EQUIVARIANT DISSIPATION IN NON-ARCHIMEDEAN GROUPS

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ABSTRACT. We prove that, if a topological group G has an open subgroup of infinite index, then every net of tight Borel probability measures on G UEB-converging to invariance dissipates in G in the sense of Gromov. In particular, this solves a 2006 problem by Pestov: for every left-invariant (or right-invariant) metric d on the infinite symmetric group Sym(\mathbb{N}), compatible with the topology of pointwise convergence, the sequence of the finite symmetric groups (Sym $(n), d \upharpoonright_{\text{Sym}(n)}, \mu_{\text{Sym}(n)})_{n \in \mathbb{N}}$ equipped with the restricted metrics and their normalized counting measures dissipates, thus fails to admit a subsequence being Cauchy with respect to Gromov's observable distance.

1. INTRODUCTION

In his seminal work on metric measure geometry [Gro99, Chapter $3\frac{1}{2}$], Gromov introduced the observable distance, $d_{\rm conc}$, a metric on the set of isomorphism classes of *mm*-spaces, i.e., separable complete metric spaces equipped with a Borel probability measure. This metric generates an interesting topology, commonly referred to as the concentration topology, which generalizes the Lévy concentration property [Lév22, Mil67, MS86] in a very natural way: a sequence of *mm*-spaces constitutes a Lévy family if and only if it converges to a singleton space with respect Gromov's concentration topology. Inspired by the work of Gromov and Milman [GM83] on applications of concentration to dynamics of topological groups, Pestov proposed to study instances of concentration to non-trivial spaces in the context of topological groups [Pes06, Section 7.4] (see also his work with Giordano [GP07, Section 7]).

In the present note, we study Gromov's observable distance with regard to non-archimedean topological groups, i.e., those whose neutral element admits a neighborhood basis consisting of (open) subgroups. More specifically, our focus will be on the topological group $\text{Sym}(\mathbb{N})$ of all permutations of the set \mathbb{N} of natural numbers, endowed with the topology of pointwise convergence. In [GW02], Glasner and Weiss showed that the closed subspace $\text{LO}(\mathbb{N}) \subseteq 2^{\mathbb{N} \times \mathbb{N}}$ of linear orders on \mathbb{N} , equipped with the natural continuous left $\text{Sym}(\mathbb{N})$ -action given by

$$x \xrightarrow{g} y \iff g^{-1}x \prec g^{-1}y \qquad (g \in \operatorname{Sym}(\mathbb{N}), \, \prec \in \operatorname{LO}(\mathbb{N}), \, x, y \in \mathbb{N})$$

constitutes the universal minimal flow of $\text{Sym}(\mathbb{N})$ and admits a unique $\text{Sym}(\mathbb{N})$ -invariant Borel probability measure μ_{LO} . Their results prompted Pestov to pose the following question.

Problem 1.1 ([Pes06], Problem 7.4.27). For each $n \in \mathbb{N}$, let us denote by μ_n the normalized counting measure on Sym $(n) \subseteq$ Sym (\mathbb{N}) . Do there exist compatible metrics d_{LO} on $\text{LO}(\mathbb{N})$ and d_{Sym} , left-invariant, on Sym (\mathbb{N}) such that

$$d_{\rm conc}(({\rm Sym}(n), d_{\rm Sym}|_{{\rm Sym}(n)}, \mu_n), ({\rm LO}(\mathbb{N}), d_{\rm LO}, \mu_{\rm LO})) \longrightarrow 0 \quad (n \longrightarrow \infty) ?$$

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The purpose of this note is to resolve Problem 1.1 in the negative. In fact, our Corollary 5.5 particularly entails that, if d is any left-invariant compatible metric on $\text{Sym}(\mathbb{N})$, then the sequence $(\text{Sym}(n), d|_{\text{Sym}(n)}, \mu_n)_{n \in \mathbb{N}}$ does not even admit a d_{conc} -Cauchy subsequence, where μ_n denotes the normalized counting measure on Sym(n) for each $n \in \mathbb{N}$. Our argument proving Corollary 5.5 does indeed establish *dissipation* (Definition 3.3), a phenomenon stronger than the negation of concentration (cf. Corollary 3.10, Remark 3.11), and moreover works in much greater generality, thus allowing us to deduce a dichotomy concerning the geometric behavior of asymptotically invariant nets of Borel probability measures in arbitrary non-archimedean topological groups (Theorem 5.1, Corollary 5.3, Corollary 5.7).

This note is organized as follows. In Section 2 we recollect some basic material from metric geometry, most importantly, the Gromov-Hausdorff distance and Gromov's compactness theorem. Then, Section 3 is devoted to a short introduction to mm-spaces and observable distance, as well as a brief comparison of the concepts of concentration and dissipation. In Section 4 we provide the background on UEB-convergence to invariance in topological groups necessary for Section 5, where we turn our attention towards non-archimedean topological groups, prove the aforementioned dichotomy, and infer the solution to Pestov's Problem 1.1.

2. Metric geometry: Gromov's compactness theorem

In this section, we recollect some very few bits of metric geometry, the most important of which will be the Gromov-Hausdorff distance and Gromov's compactness theorem. For more on this, the reader is referred to [BBI01] or [Shi16, Chapter 3].

For a start, let us briefly clarify some basic notation and terminology concerning metric spaces. By a *compatible* metric on a (metrizable) topological space X, we will mean a metric generating the topology of X. Let $\mathcal{X} = (X, d)$ be a pseudo-metric space. The *diameter* of \mathcal{X} is defined as diam $(\mathcal{X}) := \sup\{d(x, y) \mid x, y \in X\}$. Given any real number $\ell \geq 0$, let us denote by $\operatorname{Lip}_{\ell}(\mathcal{X})$ the set of all ℓ -Lipschitz real-valued functions on \mathcal{X} , and define

$$\operatorname{Lip}_{\ell}^{s}(\mathcal{X}) := \{ f \in \operatorname{Lip}_{\ell}(\mathcal{X}) \mid \sup_{x \in X} |f(x)| \le s \}$$

for any real number $s \ge 0$. For a real number $\varepsilon > 0$, a subset $B \subseteq X$ is said to be ε -discrete in \mathcal{X} if $d(x, y) > \varepsilon$ for any two distinct $x, y \in B$, and the ε -capacity of \mathcal{X} is defined as

$$\operatorname{Cap}_{\varepsilon}(\mathcal{X}) := \sup\{|B| \mid B \subseteq X \text{ }\varepsilon\text{-discrete in } \mathcal{X}\}$$

Given a subset $A \subseteq X$, we abbreviate $d \upharpoonright_A := d|_{A \times A}$. For $x \in A \subseteq X$ and $\varepsilon > 0$, we let

$$B_d(x,\varepsilon) := \{ y \in X \mid d(x,y) < \varepsilon \}, \qquad B_d(A,\varepsilon) := \{ y \in X \mid \exists a \in A \colon d(a,y) < \varepsilon \}.$$

The Hausdorff distance of two subsets $A, B \subseteq X$ in \mathcal{X} is denoted by

$$\mathrm{H}_{\mathcal{X}}(A,B) := \mathrm{H}_{d}(A,B) := \inf\{\varepsilon > 0 \mid B \subseteq B_{d}(A,\varepsilon), A \subseteq B_{d}(B,\varepsilon)\}.$$

Definition 2.1. The *Gromov-Hausdorff distance* between any two arbitrary compact metric spaces $\mathcal{X} = (X, d_X)$ and $\mathcal{Y} = (Y, d_Y)$ is defined as

 $d_{\mathrm{GH}}(\mathcal{X}, \mathcal{Y}) := \inf \left\{ \mathrm{H}_{\mathcal{Z}}(\varphi(X), \psi(Y)) \, | \, \mathcal{Z} \text{ metric space, } \varphi \colon \mathcal{X} \to \mathcal{Z}, \, \psi \colon \mathcal{Y} \to \mathcal{Z} \text{ isom. emb.} \right\}.$

The Gromov-Hausdorff distance of compact metric spaces is easily seen to be invariant under isometries, i.e., $d_{\text{GH}}(\mathcal{X}_0, \mathcal{X}_1) = d_{\text{GH}}(\mathcal{Y}_0, \mathcal{Y}_1)$ for any two pairs of isometrically isomorphic compact metric spaces $\mathcal{X}_i \cong \mathcal{Y}_i$ ($i \in \{0, 1\}$). Furthermore, d_{GH} gives a complete metric on the set of isomorphism classes of compact metric spaces [Shi16, Lemma 3.9]. In particular, two compact metric spaces \mathcal{X} and \mathcal{Y} are isometrically isomorphic if and only if $d_{\text{GH}}(\mathcal{X}, \mathcal{Y}) = 0$. Let us recall below a useful description of d_{GH} -precompactness in terms of capacities, known as Gromov's compactness theorem. We will say that a set \mathcal{C} of (isometry classes of) compact metric spaces has uniformly bounded capacity if $\sup_{\mathcal{X}\in\mathcal{C}} \operatorname{Cap}_{\varepsilon}(\mathcal{X}) < \infty$ for every $\varepsilon > 0$.

Theorem 2.2 (cf. [Shi16], Lemma 3.12; [BBI01], Section 7.4.2). A set C of isometry classes of compact metric spaces is d_{GH} -precompact if and only if C has uniformly bounded capacity and $\sup_{\mathcal{X} \in C} \operatorname{diam}(\mathcal{X}) < \infty$.

3. Metric measure geometry: concentration vs. dissipation

We now turn to the study of measured metric spaces. In this section, we will only briefly review the concepts of *concentration* and *dissipation*, two phenomena at opposite ends of the spectrum of the asymptotic behavior of measured metric spaces. A more substantial account on metric measure geometry is to be found in [Gro99, Shi16]. For more details on the classical phenomenon of measure concentration, the reader is referred to [MS86, Led01].

To begin with, let us address some few matters of notation. Let X be a measurable space. As usual, we will denote by δ_x the Dirac measure on X associated with a point $x \in X$. Furthermore, given a finite non-empty subset $F \subseteq X$, we will consider the probability measure $\delta_F := |F|^{-1} \sum_{x \in F} \delta_x$ on X. We denote by P(X) the set of all probability measures on X. Consider any $\mu \in P(X)$. If B is a measurable subset of X with $\mu(B) = 1$, then we may consider the probability measure $\mu \upharpoonright_B$ on the measurable subspace B given by $\mu \upharpoonright_B := \mu(A)$ for every measurable subset $A \subseteq B$. The *push-forward measure* $f_*(\mu)$ of μ along a measurable map $f: X \to Y$ into another measurable space Y is defined by $f_*(\mu)(B) := \mu(f^{-1}(B))$ for every measurable $B \subseteq Y$. Moreover, we obtain a pseudo-metric me_µ on the set of all measurable real-valued functions X defined by

$$\operatorname{me}_{\mu}(f,g) := \inf\{\varepsilon > 0 \mid \mu(\{x \in X \mid |f(x) - g(x)| > \varepsilon\}) \le \varepsilon\}$$

for any two measurable functions $f, g: X \to \mathbb{R}$. Finally, let ν be a Borel probability measure on a Hausdorff topological space T. Then the *support* of ν is defined as

$$\operatorname{spt} \nu := \{ x \in T \mid \forall U \subseteq T \text{ open} \colon x \in U \Longrightarrow \nu(U) > 0 \},\$$

which forms a closed subset of T. Furthermore, ν will be called *tight* [Bog07, Definition 7.1.4] if for every $\varepsilon > 0$ there exists a compact subset $K \subseteq T$ with $\nu(K) \ge 1 - \varepsilon$, and ν will be called *regular* if $\nu(B) = \sup\{\nu(K) \mid K \subseteq B, K \text{ compact}\}$ for every Borel subset $B \subseteq T$. Evidently, regularity implies tightness. Conversely, if T is metrizable and ν is tight, then ν will be regular as well (see, e.g., [Par67, Chapter II, Theorem 3.1]). For additional measuretheoretic background, we refer to [Par67, Bog07].

Definition 3.1. Let $\mathcal{X} = (X, d, \mu)$ be an *mm-space*, that is, (X, d) is a separable complete metric space and μ is a Borel probability measure on X. We will call \mathcal{X} compact if (X, d) is compact, and fully supported if spt $\mu = X$. A parametrization of \mathcal{X} is a Borel measurable map $\varphi \colon [0, 1] \to X$ such that $\varphi_*(\lambda) = \mu$, where λ denotes the Lebesgue measure on [0, 1]. We will call two *mm*-spaces $\mathcal{X}_0 = (X_0, d_0, \mu_0)$ and $\mathcal{X}_1 = (X_1, d_1, \mu_1)$ isomorphic and write $\mathcal{X}_0 \cong \mathcal{X}_1$ if there exists an isometry

$$f: (\operatorname{spt} \mu_0, d_0 \upharpoonright_{\operatorname{spt} \mu_0}) \longrightarrow (\operatorname{spt} \mu_1, d_1 \upharpoonright_{\operatorname{spt} \mu_1})$$

such that $f_*(\mu_0 \upharpoonright_{\operatorname{spt} \mu_0}) = \mu_1 \upharpoonright_{\operatorname{spt} \mu_1}$.

It is well known that any *mm*-space admits a parametrization (see e.g. [Shi16, Lemma 4.2]).

Definition 3.2. The observable distance between two mm-spaces \mathcal{X} and \mathcal{Y} is defined to be

 $d_{\rm conc}(\mathcal{X},\mathcal{Y}) := \inf \left\{ \mathrm{H}_{\mathrm{me}_{\lambda}}(\mathrm{Lip}_{1}(\mathcal{X}) \circ \varphi, \mathrm{Lip}_{1}(\mathcal{Y}) \circ \psi) \, | \, \varphi \text{ param. of } \mathcal{X}, \, \psi \text{ param. of } \mathcal{Y} \right\}.$

A sequence $(\mathcal{X}_n)_{n\in\mathbb{N}}$ of *mm*-spaces is said to *concentrate to* an *mm*-space \mathcal{X} if

$$\lim_{n\to\infty} d_{\operatorname{conc}}(\mathcal{X}_n, \mathcal{X}) = 0$$

It is easy to see that the observable distance of *mm*-spaces is invariant under isomorphisms, which means that $d_{\text{conc}}(\mathcal{X}_0, \mathcal{X}_1) = d_{\text{conc}}(\mathcal{Y}_0, \mathcal{Y}_1)$ for any two pairs of isomorphic *mm*-spaces $\mathcal{X}_i \cong \mathcal{Y}_i \ (i \in \{0,1\})$. Furthermore, d_{conc} induces a metric on the set of isomorphism classes of mm-spaces, see [Shi16, Theorem 5.16]. In particular, this entails that two mm-spaces \mathcal{X} and \mathcal{Y} are isomorphic if and only if $d_{\text{conc}}(\mathcal{X}, \mathcal{Y}) = 0$.

Let us proceed to the concept of dissipation, cf. [Gro99, Chapter $3\frac{1}{2}$.J], [Shi16, Chapter 8]. As will become evident in Corollary 3.10, dissipating sequences of mm-spaces are, in a certain sense, as far from being convergent with respect to Gromov's observable distance as possible. The observation most crucial for the proof of Corollary 3.10 will be given in Lemma 3.8.

Definition 3.3. Let $\mathcal{X} = (X, d, \mu)$ be an *mm*-space. For every $m \in \mathbb{N} \setminus \{0\}$ and real numbers $\kappa_0, \ldots, \kappa_m > 0$, the corresponding *separation distance* is defined as

$$\operatorname{Sep}(\mathcal{X};\kappa_0,\ldots,\kappa_m) := \sup_{B \in [\mathcal{X};\kappa_0,\ldots,\kappa_m]} \inf\{d(x,y) \mid i,j \in \{0,\ldots,m\}, i \neq j, x \in B_i, y \in B_j\},$$
where we abbreviate

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$$[\mathcal{X};\kappa_0,\ldots,\kappa_m] := \{ (B_0,\ldots,B_m) \mid B_0,\ldots,B_m \subseteq X \text{ Borel}, \, \mu(B_0) \ge \kappa_0,\ldots,\mu(B_m) \ge \kappa_m \}$$

and the infima and suprema are taken in the interval $[0,\infty)$.¹ For any $m \in \mathbb{N} \setminus \{0\}$ and any real number $\alpha > 0$, we define $\operatorname{Sep}_m(\mathcal{X}; \alpha) := \operatorname{Sep}(\mathcal{X}; \kappa_0, \ldots, \kappa_m)$ where $\kappa_0 = \ldots = \kappa_m = \alpha$. With regard to a real number $\delta > 0$, a sequence of *mm*-spaces $(\mathcal{X}_n)_{n \in \mathbb{N}}$ is said to δ -dissipate if, for any $m \in \mathbb{N} \setminus \{0\}$ and real numbers $\kappa_0, \ldots, \kappa_m > 0$ with $\sum_{i=0}^m \kappa_i < 1$,

$$\liminf_{n\to\infty} \operatorname{Sep}(\mathcal{X}_n; \kappa_0, \dots, \kappa_m) \geq \delta.$$

A sequence of mm-spaces is said to dissipate if it δ -dissipates for some $\delta > 0$.

Let us note the following useful reformulation of dissipation.

Lemma 3.4. Let $\delta > 0$. A sequence of mm-spaces $\mathcal{X}_n = (X_n, d_n, \mu_n)$ $(n \in \mathbb{N})$ δ -dissipates if and only if, for every $\tau \in (0, \delta)$, there exists a sequence $(\mathcal{B}_n)_{n \in \mathbb{N}}$ such that

- (0) for each $n \in \mathbb{N}$, \mathcal{B}_n is a finite set of Borel subsets of X_n ,
- (1) $\inf\{d_n(x,y) \mid x \in B, y \in C\} \ge \tau$ for every $n \in \mathbb{N}$ and any two distinct $B, C \in \mathcal{B}_n$,
- (2) $\lim_{n\to\infty} \mu_i(\bigcup \mathcal{B}_n) = 1$, and
- (3) $\lim_{n\to\infty} \sup\{\mu_n(B) \mid B \in \mathcal{B}_n\} = 0.$

Proof. (\Longrightarrow) Let $\tau \in (0, \delta)$. For each $n \in \mathbb{N}$, we define

$$k_n := \sup \left\{ k \in \{1, \dots, n\} \, | \, \operatorname{Sep}_{k-1}(\mathcal{X}_n; (k+1)^{-1}) > \tau \right\}.$$

Since $(\mathcal{X}_n)_{n\in\mathbb{N}}$ δ -dissipates, $k_n \longrightarrow \infty$ as $n \longrightarrow \infty$. For each $n \in \mathbb{N}$, let us choose Borel subsets $B_{n,0},\ldots,B_{n,k_n-1}\subseteq X_n$ such that

- $\inf\{d_n(x,y) \mid x \in B_{n,i}, y \in B_{n,j}\} \ge \tau$ for any two distinct $i, j \in \{0, \ldots, k_n 1\}$,
- $\mu_n(B_{n,i}) \ge \frac{1}{k_n+1}$ for each $i \in \{0, \dots, k_n-1\}$.

¹For any $x \in X$ there exists $n \in \mathbb{N} \setminus \{0\}$ such that $\mu(B_d(x, n)) > 1 - \min\{\kappa_0, \ldots, \kappa_m\}$, which is easily seen to imply that $\operatorname{Sep}(\mathcal{X}; \kappa_0, \ldots, \kappa_m) \leq 2n$. In particular, $\operatorname{Sep}(\mathcal{X}; \kappa_0, \ldots, \kappa_m) < \infty$.

Then (0) and (1) hold with respect to the sequence $\mathcal{B}_n := \{B_{n,i} \mid i \in \{0, \dots, k_n - 1\}\}$ $(n \in \mathbb{N})$. Moreover,

$$\mu_n \Big(\bigcup \mathcal{B}_n\Big) = \sum_{i=0}^{k_n - 1} \mu_n(B_{n,i}) \ge \frac{k_n}{k_n + 1}, \qquad \sup\{\mu_n(B) \mid B \in \mathcal{B}_n\} \le 1 - \frac{k_n - 1}{k_n + 1}$$

for every $n \in \mathbb{N}$, which entails that $(\mathcal{B}_n)_{n \in \mathbb{N}}$ also satisfies (3) and (4).

(\Leftarrow) Consider an integer $m \geq 1$ and real numbers $\kappa_0, \ldots, \kappa_m > 0$ such that $\sum_{i=0}^m \kappa_i < 1$. To prove that $\liminf_{n\to\infty} \operatorname{Sep}(\mathcal{X}_n; \kappa_0, \ldots, \kappa_n) \geq \delta$, consider any $\tau \in (0, \delta)$. By our hypothesis, there exists a sequence $(\mathcal{B}_n)_{n\in\mathbb{N}}$ satisfying (0)–(3) with respect to $(\mathcal{X}_n)_{n\in\mathbb{N}}$ and τ . Since \mathbb{Q}^{m+1} is dense in \mathbb{R}^{m+1} , we find positive integers q, p_0, \ldots, p_m such that $\sum_{i=0}^m p_i < q$ and $\kappa_i \leq p_i q^{-1}$ for each $i \in \{0, \ldots, m\}$. Upon multiplying q, p_0, \ldots, p_m by a suitable positive integer, we may furthermore assume that $m + 2 + \sum_{i=0}^m p_i \leq q$. By (2) and (3), there is $\ell \in \mathbb{N}$ such that

$$\forall n \in \mathbb{N}_{\geq \ell}: \qquad \mu_n \left(\bigcup \mathcal{B}_n\right) \geq 1 - \frac{1}{q}, \qquad \sup\{\mu_n(B) \mid B \in \mathcal{B}_n\} \leq \frac{1}{q}. \tag{*}$$

We now claim that

$$\forall n \in \mathbb{N}_{\geq \ell}: \qquad \operatorname{Sep}(\mathcal{X}_n; \kappa_0, \dots, \kappa_m) \geq \tau.$$

Let $n \in \mathbb{N}_{\geq \ell}$. Thanks to (*), there exists a subset $\mathcal{C}_0 \subseteq \mathcal{B}_n$ such that $\frac{p_0}{q} \leq \mu_n(\bigcup \mathcal{C}_0) < \frac{p_0+1}{q}$. Recursively, for $\{1, \ldots, m\}$, having chosen pairwise disjoint subsets $\mathcal{C}_0, \ldots, \mathcal{C}_{i-1} \subseteq \mathcal{B}_n$ so that

$$\frac{p_0}{q} \le \mu_n(\mathcal{C}_0) < \frac{p_0+1}{q}, \quad \dots, \quad \frac{p_{i-1}}{q} \le \mu_n(\mathcal{C}_{i-1}) < \frac{p_{i-1}+1}{q}$$

we may apply (*) to find a subset $C_i \subseteq \mathcal{B}_n \setminus (C_0 \cup \ldots \cup C_{i-1})$ such that $\frac{p_i}{q} \leq \mu_n(\bigcup C_i) < \frac{p_i+1}{q}$. For each $i \in \{0, \ldots, m\}$, we obtain a Borel subset $D_i := \bigcup C_i \subseteq X_n$ with $\mu_n(D_i) \geq p_i q^{-1} \geq \kappa_i$. Moreover, $\inf\{d_n(x, y) \mid x \in D_i, y \in D_j\} \geq \tau$ for any two distinct $i, j \in \{0, \ldots, m\}$, which entails that $\operatorname{Sep}(\mathcal{X}_n; \kappa_0, \ldots, \kappa_m) \geq \tau$, as desired. This shows that $(\mathcal{X}_n)_{n \in \mathbb{N}} \delta$ -dissipates. \Box

As mentioned above, dissipation is a strong form of non-concentration. Making this evident will require some preliminary considerations. We start off with a fairly general fact.

Lemma 3.5. Let (X, d, μ) be a fully supported mm-space. Then $(\text{Lip}_{\ell}^{s}(X, d), \text{me}_{\mu})$ is a compact metric space for any two real numbers $\ell, s \geq 0$.

Proof. Note that me_{μ} is a metric on $\operatorname{Lip}_{\ell}^{s}(X,d)$, because $\operatorname{spt} \mu = X$. Since $\operatorname{Lip}_{\ell}^{s}(X,d)$ is a compact subset of the product space \mathbb{R}^{X} and equicontinuous, the Arzelà-Ascoli theorem, in the form of [Kel75, 7.15, pp. 232], asserts that $\operatorname{Lip}_{\ell}^{s}(X,d)$ is compact with respect to the topology τ_{C} of uniform convergence on compact subsets of X. To prove compactness of the metric space $(\operatorname{Lip}_{\ell}^{s}(X,d), \operatorname{me}_{\mu})$, it thus suffices to show that the topology τ_{M} generated by the metric me_{μ} on $\operatorname{Lip}_{\ell}^{s}(X,d)$ is contained in τ_{C} . To this end, let $U \in \tau_{M}$. Consider any $f \in U$. As $U \in \tau_{M}$, there exists $\varepsilon > 0$ such that $B_{\operatorname{me}_{\mu}}(f, \varepsilon) \subseteq U$. Being a Borel probability measure on a Polish space, μ must be tight (see, e.g., [Par67, Chapter II, Theorem 3.2]). Hence, there exists a compact subset $K \subseteq X$ with $\mu(K) > 1 - \varepsilon$. In turn,

$$\{g \in \operatorname{Lip}_{\ell}^{s}(X,d) \mid \sup_{x \in K} |f(x) - g(x)| < \varepsilon\} \subseteq B_{\operatorname{me}_{\mu}}(f,\varepsilon) \subseteq U,$$

which entails that U is a neighborhood of f in τ_C . This shows that $U \in \tau_C$. Thus, $\tau_M \subseteq \tau_C$ as desired. (In fact, $\tau_M = \tau_C$ since τ_M is Hausdorff and τ_C is compact.)

Our next observation relates the observable distance of mm-spaces with the Gromov-Hausdorff distance of the corresponding spaces of bounded Lipschitz functions.

Lemma 3.6. Let $\mathcal{X}_0 = (X_0, d_0, \mu_0)$ and $\mathcal{X}_1 = (X_1, d_1, \mu_1)$ be two fully supported mm-spaces. For any two real numbers $\ell \geq 1$ and $s \geq 0$,

 $d_{\mathrm{GH}}\left(\left(\mathrm{Lip}_{\ell}^{s}(X_{0}, d_{0}), \mathrm{me}_{\mu_{0}}\right), \left(\mathrm{Lip}_{\ell}^{s}(X_{1}, d_{1}), \mathrm{me}_{\mu_{1}}\right)\right) \leq \ell d_{\mathrm{conc}}(\mathcal{X}_{0}, \mathcal{X}_{1}).$

Proof. For each $i \in \{0, 1\}$, consider an arbitrary parametrization φ_i of \mathcal{X}_i . Let

$$Z := (\operatorname{Lip}^{s}_{\ell}(X_{0}, d_{0}) \circ \varphi_{0}) \cup (\operatorname{Lip}^{s}_{\ell}(X_{1}, d_{1}) \circ \varphi_{1})$$

For each $i \in \{0, 1\}$, since $\mu_i = (\varphi_i)_*(\lambda)$, the map

$$\Phi_i \colon (\operatorname{Lip}_{\ell}^s(X_i, d_i), \operatorname{me}_{\mu_i}) \longrightarrow (Z, \operatorname{me}_{\lambda}), \quad f \longmapsto f \circ \varphi_i$$

is an isometric embedding, cf. [Shi16, Lemma 5.31(1)]. Furthermore,

 $\mathrm{H}_{\mathrm{me}_{\lambda}}(\mathrm{Lip}_{\ell}^{s}(X_{0}, d_{0}) \circ \varphi_{0}, \mathrm{Lip}_{\ell}^{s}(X_{1}, d_{1}) \circ \varphi_{1}) \leq \ell \mathrm{H}_{\mathrm{me}_{\lambda}}(\mathrm{Lip}_{1}(X_{0}, d_{0}) \circ \varphi_{0}, \mathrm{Lip}_{1}(X_{1}, d_{1}) \circ \varphi_{1}).$

Indeed, if $\operatorname{H}_{\operatorname{me}_{\lambda}}(\operatorname{Lip}_{1}(X_{0}, d_{0}) \circ \varphi_{0}, \operatorname{Lip}_{1}(X_{1}, d_{1}) \circ \varphi_{1}) < \delta$ for some number $\delta \in \mathbb{R}$, and we let $\{i, j\} = \{0, 1\}$ and $f \in \operatorname{Lip}_{\ell}^{s}(X_{i}, d_{i})$, then $\ell^{-1}f \in \operatorname{Lip}_{1}(X_{i}, d_{i})$, thus there is $g \in \operatorname{Lip}_{1}(X_{j}, d_{j})$ with $\operatorname{me}_{\lambda}((\ell^{-1}f) \circ \varphi_{i}, g \circ \varphi_{j}) < \delta$, wherefore $h := ((\ell g) \land s) \lor (-s) \in \operatorname{Lip}_{\ell}^{s}(X_{j}, d_{j})$ and

$$\mathrm{me}_{\lambda}(f \circ \varphi_i, h \circ \varphi_j) \leq \mathrm{me}_{\lambda}(f \circ \varphi_i, (\ell g) \circ \varphi_j) \leq \ell \mathrm{me}_{\lambda}((\ell^{-1} f) \circ \varphi_i, g \circ \varphi_j) < \ell \delta,$$

which shows that $\operatorname{H}_{\operatorname{me}_{\lambda}}(\operatorname{Lip}_{\ell}^{s}(X_{0}, d_{0}) \circ \varphi_{0}, \operatorname{Lip}_{\ell}^{s}(X_{1}, d_{1}) \circ \varphi_{1}) \leq \ell \delta$. Consequently,

$$d_{\mathrm{GH}}\left(\left(\mathrm{Lip}_{\ell}^{s}(X_{0},d_{0}),\mathrm{me}_{\mu_{0}}\right),\left(\mathrm{Lip}_{\ell}^{s}(X_{1},d_{1}),\mathrm{me}_{\mu_{1}}\right)\right) \\ \leq \mathrm{H}_{\mathrm{me}_{\lambda}}(\Phi_{0}(\mathrm{Lip}_{\ell}^{s}(X_{0},d_{0})),\Phi_{1}(\mathrm{Lip}_{\ell}^{s}(X_{1},d_{1}))) \\ = \mathrm{H}_{\mathrm{me}_{\lambda}}(\mathrm{Lip}_{\ell}^{s}(X_{0},d_{0})\circ\varphi_{0},\mathrm{Lip}_{\ell}^{s}(X_{1},d_{1})\circ\varphi_{1}) \\ \leq \ell\mathrm{H}_{\mathrm{me}_{\lambda}}(\mathrm{Lip}_{1}(X_{0},d_{0})\circ\varphi_{0},\mathrm{Lip}_{1}(X_{1},d_{1})\circ\varphi_{1}).$$

According to the definition of $d_{\rm conc}$, this completes the proof.

Our proof of Lemma 3.8 will involve Rademacher functions. Recall that, for any $n \in \mathbb{N} \setminus \{0\}$, the *n*-th Rademacher function is defined as

$$r_n \colon [0,1) \longrightarrow \{-1,1\}, \quad t \longmapsto (-1)^{\lfloor 2^n t \rfloor},$$

that is, $r_n(t) = (-1)^k$ whenever $t \in [2^{-n}k, 2^{-n}(k+1))$ and $k \in \{0, \ldots, 2^n - 1\}$. Let us note a well-known, elementary fact about this family of functions. Given any $n \in \mathbb{N} \setminus \{0\}$, we will henceforth abbreviate $T_n := \{2^{-n}k \mid k \in \{0, \ldots, 2^n - 1\}\}.$

Lemma 3.7. Let $n \in \mathbb{N} \setminus \{0\}$. For any two distinct $i, j \in \{1, \ldots, n\}$,

$$|\{t \in T_n \mid r_i(t) = r_j(t)\}| = 2^{n-1} = \frac{|T_n|}{2}.$$

Proof. We include a proof for the sake of completeness. Let us briefly agree on some convenient notation: given a finite subset $T \subseteq [0, 1)$, let $[x, y)_T := \{t \in T \mid x \leq t < y\}$ for $x, y \in [0, 1]$, and define $\sigma_T : [0, 1) \to [0, 1], x \mapsto \min\{t \in T \cup \{1\} \mid x < t\}$. Without loss of generality, we may assume that i < j. For every $x \in T_i$,

(1)
$$|\{t \in [x, \sigma_{T_i}(x))_{T_n} | r_i(t) = r_j(t)\}| = 2^{n-j} |\{y \in [x, \sigma_{T_i}(x))_{T_j} | r_i(y) = r_j(y)\}|,$$

(2)
$$|\{y \in [x, \sigma_{T_i}(x))_{T_j} | r_i(y) = r_j(y)\}| = 2^{j-i-1}$$

In order to prove (1) and (2), let $x \in T_i$. Then (2) follows by observing that r_i is constant on the 2^{j-i} -element set $[x, \sigma_{T_i}(x))_{T_j}$, whereas $r_j(\sigma_{T_j}(y)) = -r_j(y)$ for every $y \in [x, \sigma_{T_i}(x))_{T_j}$. With regards to (1), we note that $[x, \sigma_{T_i}(x))_{T_n} = \bigcup \{[y, \sigma_{T_j}(y))_{T_n} | y \in [x, \sigma_{T_i}(x))_{T_j}\}$ and that

both r_i and r_j are constant on each of the 2^{n-j} -element sets $[y, \sigma_{T_j}(y))_{T_n}$ $(y \in [x, \sigma_{T_i}(x))_{T_j})$, which entails that

$$\begin{aligned} |\{t \in [x, \sigma_{T_i}(x))_{T_n} \mid r_i(t) = r_j(t)\}| &= \sum_{y \in [x, \sigma_{T_i}(x))_{T_j}} |\{t \in [y, \sigma_{T_j}(y))_{T_n} \mid r_i(t) = r_j(t)\}| \\ &= 2^{n-j} |\{y \in [x, \sigma_{T_i}(x))_{T_j} \mid r_i(y) = r_j(y)\}|. \end{aligned}$$

Combining (1) and (2) with the fact that $T_n = \bigcup \{ [x, \sigma_{T_i}(x))_{T_n} \mid x \in T_i \}$, we arrive at

$$\begin{aligned} |\{t \in T_n \mid r_i(t) = r_j(t)\}| &= \sum_{x \in T_i} |\{t \in [x, \sigma_{T_i}(x))_{T_n} \mid r_i(t) = r_j(t)\}| \\ &\stackrel{(1)}{=} 2^{n-j} \sum_{x \in T_i} |\{y \in [x, \sigma_{T_i}(x))_{T_j} \mid r_i(y) = r_j(y)\}| \\ &\stackrel{(2)}{=} 2^{n-j} 2^{j-i-1} |T_i| = 2^{n-1} = \frac{|T_n|}{2}. \end{aligned}$$

Now we are prepared for proving the key Lemma 3.8.

Lemma 3.8. Let $\mathcal{X} = (X, d, \mu)$ be a fully supported mm-space. For all $n \in \mathbb{N} \setminus \{0\}$, $\delta \in (0, \infty)$, $\varepsilon \in (0, 1)$ and $\tau \in (0, \frac{1}{2})$,

$$\operatorname{Sep}_{2^{n}-1}(\mathcal{X};(1-\varepsilon)2^{-n}) \ge \delta \quad \Longrightarrow \quad \operatorname{Cap}_{(1-\varepsilon)\tau}\left(\operatorname{Lip}_{\delta^{-1}}^{1}(X,d),\operatorname{me}_{\mu}\right) \ge n.$$

Proof. Assume that $\operatorname{Sep}_{2^n-1}(\mathcal{X}; (1-\varepsilon)2^{-n}) \geq \delta$. Then there exist (necessarily disjoint) Borel subsets $B_0, \ldots, B_{2^n-1} \subseteq X$ such that

- (1) $\mu(B_i) \ge (1-\varepsilon)2^{-n}$ for each $i \in \{0, \dots, 2^n 1\}$, and
- (2) $\inf\{d(x,y) \mid x \in B_i, y \in B_j\} \ge \delta$ for any two distinct $i, j \in \{0, \dots, 2^n 1\}$.

Consider the Borel set $B := \bigcup_{i=0}^{2^n-1} B_i \subseteq X$, and note that $\mu(B) \ge 1 - \varepsilon$, as follows from (1) and the fact that B_0, \ldots, B_{2^n-1} are pairwise disjoint. Let $\pi \colon B \to \{0, \ldots, 2^n - 1\}$ be the unique map with $\pi^{-1}(i) = B_i$ for all $i \in \{0, \ldots, 2^n - 1\}$. For each $i \in \{1, \ldots, n\}$, consider the *i*-th Rademacher function $r_i \colon [0, 1) \to \{-1, 1\}$ and let

$$f_i: B \longrightarrow \{0, 1\}, \quad x \longmapsto \frac{1}{2}(1 + r_i(2^{-n}\pi(x))).$$

As each of the functions $f_1, \ldots, f_n \colon B \to \{0, 1\}$ is constant on each of the sets B_0, \ldots, B_{2^n-1} , assertion (2) implies that $\{f_1, \ldots, f_n\} \subseteq \operatorname{Lip}_{\delta^{-1}}^1(B, d \upharpoonright_B)$. Utilizing a standard construction, for each $i \in \{1, \ldots, n\}$ we define

$$f_i^* \colon X \longrightarrow [0,1], \quad x \longmapsto \left(\inf_{y \in B} f_i(y) + \delta^{-1} d(x,y)\right) \wedge 1$$

and observe that $f_i^* \in \operatorname{Lip}_{\delta^{-1}}^1(X, d)$ and $f_i^*|_B = f_i$. Define $T_n := \{2^{-n}k \mid k \in \{0, \dots, 2^n - 1\}\}$ as in Lemma 3.7. For any two distinct $i, j \in \{1, \dots, n\}$, we consider the Borel set

$$N_{ij} := \{ x \in B \mid |f_i(x) - f_j(x)| = 1 \} = \{ x \in B \mid f_i(x) \neq f_j(x) \}$$
$$= \bigcup \{ B_{2^n t} \mid t \in T_n, r_i(t) \neq r_j(t) \}$$

and conclude that $\mu(N_{ij}) \geq 2^{n-1}(1-\varepsilon)2^{-n} = (1-\varepsilon)2^{-1}$, taking into account assertion (1), the pairwise disjointness of B_0, \ldots, B_{2^n-1} , and Lemma 3.7. It follows that

$$\operatorname{me}_{\mu}(f_i^*, f_j^*) \ge \mu(N_{ij}) \ge 2^{-1}(1-\varepsilon) > (1-\varepsilon)\tau$$

for any two distinct $i, j \in \{1, \ldots, n\}$, i.e., $\{f_1^*, \ldots, f_n^*\}$ is $(1-\varepsilon)\tau$ -discrete in $(\operatorname{Lip}_{\delta^{-1}}^1(X, d), \operatorname{me}_{\mu})$. Thus, $\operatorname{Cap}_{(1-\varepsilon)\tau}(\operatorname{Lip}_{\delta^{-1}}^1(X, d), \operatorname{me}_{\mu}) \ge n$ as desired. \Box Everything is prepared to show that dissipation does indeed constitute a strong opposite to concentration.

Proposition 3.9. Let $\mathcal{X}_n = (X_n, d_n, \mu_n)$ $(n \in \mathbb{N})$ be a sequence of fully supported mm-spaces δ -dissipating for some $\delta > 0$. Then, for all $\ell \in (\delta^{-1}, \infty)$ and $\alpha \in (0, \frac{1}{2})$,

$$\operatorname{Cap}_{\alpha}\left(\operatorname{Lip}_{\ell}^{1}(X_{n}, d_{n}), \operatorname{me}_{\mu_{n}}\right) \longrightarrow \infty \quad (n \longrightarrow \infty).$$

Proof. Let $\ell \in (\delta^{-1}, \infty)$, $\alpha \in (0, \frac{1}{2})$, $m \in \mathbb{N} \setminus \{0\}$. Choose $\varepsilon \in (0, 1)$ and $\tau \in (0, \frac{1}{2})$ such that $(1-\varepsilon)\tau \ge \alpha$. Since $(\mathcal{X}_n)_{n\in\mathbb{N}}$ δ -dissipates, we find $n_0 \in \mathbb{N}$ with $\operatorname{Sep}_{2^m-1}(\mathcal{X}_n; (1-\varepsilon)2^{-m}) \ge \ell^{-1}$ for all $n \in \mathbb{N}$, $n \ge n_0$. By Lemma 3.8, it follows that

$$\operatorname{Cap}_{\alpha}\left(\operatorname{Lip}_{\ell}^{1}(X_{n}, d_{n}), \operatorname{me}_{\mu_{n}}\right) \geq \operatorname{Cap}_{(1-\varepsilon)\tau}\left(\operatorname{Lip}_{\ell}^{1}(X_{n}, d_{n}), \operatorname{me}_{\mu_{n}}\right) \geq m$$

for every $n \in \mathbb{N}$ with $n \ge n_0$, which proves our claim.

Combining Proposition 3.9 with Theorem 2.2, we arrive at the following.

Corollary 3.10. If a sequence of mm-spaces dissipates, then it does not have a d_{conc} -Cauchy subsequence.

Proof. Since dissipation is inherited by subsequences, it suffices to check that no dissipating sequence of mm-spaces can possibly be d_{conc} -Cauchy. Moreover, since both dissipation and being Cauchy with respect to d_{conc} are invariant under mm-space isomorphisms and every mm-space is isomorphic to a fully supported one, it is sufficient to consider a sequence of fully supported mm-spaces $\mathcal{X}_n = (X_n, d_n, \mu_n)$ $(n \in \mathbb{N})$. If $(\mathcal{X}_n)_{n \in \mathbb{N}} \delta$ -dissipates for some $\delta > 0$, then Proposition 3.9 along with Theorem 2.2 asserts that $(\text{Lip}_{1+\delta^{-1}}^1(X_n, d_n), \text{me}_{\mu_n})_{n \in \mathbb{N}}$ is not d_{GH} -Cauchy, which, according to Lemma 3.6, implies that $(\mathcal{X}_n)_{n \in \mathbb{N}}$ is not d_{conc} -Cauchy. \Box

Remark 3.11. The converse of Corollary 3.10 does not hold. In fact, letting $\lambda_n := \lambda^{\otimes n}$ and

$$d_n \colon [0,1]^n \times [0,1]^n \longrightarrow \mathbb{R}, \quad (x,y) \longmapsto \sup\{|x_i - y_i| \mid i \in \{1,\dots,n\}\}$$

for each $n \in \mathbb{N}$, the sequence $([0, 1]^n, d_n, \lambda_n)_{n \in \mathbb{N}}$ does not dissipate [Shi16, Theorem 8.8], but does not contain a d_{conc} -Cauchy subsequence either, as follows by the argument in the proof of [Shi16, Proposition 7.36].

Since the present note is aimed at exhibiting instances of dissipation in topological groups, we will subsequently add some remarks about dissipation in the very general framework of uniform spaces, thus following the lead of Pestov, who in [Pes02, Definition 2.6] adapted the Lévy concentration property from the realm of mm-spaces to Borel probability measures in uniform spaces. We provide a corresponding generalization of Gromov's concept of dissipation. Throughout the present note, all uniform spaces are assumed to be Hausdorff.

Definition 3.12. Let X be a uniform space. For an entourage U in X, we will say that a net $(\mu_i)_{i \in I}$ of Borel probability measures on X U-dissipates in X if there exists a family $(\mathcal{B}_i)_{i \in I}$ of finite sets of Borel subsets of X such that

- (1) $(B \times C) \cap U = \emptyset$ for every $i \in I$ and any two distinct $B, C \in \mathcal{B}_i$,
- (2) $\lim_{i\to I} \mu_i(\bigcup \mathcal{B}_i) = 1$, and
- (3) $\lim_{i \to I} \sup\{\mu_i(B) \mid B \in \mathcal{B}_i\} = 0.$

A net of Borel probability measures on X will be said to *dissipate in* X if it U-dissipates for some entourage U of X.

Of course, dissipation and Lévy concentration [Pes02, Definition 2.6] are two mutually exclusive properties for a net of Borel probability measures on a uniform space to have. Let us point out the connection with dissipation of mm-spaces.

Proposition 3.13. Let (X, d) be any metric space and let $\mathcal{X}_n = (X_n, d_n, \mu_n)$ $(n \in \mathbb{N})$ be a sequence of mm-spaces. For each $n \in \mathbb{N}$, let $\varphi_n : (X_n, d_n) \to (X, d)$ be an isometric embedding. Then $(\mathcal{X}_n)_{n \in \mathbb{N}}$ dissipates if and only if the sequence $((\varphi_n)_*(\mu_n))_{n \in \mathbb{N}}$ dissipates in (X, d).

Proof. This is an immediate consequence of Lemma 3.4.

We conclude this section with a clarifying remark about the existence of dissipating families of Borel probability measures in uniform spaces. To agree on some terminology and notation, let X be a uniform space. For an entourage U of X, we let

$$U[A] := \{ y \in X \mid \exists x \in A \colon (x, y) \in U \} \qquad (A \subseteq X).$$

Recall that X is precompact if, for every entourage U of X, there exists a finite subset $F \subseteq X$ such that X = U[F]. Moreover, for an entourage U of X, a subset $B \subseteq X$ is called U-discrete if $(B \times B) \cap U = \Delta_B$. It is easy to see that X is precompact if and only if the quantity

 $\operatorname{Cap}_U(X) := \sup\{|B| \mid B \subseteq X \text{ } U\text{-discrete}\}$

is finite for every entourage U of X.

Proposition 3.14. A uniform space X is non-precompact if and only if X admits a dissipating net of (finitely supported regular) Borel probability measures.

Proof. (\Longrightarrow) Being non-precompact, X admits an entourage U with $\operatorname{Cap}_U(X) = \infty$, whence we find a sequence $(F_n)_{n \in \mathbb{N}}$ of finite non-empty U-discrete subsets of X such that $|F_n| \longrightarrow \infty$ as $n \longrightarrow \infty$. The sequence of finitely supported regular Borel probability measures $(\delta_{F_n})_{n \in \mathbb{N}}$ thus U-dissipates in X, as witnessed by the sequence $\mathcal{B}_n := \{\{x\} \mid x \in F_n\} \ (n \in \mathbb{N}).$

 (\Leftarrow) Suppose that X admits a net $(\mu_i)_{i \in I}$ of Borel probability measures U-dissipating in X for some entourage U of X, i.e., there is a family $(\mathcal{B}_i)_{i \in I}$ of finite sets of Borel subsets of X such that

- (1) $(B \times C) \cap U = \emptyset$ for every $i \in I$ and any two distinct $B, C \in \mathcal{B}_i$,
- (2) $\lim_{i\to I} \mu_i(\bigcup \mathcal{B}_i) = 1$, and
- (3) $\lim_{i \to I} \sup\{\mu_i(B) \mid B \in \mathcal{B}_i\} = 0.$

We show that $\operatorname{Cap}_U(X) = \infty$. To this end, let $m \in \mathbb{N} \setminus \{0\}$. There exists $i \in I$ such that

$$\mu_i\left(\bigcup \mathcal{B}_i\right) > 1 - \frac{1}{m}, \qquad \sup\{\mu_i(B) \mid B \in \mathcal{B}_i\} \le \frac{1}{m}. \tag{*}$$

Select a subset $F \subseteq \bigcup \mathcal{B}_i$ such that $|F \cap B| = 1$ for each $B \in \mathcal{B}_i$. Clearly, F is U-discrete by (1). Moreover, $|F| \ge m$ due to (*), and thus $\operatorname{Cap}_U(X) \ge m$. Hence, X is not precompact. \Box

4. TOPOLOGICAL GROUPS: CONVERGENCE TO INVARIANCE

The purpose of this section is to recompile some results of [ST18] concerning amenability of topological groups and UEB-convergence to invariance. Throughout the present note, all topological groups are assumed to be *Hausdorff*.

Let X be a uniform space and consider the commutative unital real Banach algebra UCB(X) of all bounded uniformly continuous real-valued functions on X carrying the supremum norm. The collection M(X) of all *means* on UCB(X), that is, (necessarily continuous) positive unital

linear forms on UCB(X), endowed with the weak-* topology, i.e., the initial topology generated by all maps $M(X) \to \mathbb{R}$, $\mu \mapsto \mu(f)$ where $f \in UCB(X)$, is a compact Hausdorff space. The set S(X) of all (necessarily positive and linear) unital ring homomorphisms from UCB(X) into \mathbb{R} is a closed subspace thereof, commonly referred to as the *Samuel compactification* of X. A subset $H \subseteq UCB(X)$ is called *UEB* (short for *uniformly equicontinuous bounded*) if H is bounded in the supremum norm and *uniformly equicontinuous*, i.e., for every $\varepsilon > 0$ there exists an entourage U of X such that

$$\forall f \in H \; \forall (x, y) \in U \colon \qquad |f(x) - f(y)| \le \varepsilon.$$

The set UEB(X) of all UEB subsets of UCB(X) is a convex vector bornology on the vector space UCB(X). The *UEB topology* on the continuous dual $UCB(X)^*$ is defined as the topology of uniform convergence on UEB subsets of UCB(X), which is a locally convex linear topology on the vector space $UCB(X)^*$ containing the weak-* topology, i.e., the initial topology generated by the maps $UCB(X)^* \to \mathbb{R}$, $\mu \mapsto \mu(f)$ where $f \in UCB(X)$. A comprehensive account on the UEB topology is given in Pachl's book [Pac13].

Let G be a topological group and denote by $\mathcal{U}(G)$ the neighborhood filter of the neutral element in G. We endow G with its *right uniformity* defined by the basic entourages

$$U_{\vec{r}} := \left\{ (x, y) \in G \times G \, \big| \, yx^{-1} \in U \right\} \qquad (U \in \mathcal{U}(G)).$$

In reference to this uniformity, we denote by $\operatorname{RUCB}(G)$ the set of all bounded uniformly continuous real-valued function on G and by $\operatorname{RUEB}(G)$ the set of all UEB subsets of $\operatorname{RUCB}(G)$. Letting $\lambda_q \colon G \to G, x \mapsto gx$ for any $g \in G$, we note that G acts continuously on $\operatorname{M}(G)$ by

$$(g\mu)(f) := \mu(f \circ \lambda_g) \qquad (g \in G, \ \mu \in \mathcal{M}(G), \ f \in \mathcal{RUCB}(G)),$$

and that S(G) is a *G*-invariant subspace of M(G). Recall that *G* is *amenable* (resp., *extremely amenable*) if M(G) (resp., S(G)) contains a *G*-fixed point. It is well known that *G* is amenable (resp., extremely amenable) if and only if every continuous action of *G* on a non-void compact Hausdorff space admits an invariant regular Borel probability measure (resp., a fixed point). For more on (extreme) amenability of topological groups, we refer to [Pes06].

We will need a characterization of amenability in terms of asymptotically invariant finitely supported probability measures from [ST18].

Definition 4.1. Let G be a topological group. A net $(\mu_i)_{i \in I}$ of Borel probability measures on G is said to UEB-converge to invariance (over G) if

$$\forall g \in G \ \forall H \in \text{RUEB}(G): \qquad \sup_{f \in H} \left| \int f \circ \lambda_g \, d\mu_i - \int f \, d\mu_i \right| \longrightarrow 0 \quad (i \longrightarrow I).$$

Theorem 4.2 ([ST18], Theorem 3.2). A topological group is amenable if and only if it admits a net of (finitely supported, regular) Borel probability measures UEB-converging to invariance.

A very useful fact about UEB-convergence to invariance is the stability with respect to convolution of measures from the right, see Lemma 4.4 below. For the proof, we will use the following Lemma 4.3. For a Borel probability measure μ on a topological group G and any bounded Borel function $f: G \to \mathbb{R}$, let us define $\mu f: G \to \mathbb{R}, x \mapsto \int f(xy) d\mu(y)$.

Lemma 4.3 (cf. [Pac13], Lemma 9.1). Let G be a topological group. For every $H \in \text{RUEB}(G)$,

 $\{\mu f \mid f \in H, \mu \in \mathcal{P}(G)\} \in \mathrm{RUEB}(G).$

Proof. Let $H \in \text{RUEB}(G)$. Evidently, as H is norm-bounded, $K := \{\mu f \mid f \in H, \mu \in P(G)\}$ must be, too. To prove uniform equicontinuity, let $\varepsilon > 0$. Since H is a member of RUEB(G), there exists $U \in \mathcal{U}(G)$ such that

$$\forall f \in H \,\forall x, y \in G: \qquad xy^{-1} \in U \implies |f(x) - f(y)| \le \varepsilon.$$

Therefore, if $f \in H$ and $x, y \in G$ with $xy^{-1} \in U$, then $(xz)(yz)^{-1} = xy^{-1} \in U$ and hence $|f(xz) - f(yz)| \le \varepsilon$ for all $z \in G$, which implies that

$$\left|\left(_{\mu}f\right)(x) - \left(_{\mu}f\right)(y)\right| = \left|\int f(xz) - f(yz)\,d\mu(z)\right| \le \int \left|f(xz) - f(yz)\right|d\mu(z) \le \varepsilon$$

for any $\mu \in P(G)$. This shows that K indeed belongs to RUEB(G).

Given a topological group G, we define the *convolution* of two measures $\mu, \nu \in P(G)$ to be the Borel probability measure $\mu * \nu := m_*(\mu \otimes \nu)$, where $m : G \times G \to G$, $(x, y) \mapsto xy$.

Lemma 4.4. Let $(\mu_i)_{i \in I}$ and $(\nu_i)_{i \in I}$ be two nets of Borel probability measures on a topological group G. If $(\mu_i)_{i \in I}$ UEB-converges to invariance over G, then so does $(\mu_i * \nu_i)_{i \in I}$.

Proof. Let $H \in \text{RUEB}(G)$. Due to Lemma 4.3, $\{\nu f \mid f \in H, \nu \in P(G)\}$ belongs to RUEB(G). Hence, as $(\mu_i)_{i \in I}$ UEB-converges to invariance over G,

$$\sup_{\nu \in \mathcal{P}(G)} \sup_{f \in H} \left| \int (\nu f) \circ \lambda_g \, d\mu_i - \int \nu f \, d\mu_i \right| \longrightarrow 0 \quad (i \longrightarrow I)$$

for every $g \in G$. Since Fubini's theorem yields that

$$\int (\nu_i f) \circ \lambda_g \, d\mu_i = \int \int f(gxy) \, d\nu_i(y) \, d\mu_i(x) = \int \int f(gxy) \, d\mu_i(x) \, d\nu_i(y) = \int f \circ \lambda_g \, d(\mu_i * \nu_i)$$

for all $i \in I, f \in H$ and $g \in G$, it follows that

$$\sup_{f \in H} \left| \int f \circ \lambda_g \, d(\mu_i * \nu_i) - \int f \, d(\mu_i * \nu_i) \right| = \sup_{f \in H} \left| \int (\nu_i f) \circ \lambda_g \, d\mu_i - \int \nu_i f \, d\mu_i \right|$$
$$\leq \sup_{\nu \in \mathcal{P}(G)} \sup_{f \in H} \left| \int (\nu_i f) \circ \lambda_g \, d\mu_i - \int \nu_i f \, d\mu_i \right| \longrightarrow 0 \quad (i \to I).$$

for every $g \in G$. This shows that $(\mu_i * \nu_i)_{i \in I}$ UEB-converges to invariance over G.

5. Equivariant dissipation

The main result of this note reads as follows.

Theorem 5.1. If a topological group G admits an open subgroup of infinite index, then every net of tight Borel probability measures on G UEB-converging to invariance dissipates in G.

Proof. Let H be any open subgroup of G. For any tight Borel probability measure μ on G,

$$\sup\{\mu(HF) \mid F \subseteq G \text{ finite}\} = 1. \tag{(*)}$$

To see this, let $\varepsilon > 0$. As μ is tight, there exists a compact subset $K \subseteq G$ with $\mu(K) \ge 1 - \varepsilon$. Since K is compact and H is open, we find a finite subset $F \subseteq G$ such that $K \subseteq HF$, which readily implies that $\mu(HF) \ge \mu(K) \ge 1 - \varepsilon$, as desired.

Now suppose that H has infinite index in G. We claim that, if $(\mu_i)_{i \in I}$ is any net of Borel probability measures on G UEB-converging to invariance, then

$$\lim_{i \to I} \sup\{\mu_i(Hg) \mid g \in G\} = 0. \tag{(**)}$$

For a subset $A \subseteq G$, we denote by $\chi_A \colon G \to \{0, 1\}$ the corresponding characteristic function. Since H is an open subgroup of G, we have $\{\chi_{Hg} \mid g \in G\} \in \operatorname{RUEB}(G)$. Indeed, $\{\chi_{Hg} \mid g \in G\}$ is clearly norm-bounded, and if $x, y \in G$ with $xy^{-1} \in H$, then $\chi_{Hg}(x) = \chi_{Hg}(y)$ for all $g \in G$. Thus, as $(\mu_i)_{i \in I}$ UEB-converges to invariance over G,

$$\sup_{g \in G} |\mu_i(fHg) - \mu_i(Hg)| = \sup_{g \in G} \left| \int \chi_{Hg} \circ \lambda_{f^{-1}} d\mu_i - \int \chi_{Hg} d\mu_i \right| \longrightarrow 0 \quad (i \longrightarrow I)$$

for all $f \in G$. In order to prove (******), let $\varepsilon > 0$. Pick any $n \in \mathbb{N} \setminus \{0\}$ with $\frac{2}{n} \leq \varepsilon$. Due to H having infinite index in G, we find $F \subseteq G$ such that |F| = n and $fH \cap f'H = \emptyset$ for any two distinct $f, f' \in F$. By the above, there exists $i_0 \in I$ such that, for all $i \in I$ with $i \geq i_0$,

 $\sup_{f \in F} \sup_{g \in G} |\mu_i(fHg) - \mu_i(Hg)| \le \frac{1}{n}.$

For all $g \in G$ and $i \in I$ with $i \ge i_0$, it follows that

$$1 = \mu_i(G) \ge \sum_{f \in F} \mu_i(fHg) \ge n \cdot \left(\mu_i(Hg) - \frac{1}{n}\right) = n\mu_i(Hg) - 1,$$

whence $\mu_i(Hg) \leq \frac{2}{n} \leq \varepsilon$. That is, $\sup\{\mu_i(Hg) \mid g \in G\} \leq \varepsilon$ for all $i \in I$ with $i \geq i_0$, which proves the desired assertion (**).

To conclude, let $(\mu)_{i\in I}$ be any net of tight Borel probability measures on G UEB-converging to invariance over G. Thanks to (*), we may choose finite $H_{\vec{r}}$ -discrete subsets $F_i \subseteq G$ $(i \in I)$ such that $\lim_{i\to I} \mu_i(HF_i) = 1$. Moreover, $\lim_{i\to I} \sup\{\mu_i(Hg) \mid g \in F_i\} = 0$ due to (**). The family $\mathcal{B}_i := \{Hg \mid g \in F_i\}$ $(i \in I)$ hence witnesses that $(\mu_i)_{i\in I}$ dissipates in G.

Remarks 5.2. Let G be a topological group.

(1) Concerning the hypothesis of Theorem 5.1, note that G admitting an open subgroup of infinite index is equivalent to the existence of a surjective continuous homomorphism from G onto some non-precompact, non-archimedean topological group.

(2) Provided that G is separable, the tightness condition for the measures in Theorem 5.1 may be removed, for if H is any open subgroup of G, then $\{Hg \mid g \in G\}$ is countable and thus (*) in the proof of Theorem 5.1 holds for any Borel probability measure on G, by σ -additivity. This works more generally in case G is τ -narrow [AT08, Section 5.1, p. 286] for some cardinal τ not real-valued measurable [BB00, Appendix 3, Definition 1]: if H is an open subgroup of G, then the cardinality of $\{Hg \mid g \in G\}$ is not real-valued measurable either, and consequently [BB00, Chapter 8, Theorem 1] asserts that, for any Borel probability measure μ on G, there exists a countable subset $C \subseteq G$ with $\mu(HC) = 1$, which implies (*) by σ -additivity of μ .

We proceed to consequences of Theorem 5.1.

Corollary 5.3. Let G be a metrizable topological group along with a right-invariant compatible metric d and let $(K_n)_{n\in\mathbb{N}}$ be an ascending chain of compact subgroups such that $G = \bigcup_{n\in\mathbb{N}} K_n$. For each $n \in \mathbb{N}$, we define $d_n := d|_{K_n}$ and denote by μ_n the normalized Haar measure on K_n . If G admits an open subgroup of infinite index, then $(K_n, d_n, \mu_n)_{n\in\mathbb{N}}$ dissipates, thus fails to have a d_{conc} -Cauchy subsequence.

Proof. Due to Theorem 5.1, the measures $(\mu_n)_{n \in \mathbb{N}}$, pushed forward to G along the respective inclusion maps, dissipate in G. Being a compatible right-invariant metric, d generates the right uniformity of G. Consequently, $(K_n, d_n, \mu_n)_{n \in \mathbb{N}}$ must dissipate by Proposition 3.13. By Corollary 3.10, this entails that $(K_n, d_n, \mu_n)_{n \in \mathbb{N}}$ cannot have a d_{conc} -Cauchy subsequence. \Box

Remark 5.4. Let G be a metrizable topological group together with a left-invariant compatible metric d. Then

$$d^-: G \times G \longrightarrow \mathbb{R}, \quad (x,y) \longmapsto d(x^{-1}, y^{-1})$$

is a right-invariant compatible metric on G. Moreover, if K is any compact subgroup of G and μ denotes its normalized Haar measure, then $(K, d^{\uparrow} \upharpoonright_{K}, \mu) \to (K, d \upharpoonright_{K}, \mu), x \mapsto x^{-1}$ constitutes an isomorphism of *mm*-spaces. It follows that, in Corollary 5.3, the word *right-invariant* may equivalently be replaced by *left-invariant*.

In view of Remark 5.4, our Corollary 5.3 resolves Problem 1.1 in the negative.

Corollary 5.5. For each $n \in \mathbb{N}$, let μ_n denote the normalized counting measure on Sym(n). If d is a left-invariant metric on Sym(\mathbb{N}), compatible with the topology of pointwise convergence, then $(\text{Sym}(n), d|_{\text{Sym}(n)}, \mu_n)_{n \in \mathbb{N}}$ dissipates, thus fails to admit a d_{conc} -Cauchy subsequence.

Corollary 5.5 is to be compared with the following well-known result due to Maurey [Mau79] (see also [MS86, Pes06]): with regard to the *normalized Hamming distances*

$$d_{\text{Ham},n}(g,h) := \frac{|\{i \in \{1,\dots,n\} \mid g(i) \neq h(i)\}|}{n} \qquad (g,h \in \text{Sym}(n))$$

and the normalized counting measures μ_n on $\operatorname{Sym}(n)$, the sequence $(\operatorname{Sym}(n), d_{\operatorname{Ham},n}, \mu_n)_{n\geq 1}$ constitutes a normal Lévy family, thus concentrates to a singleton space.

Pestov's Problem 1.1 has an interesting sibling, namely [Pes06, Conjecture 7.4.26], which has been confirmed recently in [Sch17] as part of the following more general result.

Theorem 5.6 ([Sch17], Theorem 1.1). Let G be a second-countable topological group equipped with a right-invariant compatible metric d. Suppose that there exists a sequence $(\mu_n)_{n \in \mathbb{N}}$ of Borel probability measures on G with compact supports $K_n := \operatorname{spt} \mu_n \ (n \in \mathbb{N})$ such that

(1) $(\mu_n)_{n\in\mathbb{N}}$ UEB-converges to invariance over G, and

(2) $(K_n, d\restriction_{K_n}, \mu_n\restriction_{K_n})_{n\in\mathbb{N}}$ concentrates to a fully supported, compact mm-space (X, d_X, μ_X) . Then there exists a topological embedding $\psi \colon X \to S(G)$ such that the push-forward measure $\psi_*(\mu_X)$ is G-invariant. In particular, $\psi(X)$ is a G-invariant subspace of S(G).

Combining Theorem 5.1 with the results of [Sch17], we subsequently deduce a dichotomy between concentration and dissipation in the context of non-archimedean second-countable topological groups, that is, Corollary 5.7. This dichotomy will distinguish precompact topological groups from non-precompact ones. Preparing the statement of Corollary 5.7, let us briefly clarify some notions. For a topological group G, we consider its *Bohr compactification* $\kappa_G: G \to \kappa G$ (see [Hol64, dV93]), i.e., κG is the Gelfand spectrum of the C^* -algebra AP(G) of all almost periodic continuous bounded complex-valued functions on G equipped with the continuous group structure given by

$$(\mu\nu)(f) := \mu(g \mapsto \nu(f \circ \lambda_g)) \qquad (\mu, \nu \in \kappa G, \ f \in \operatorname{AP}(G)),$$

and $\kappa_G \colon G \to \kappa G$ is the continuous homomorphism defined by

$$\kappa_G(x)(f) := f(x) \qquad (x \in G, f \in AP(G)),$$

which has dense image in κG . It is is well known that a topological group G is precompact if and only if κ_G is a topological embedding. This fact readily implies that, if G is a metrizable precompact topological group and d is any right-invariant compatible metric on G, then there exists a uniquely determined – necessarily right-invariant – compatible metric $d_{\kappa G}$ on κG such that $d_{\kappa G}(\kappa_G(x), \kappa_G(y)) = d(x, y)$ for all $x, y \in G$ (see e.g. [Sch17, Lemma 4.5]). **Corollary 5.7.** Let G be a non-archimedean second-countable topological group together with a right-invariant compatible metric d and let $(\mu_n)_{n \in \mathbb{N}}$ be a sequence of Borel probability measures on G with compact supports $K_n := \operatorname{spt} \mu_n \ (n \in \mathbb{N})$. Suppose that $(\mu_n)_{n \in \mathbb{N}}$ UEB-converges to invariance over G. Then, either

- (1) G is precompact, and then $(K_n, d\restriction_{K_n}, \mu_n\restriction_{K_n})_{n\in\mathbb{N}}$ concentrates to $(\kappa G, d_{\kappa G}, \mu_{\kappa G})$, or
- (2) G is not precompact, and then $(K_n, d\restriction_{K_n}, \mu_n \restriction_{K_n})_{n \in \mathbb{N}}$ dissipates.

Proof. The first assertion is proved in [Sch17]: if G is precompact, then $(K_n, d \upharpoonright_{K_n}, \mu_n \upharpoonright_{K_n})_{n \in \mathbb{N}}$ concentrates to $(\kappa G, d_{\kappa G}, \mu_{\kappa G})$ (see [Sch17, Proof of Theorem 1.1, first case, pages 10–11]). If G is not precompact, then G, being non-archimedean, must admit an open subgroup of infinite index, in which case the desired conclusion follows by Theorem 5.1 and Proposition 3.13.

In particular, Corollary 5.7 substantiates that the only manifestations of the kind of concentration described by Theorem 5.6 to be found in the realm of non-archimedean groups are those occuring – in a trivial fashion – in precompact topological groups.

Extending earlier work of Pestov [Pes10] as well as Glasner, Tsirelson and Weiss [GTW05], it was shown in [PS17, Theorem 3.9] that if G is any topological group admitting a net of Borel probability measures that concentrates in G (see [Pes02, Definition 2.6]) and, at the same time, UEB-converges to invariance over G, then G is *whirly amenable*, i.e.,

- G is amenable, and
- every invariant regular Borel probability measure on a compact G-space is supported on the set of fixed points,

which particularly entails that no non-trivial compact G-space can possibly admit an ergodic regular Borel probability measure. In the light of these results, one might be tempted to conjecture that the mere existence of *some* net of Borel probability measures on a topological group, simultaneously dissipating and UEB-converging to invariance, would have interesting dynamical or ergodic-theoretical consequences, beyond the obvious non-precompactness and amenability. We will finish the present note by disposing of this hope: in fact, every nonprecompact amenable topological group admits such a net of measures (Proposition 5.10).

Recall that a topological group G is said to be *precompact* if G is precompact with respect to its right uniformity, that is, for every $U \in \mathcal{U}(G)$ there exists a finite subset $F \subseteq G$ such that G = UF. The following characterization of precompact groups was found independently by Uspenskij (unpublished, cf. a footnote in [Usp08]) and Solecki [Sol00]. For a short proof, the reader is referred to [BT07, Proposition 4.3].

Lemma 5.8. A topological group G is precompact if and only if, for every $U \in \mathcal{U}(G)$, there exists a finite subset $F \subseteq G$ with G = FUF.

A proof of the following easy consequence of Lemma 5.8 may be found in [Sch17].

Corollary 5.9 (see Corollary 4.3 in [Sch17]). Let G be a non-precompact topological group. Then there exists some $U \in \mathcal{U}(G)$ such that, for every sequence $(F_n)_{n \in \mathbb{N}}$ of finite subsets of G, there exists $(g_n)_{n \in \mathbb{N}} \in G^{\mathbb{N}}$ such that

 $\forall m, n \in \mathbb{N}, \, m \neq n \colon \qquad UF_m g_m \cap UF_n g_n \, = \, \emptyset.$

We may now prove the aforementioned result.

Proposition 5.10. Let G be a topological group. The following are equivalent.

(1) G is amenable and not precompact.

- (2) G admits a net of Borel probability measures simultaneously dissipating in G and UEB-converging to invariance over G.
- (3) G admits a net of finitely supported regular Borel probability measures simultaneously dissipating in G and UEB-converging to invariance over G.

Proof. $(2) \Longrightarrow (1)$. This is due to Theorem 4.2 and Proposition 3.14.

 $(3) \Longrightarrow (2)$. Trivial.

(1) \Longrightarrow (3). Suppose that G is not precompact. Let $U \in \mathcal{U}(G)$ be as in Corollary 5.9. According to Theorem 4.2, there exists a net $(\mu_i)_{i \in I}$ of finitely supported regular Borel probability measures on G UEB-converging to invariance over G. Let $S_i := \operatorname{spt} \mu_i$ for all $i \in I$. Consider the directed set $J := I \times (\mathbb{N} \setminus \{0\})$ endowed with the partial order \leq_J given by

 $(i_0, n_0) \leq_J (i_1, n_1) \quad :\iff \quad i_0 \leq_I i_1, \ n_0 \leq n_1 \qquad ((i_0, n_0), (i_1, n_1) \in J).$

Due to Corollary 5.9, for each $(i, n) \in J$ there exists a subset $F_{i,n} \subseteq G$ such that

- (i) $|F_{i,n}| = n$, and
- (ii) $US_ig \cap US_ih = \emptyset$ for any two distinct $g, h \in F_{i,n}$.

For every $(i, n) \in J$, let us consider the regular Borel probability measure $\nu_{i,n} := \mu_i * \delta_{F_{i,n}}$ on G, and note that $\operatorname{spt} \nu_{i,n} = \bigcup_{g \in F_{i,n}} S_i g$ is finite. Since the net $(\mu_i)_{(i,n) \in J}$ UEB-converges to invariance over G, so does $(\nu_{i,n})_{(i,n) \in J}$ according to Lemma 4.4. Therefore, it only remains to argue that $(\nu_{i,n})_{(i,n) \in J} U_{\Gamma}$ -dissipates in G. Indeed, if $(i, n) \in J$, then $\mathcal{B}_{i,n} := \{S_i g \mid g \in F_{i,n}\}$ is a finite collection of finite (thus Borel) subsets of G, and moreover

- $(B \times C) \cap U_{\vec{r}} = \emptyset$ for any two distinct $B, C \in \mathcal{B}_{i,n}$ by (ii),
- $\nu_i(\bigcup \mathcal{B}_{i,n}) = \nu_i(S_i F_{i,n}) = 1$ as spt $\nu_{i,n} = S_i F_{i,n}$,
- $\nu_i(S_ig) = \frac{1}{n}$ for every $g \in F_{i,n}$ by (i) and (ii).

This completes the proof.

We conclude with an additional remark concerning the proof of Proposition 5.10.

Remark 5.11. Let G be a topological group and $(\mu_i)_{i \in I}$ be a net of finitely supported regular Borel probability measures on G UEB-converging to invariance. Let $S_i := \operatorname{spt} \mu_i$ for $i \in I$.

(1) If G is infinite, then $|S_i| \longrightarrow \infty$ as $i \to I$. To see this, let $m \in \mathbb{N}$. As G is infinite, we find a finite subset $E \subseteq G$ with $|E| > m^2$. Pick $U \in \mathcal{U}(G)$ so that $(UU^{-1}) \cap (E^{-1}E) = \{e\}$. By Urysohn's lemma for uniform spaces, there exists $f \in \text{RUCB}(G)$ such that $f(G) \subseteq [0, 1]$, f(e) = 1 and f(x) = 0 whenever $x \in G \setminus U$. For every subset $S \subseteq G$, define $f_S \colon G \to [0, 1]$ by

$$f_S(x) := \sup_{s \in S} f(xs^{-1}) \qquad (x \in G).$$

It is not difficult to see that $\{f_S \mid S \subseteq G\} \in \text{RUEB}(G)$. Since the net $(\mu_i)_{i \in I}$ UEB-converges to invariance over G, there exists $i_0 \in I$ such that

$$\forall i \in I, i \ge i_0: \quad \sup_{g \in E} \sup_{S \subseteq G} \left| \int f_S \, d\mu_i - \int f_S \circ \lambda_{g^{-1}} \, d\mu_i \right| \le \frac{1}{2}.$$

We show that $|S_i| > m$ for all $i \in I$ with $i \ge i_0$. To this end, let $i \in I$ with $i \ge i_0$. Note that $\int f_{S_i} d\mu_i = 1$, because $\mu_i(S_i) = 1$ and $f_{S_i}(S_i) = \{1\}$. Hence, for each $g \in E$, the above implies that $\int f_{S_i} \circ \lambda_{g^{-1}} d\mu_i \ge \frac{1}{2}$, and so $(f_{S_i} \circ \lambda_{g^{-1}}) |_{S_i} \ne \mathbf{0}$, which entails that $gUS_i \cap S_i \ne \emptyset$, i.e., $g \in S_i S_i^{-1} U^{-1}$. Since $(UU^{-1}) \cap (E^{-1}E) = \{e\}$, we conclude that $|E| \le |S_i S_i^{-1}| \le |S_i|^2$, and therefore $m < |S_i|$ as desired.

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(2) Let $n_i := |S_i|$ for $i \in I$. Suppose that G is not precompact. Then $n_i \to \infty$ $(i \to I)$ by (1). Moreover, by Corollary 5.9, for each $i \in I$ there exists an n_i -element subset $F_i \subseteq G$ such that $US_{ig} \cap US_i h = \emptyset$ for any two distinct elements $g, h \in F_i$. An argument analogous of the one given in the proof of Proposition 5.10 now shows that $(\mu_i * \delta_{F_i})_{i \in I}$ constitutes a net of finitely supported regular Borel probability measures on G, simultaneously dissipating in and UEB-converging to invariance over G.

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