mathbb{Z}) \mapsto \ell^2_{\vec{\mathcal{Y}}}(\mathbb{Z}).$$

Then one can check that $\sigma_{k,\mathcal{U}}$, $\sigma_{k,\mathcal{X}}$ and $\sigma_{k,\mathcal{Y}}$ are all isometries. Furthermore, the operators

$$\begin{aligned}
\sigma_{\vec{\mathcal{U}}} &= \begin{bmatrix} \cdots & \sigma_{-1,\mathcal{U}} & \sigma_{0,\mathcal{U}} & \sigma_{1,\mathcal{U}} & \cdots \end{bmatrix} : \ell^2_{\vec{\mathcal{U}}}(\mathbb{Z}) \to \ell^2_{\vec{\mathcal{U}}}(\mathbb{Z}),\\
\sigma_{\vec{\mathcal{X}}} &= \begin{bmatrix} \cdots & \sigma_{-1,\mathcal{X}} & \sigma_{0,\mathcal{X}} & \sigma_{1,\mathcal{X}} & \cdots \end{bmatrix} : \ell^2_{\vec{\mathcal{X}}}(\mathbb{Z}) \to \ell^2_{\vec{\mathcal{X}}}(\mathbb{Z}),\\
\sigma_{\vec{\mathcal{Y}}} &= \begin{bmatrix} \cdots & \sigma_{-1,\mathcal{Y}} & \sigma_{0,\mathcal{Y}} & \sigma_{1,\mathcal{Y}} & \cdots \end{bmatrix} : \ell^2_{\vec{\mathcal{Y}}}(\mathbb{Z}) \to \ell^2_{\vec{\mathcal{Y}}}(\mathbb{Z}),
\end{aligned}$$

are all unitary. The relationship between the input-output map $T_{\mathbf{\Sigma}}$ for the stationary system $\mathbf{\Sigma}$ and the input-output map T_Σ for the original nonstationary system is encoded in the identity

$$T_{\mathbf{\Sigma}} = \bigoplus_{k=-\infty}^{\infty} \sigma_{k,\mathcal{Y}}\, \mathcal{S}_{\mathcal{Y}}^{*k}\, T_\Sigma\, \mathcal{S}_{\mathcal{U}}^{k}\, \sigma_{k,\mathcal{U}}^* = \sigma_{\vec{\mathcal{Y}}} \operatorname{diag}_{k\in\mathbb{Z}} [T_\Sigma]\, \sigma_{\vec{\mathcal{U}}}^*$$

where $\mathcal{S}_{\mathcal{U}}$ is the bilateral shift on $\ell^2_{\mathcal{U}}(\mathbb{Z})$ and $\mathcal{S}_{\mathcal{Y}}$ is the bilateral shift on $\ell^2_{\mathcal{Y}}(\mathbb{Z})$, i.e., the input-output map $T_{\mathbf{\Sigma}}$ is unitarily equivalent to the infinite inflation $T_\Sigma \otimes I_{\ell^2(\mathbb{Z})} = \operatorname{diag}_{k\in\mathbb{Z}}[T_\Sigma]$ of T_Σ. In particular, it follows that $\|T_{\mathbf{\Sigma}}\| = \|T_\Sigma\|$, and hence the hypothesis that $\|T_\Sigma\| < 1$ implies that also $\|T_{\mathbf{\Sigma}}\| < 1$.

We may now apply Theorem 7.2 to conclude that there is a bounded invertible selfadjoint operator $\mathbf{H}$ on $\vec{\mathcal{X}}$ such that

$$\begin{bmatrix} \mathbf{A}^* & \mathbf{C}^* \\ \mathbf{B}^* & \mathbf{D}^* \end{bmatrix} \begin{bmatrix} \mathbf{H} & 0 \\ 0 & I_{\vec{\mathcal{Y}}} \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} - \begin{bmatrix} \mathbf{H} & 0 \\ 0 & I_{\vec{\mathcal{U}}} \end{bmatrix} \prec 0. \tag{8.8}$$

Conjugate this identity with the isometry σ_0:

$$\begin{aligned}
&\begin{bmatrix} \sigma_0^* & 0 \\ 0 & \sigma_0^* \end{bmatrix} \begin{bmatrix} \mathbf{A}^* & \mathbf{C}^* \\ \mathbf{B}^* & \mathbf{D}^* \end{bmatrix} \begin{bmatrix} \mathbf{H} & 0 \\ 0 & I_{\vec{\mathcal{Y}}} \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \sigma_0 & 0 \\ 0 & \sigma_0 \end{bmatrix}\\
&\qquad - \begin{bmatrix} \sigma_0^* & 0 \\ 0 & \sigma_0^* \end{bmatrix} \begin{bmatrix} \mathbf{H} & 0 \\ 0 & I_{\vec{\mathcal{U}}} \end{bmatrix} \begin{bmatrix} \sigma_0 & 0 \\ 0 & \sigma_0 \end{bmatrix} \prec 0.
\end{aligned} \tag{8.9}$$

If we define $H_n \in \mathcal{L}(\mathcal{X})$ by

$$H_n = \iota_n^*\, \sigma_0^*\, \mathbf{H}\, \sigma_0\, \iota_n$$

where ι_n is the embedding of $\mathcal{X}$ into the n-th coordinate subspace of $\ell^2_{\mathcal{X}}(\mathbb{Z})$, then one can check that the identity (8.8) collapses to the identity (8.6) holding for all $n \in \mathbb{Z}$. This completes the proof of Theorem 8.1. □

Acknowledgement. This work is based on the research supported in part by the National Research Foundation of South Africa. Any opinion, finding and conclusion or recommendation expressed in this material is that of the authors and the NRF does not accept any liability in this regard.

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