

PLANARLY BRANCHED ROUGH PATHS AND ROUGH DIFFERENTIAL EQUATIONS ON HOMOGENEOUS SPACES

C. CURRY, K. EBRAHIMI-FARD, D. MANCHON, AND H. Z. MUNTHE-KAAS

ABSTRACT. This work studies rough differential equations (RDEs) on homogeneous spaces. We provide an explicit expansion of the solution at each point of the real line using decorated planar forests. The notion of planarly branched rough path is developed, following Gubinelli’s branched rough paths. The main difference being the replacement of the Butcher–Connes–Kreimer Hopf algebra of non-planar rooted trees by the Munthe-Kaas–Wright Hopf algebra of planar rooted forests. The latter underlies the extension of Butcher’s B -series to Lie–Butcher series known in Lie group integration theory. Planarly branched rough paths admit the study of RDEs on homogeneous spaces, the same way Gubinelli’s branched rough paths are used for RDEs on finite-dimensional vector spaces. An analogue of Lyons’ extension theorem is proven. Under analyticity assumptions on the coefficients and when the Hölder index of the driving path is one, we show convergence of the planar forest expansion in a small time interval.

Keywords: rough paths; rough differential equations; homogeneous spaces; Lie–Butcher series; Hopf algebras; post-Lie algebras; planar rooted forests.

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1. INTRODUCTION

Given a set of vector fields $\{f_i\}_{i=1}^d$ on some n -dimensional smooth manifold \mathcal{M} , we are interested in the controlled differential equation:

$$(1) \quad dY_{st} = \sum_{i=1}^d f_i(Y_{st}) dX_t^i,$$

with initial condition $Y_{ss} = y$, where the controls $t \mapsto X_t^i$ are differentiable, or even only Hölder-continuous real-valued functions¹. When \mathcal{M} is an affine space \mathbb{R}^n , rough path theory on \mathbb{R}^d , together with its branched version introduced by M. Gubinelli [29], is the correct setting to express the solutions of (1) when the controls are not differentiable. An important case of the latter situation is given by Brownian motion on \mathbb{R}^d , of which sample paths are almost surely nowhere differentiable². The existence of a solution in a small interval around the point s has been proven by Gubinelli, using the notion of controlled path in the branched setting [29, Section 8] (see also [30, Section 3]). The Taylor expansion of such a solution at any point is expressed by means of choosing a branched rough path \mathbb{X} over the driving path $X = (X^1, \dots, X^d)$. See, for example, the introduction of reference [30] by M. Hairer and D. Kelly for a concise account.

The theory of rough paths was introduced and developed by T. Lyons' [35]. It is based on Chen's theory of iterated integrals [11] and provides an integration theory for solving differential equations driven by irregular signals. The intuitive idea of prescribing the path together with its iterated integrals is encapsulated by the definition of a rough path as a two-parameter family of Hopf algebra characters of the shuffle Hopf algebra \mathcal{H}_{\square}^A over the finite alphabet $A = \{a_1, \dots, a_d\}$, subject to precise estimates as well as to Chen's lemma. The latter is a lifting of the chain rule for integration. Gubinelli's branched rough paths are based on J. Butcher's B -series from numerical integration theory, and are defined similarly to Lyons' rough paths, with the exception that the Hopf algebra at hand is the Butcher–Connes–Kreimer Hopf algebra $\mathcal{H}_{\text{BCK}}^A$ of A -decorated non-planar rooted trees.

¹We have chosen any real s as initial time rather than zero, whence the two-variable notation. Derivation is always understood with respect to the variable t , the first variable s remaining inert. The two positive integers n and d are a priori unrelated.

²Brownian motion is almost surely of Hölder regularity γ for any $\gamma < 1/2$.

In a first step, this article introduces and develops the theory of rough paths on \mathbb{R}^d for any connected graded Hopf algebra fulfilling rather mild assumptions with respect to its combinatorics. An analogue of Lyons' extension theorem is proven (Theorem 4.4), using the Sewing Lemma as in the classical case (Proposition 8.11). In particular, following Gubinelli's approach we use the notion of Lie–Butcher series from Lie group integration theory to define *planarly branched rough paths* on \mathbb{R}^d as rough paths in that generalised sense, for which the Hopf algebra at hand is the Hopf algebra of Lie group integrators $\mathcal{H}_{\text{MKW}}^A$ introduced in [38] by W. Wright and one of the current authors. In a nutshell, this combinatorial Hopf algebra is linearly spanned by planar ordered rooted forests, possibly with decorations on the vertices. The product in this commutative Hopf algebra is the shuffle product of the forests, which are considered as words with planar rooted trees as letters. The coproduct is based on the notion of *left admissible cuts* on forests. We then argue that planarly branched rough paths provide the correct setting for understanding controlled differential equations on a homogeneous space, i.e., a manifold acted upon transitively by a finite-dimensional Lie group. To be more precise, it provides the means to write the Taylor expansion of a solution at each time, particularly suited to the underlying geometric setting.

We conclude with a first discussion of the analytic aspects of differential equations driven by planarly branched rough paths. In this article, we restrict to considering the convergence of the full Taylor series on a small time interval (Corollary 8.10). This necessarily assumes analyticity of the vector fields, and makes use of Cauchy estimates in a similar manner to [29, Section 5]. On the other hand, this method is limited to considering driving paths for which the Hölder index γ of the control path X is equal to one (Lipschitz case). A much more promising approach is to consider instead truncations of the Taylor expansion with controlled remainder, following Davie [16], see also [7] for the extension of this method to Lie series for the pullback flow. The main obstacle to this technique is the lack of results showing that iterative applications of approximate flows can be concatenated to give an approximation of controlled error on a larger compact time interval. This is equivalent to the existence of global error estimates for Lie group integrators. Such results are established in the forthcoming work [14], the ramifications of which will be explored in a future sequel on the existence and uniqueness of solutions under much less restrictive assumptions.

The paper is organised as follows: in Section 2 we write down the Taylor expansion of the solution of (1) on a homogeneous space in the case of differentiable controls, using a Picard iteration. **We then introduce** a suitable class of combinatorial Hopf algebras in Section 3, defining a notion of factorial adapted to this general setting. Then we define in Section 4 a functorial notion of γ -regular rough path associated to any combinatorial Hopf algebra in the above sense, and we prove Lyons' extension theorem in this setting, along the lines of reference [23]. After giving a brief account on Lie–Butcher theory in Section 5, we recall the Munthe-Kaas–Wright Hopf algebra $\mathcal{H}_{\text{MKW}}^A$ of Lie group integrators. We recall in Section 6 two relevant combinatorial notions associated with planar forests, namely three partial orders on the set of vertices [1], and the planar forest factorial $\sigma \mapsto \sigma!$ of [27], which matches the general notion of factorial mentioned above. Planarly branched rough paths are then defined as rough paths associated to the particular Hopf algebra $\mathcal{H}_{\text{MKW}}^A$. Section 7 is devoted to a canonical surjective Hopf algebra morphism \mathfrak{a}_{\ll} from $\mathcal{H}_{\text{MKW}}^A$ onto the shuffle Hopf algebra $\mathcal{H}_{\text{III}}^A$, the *planar arborification*, in the sense of J. Ecalle's notion of arborification. A contracting version of planar arborification is also given, where the shuffle Hopf algebra is replaced by a quasi-shuffle Hopf algebra. Finally, Section 8 deals with rough differential equations on homogeneous spaces driven by a Hölder-continuous path

X . Any planarly branched rough path above X yields a corresponding formal solution. Following the lines of thought of [29, Section 5] (see also [4, Proposition 1.8]), we prove convergence of the planar forest expansion in a small interval at each time, under an appropriate analyticity assumption on the coefficients f_i , when the driving path is Lipschitz, i.e., of Hölder regularity $\gamma = 1$. [An account of the sewing lemma is given in the Appendix.](#)

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2. FORMAL SERIES EXPANSION OF THE SOLUTION

The theory of numerical integration algorithms on Lie groups and manifolds [33] has been developed over the last two decades. In this context new algebraic structures were revealed which combine Butcher’s B -series [31] and Lie-series into Lie–Butcher series on manifolds [36]. Brouder’s work [5] showed that Hopf and pre-Lie algebras of non-planar rooted trees provide the algebraic foundation of B -series. For Lie–Butcher series the new concepts of post-Lie algebras and the Munthe-Kaas–Wright Hopf algebra are the foundations. These are examples of algebraic combinatorial structures which arise naturally from the geometry of connections on homogenous spaces.

We rewrite the differential equation (1) in the following form:

$$(2) \quad dY_{st} = \sum_{i=1}^d \#f_i(Y_{st}) dX_t^i$$

with initial condition $Y_{ss} = y$, where the unknown is a path $Y_s: t \mapsto Y_{st}$ in a homogeneous space \mathcal{M} , with transitive action $(g, x) \mapsto g.x$ of a Lie group G on it. The control path $X: t \mapsto X_t = (X_t^1, \dots, X_t^d)$ with values in \mathbb{R}^d is given, and the f_i ’s are smooth maps from \mathcal{M} into the Lie algebra $\mathfrak{g} = \text{Lie}(G)$, which in turn define smooth vector fields $x \mapsto \#f_i(x)$ on \mathcal{M} :

$$\#f_i(x) := \left. \frac{d}{dt} \right|_{t=0} \exp(t f_i(x)).x \in T_x \mathcal{M}.$$

In the language of Lie algebroids, considering the tangent vector bundle and the trivial vector bundle $E = \mathcal{M} \times \mathfrak{g}$, the map $\#: C^\infty(\mathcal{M}, \mathfrak{g}) \rightarrow \chi(\mathcal{M})$ is the composition on the left with the anchor map $\rho: E \rightarrow T\mathcal{M}$ defined by $\rho(x, X) := \left. \frac{d}{dt} \right|_{t=0} (\exp tX).x$.

The central point of our approach is based on formally lifting the differential equation (2) to the space $C^\infty(\mathcal{M}, \mathcal{U}(\mathfrak{g}))[[h]]$, where $\mathcal{U}(\mathfrak{g})$ is the universal enveloping algebra of \mathfrak{g} . This is achieved as follows: setting $t = s + h$, we denote by φ_{st} the formal diffeomorphism defined by $\varphi_{st}(Y_{ss}) := Y_{st}$, where $t \mapsto Y_{st}$ is the solution of the initial value problem (2). This formal diffeomorphism can be expressed as:

$$\varphi_{st} = \#Y_{st}$$

with $\mathbf{Y}_{st} \in C^\infty(\mathcal{M}, \mathcal{U}(\mathfrak{g}))[[h]]$. It turns out that there exists a non-commutative associative product $*$ on $C^\infty(\mathcal{M}, \mathcal{U}(\mathfrak{g}))$, distinct from the pointwise product in $\mathcal{U}(\mathfrak{g})$, which reflects the composition product of differential operators on \mathcal{M} , in the sense that:

$$\#(u * v) = \#u \circ \#v$$

for any $u, v \in C^\infty(\mathcal{M}, \mathcal{U}(\mathfrak{g}))$. See reference [38] for details. The unit is the constant function $\mathbf{1}$ on \mathcal{M} equal to $1 \in \mathcal{U}(\mathfrak{g})$, and $\#\mathbf{1}$ is the identity operator. The existence of this product is a direct consequence of the post-Lie algebra structure on $C^\infty(\mathcal{M}, \mathfrak{g})$. The reader may consult [18] for details. Extending this product to formal series, our lifting of (2) is written as:

$$(3) \quad d\mathbf{Y}_{st} = \sum_{i=1}^d \mathbf{Y}_{st} * f_i dX_t^i$$

with initial condition $\mathbf{Y}_{ss} = \mathbf{1}$. The non-commutative product $*$ is the extension of the Grossman–Larson product on the post-associative algebra $C^\infty(\mathcal{M}, \mathcal{U}(\mathfrak{g}))$ to formal series, which reflects the composition of differential operators [38]. A full account of the post-Lie algebra structure on $C^\infty(\mathcal{M}, \mathfrak{g})$ and the post-associative algebra structure on $C^\infty(\mathcal{M}, \mathcal{U}(\mathfrak{g}))$ will be provided further below in Section 5. Let us just mention at this stage that for any $f, g \in C^\infty(\mathcal{M}, \mathfrak{g})$ we have (Leibniz' rule):

$$(4) \quad f * g = fg + f \triangleright g,$$

where fg stands for the pointwise product in $C^\infty(\mathcal{M}, \mathcal{U}(\mathfrak{g}))$, and where $f \triangleright g$ stands for $\#(f).g$. The solution of (3) is a formal diffeomorphism, i.e., it verifies $\mathbf{Y}_{st} \triangleright (\rho\psi) = (\mathbf{Y}_{st} \triangleright \rho)(\mathbf{Y}_{st} \triangleright \psi)$ for any $\rho, \psi \in C^\infty(\mathcal{M})$. The formal path Y_{st} solving the initial value problem (2), with initial condition $Y_{ss} = y$, is then the character of $C^\infty(\mathcal{M})$ with values in $\mathbb{R}[[h]]$ given for any $\psi \in C^\infty(\mathcal{M})$ by:

$$(5) \quad \begin{aligned} Y_{st}: C^\infty(\mathcal{M}) &\longrightarrow \mathbb{R}[[h]] \\ \psi &\longmapsto \psi(Y_{st}) = (\mathbf{Y}_{st} \triangleright \psi)(y). \end{aligned}$$

Plugging (3) into (5) yields:

$$\begin{aligned} \frac{d}{dt}\psi(Y_{st}) &= \frac{d}{dt}(\mathbf{Y}_{st} \triangleright \psi)(y) \\ &= ((\mathbf{Y}_{st} * F(t)) \triangleright \psi)(y) \\ &= (\mathbf{Y}_{st} \triangleright (F(t) \triangleright \psi))(y) \\ &= (F(t) \triangleright \psi)(Y_{st}), \end{aligned}$$

which proves this assertion, and therefore justifies viewing (3) as a lift of (2). We refer to $\psi(Y_{st})$ as the evaluation of ψ on the formal path Y_{st} . Equation (3) can be written in integral form:

$$(6) \quad \begin{aligned} \mathbf{Y}_{st} &= \mathbf{1} + \int_s^t \mathbf{Y}_{su} * F(u) du \\ &= \mathbf{1} + \sum_{i=1}^d \int_s^t \mathbf{Y}_{su} * f_i dX_u^i. \end{aligned}$$

A simple Picard iteration gives the formal expansion:

$$\mathbf{Y}_{st} = \mathbf{1} + \sum_{n \geq 1} \sum_{1 \leq i_1, \dots, i_n \leq d} \left(\int \cdots \int_{s \leq t_n \leq \dots \leq t_1 \leq t} f_{i_n} * \cdots * f_{i_1} dX_{t_1}^{i_1} \cdots dX_{t_n}^{i_n} \right)$$

$$(7) \quad = \mathbf{1} + \sum_{n \geq 1} \sum_{1 \leq i_1, \dots, i_n \leq d} \left(\int \cdots \int_{s \leq t_n \leq \dots \leq t_1 \leq t} dX_{t_1}^{i_1} \cdots dX_{t_n}^{i_n} \right) f_{i_n} * \cdots * f_{i_1}.$$

Using word notation, where f_w stands for the monomial $f_{i_n} * \cdots * f_{i_1}$ when the word w is given by $a_{i_1} \cdots a_{i_n}$, the formal expansion (7) will be written as a word series

$$(8) \quad \mathbf{Y}_{st} = \sum_{w \in A^*} \langle \mathbb{X}_{st}, w \rangle f_w.$$

Using (4), the first terms of the expansion are:

$$\begin{aligned} \mathbf{Y}_{st} &= \mathbf{1} + \sum_{i=1}^d \langle \mathbb{X}_{st}, a_i \rangle f_i + \sum_{i,j=1}^d \langle \mathbb{X}_{st}, a_i a_j \rangle (f_j f_i + f_j \triangleright f_i) \\ &+ \sum_{i,j,k=1}^d \langle \mathbb{X}_{st}, a_i a_j a_k \rangle \left(f_k f_j f_i + (f_k \triangleright f_j) f_i + f_k (f_j \triangleright f_i) + f_j (f_k \triangleright f_i) + (f_k f_j) \triangleright f_i + (f_k \triangleright f_j) \triangleright f_i \right) \\ &+ O(h^4). \end{aligned}$$

We observe that the number of components in the term of order three on the right-hand side can be reduced from six to five:

$$(9) \quad \sum_{i,j,k=1}^d \left[\langle \mathbb{X}_{st}, a_i a_j a_k \rangle \left(f_k f_j f_i + (f_k \triangleright f_j) f_i + (f_k f_j) \triangleright f_i + (f_k \triangleright f_j) \triangleright f_i \right) + \langle \mathbb{X}_{st}, a_i a_j a_k + a_i a_k a_j \rangle f_j (f_k \triangleright f_i) \right],$$

which corresponds to the five planar rooted decorated forests with three vertices, displayed in the following order:

$$(10) \quad \bullet_k \bullet_j \bullet_i \quad \begin{array}{c} \bullet_k \\ \vdots \\ \bullet_j \end{array} \bullet_i \quad \begin{array}{c} k \\ \bullet \\ \vdots \\ \bullet_j \end{array} \bullet_i \quad \begin{array}{c} \bullet_k \\ \vdots \\ \bullet_j \end{array} \bullet_i \quad \bullet_j \begin{array}{c} \bullet_k \\ \vdots \\ \bullet_i \end{array}.$$

The appearance of planar rooted forests relates to a natural further step in abstraction, namely using the *Lie–Butcher series formalism*. It consists in an additional lifting of equation (3) to the free post-associative algebra, i.e., the universal enveloping algebra over the free post-Lie algebra in d generators, more precisely to its completion $(\mathcal{H}_{\text{MKW}}^A)^*$. We obtain then the so-called fundamental differential equation:

$$(11) \quad d\mathbb{Y}_{st} = \sum_{i=1}^d \mathbb{Y}_{st} * \bullet_i dX_t^i$$

with initial condition $\mathbb{Y}_{ss} = \mathbf{1}$, where $*$ is now the non-commutative convolution (Grossman–Larson) product of two linear forms on $\mathcal{H}_{\text{MKW}}^A$. Suppose for the moment that the path X in \mathbb{R}^d is differentiable. Equation (11) can then be re-written as:

$$(12) \quad \dot{\mathbb{Y}}_{st} = \frac{d}{dt} \mathbb{Y}_{st} = \sum_{i=1}^d \dot{X}_t^i \mathbb{Y}_{st} * \bullet_i,$$

with initial condition $\mathbb{Y}_{ss} = \mathbf{1}$. For any s, t the so-called fundamental solution \mathbb{Y}_{st} of (11) is given by

$$(13) \quad \mathbb{Y}_{st} = \sum_{\ell \geq 0} \sum_{w = a_1 \cdots a_\ell \in A^*} \langle \mathbb{X}_{st}, w \rangle \bullet_{a_\ell} * \cdots * \bullet_{a_1}.$$

The coefficient of the last component in (9) is obtained by integrating $dX_{t_1}^{i_1} dX_{t_2}^{i_2} dX_{t_3}^{i_3}$ on the union of two simplices $\{(t_1, t_2, t_3), s \leq t_3 \leq t_1, t_2 \leq t\}$. This domain is associated to the decorated forest $\bullet_j \bullet_i^k$ by means of a partial order \ll on the vertices described in Subsection 5.5, which is closely related to the notion of left-admissible cuts for the coproduct in $\mathcal{H}_{\text{MKW}}^A$. The order \ll is total on the four other planar forests of degree three appearing in (10), hence the corresponding coefficients are obtained by integrating on a single simplex. Integrating over these domains lifts \mathbb{X}_{st} to a two-parameter family of characters of the Hopf algebra $\mathcal{H}_{\text{MKW}}^A$, which still verifies Chen's lemma. This calls for considering rough differential equations defined on the homogeneous space \mathcal{M} driven by planarly branched rough paths.

Further below we will use J. Ecalle's notion of arborification to write the Taylor expansion of the solution (8), or rather its abstract counterpart (13) in its planar arborified form:

$$(14) \quad \mathbb{Y}_{st} = \sum_{\sigma \in F_{\text{pl}}^A} \langle \widetilde{\mathbb{X}}_{st}, \sigma \rangle \sigma$$

with $\widetilde{\mathbb{X}}_{st} := \mathbb{X}_{st} \circ \mathbf{a}_{\ll}$ where $(\mathbb{X}_{st})_{s,t \in \mathbb{R}}$ is the signature of the path X , and where F_{pl}^A stands for the set of A -decorated planar rooted forests.

3. FACTORIALS IN COMBINATORIAL HOPF ALGEBRAS

We consider the notion of factorial in the context of a fairly general class of combinatorial algebras. This concept will encompass the usual factorial of positive integers, the tree and forest factorials as well as a planar version of the latter.

3.1. Inverse-factorial characters in connected graded Hopf algebras. Let $\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n$ be any connected graded Hopf algebra over some field \mathbf{k} of characteristic zero, and let $\alpha: \mathcal{H}_1 \rightarrow \mathbf{k}$ be a nonzero linear map. The degree of an element $x \in \mathcal{H}$ is denoted $|x|$. The *inverse-factorial character* q_α associated to these data is defined by

- $q_\alpha(\mathbf{1}) = 1$,
- $q_\alpha|_{\mathcal{H}_1} = \alpha$,
- $q_\alpha * q_\alpha(x) = 2^{|x|} q_\alpha(x)$ for any $x \in \mathcal{H}$.

It is indeed given for any homogeneous x of degree $|x| \geq 2$ by the recursive procedure:

$$(15) \quad q_\alpha(x) = \frac{1}{2^{|x|} - 2} \sum'_{(x)} q_\alpha(x') q_\alpha(x'').$$

See [29, Section 7] in the particular case of the Butcher–Connes–Kreimer Hopf algebra. Here $\Delta'(x) := \sum'_{(x)} x' \otimes x''$ denotes the reduced coproduct of \mathcal{H} (in Sweedler's notation), and the full coproduct is $\Delta(x) = \Delta'(x) + x \otimes \mathbf{1} + \mathbf{1} \otimes x = \sum_{(x)} x_1 \otimes x_2$. The multiplicativity property $q_\alpha(xy) = q_\alpha(x)q_\alpha(y)$ is verified recursively with respect to $\ell = |x| + |y| \geq 2$ (the cases $\ell = 0$ and $\ell = 1$ are immediately checked):

$$\begin{aligned} q_\alpha(xy) &= \frac{1}{2^{|x|+|y|}} \sum_{(x),(y)} q_\alpha(x_1 y_1) q_\alpha(x_2 y_2) \\ &= \frac{1}{2^{|x|+|y|}} \left(\sum_{(x),(y)} q_\alpha(x_1) q_\alpha(y_1) q_\alpha(x_2) q_\alpha(y_2) - 2q_\alpha(x) q_\alpha(y) + 2q_\alpha(xy) \right) \end{aligned}$$

$$= q_\alpha(x)q_\alpha(y) - \frac{1}{2^{|x|+|y|-1}} \left(q_\alpha(x)q_\alpha(y) - q_\alpha(xy) \right),$$

hence $q_\alpha(x)q_\alpha(y) - q_\alpha(xy) = 0$.

In general the linear form α is fixed once and for all, and q_α will be abbreviated to q . We remark that in concrete situations there is a natural linear basis for the degree one component \mathcal{H}_1 (and more generally for \mathcal{H} , see Paragraph 3.2 below for a precise setting), and α will be the linear form on \mathcal{H}_1 which takes the value 1 on each element of the basis. Taking as a simple example the shuffle algebra on an alphabet A , the binomial formula:

$$\frac{2^n}{n!} = \sum_{p=0}^n \frac{1}{p!} \frac{1}{(n-p)!}$$

shows that $q_\alpha(w) = 1/|w|!$ where $\alpha(a) = 1$ for each letter $a \in A$. This example justifies the terminology chosen.

Proposition 3.1. *For any $h \in \mathbf{k}$ the h -th power convolution of $q = q_\alpha$ makes sense as a character of \mathcal{H} , and admits the following explicit expression:*

$$(16) \quad q^{*h}(x) = h^{|x|}q(x).$$

Proof. One can express q as $\varepsilon + \kappa$ with $\kappa(\mathbf{1}) = 0$. Then we have for any $x \in \mathcal{H}$

$$q^{*h}(x) = (\varepsilon + \kappa)^{*h}(x) = \sum_{p \geq 0} \binom{h}{p} \kappa^{*p}(x).$$

The right-hand side is a finite sum, owing to the co-nilpotence of the coproduct. The expression $q^{*h}(xy) - q^{*h}(x)q^{*h}(y)$ is polynomial in h and vanishes at any non-negative integer h , hence vanishes identically. Similarly, the expression $q^{*h}(x) - h^{|x|}q(x)$ is polynomial in h and vanishes at any $h = 2^N$ where N is a non-negative integer, hence vanishes identically. \square

The following corollary generalises both the binomial formula and Gubinelli's branched binomial formula [29, Lemma 4.4].

Corollary 3.2. *For any $h, k \in \mathbf{k}$ and for any homogeneous element $x \in \mathcal{H}$, the following Hopf-algebraic binomial formula holds:*

$$q(x)(h+k)^{|x|} = \sum_{(x)} q(x_1)q(x_2)h^{|x_1|}k^{|x_2|}.$$

Proof. It is a straightforward application of (16) together with the group property $q^{*h} * q^{*k} = q^{*(h+k)}$. \square

Inverse-factorial characters are functorial, that is, if \mathcal{H} and \mathcal{H}' are two connected graded Hopf algebras and if $\Phi : \mathcal{H} \rightarrow \mathcal{H}'$ is a morphism of Hopf algebras preserving the degree, then for any linear map $\alpha' : \mathcal{H}' \rightarrow \mathbf{k}$ we have:

$$(17) \quad q_\alpha = q_{\alpha'} \circ \Phi$$

where $\alpha := \alpha' \circ \Phi|_{\mathcal{H}_1}$.

3.2. A suitable category of combinatorial Hopf algebras. Although the theory of combinatorial Hopf algebras constitutes an active field of research, with duly acknowledged applications in discrete mathematics, analysis, probability, control, and quantum field theory, no general consensus has yet emerged on a proper definition of those Hopf algebras. Saying this, we propose here a definition which will match our purpose, i.e., give estimates which will ensure convergence of the formal solutions of our singular differential equations in some particular cases. A different proposal for a definition of combinatorial Hopf algebras can also be found in [15] (see Definition 3.1 therein). In both definitions, a privileged linear basis is part of the initial data.

Definition 3.3. *A combinatorial Hopf algebra is a graded connected Hopf algebra $\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n$ over a field \mathbf{k} of characteristic zero, together with a basis $\mathcal{B} = \bigsqcup_{n \geq 0} \mathcal{B}_n$ of homogeneous elements, such that*

- (1) *There exist two positive constants B and C such that the dimension of \mathcal{H}_n is bounded by BC^n (in other words, the Poincaré–Hilbert series of \mathcal{H} converges in a small disk around the origin).*
- (2) *The structure constants $c_{\sigma\tau}^\rho$ and $c_\rho^{\sigma\tau}$ of the product and the coproduct, defined for any $\sigma, \tau, \rho \in \mathcal{B}$ respectively by*

$$\sigma\tau = \sum_{\rho \in \mathcal{B}} c_{\sigma\tau}^\rho \rho, \quad \Delta\rho = \sum_{\sigma, \tau \in \mathcal{B}} c_\rho^{\sigma\tau} \sigma \otimes \tau$$

are non-negative integers (which vanish unless $|\sigma| + |\tau| = |\rho|$).

In any combinatorial Hopf algebra in the above sense, the inverse-factorial character q will be chosen such that $q(\tau) = 1$ for any $\tau \in \mathcal{B}$ of degree one. We adopt the natural shorthand notation:

$$(18) \quad \tau! = \frac{1}{q(\tau)}$$

for any $\tau \in \mathcal{B}$. The two main examples are the shuffle Hopf algebra $\mathcal{H}_{\text{III}}^A$ and the Butcher–Connes–Kreimer Hopf algebra $\mathcal{H}_{\text{BCK}}^A$ over a finite alphabet A . In the former case the basis \mathcal{B} is given by words with letters in A , whereas in the latter case we have non-planar rooted forests decorated by A . On $\mathcal{H}_{\text{III}}^A$ the corresponding factorial is the usual factorial of the length of a word. On $\mathcal{H}_{\text{BCK}}^A$ it is the usual forest factorial [29, Lemma 4.4]. A third major example is the Hopf algebra $\mathcal{H}_{\text{MKW}}^A$ of Lie group integrators described in Paragraph 5.6 below.

Let $(\mathcal{H}, \mathcal{B})$ and $(\mathcal{H}', \mathcal{B}')$ be two combinatorial hopf algebras in the above sense. A Hopf algebra morphism $\Phi : (\mathcal{H}, \mathcal{B}) \rightarrow (\mathcal{H}', \mathcal{B}')$ is *combinatorial* if it is of degree zero and if, for any $\tau \in \mathcal{B}$, the element $\Phi(\tau) \in \mathcal{H}'$ is a linear combination of elements of the basis \mathcal{B}' with non-negative integer coefficients. Combinatorial Hopf algebras in the above sense together with combinatorial morphisms form a category. The forgetful functor $(\mathcal{H}, \mathcal{B}) \mapsto \mathcal{H}$ into the category of connected graded Hopf algebras is given by forgetting the basis.

Remark 3.4. The inverse-factorial character q may vanish on some elements τ of the basis, yielding $\tau! = +\infty$. This happens if and only if τ is primitive of degree $n \geq 2$. We therefore call a combinatorial Hopf algebra *non-degenerate* if

$$(19) \quad \mathcal{B} \cap \text{Prim}(\mathcal{H}) = \mathcal{B}_1.$$

The three combinatorial Hopf algebras $\mathcal{H}_{\text{III}}^A$, $\mathcal{H}_{\text{BCK}}^A$ and $\mathcal{H}_{\text{MKW}}^A$ happen to be non-degenerate. Examples of degenerate combinatorial Hopf algebras can easily be found among Hopf algebras of Feynman graphs, as primitive multiloop Feynman graphs do exist.

4. ROUGH PATHS AND CONNECTED GRADED HOPF ALGEBRAS

We show that Lyons' definition of rough paths [35] extends straightforwardly when replacing the shuffle Hopf algebra with any commutative connected graded Hopf algebra. In particular a naturally extended version of the extension theorem [35, Theorem 2.2.1] is available.

4.1. Chen iterated integrals and rough paths. Let d be a positive integer, and let us consider a smooth path in \mathbb{R}^d

$$\begin{aligned} X : \mathbb{R} &\longrightarrow \mathbb{R}^d \\ t &\longmapsto X(t) = (X_1(t), \dots, X_d(t)). \end{aligned}$$

Let \mathcal{H}^A be the algebra of the free monoid A^* generated by the alphabet $A := \{a_1, \dots, a_d\}$, and augmented with the empty word $\mathbf{1}$ as unit. Let $\mathbb{X}_{st} \in (\mathcal{H}^A)^*$ be defined for any $s, t \in \mathbb{R}$ and word $w = a_{j_1} \cdots a_{j_n} \in A^*$ by n -fold iterated integrals:

$$\begin{aligned} \langle \mathbb{X}_{st}, a_{j_1} \cdots a_{j_n} \rangle &:= \int \cdots \int_{s \leq t_n \leq \cdots \leq t_1 \leq t} \dot{X}_{j_1}(t_1) \cdots \dot{X}_{j_n}(t_n) dt_1 \cdots dt_n \\ (20) \qquad \qquad \qquad &= \int \cdots \int_{s \leq t_n \leq \cdots \leq t_1 \leq t} dX_{j_1}(t_1) \cdots dX_{j_n}(t_n). \end{aligned}$$

This is extended to the empty word $\mathbf{1}$ by $\langle \mathbb{X}_{st}, \mathbf{1} \rangle := 1$. Suppose moreover that the derivative \dot{X} is bounded, i.e., $\sup_{j=1}^d \sup_{t \in \mathbb{R}} |\dot{X}_j(t)| = C < +\infty$. The volume of the simplex

$$\Delta_{[s,t]}^n := \{(t_1, \dots, t_n), s \leq t_n \leq \cdots \leq t_1 \leq t\}$$

over which the iterated integration (20) of length n is performed is equal to $|t-s|^n/n!$, which yields the following estimate for any word $w \in A^*$:

$$(21) \qquad \sup_{s \neq t} \frac{|\langle \mathbb{X}_{st}, w \rangle|}{|t-s|^{|w|}} \leq \frac{C^{|w|}}{|w|!},$$

where $|w|$ stands for the length of the word w , i.e., its number of letters. It turns out [8, 9, 10, 11, 41] that \mathbb{X}_{st} is a two-parameter family of characters with respect to the shuffle product of words, namely:

$$(22) \qquad \langle \mathbb{X}_{st}, v \rangle \langle \mathbb{X}_{st}, w \rangle = \langle \mathbb{X}_{st}, v \amalg w \rangle.$$

The shuffle product \amalg is defined inductively by $w \amalg \mathbf{1} = \mathbf{1} \amalg w = w$ and

$$(23) \qquad (a_i v) \amalg (a_j w) = a_i (v \amalg a_j w) + a_j (a_i v \amalg w),$$

for all words $v, w \in A^*$ and letters $a_i, a_j \in A$. The resulting shuffle algebra is denoted \mathcal{H}_{\amalg}^A . For instance, $a_i \amalg a_j = a_i a_j + a_j a_i$ and

$$a_{i_1} a_{i_2} \amalg a_{i_3} a_{i_4} = a_{i_1} a_{i_2} a_{i_3} a_{i_4} + a_{i_1} a_{i_3} a_{i_2} a_{i_4} + a_{i_1} a_{i_3} a_{i_4} a_{i_2} + a_{i_3} a_{i_1} a_{i_2} a_{i_4} + a_{i_3} a_{i_1} a_{i_4} a_{i_2} + a_{i_3} a_{i_4} a_{i_1} a_{i_2}.$$

Moreover, the following property, now widely referred to as ‘‘Chen’s lemma’’, is verified:

$$(24) \qquad \langle \mathbb{X}_{st}, v \rangle = \sum_{v'v''=v} \langle \mathbb{X}_{su}, v' \rangle \langle \mathbb{X}_{ut}, v'' \rangle.$$

The sum on the righthand side extends over all splittings of the word $v \in A^*$ into two words, v' and v'' , such that the concatenation $v'v''$ equals v . Both properties are easily shown by a suitable decomposition of the integration domain into smaller pieces with Lebesgue-negligible mutual intersections: a product of two simplices is written

as a union of simplices for proving (22), and a simplex of size $t - s$ is written as a union of products of simplices of respective size $u - s$ and $t - u$ for proving (24) when $s \leq u \leq t$. The latter is advantageously re-written in terms of the convolution product associated to the deconcatenation coproduct $\Delta : \mathcal{H}_{\text{III}}^A \rightarrow \mathcal{H}_{\text{III}}^A \otimes \mathcal{H}_{\text{III}}^A$ defined on words, $w \mapsto \Delta(w) = \sum_{v'v''=w} v' \otimes v''$. The latter turns the shuffle algebra $\mathcal{H}_{\text{III}}^A$ into a connected graded commutative Hopf algebra $(\mathcal{H}_{\text{III}}^A, \text{III}, \Delta)$ with convolution product defined on the dual $(\mathcal{H}_{\text{III}}^A)^*$:

$$(25) \quad \mathbb{X}_{st} = \mathbb{X}_{su} * \mathbb{X}_{ut} = m_{\mathbb{R}}(\mathbb{X}_{su} \otimes \mathbb{X}_{ut})\Delta.$$

It has long ago been proposed by K. T. Chen to call “generalized path” [10] any two-parameter family of characters of the shuffle algebra $\mathcal{H}_{\text{III}}^A$ verifying (22) and (24) together with a mild continuity assumption. Lyons introduced the seminal notion of rough path [35], which can be defined as follows [30, Definition 1.2]: a *geometric rough path*³ of regularity γ , with $0 < \gamma \leq 1$, is a generalized path in the sense of Chen, satisfying moreover the estimates:

$$(26) \quad \sup_{s \neq t} \frac{|\langle \mathbb{X}_{st}, w \rangle|}{|t - s|^{\gamma|w|}} < C(w)$$

for any word w of length $|w|$, where $C(w)$ is some positive constant. The evaluation on length one words is then given by the increments of a γ -Hölder continuous path:

$$(27) \quad \langle \mathbb{X}_{st}, a_j \rangle := X_j(t) - X_j(s)$$

with $X_j(t) := \langle \mathbb{X}_{t_0t}, a_j \rangle$ for some arbitrary choice of $t_0 \in \mathbb{R}$. Iterated integrals (20) cannot be given any sense for any $n > 0$ if the path is only of regularity $\gamma \leq 1/2$. Lyons’ extension theorem [35, 30], however, stipulates that the collection of coefficients $\langle \mathbb{X}_{st}, w \rangle$ for the words w of length up to $[1/\gamma]$ completely determines the γ -regular rough path \mathbb{X} . This result is a particular case of Theorem 4.4 below, the proof of which also uses the sewing lemma (Proposition 8.11).

4.2. Rough paths generalized to commutative combinatorial Hopf algebras. We have briefly indicated in the Introduction how to adapt the notion of rough path to any connected graded Hopf algebra. Here is the precise definition:

Definition 4.1. *Let $\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n$ be a commutative graded Hopf algebra with unit $\mathbf{1}$, connected in the sense that \mathcal{H}_0 is one-dimensional, and let $\gamma \in]0, 1]$. We suppose that \mathcal{H} is endowed with a homogeneous basis \mathcal{B} making it combinatorial and non-degenerate in the sense of Section 3.2. A γ -regular \mathcal{H} -rough path is a two-parameter family $\mathbb{X} = (\mathbb{X}_{st})_{s,t \in \mathbb{R}}$ of linear forms on \mathcal{H} such that $\langle \mathbb{X}_{st}, \mathbf{1} \rangle = 1$ and*

I) for any $s, t \in \mathbb{R}$ and for any σ, τ in \mathcal{H} , the following equality holds

$$\langle \mathbb{X}_{st}, \sigma\tau \rangle = \langle \mathbb{X}_{st}, \sigma \rangle \langle \mathbb{X}_{st}, \tau \rangle,$$

II) for any $s, t, u \in \mathbb{R}$, Chen’s lemma holds

$$\mathbb{X}_{su} * \mathbb{X}_{ut} = \mathbb{X}_{st},$$

where the convolution $$ is the usual one defined in terms to the coproduct on \mathcal{H} ,*

³To be precise: weak geometric rough path, see [30, Remark 1.3].

III) for any $n \geq 0$ and for any $\sigma \in \mathcal{B}_n$, we have the estimates

$$(28) \quad \sup_{s \neq t} \frac{|\langle \mathbb{X}_{st}, \sigma \rangle|}{|t - s|^{\gamma|\sigma|}} < C(\sigma).$$

The notion of γ -regular branched rough path [29, 30] is recovered by choosing for \mathcal{H} the Butcher–Connes–Kreimer Hopf algebra $\mathcal{H}_{\text{BCK}}^A$ of A -decorated (non-planar) rooted forests. Recall that the product in this Hopf algebra is given by the disjoint union of rooted trees.

Remark 4.2. Theorem 4.4 below will permit to give precise expressions of the constants $C(w)$ and $C(\sigma)$ in Estimates (26) and (28), respectively.

The truncated counterpart of \mathcal{H} -rough paths is defined as follows.

Definition 4.3. Let N be a positive integer and let $\mathcal{H}^{(N)} := \bigoplus_{k=0}^N \mathcal{H}_k$. Let $\gamma \in]0, 1]$. A γ -regular N -truncated \mathcal{H} -rough path is a two-parameter family $\mathbb{X} = (\mathbb{X}_{st})_{s,t \in \mathbb{R}}$ of linear forms on $\mathcal{H}^{(N)}$ such that:

- i) the multiplicativity property (I) above holds for any $\sigma \in \mathcal{H}_p$ and $\tau \in \mathcal{H}_q$ with $p + q \leq N$,
- ii) Chen's lemma (II) holds, where the convolution refers to the restriction of the coproduct to $\mathcal{H}^{(N)}$,
- iii) the estimates (III) hold for any $\sigma \in \mathcal{H}_n$ with $n \leq N$.

For later use we also recall Sweedler's notation $\Delta(\sigma) = \sum_{(\sigma)} \sigma_1 \otimes \sigma_2$, for the full coproduct Δ in \mathcal{H} , as well as its iterated versions $\Delta^{(k-1)}(\sigma) = \sum_{(\sigma)} \sigma_1 \otimes \cdots \otimes \sigma_k$. For the reduced coproduct, we also adopted a Sweedler-type notation:

$$\Delta'(\sigma) := \Delta(\sigma) - \sigma \otimes \mathbf{1} - \mathbf{1} \otimes \sigma = \sum'_{(\sigma)} \sigma' \otimes \sigma''.$$

Lyons' extension theorem [35, Theorem 2.2.1] can be generalised to this setting, with basically the same proof:

Theorem 4.4. Let $\gamma \in]0, 1]$, and let $N := \lceil 1/\gamma \rceil$. Any γ -regular N -truncated \mathcal{H} -rough path admits a unique extension to a γ -regular \mathcal{H} -rough path. Moreover, there exists a positive constant c such that the following estimate holds:

$$(29) \quad |\langle \mathbb{X}_{st}, \sigma \rangle| \leq c^{|\sigma|} q_\gamma(\sigma) |t - s|^{\gamma|\sigma|}$$

for any $\sigma \in \mathcal{B}$, with $q_\gamma(\sigma) = q(\sigma)$ for $|\sigma| \leq N$ and

$$(30) \quad q_\gamma(\sigma) := \frac{1}{2^{\gamma|\sigma|} - 2} \sum'_{(\sigma)} q_\gamma(\sigma') q_\gamma(\sigma'')$$

for $|\sigma| \geq N + 1$.

Proof. Notice that $\varepsilon := \gamma(N + 1) - 1$ is (strictly) positive. If the element $\sigma \in \mathcal{H}$ is homogeneous of degree n , then σ' and σ'' in its reduced coproduct, $\Delta'(\sigma)$, can be taken homogeneous with respective degree p and q with $p + q = n$ and $p, q \leq n - 1$. Now let $(\mathbb{X}_{st})_{s,t \in \mathbb{R}}$ be a γ -regular N -truncated \mathcal{H} -rough path. We extend it trivially to $\mathcal{H}^{(N+1)}$ by setting $\langle \mathbb{X}_{st}, \sigma \rangle = 0$ for any $\sigma \in \mathcal{H}_{N+1}$.

Now let $\sigma \in \mathcal{B}_{N+1}$. Fix an arbitrary $o \in \mathbb{R}$, and consider the function of two real variables:

$$(31) \quad \mu(s, t) := \sum'_{(\sigma)} \langle \mathbb{X}_{os}, \sigma' \rangle \langle \mathbb{X}_{st}, \sigma'' \rangle.$$

Let $s, t, u \in \mathbb{R}$. A simple computation yields:

$$\begin{aligned}
\mu(s, u) + \mu(u, t) - \mu(s, t) &= \sum'_{(\sigma)} -\langle \mathbb{X}_{os}, \sigma' \rangle \langle \mathbb{X}_{st}, \sigma'' \rangle + \langle \mathbb{X}_{os}, \sigma' \rangle \langle \mathbb{X}_{su}, \sigma'' \rangle + \langle \mathbb{X}_{ou}, \sigma' \rangle \langle \mathbb{X}_{ut}, \sigma'' \rangle \\
&= -\sum'_{(\sigma)} \langle \mathbb{X}_{os}, \sigma' \rangle \langle \mathbb{X}_{su}, \sigma'' \rangle \langle \mathbb{X}_{ut}, \sigma''' \rangle - \sum'_{(\sigma)} \langle \mathbb{X}_{os}, \sigma' \rangle \langle \mathbb{X}_{su}, \sigma'' \rangle - \sum'_{(\sigma)} \langle \mathbb{X}_{os}, \sigma' \rangle \langle \mathbb{X}_{ut}, \sigma'' \rangle \\
&\quad + \sum'_{(\sigma)} \langle \mathbb{X}_{os}, \sigma' \rangle \langle \mathbb{X}_{su}, \sigma'' \rangle \\
&\quad + \sum'_{(\sigma)} \langle \mathbb{X}_{os}, \sigma' \rangle \langle \mathbb{X}_{su}, \sigma'' \rangle \langle \mathbb{X}_{ut}, \sigma''' \rangle + \sum'_{(\sigma)} \langle \mathbb{X}_{os}, \sigma' \rangle \langle \mathbb{X}_{ut}, \sigma'' \rangle + \sum'_{(\sigma)} \langle \mathbb{X}_{su}, \sigma' \rangle \langle \mathbb{X}_{ut}, \sigma'' \rangle \\
(32) \quad &= \sum'_{(\sigma)} \langle \mathbb{X}_{su}, \sigma' \rangle \langle \mathbb{X}_{ut}, \sigma'' \rangle.
\end{aligned}$$

Hence if $s \leq u \leq t$ or $t \leq u \leq s$, we can estimate, using (30):

$$\begin{aligned}
|\mu(s, t) - \mu(s, u) - \mu(u, t)| &\leq \sum'_{(\sigma)} |\langle \mathbb{X}_{su}, \sigma' \rangle \langle \mathbb{X}_{ut}, \sigma'' \rangle| \\
&\leq \sum'_{(\sigma)} q_\gamma(\sigma') c^{|\sigma'|} |u - s|^{\gamma|\sigma'|} q_\gamma(\sigma'') c^{|\sigma''|} |t - u|^{\gamma|\sigma''|}.
\end{aligned}$$

Here we have chosen the constant c such that the estimates (29) hold for any $\sigma \in \mathcal{B}_n$, $n \leq N$. This is possible since the sets \mathcal{B}_n are finite. It follows from $\gamma|\sigma| = \gamma(N+1) = 1 + \varepsilon$ and the Sewing Lemma (Proposition 8.11) that there exists a unique map φ defined on \mathbb{R} , up to an additive constant, such that:

$$(33) \quad |\varphi(t) - \varphi(s) - \mu(s, t)| \leq \frac{c^{|\sigma|}}{2^{\gamma|\sigma|} - 2} \sum'_{(\sigma)} q_\gamma(\sigma') q_\gamma(\sigma'') |t - s|^{\gamma|\sigma|}$$

$$(34) \quad = c^{|\sigma|} q_\gamma(\sigma) |t - s|^{\gamma|\sigma|}.$$

Now defining:

$$(35) \quad \langle \tilde{\mathbb{X}}_{st}, \sigma \rangle := \begin{cases} \langle \mathbb{X}_{st}, \sigma \rangle, & \text{for } \sigma \in \mathcal{H}_n, n \leq N, \\ \varphi(s) - \varphi(t) - \mu(s, t), & \text{for } \sigma \in \mathcal{H}_{N+1}, \end{cases}$$

we immediately get from (32):

$$(36) \quad \langle \tilde{\mathbb{X}}_{st} - \tilde{\mathbb{X}}_{su} - \tilde{\mathbb{X}}_{ut}, \sigma \rangle = \sum'_{(\sigma)} \langle \mathbb{X}_{su}, \sigma' \rangle \langle \mathbb{X}_{ut}, \sigma'' \rangle.$$

From (33) and (35) we then have

$$(37) \quad |\langle \tilde{\mathbb{X}}_{st}, \sigma \rangle| \leq c^{|\sigma|} q_\gamma(\sigma) |t - s|^{\gamma|\sigma|}.$$

Let us now check item (i) in Definition 4.3 for $\tilde{\mathbb{X}}_{st}$ for any $\sigma \in \mathcal{H}_p$ and $\tau \in \mathcal{H}_q$ with $p + q = N + 1$. We split the interval $[s, t]$ (or $[t, s]$) into k sub-intervals $[s_j, s_{j+1}]$ of equal length $|t - s|/k$, with $s_0 := \inf(s, t)$ and $s_k := \sup(s, t)$. From Chen's lemma up to degree $N + 1$ stemming from (36), we can compute, supposing $s \leq t$ here:

$$\langle \tilde{\mathbb{X}}_{st}, \sigma\tau \rangle - \langle \tilde{\mathbb{X}}_{st}, \sigma \rangle \langle \tilde{\mathbb{X}}_{st}, \tau \rangle$$

$$\begin{aligned}
&= \langle \tilde{\mathbb{X}}_{s_0 s_1} * \cdots * \tilde{\mathbb{X}}_{s_{k-1} s_k}, \sigma \tau \rangle - \langle \tilde{\mathbb{X}}_{s_0 s_1} * \cdots * \tilde{\mathbb{X}}_{s_{k-1} s_k}, \sigma \rangle \langle \tilde{\mathbb{X}}_{s_0 s_1} * \cdots * \tilde{\mathbb{X}}_{s_{k-1} s_k}, \tau \rangle \\
&= \sum_{(\sigma), (\tau)} \left(\prod_{j=0}^{k-1} \langle \tilde{\mathbb{X}}_{s_j s_{j+1}}, \sigma_j \tau_j \rangle - \prod_{j=0}^{k-1} \langle \tilde{\mathbb{X}}_{s_j s_{j+1}}, \sigma_j \rangle \langle \tilde{\mathbb{X}}_{s_j s_{j+1}}, \tau_j \rangle \right).
\end{aligned}$$

The term under the summation sign vanishes unless there is a $j \in \{0, \dots, k-1\}$ such that $\sigma_j = \sigma$ and $\tau_j = \tau$, in which case we have $\sigma_i = \tau_i = \mathbf{1}$ for $i \neq j$. Hence,

$$\langle \tilde{\mathbb{X}}_{st}, \sigma \tau \rangle - \langle \tilde{\mathbb{X}}_{st}, \sigma \rangle \langle \tilde{\mathbb{X}}_{st}, \tau \rangle = \sum_{j=0}^{k-1} \langle \tilde{\mathbb{X}}_{s_j s_{j+1}}, \sigma \tau \rangle - \langle \tilde{\mathbb{X}}_{s_j s_{j+1}}, \sigma \rangle \langle \tilde{\mathbb{X}}_{s_j s_{j+1}}, \tau \rangle.$$

From (37) and item (iii) of Definition 4.3 we get

$$|\langle \tilde{\mathbb{X}}_{st}, \sigma \tau \rangle - \langle \tilde{\mathbb{X}}_{st}, \sigma \rangle \langle \tilde{\mathbb{X}}_{st}, \tau \rangle| \leq kc^{|\sigma|} q_\gamma(\sigma) \left| \frac{t-s}{k} \right|^{1+\varepsilon} = k^{-\varepsilon} c^{|\sigma|} q_\gamma(\sigma) |t-s|^{1+\varepsilon}.$$

Hence $\langle \tilde{\mathbb{X}}_{st}, \sigma \tau \rangle = \langle \tilde{\mathbb{X}}_{st}, \sigma \rangle \langle \tilde{\mathbb{X}}_{st}, \tau \rangle$ by letting k go to $+\infty$. The same argument works *mutatis mutandis* in the case $s > t$. Hence, estimate (29) is proven for $\langle \tilde{\mathbb{X}}, \sigma \rangle$ for any $\sigma \in \mathcal{B}_n$, $n \leq N+1$.

Uniqueness can be proven by a similar argument. Indeed, suppose that $\bar{\mathbb{X}}$ is another $(N+1)$ -truncated γ -regular \mathcal{H} -rough path extending \mathbb{X} , and let $\delta_{st} := \tilde{\mathbb{X}}_{st} - \bar{\mathbb{X}}_{st}$ for any $s, t \in \mathbb{R}$. For any $\sigma \in \mathcal{H}_{N+1}$ we have then the following:

$$\begin{aligned}
\langle \delta_{st}, \sigma \rangle &= \langle \tilde{\mathbb{X}}_{s s_1} * \cdots * \tilde{\mathbb{X}}_{s_{k-1} t} - \bar{\mathbb{X}}_{s s_1} * \cdots * \bar{\mathbb{X}}_{s_{k-1} t}, \sigma \rangle \\
&= \langle \delta_{s s_1} + \cdots + \delta_{s_{k-1} t}, \sigma \rangle.
\end{aligned}$$

As we have $|\langle \delta_{s_j s_{j+1}}, \sigma \rangle| \leq \bar{C} \left| \frac{t-s}{k} \right|^{1+\varepsilon}$ for some constant \bar{C} , we get $|\langle \delta_{st}, \sigma \rangle| \leq \bar{C} |t-s|^{1+\varepsilon} k^{-\varepsilon}$, hence $\langle \delta_{st}, \sigma \rangle = 0$ by letting k go to infinity.

Iterating this process at any order finally yields a fully-fledged γ -regular \mathcal{H} -rough path $\tilde{\mathbb{X}}$ extending \mathbb{X} . \square

Remark 4.5. Considering the striking similarity of the map q_γ with the inverse-factorial character, which is nothing but q_γ for $\gamma = 1$, Gubinelli conjectured [29, Remark 7.4] the following comparison, in the special case of the Butcher–Connes–Kreimer Hopf algebra (corresponding to branched rough paths):

$$(38) \quad q_\gamma(\sigma) \leq \frac{BC^{|\sigma|}}{(\sigma!)^\gamma}$$

for any σ in \mathcal{B} (i.e., any decorated forest in this particular case of $\mathcal{H}_{\text{BCK}}^A$), where B and C are positive constants. This conjecture has been recently proven by H. Boedihardjo [3]. In the case of the shuffle Hopf algebra (corresponding to geometric rough paths), it happens to be a consequence of Lyons' neoclassical inequality ([35, Theorem 2.1.1], see also [29, Remark 7.4]). It would be interesting to prove a similar result for a general class of combinatorial Hopf algebra, in particular for the Hopf algebra of Lie group integrators $\mathcal{H}_{\text{MKW}}^A$ defined in paragraph 5.6 below, corresponding to the notion of rough paths we call *planarly branched rough paths*, defined in Paragraph 6.2. Our definition of q_γ differs from that of Gubinelli's in the initial conditions $q_\gamma(\sigma) = q(\sigma)$ versus $q_\gamma^{\text{Gub}}(\sigma) = 1$ for any $\sigma \in \mathcal{B}_n$, $n \leq N$. This choice is dictated by the functorial considerations of Paragraph 4.4 below. In practice, one has very often $q(\sigma) \leq 1$ for any element $\sigma \in \mathcal{B}_n$, $n \leq N$, which yields $q_\gamma(\sigma) \leq q_\gamma^{\text{Gub}}(\sigma)$ for any $\sigma \in \mathcal{B}$ by induction. Hence the majorations for q_γ^{Gub} obtained in [3] in the branched case also hold for our q_γ .

4.3. Factorial decay estimates. The linear map q_γ defined in the statement of Theorem 4.4 is uniquely defined by $q_\gamma(\sigma) = 1$ for any $\sigma \in \mathcal{B}_1 \cup \{\mathbf{1}\}$ and the recursive equations

$$q_\gamma(\sigma) := \begin{cases} \frac{1}{2^{|\sigma|}} q_\gamma * q_\gamma(\sigma), & \text{for } 2 \leq |\sigma| \leq N, \\ \frac{1}{2^{|\sigma|}} q_\gamma * q_\gamma(\sigma), & \text{for } |\sigma| \geq N + 1. \end{cases}$$

As a consequence, q_γ has the same functorial properties than the inverse-factorial character q , namely if $(\mathcal{H}, \mathcal{B})$ and $(\mathcal{H}', \mathcal{B}')$ are two connected graded Hopf algebras and if $\Phi : \mathcal{H} \rightarrow \mathcal{H}'$ is a morphism of Hopf algebras preserving the degree, then we have for any $\gamma \in]0, 1]$, with self-explanatory notations:

$$(39) \quad q_\gamma = q'_\gamma \circ \Phi.$$

Proposition 4.6. [29, Remark 7.4] *Let $\mathcal{H}_{\text{III}}^A$ be the shuffle Hopf algebra on a finite alphabet A . Then the following estimates hold: for any word $w \in A^*$,*

$$(40) \quad |q_\gamma(w)| \leq \frac{C_\gamma^{|w|-1}}{(|w|!)^\gamma}$$

where C_γ is a positive real number depending only on γ .

Proof. Recall Gubinelli's variant of Lyons' neo-classical inequality: there exists $c_\gamma > 0$ such that

$$(41) \quad \sum_{k=0}^n \frac{a^{\gamma k} b^{\gamma(n-k)}}{(k!)^\gamma [(n-k)!]^\gamma} \leq c_\gamma \frac{(a+b)^{n\gamma}}{(n!)^\gamma}.$$

Now set

$$C_\gamma := \sup \left(1, \frac{2^{\gamma(N+1)}}{2^{\gamma(N+1)} - 2} c_\gamma \right),$$

and proceed by induction on the length of the word w . The case $|w| \leq N$ being obvious, suppose $|w| \geq N + 1$. We can compute, using (41) in the particular case $a = b = 1$:

$$\begin{aligned} q_\gamma(w) &= \frac{1}{2^{\gamma|w|} - 2} \sum'_{(w)} q_\gamma(w') q_\gamma(w'') \\ &\leq \frac{1}{2^{\gamma|w|} - 2} \sum'_{(w)} \frac{C_\gamma^{|w'|-1}}{(|w'|!)^\gamma} \frac{C_\gamma^{|w''|-1}}{(|w''|!)^\gamma} \\ &\leq \frac{1}{2^{\gamma|w|} - 2} \sum_{(w)} \frac{C_\gamma^{|w_1|-1}}{(|w_1|!)^\gamma} \frac{C_\gamma^{|w_2|-1}}{(|w_2|!)^\gamma} \\ &\leq \frac{C_\gamma^{|w|-2}}{2^{\gamma|w|} - 2} c_\gamma \frac{2^{|w|\gamma}}{(|w|!)^\gamma} \\ &\leq \frac{C_\gamma^{|w|-1}}{(|w|!)^\gamma}. \end{aligned}$$

□

Corollary 4.7. *Let $(\mathcal{H}, \mathcal{B})$ be a combinatorial Hopf algebra endowed with a combinatorial morphism $\Phi : (\mathcal{H}, \mathcal{B}) \rightarrow (\mathcal{H}_{\text{III}}^A, \mathcal{B}')$, where A is a finite alphabet and $\mathcal{B}' = A^*$ is the standard basis of words. Then for any $\sigma \in \mathcal{B}$ the following estimate holds:*

$$(42) \quad |q_\gamma(\sigma)| \leq C_\gamma^{|\sigma|-1} \frac{(|\sigma|!)^{1-\gamma}}{\sigma!}$$

with the same C_γ as above.

Proof. For any $\sigma \in \mathcal{B}$ we have $\Phi(\sigma) = \sum_{w \in A^*} b_w^\sigma w$, and we have by functoriality of the inverse factorial:

$$\sum_{w \in A^*} b_w^\sigma = \frac{|\sigma|!}{\sigma!}.$$

The proof relies on a simple computation using functoriality of q_γ as well as the non-negativity of the coefficients b_w^σ , together with the fact that the L^p norms are nondecreasing with respect to $p \in]0, 1]$ for probability measures:

$$\begin{aligned} |q_\gamma(\sigma)|^{1/\gamma} &= \left(\sum_{w \in A^*} b_w^\sigma q_\gamma(w) \right)^{1/\gamma} \\ &= \left(\frac{|\sigma|!}{\sigma!} \right)^{1/\gamma} \left(\frac{\sigma!}{|\sigma|!} \sum_{w \in A^*} b_w^\sigma q_\gamma(w) \right)^{1/\gamma} \\ &\leq \left(\frac{|\sigma|!}{\sigma!} \right)^{1/\gamma-1} \sum_{w \in A^*} b_w^\sigma q_\gamma(w)^{1/\gamma} \\ &\leq \left(\frac{|\sigma|!}{\sigma!} \right)^{1/\gamma-1} \sum_{w \in A^*} C_\gamma^{(|w|-1)/\gamma} \frac{b_w^\sigma}{|w|!} \\ &\leq C_\gamma^{(|\sigma|-1)/\gamma} \left(\frac{|\sigma|!}{\sigma!} \right)^{1/\gamma-1} \frac{1}{\sigma!} \\ &\leq \left(C_\gamma^{|\sigma|-1} \frac{(|\sigma|!)^{1-\gamma}}{\sigma!} \right)^{1/\gamma}. \end{aligned}$$

□

We remark that H. Boedihardjo recently obtained a much better estimate in the context of Gubinelli's branched rough paths, i.e., for the Butcher–Connes–Kreimer Hopf algebra, see [3, Theorem 4].

4.4. Rough paths and combinatorial Hopf algebras. We shall examine further properties of rough paths in the generalised sense given in Paragraph 4.2, i.e., when the Hopf algebra at hand is combinatorial.

Proposition 4.8. (1) *Let $(\mathcal{H}, \mathcal{B})$ be a combinatorial Hopf algebra in the sense of Paragraph 3.2 and let q be the associated inverse factorial character. Then $q(x) = \frac{1}{x!}$ is a (possibly vanishing) non-negative rational number for any $x \in \mathcal{B}$.*

(2) *Let $(\mathcal{H}, \mathcal{B})$ and $(\mathcal{H}', \mathcal{B}')$ be two combinatorial Hopf algebras, and let $\Phi : (\mathcal{H}, \mathcal{B}) \rightarrow (\mathcal{H}', \mathcal{B}')$ be a combinatorial Hopf algebra morphism. Then the pull-back $\mathbb{X}_{st} := \mathbb{X}'_{st} \circ \Phi$ of any γ -regular \mathcal{H}' -rough path \mathbb{X}'_{st} is a γ -regular \mathcal{H} -rough path.*

Proof. Recall that $q(x) = 1$ for any $x \in \mathcal{B}_1$. The first assertion is then recursively derived from equation (15). Multiplicativity as well as Chen's Lemma are immediate consequences of the fact that Φ is a Hopf algebra morphism. We now check the estimate for any $x \in \mathcal{B}_n$ with $n \geq 0$:

$$\begin{aligned} |\langle \mathbb{X}_{st}, x \rangle| &= |\langle \mathbb{X}'_{st}, \Phi(x) \rangle| \\ &\leq \sum_{y \in \mathcal{B}'_n} b_y^x |\langle \mathbb{X}'_{st}, y \rangle| \\ &\leq C^n \sum_{y \in \mathcal{B}'_n} q_\gamma(y) b_y^x |t - s|^{\gamma n}. \end{aligned}$$

The proof of functoriality of the inverse factorial character (17) can be easily adapted to its counterpart q_γ , although it is generally not a character when γ differs from 1. Hence we can derive the desired estimate:

$$(43) \quad |\langle \mathbb{X}_{st}, x \rangle| \leq C^n q_\gamma(x) |t - s|^{\gamma n}.$$

Note that we have used the non-negativity of the coefficients b_y^x of the matrix of Φ expressed in the bases \mathcal{B} and \mathcal{B}' . \square

5. LIE-BUTCHER THEORY

Butcher's *B-series* are a special form of Taylor expansion indexed by trees. They have become a fundamental tool for analysing numerical integration algorithms. The numerical analysis of general Lie group methods requires the generalisation of the *B-series* theory to so-called Lie-Butcher series, which are based on planar rooted forest, possibly decorated.

5.1. Rooted trees and forests. For any positive integer n , a rooted tree of degree n is a finite oriented tree with n vertices. One of them, called the root, is a distinguished vertex without any outgoing edge. Any vertex can have arbitrarily many incoming edges, and any vertex other than the root has exactly one outgoing edge. Vertices with no incoming edges are called leaves. A planar rooted tree is a rooted tree together with an embedding in the plane. A planar rooted forest is a finite ordered collection of planar rooted trees. Here are the planar rooted forests up to four vertices:



Let A be any set. An A -decorated planar rooted forest is a pair $\sigma = (\bar{\sigma}, \varphi)$ where $\bar{\sigma}$ is a planar forest, and where φ is a map from the vertex set $V(\bar{\sigma})$ into A . We denote by T_A^{pl} (respectively F_A^{pl}) the set of all A -decorated planar rooted trees (respectively forests), and by $\mathcal{T}_A^{\text{pl}}$ (respectively $\mathcal{F}_A^{\text{pl}}$) the linear space spanned by the elements of T_A^{pl} (respectively F_A^{pl}).

A -decorated non-planar rooted forests are denoted by $\tilde{\sigma} = (\bar{\tilde{\sigma}}, \varphi)$, where φ is the decoration and $\bar{\tilde{\sigma}}$ is the underlying non-planar forest. When A is reduced to one element the notion of A -decoration is superfluous. Hence any A -decorated forest can be identified with its overlined counterpart.

Every A -decorated planar rooted tree can be written as:

$$(44) \quad \sigma = B_a^+(\sigma_1 \cdots \sigma_k),$$

where B_a^+ is the operation on forests which grafts each connected component σ_i of a planar rooted forest $\sigma_1 \cdots \sigma_k$ on a common root decorated by $a \in A$. Note that in numerical analysis the bracket notation for $\sigma = [\sigma_1 \cdots \sigma_k]_a$ is often used instead of the B_a^+ operator.

5.2. Post-Lie and post-associative algebras. A *left post-Lie algebra* [43, 18] is a vector space A (over some field \mathbf{k}) together with two bilinear maps $[-, -]$ and \triangleright from $A \otimes A$ to A such that

- $[-, -]$ is a Lie bracket, i.e., it is antisymmetric and verifies the Jacobi identity.
- For any $a, b, c \in A$ we have

$$a \triangleright [b, c] = [a \triangleright b, c] + [a, b \triangleright c].$$

- For any $a, b, c \in A$ we have

$$[a, b] \triangleright c = a \triangleright (b \triangleright c) - (a \triangleright b) \triangleright c - b \triangleright (a \triangleright c) + (b \triangleright a) \triangleright c.$$

The bracket $\llbracket -, - \rrbracket$ defined by $\llbracket a, b \rrbracket := [a, b] + a \triangleright b - b \triangleright a$ is another Lie bracket on A . The particular case when the Lie bracket $[-, -]$ vanishes on A is referred to as *left pre-Lie algebra*. See [37] for details. Associative counterparts of post-Lie algebras are referred to as *post-associative algebras*. They first appear under the terminology "D-algebras" in [38]. A post-associative algebra is a vector space B endowed with two linear maps \cdot and \triangleright from $B \otimes B$ to B , a filtration $B^0 = \mathbf{k} \cdot \mathbf{1} \subset B^1 \subset B^2 \subset \cdots$ with $B = \bigcup_j B^j$, and an augmentation $\varepsilon : B \rightarrow \mathbf{k}$ such that

- (1) $L_{\mathbf{1}} = \text{Id}_B$, and $a \triangleright \mathbf{1} = 0$ for any $a \in \text{Ker } \varepsilon$.
- (2) The product \cdot is associative with unit $\mathbf{1}$, and $B^p \cdot B^q \subset B^{p+q}$ for any $p, q \geq 0$.
- (3) $A := B^1 \cap \text{Ker } \varepsilon$ is stable under the product \triangleright as well as under the Lie bracket obtained by antisymmetrisation of the associative product, and generates the unital associative algebra (B, \cdot) .
- (4) For any $a, b, c \in B$ with $a \in A$ we have

$$a \triangleright (b \cdot c) = (a \triangleright b) \cdot c + b \cdot (a \triangleright c).$$

- (5) For any $a, b, c \in B$ with $a \in A$ we have

$$(a \cdot b) \triangleright c = a \triangleright (b \triangleright c) - (a \triangleright b) \triangleright c.$$

In particular, A is a post-Lie algebra. The other way round, the enveloping algebra of a post-Lie algebra is a post-associative algebra.

The *Grossman–Larson product* on a post-associative algebra is characterised by the identity:

$$(45) \quad (a * b) \triangleright c = a \triangleright (b \triangleright c)$$

for any $a, b, c \in B$, in other words $L_{a*b} = L_a \circ L_b$. It is defined as follows: the map $M : A \rightarrow \text{Der } \mathcal{U}(A, \llbracket -, - \rrbracket)$ defined by $M_a b := ab + a \triangleright b$ is easily seen to verify

$$\llbracket M_a, M_b \rrbracket = M_{\llbracket a, b \rrbracket}.$$

Hence M yields an associative algebra morphism, still denoted by M , from $\mathcal{U}(A, \llbracket -, - \rrbracket)$ to $\text{End } \mathcal{U}(A, \llbracket -, - \rrbracket)$. Using for example a Poincaré–Birkhoff–Witt basis, one can see that the morphism

$$\begin{aligned} \Theta : \mathcal{U}(A, \llbracket -, - \rrbracket) &\longrightarrow \mathcal{U}(A, \llbracket -, - \rrbracket) \\ v &\longmapsto M_v \cdot \mathbf{1} \end{aligned}$$

of $\mathcal{U}(A, \llbracket -, - \rrbracket)$ -modules is bijective, which yields a new associative product $*$ on $\mathcal{U}(A, \llbracket -, - \rrbracket)$ given by $u * v := \Theta(\Theta^{-1}(u) \cdot \Theta^{-1}(v))$. Now the identity of A extends to a unique surjective post-associative algebra morphism $\kappa : \mathcal{U}(A, \llbracket -, - \rrbracket) \rightarrow B$. The ideal $\text{Ker } \kappa$ is stable by both products \cdot and \triangleright , hence by the product $*$, which thus descends to the quotient B . In particular, for any $a, b, c \in B^1$ we have $a * b = ab + a \triangleright b$, and

$$a * b * c = a \cdot b \cdot c + (a \triangleright b) \cdot c + b \cdot (a \triangleright c) + a \cdot (b \triangleright c) + (a \cdot b) \triangleright c + (a \triangleright b) \triangleright c.$$

An important example of post-Lie algebra is given by $C^\infty(\mathcal{M}, \mathfrak{g})$. We suppose that the Lie group G , with Lie algebra \mathfrak{g} , acts transitively on the smooth manifold \mathcal{M} . Any smooth map $f \in C^\infty(\mathcal{M}, \mathfrak{g})$ defines a smooth vector field $\#f$ on \mathcal{M} (i.e., a derivation of $C^\infty(\mathcal{M})$) via:

$$(46) \quad \#f(g) := \left. \frac{d}{dt} \right|_{t=0} g(\exp(tf(x)).x).$$

In the language of Lie algebroids, considering the tangent vector bundle and the trivial vector bundle $E = \mathcal{M} \times \mathfrak{g}$, the map $\# : C^\infty(\mathcal{M}, \mathfrak{g}) \rightarrow \text{Der } C^\infty(\mathcal{M})$ is the composition on the left with the anchor map $\rho : E \rightarrow T\mathcal{M}$ defined by $\rho(x, X) := \left. \frac{d}{dt} \right|_{t=0} (\exp tX).x$.

Formula (46) also makes sense for $g \in C^\infty(\mathcal{M}, \mathfrak{g})$ or $g \in C^\infty(\mathcal{M}, \mathcal{U}(\mathfrak{g}))$. It is shown in [38] that $C^\infty(\mathcal{M}, \mathcal{U}(\mathfrak{g}))$, endowed with the pointwise product in $\mathcal{U}(\mathfrak{g})$ as well as the product \triangleright given by $f \triangleright g := \#f(g)$ is a post-associative algebra. The Grossman–Larson product reflects the composition of differential operators, in the sense that we have

$$\#(f * g) = \#f \circ \#g.$$

Similarly $C^\infty(\mathcal{M}, \mathfrak{g})$, endowed with the pointwise Lie bracket in \mathfrak{g} and the product \triangleright given by $f \triangleright g := \#f(g)$ is a post-Lie algebra.

5.3. Free post-Lie algebras. It is proven in [38] that the free post-associative algebra \mathcal{D}_A generated by the set A is the algebra of A -decorated planar forests endowed with concatenation and left grafting. The latter is defined for any A -decorated planar rooted tree σ and forest τ :

$$(47) \quad \sigma \triangleright \tau = \sum_{v \text{ vertex of } \tau} \sigma \searrow_v \tau,$$

where $\sigma \searrow_v \tau$ is the decorated forest obtained by grafting the planar tree σ on the vertex v of the planar forest τ , such that σ becomes the leftmost branch, starting from vertex v , of this new tree. It is also well-known that the usual grafting product “ \rightarrow ” given for *non-planar* rooted trees $\tilde{\tau}$ by the same formula (47), satisfies the left pre-Lie identity:

$$\tilde{\sigma}_1 \rightarrow (\tilde{\sigma}_2 \rightarrow \tilde{\tau}) - (\tilde{\sigma}_1 \rightarrow \tilde{\sigma}_2) \rightarrow \tilde{\tau} = \tilde{\sigma}_2 \rightarrow (\tilde{\sigma}_1 \rightarrow \tilde{\tau}) - (\tilde{\sigma}_2 \rightarrow \tilde{\sigma}_1) \rightarrow \tilde{\tau}.$$

One the other hand, the linear span of A -decorated planar rooted trees endowed with the operation \triangleright of (47) is the free magmatic algebra⁴, see [17]. The definition of $\sigma \triangleright \tau$ when σ is a decorated forest is given recursively with respect to the number of connected components of σ , using axiom (5).

⁴The free magma $(\mathcal{M}_A, *)$ on a set A is the set of well-parenthesised words with letters in A . The binary operation $*$ consists in putting each component between an extra pair of parentheses and concatenating them, e.g. $ab * c(de) = (ab)(c(de))$. The free magmatic algebra $\langle \mathcal{M}_A \rangle$ is the vector space freely generated by \mathcal{M}_A , endowed by the bilinear extension of the product $*$. It is determined up to isomorphism by the universal property so that for any vector space V endowed with a bilinear map $\# : V \times V \rightarrow V$ and for any set map $f : A \rightarrow V$, there is a unique linear map $\bar{f} : \langle \mathcal{M}_A \rangle \rightarrow V$ respecting both bilinear products.

As a result, the free post-Lie algebra \mathcal{P}_A generated by A is the free Lie algebra generated by the linear span of A -decorated planar rooted trees (see also [43]).

5.4. Lie–Butcher series. Let \mathcal{M} be a homogeneous space under the action of a Lie group G with Lie algebra \mathfrak{g} , and let $f := \{f_i\}_{i \in A}$ be a collection of smooth maps from \mathcal{M} to \mathfrak{g} indexed by a set A . By freeness property, there is a unique post-Lie algebra morphism $\mathcal{F}_f : \mathcal{P}_A \rightarrow C^\infty(\mathcal{M}, \mathfrak{g})$ such that $\mathcal{F}_f(\bullet_i) = f_i$. The vector fields $\#\mathcal{F}_f(\sigma)$, where σ is a planar rooted A -decorated tree, are the so-called *elementary differentials*. Similarly, \mathcal{F}_f extends uniquely to a post-associative algebra morphism $\mathcal{F}_f : \mathcal{D}_A \rightarrow C^\infty(\mathcal{M}, \mathcal{U}(\mathfrak{g}))$. This extended morphism also respects the Grossman–Larson product of both sides.

A *Lie–Butcher series* is an element of $C^\infty(\mathcal{M}, \mathcal{U}(\mathfrak{g}))[[\hbar]]$ given by

$$(48) \quad LB(\alpha, hf) := \sum_{k \geq 0} \sum_{\sigma \in F_{A,k}^{\text{pl}}} \hbar^k \alpha(\sigma) \mathcal{F}_{hf}(\sigma),$$

where $F_{A,k}^{\text{pl}}$ is the set of A -decorated planar rooted forests with k vertices and α is a linear map from \mathcal{P}_A to the field \mathbf{k} .

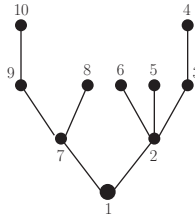
5.5. Three partial orders on planar forests. Let $\bar{\sigma}$ be any planar rooted forest, and v, w be two elements in its vertex set $V(\bar{\sigma})$. Define a partial order $<$ on $V(\bar{\sigma})$ as follows: $v < w$ if there is a path from one root to w passing through v . Roots are the minimal elements, and leaves are the maximal elements.

Following [1], we define a refinement \ll of this order to be the transitive closure of the relation R defined by: vRw if $v < w$, or both v and w are linked to a third vertex $u \in V(\bar{\sigma})$, such that v lies on the right of w , like this:



or both v and w are roots, with v on the right of w .

A further refinement \lll on $V(\bar{\sigma})$ is the total order defined as follows: $v \lll w$ if and only if v occurs before w on a path exploring the rooted forest from right to left, starting from the root of the rightmost connected component:



A planar rooted tree with its vertices labelled according to total order \lll .

5.6. The Hopf algebra of Lie group integrators. The universal enveloping algebra over the free post-Lie algebra \mathcal{P}_A endowed with the Grossman–Larson product and deshuffle coproduct⁵ is a connected Hopf algebra, graded by the number of vertices. Its graded dual is the Hopf algebra of Lie group integrators $\mathcal{H}_{\text{MKW}}^A$

⁵Uniquely determined by the fact that any A -decorated planar rooted tree is primitive.

introduced by Munthe-Kaas and Wright [38]. The convolution product on $\mathcal{L}(\mathcal{H}_{\text{MKW}}^A, \mathbf{k})$ is then the Grossman–Larson product naturally extended to series. The product is the shuffle product of planar forests (where the trees are the letters), and the coproduct is given in terms of left-admissible cuts [38]:

$$(49) \quad \Delta(\tau) = \sum_{\substack{V' \sqcup V'' = V(\tau) \\ V'' \ll V'}} (\tau|_{V'})^{\text{III}} \otimes \tau|_{V''}.$$

Here $(\tau|_{V'})^{\text{III}}$ is the shuffle product of the connected components of the poset $(V', \ll|_{V'})$. Note that the restriction of the partial order \ll to V' is generally weaker than the partial order \ll of the forest $\tau|_{V'}$: the latter makes the poset V' connected, which is generally not the case for the former.

We note that the Hopf algebra of Lie group integrators $\mathcal{H}_{\text{MKW}}^A$, endowed with the basis of A -decorated planar forests, is a combinatorial Hopf algebra in the sense of Paragraph 3.2.

6. PLANARLY BRANCHED ROUGH PATHS

We prove in this section the non-degeneracy of the combinatorial Hopf algebra $\mathcal{H}_{\text{MKW}}^A$ of Lie group integrators endowed with the basis of A -decorated rooted forests, and we define planarly branched rough paths as $\mathcal{H}_{\text{MKW}}^A$ -rough paths.

6.1. Tree and forest factorials, volume computations. For any $s \leq t \in \mathbb{R}$ and any finite poset P we consider the domain

$$\Omega_P^{st} := \{(t_v)_{v \in P}, s \leq t_v \leq t \text{ and } t_v \geq t_w \text{ for } v < w\} \subset \mathbb{R}^P.$$

The factorial of the poset P is uniquely determined by

$$(50) \quad \text{Volume}(\Omega_P^{st}) = \frac{|t - s|^{|P|}}{P!}.$$

For any *planar* rooted forest τ (decorated or not), we set [27]:

$$\tau! := (V(\tau), \ll)!.$$

In particular, the factorial of a poset is the product of factorials of its connected components. Note, however, that our definition differs from L. Foissy's definition given in [26, Definition 33]. In particular, our notion of factorial is invariant under the canonical involution reversing the partial order, which is not the case for the poset factorial of [26].

The factorial $\tilde{\tau}!$ of a *non-planar* rooted forest $\tilde{\tau}$ is the factorial of the underlying poset $(V(\tilde{\tau}), <)$. The notations will always make clear whether a planar or non-planar forest factorial is considered, hence the common notation $-!$ should not cause any confusion.

Lemma 6.1 ([27]). *Let $\tilde{\tau}$ (respectively σ) be a non-planar (respectively planar) rooted forest. Then:*

- (1) *For any $s \leq t \in \mathbb{R}$, the volume of the domain $\Omega_{\tilde{\tau}}^{st} := \Omega_{V(\tilde{\tau}), <}^{st}$ is equal to $\frac{1}{\tilde{\tau}!}(t - s)^{|\tilde{\tau}|}$.*
- (2) *For any $s \leq t \in \mathbb{R}$, the volume of the domain $\Omega_{V(\sigma), \ll}^{st}$ is equal to $\frac{1}{\sigma!}(t - s)^{|\sigma|}$.*
- (3) *The following identity holds:*

$$\frac{1}{\tilde{\tau}!} = \text{Sym}(\tilde{\tau}) \sum_{\sigma \twoheadrightarrow \tilde{\tau}} \frac{1}{\sigma!},$$

where the sum runs over all the planar representatives σ of $\tilde{\tau}$, and where $\text{Sym}(\sigma)$ is the symmetry factor of the planar rooted forest σ .

Proof. The volume of $\Omega_{\tilde{\tau}}^{st}$ is multiplicative, i.e., it is the product of the volumes of Ω_c^{st} where c runs over the connected components of $\tilde{\tau}$. The inverse of the forest factorial shares the same property. Hence it is sufficient to check the result on trees. We proceed by induction on the number of vertices. The case of one vertex boils down to:

$$\text{Vol}(\Omega_{\bullet}^{st}) = t - s = \frac{1}{\bullet!}(t - s).$$

Suppose that $\tilde{\tau} = B^+(f)$ is a tree with at least two vertices. From (50) we get:

$$\frac{1}{\tilde{\tau}!} = \frac{1}{|\tilde{\tau}|} \frac{1}{f!}.$$

On the other hand, using the induction hypothesis, we have

$$\begin{aligned} \text{Vol}(\Omega_{\tilde{\tau}}^{st}) &= \int_s^t \text{Vol}(\Omega_f^{sz}) dz \\ &= \frac{1}{f!} \int_s^t (z - s)^{|f|} dz \\ &= \frac{1}{f!} \frac{1}{|\tilde{\tau}|} (t - s)^{|\tilde{\tau}|} \\ &= \frac{1}{\tilde{\tau}!} (t - s)^{|\tilde{\tau}|}. \end{aligned}$$

Now let σ be a planar rooted forest. The volume of $\Omega_{V(\sigma), \ll}^{st}$ is multiplicative for the shuffle product but not for the concatenation. However, any planar rooted forest σ admits a natural unique decomposition:

$$\sigma = \sigma' \times \sigma'' = \sigma' B^+(\sigma''),$$

where σ' and σ'' are again (possibly empty) planar rooted forests. The poset $(V(\sigma), \ll)$ is obtained by considering the direct product of the two posets $(V(\sigma'), \ll)$ and $(V(\sigma''), \ll)$, and adding an extra element (the root of σ) smaller than any other element. From (50) again we get:

$$(51) \quad \sigma! = |\sigma| \sigma'! \sigma''!$$

The second assertion is then proved recursively with a computation analogous to one above:

$$\begin{aligned} \text{Vol}(\Omega_{V(\sigma), \ll}^{st}) &= \int_{z=s}^t \text{Vol}(\Omega_{V(\sigma'), \ll}^{sz}) \text{Vol}(\Omega_{V(\sigma''), \ll}^{sz}) dz \\ &= \frac{1}{\sigma'! \sigma''!} \int_{z=s}^t (s - z)^{|\sigma'| + |\sigma''|} dz \\ &= \frac{1}{|\sigma| \sigma'! \sigma''!} (t - s)^{|\sigma|} \\ &= \frac{1}{\sigma!} (t - s)^{|\sigma|}. \end{aligned}$$

Now let $\tilde{\tau}$ be a non-planar rooted forest. For any $v \in V(\tilde{\tau})$, let $\text{St}(v)$ be the set of vertices immediately above v , let S_v be the set of total orders on $\text{St}(v)$, and finally let $S_{\tilde{\tau}}$ be the product of the sets S_v for $v \in V(\tilde{\tau})$. Any element $\prec \in S_{\tilde{\tau}}$ obviously defines a binary relation on $V(\tilde{\tau})$, also denoted by \prec : to be precise, $w' \prec w''$ if and

only if there exists $v \in V(\tilde{\tau})$ such that $w', w'' \in \text{St}_v$ and $w' \prec w''$ inside St_v . For any element $\prec \in S_{\tilde{\tau}}$, let \mathcal{R}_{\prec} be the binary relation on $V(\tilde{\tau})$ defined by:

$$w' \mathcal{R} w'' \text{ if and only if } w' < w'' \text{ or } w' \prec w'',$$

and let \ll_{\prec} be the transitive closure of \mathcal{R}_{\prec} . This is a partial order refining the forest order $<$, which, by ordering the branches at any vertex, defines a unique planar representative σ_{\prec} of $\tilde{\tau}$.

The third assertion comes then from the following fact: the domain $\Omega_{\tilde{\tau}}^{st}$ is the union of the domains $\Omega_{V(\tilde{\tau}), \ll_{\prec}}^{st}$ (mutually disjoint apart from a Lebesgue-negligible intersection) where \prec runs over $S_{\tilde{\tau}}$. Now two elements \prec and \prec' give rise to the same planar representative σ if and only if the unique permutation of $V(\tau)$ which induces an increasing map from $(\text{St}(v), \prec)$ onto $(\text{St}(v), \prec')$ is an automorphism of $\tilde{\tau}$. \square

As a consequence, we easily obtain an analogue of Lemma 4.4 in reference [29]:

Corollary 6.2. *For any rooted planar forest τ and for any $h, k \geq 0$ the following holds:*

$$(h+k)^{|\tau|} = \sum_{\substack{V' \sqcup V'' = V(\tau), \\ V'' \ll_{V'} V'}} \frac{\tau!}{(V', \ll_{|V'|})! \tau_{|V''}|} h^{|V'|} k^{|V''|}.$$

Proof. Let $s \leq u \leq t$ three real numbers, with $u - s = h$ and $t - u = k$. From the definition of the domain $\Omega_{\tau, \ll}^{st}$ we immediately can express it as the following union with Lebesgue-negligible pairwise intersections:

$$\Omega_{V(\tau), \ll}^{st} = \bigcup_{\substack{V(\tau) = V' \sqcup V'', \\ V'' \ll_{V'} V'}} \Omega_{(V', \ll_{|V'|})}^{su} \times \Omega_{(V'', \ll_{|V''|})}^{ut}.$$

The conclusion then follows from item 2 of Lemma 6.1. \square

Remark 6.3. The inversion of the order in the definition of Ω_{τ}^{st} is not really necessary, as this inversion amounts to a change of variables $t_v \mapsto s + t - t_v$, which does not change the volume. But it makes the proof slightly more direct.

Remark 6.4. Applying Corollary 6.2 in the special case $h = k = 1$ shows that $\tau \mapsto 1/\tau!$ extends linearly to the unique inverse-factorial character of the Hopf algebra $\mathcal{H}_{\text{MKW}}^A$ taking value 1 on the letters of A . As a consequence, the combinatorial Hopf algebra $\mathcal{H}_{\text{MKW}}^A$ endowed with the decorated forest basis is non-degenerate. The analogue is true for non-planar forests and the A -decorated Butcher–Connes–Kreimer Hopf algebra, due to Lemma 4.4 in [29]. As a consequence, assertion (3) of Lemma 6.1 can be derived in a purely algebraic way, using the Hopf algebra morphism $\Omega : \mathcal{H}_{\text{BCK}}^A \rightarrow \mathcal{H}_{\text{MKW}}^A$ of [38] given by (53) below, as well as the functoriality of inverse-factorial characters.

Remark 6.5. Another interesting example comes from the extraction-contraction Hopf algebra $\mathcal{H}_{\text{EC}}^A$ of reference [6], in which the grading is given by the number of edges. In view of the recursive definition of the inverse factorial character given in Paragraph 3.1, and due to the fact that the coproduct of a forest with n edges contains exactly 2^n elements, the inverse factorial character is identically equal to 1 on any forest. It is however not clear whether one can consider the rather exotic corresponding notion of rough path as a driving object for some kind of rough differential equation.

By a straightforward iteration of (51) one obtains another recursive formula for the planar factorial:

Proposition 6.6. *Let $\sigma = \sigma_1 \cdots \sigma_k$ be a planar forest, decorated or not, with connected components $\sigma_j = B_+(\tau_j)$, $j = 1, \dots, k$. Then*

$$(52) \quad \sigma! = |\sigma_1| \cdot |\sigma_1 \sigma_2| \cdots |\sigma_1 \cdots \sigma_k| \tau_1! \cdots \tau_k!$$

6.2. Planarly branched rough paths. The notion of planarly branched rough paths is given in the next definition. It is motivated from a Hopf algebraic point of view. Its significance for controlled rough differential equations will become clear further below.

Definition 6.7. *Let $\gamma \in]0, 1]$ and let A be a finite alphabet. A γ -regular planarly branched rough path is a γ -regular $\mathcal{H}_{\text{MKW}}^A$ -rough path.*

7. SIMPLE AND CONTRACTING ARBORIFICATION IN THE PLANAR SETTING

7.1. A projection onto the shuffle Hopf algebra: planar arborification. Let A be an alphabet, and let $\mathcal{H}_{\text{III}}^A$ (resp. $\mathcal{H}_{\text{MKW}}^A$) be the shuffle Hopf algebra with letters in A (resp. the Hopf algebra of A -decorated planar forests). The *planar arborification map* $\mathbf{a}_{\ll} : \mathcal{H}_{\text{MKW}}^A \rightarrow \mathcal{H}_{\text{III}}^A$ sends any planar decorated forest to the sum of its linear extensions. It is defined for any degree n planar A -decorated forest $\tau = (\bar{\tau}, \varphi)$ as follows:

$$\mathbf{a}_{\ll}(\bar{\tau}, \varphi) := \sum_{\alpha: (\mathcal{V}(\bar{\tau}), \ll) \nearrow \{1, \dots, n\}} \varphi \circ \alpha^{-1}(1) \cdots \varphi \circ \alpha^{-1}(n),$$

where the sum runs over the increasing bijections from the poset $(\mathcal{V}(\bar{\tau}), \ll)$ onto $\{1, \dots, n\}$. As an example, we have:

$$\mathbf{a}_{\ll}(\begin{array}{c} b \\ | \\ \bullet \\ | \\ a \bullet c \end{array}) = bac, \quad \mathbf{a}_{\ll}(\begin{array}{c} b \\ | \\ \bullet \\ | \\ c \bullet a \end{array}) = bca + cba.$$

This definition is directly inspired from the simple arborification map $\mathbf{a} : \mathcal{H}_{\text{BCK}}^A \rightarrow \mathcal{H}_{\text{III}}^A$ which sends any (non-planar) decorated forest to the sum of its linear extensions [20, 22, 25]. It is defined for any degree n non-planar A -decorated forest by:

$$\mathbf{a}(f, \varphi) := \sum_{\alpha: (\mathcal{V}(f), <) \nearrow \{1, \dots, n\}} \varphi \circ \alpha^{-1}(1) \cdots \varphi \circ \alpha^{-1}(n),$$

where the sum runs over the increasing bijections from the poset $(\mathcal{V}(f), <)$ onto $\{1, \dots, n\}$.

Lemma 7.1. *(Canonical decomposition of planar forests)*

(1) *Any non-empty A -decorated planar forest τ admits a unique decomposition:*

$$\tau = \tau' \times_a \tau'',$$

where τ' and τ'' are A -decorated planar forests, $a \in A$ and $\tau' \times_a \tau''$ stands for $\tau' B_a^+(\tau'')$.

(2) *The planar arborification map can be recursively defined by $\mathbf{a}_{\ll}(\mathbf{1}) = \mathbf{1}$ and*

$$\mathbf{a}_{\ll}(\tau' \times_a \tau'') = [\mathbf{a}_{\ll}(\tau') \text{III} \mathbf{a}_{\ll}(\tau'')]a.$$

Proof. The first assertion is straightforward, the second is a direct consequence of the poset structure of $V(\tau' \times_a \tau'') = V(\tau') \sqcup V(\tau'') \sqcup \{a\}$ under the partial order \ll , which is entirely determined by the fact that

(1) the restriction of \ll to $V(\tau')$ is the partial order \ll determined by the planar forest τ' , and similarly for τ'' ,

- (2) $a \in A$ is the unique minimum,
- (3) vertices of τ' are incomparable with vertices of τ'' .

□

Theorem 7.2. *The planar arborification map \mathbf{a}_{\ll} is a surjective Hopf algebra morphism, combinatorial if the alphabet A is finite, and the diagram below commutes.*

$$\begin{array}{ccc}
 \mathcal{H}_{\text{BCK}}^A & \xrightarrow{\Omega} & \mathcal{H}_{\text{MKW}}^A \\
 & \searrow \mathbf{a} & \downarrow \mathbf{a}_{\ll} \\
 & & \mathcal{H}_{\text{III}}^A
 \end{array}$$

where Ω is the symmetrization map [38, Definition 8].

Proof. It is well-known that \mathbf{a} and Ω are Hopf algebra morphisms [25, 38]. The map Ω is given by:

$$(53) \quad \Omega(f) = \text{Sym}(f) \sum_{\tau \twoheadrightarrow f} \tau,$$

from which the commutation of the diagram easily follows. It only remains to prove by direct checking that \mathbf{a}_{\ll} respects the Hopf algebra structures. For any A -decorated planar forests τ, ω which admit canonical decompositions $\tau = \tau' \times_a \tau''$ and $\omega = \omega' \times_b \omega''$ according to Lemma 7.1, we compute, using induction on the sum of degrees $|\tau| + |\omega|$:

$$\begin{aligned}
 \mathbf{a}_{\ll}(\tau \text{ III } \omega) &= \mathbf{a}_{\ll}((\tau' \times_a \tau'') \text{ III } (\omega' \times_b \omega'')) \\
 &= \mathbf{a}_{\ll}(\tau' B_a^+(\tau'') \text{ III } \omega' B_b^+(\omega'')) \\
 &= \mathbf{a}_{\ll}([\tau' \text{ III } \omega' B_b^+(\omega'')] B_a^+(\tau'') + [\tau' B_a^+(\tau'') \text{ III } \omega'] B_b^+(\omega'')) \\
 &= \mathbf{a}_{\ll}([\tau' \text{ III } (\omega' \times_b \omega'')] \times_a \tau'' + [(\tau' \times_a \tau'') \text{ III } \omega'] \times_b \omega'') \\
 &= [\mathbf{a}_{\ll}(\tau' \text{ III } (\omega' \times_b \omega'')) \text{ III } \mathbf{a}_{\ll}(\tau'')] a + [\mathbf{a}_{\ll}((\tau' \times_a \tau'') \text{ III } \omega') \text{ III } \mathbf{a}_{\ll}(\omega'')] b \\
 &= [\mathbf{a}_{\ll}(\tau') \text{ III } \mathbf{a}_{\ll}(\omega' \times_b \omega'') \text{ III } \mathbf{a}_{\ll}(\tau'')] a + [\mathbf{a}_{\ll}(\tau' \times_a \tau'') \text{ III } \mathbf{a}_{\ll}(\omega') \text{ III } \mathbf{a}_{\ll}(\omega'')] b \\
 &= [\mathbf{a}_{\ll}(\tau') \text{ III } [\mathbf{a}_{\ll}(\omega') \text{ III } \mathbf{a}_{\ll}(\omega'')] b \text{ III } \mathbf{a}_{\ll}(\tau'')] a + [[\mathbf{a}_{\ll}(\tau') \text{ III } \mathbf{a}_{\ll}(\tau'')] a \text{ III } \mathbf{a}_{\ll}(\omega') \text{ III } \mathbf{a}_{\ll}(\omega'')] b \\
 &= [\mathbf{a}_{\ll}(\tau') \text{ III } \mathbf{a}_{\ll}(\tau'')] a \text{ III } [\mathbf{a}_{\ll}(\omega') \text{ III } \mathbf{a}_{\ll}(\omega'')] b \\
 &= \mathbf{a}_{\ll}(\tau) \text{ III } \mathbf{a}_{\ll}(\omega).
 \end{aligned}$$

To check compatibility with coproducts, we introduce the linear operator of left concatenation, $L_a : \mathcal{H}_{\text{III}}^A \rightarrow \mathcal{H}_{\text{III}}^A$, defined by $L_a(w) = wa$ for any word $w \in A^*$. It clearly verifies:

$$L_a \circ \mathbf{a}_{\ll} = \mathbf{a}_{\ll} \circ B_a^+.$$

For any A -decorated planar forest $\tau = \tau' \times_a \tau''$ we compute, using induction on the degree $|\tau|$:

$$\begin{aligned}
\Delta \mathbf{a}_{\ll}(\tau) &= \Delta\left([\mathbf{a}_{\ll}(\tau') \amalg \mathbf{a}_{\ll}(\tau'')]a\right) \\
&= (\text{Id} \otimes L_a)\Delta\left(\mathbf{a}_{\ll}(\tau') \amalg \mathbf{a}_{\ll}(\tau'')\right) + \mathbf{a}_{\ll}(\tau) \otimes \mathbf{1} \\
&= (\text{Id} \otimes L_a)\left(\Delta \mathbf{a}_{\ll}(\tau') \amalg \Delta \mathbf{a}_{\ll}(\tau'')\right) + \mathbf{a}_{\ll}(\tau) \otimes \mathbf{1} \\
&= (\text{Id} \otimes L_a)\left((\mathbf{a}_{\ll} \otimes \mathbf{a}_{\ll})\Delta \tau' \amalg (\mathbf{a}_{\ll} \otimes \mathbf{a}_{\ll})\Delta \tau''\right) + \mathbf{a}_{\ll}(\tau) \otimes \mathbf{1} \\
&= (\mathbf{a}_{\ll} \otimes \mathbf{a}_{\ll})\left((\text{Id} \otimes B_a^+)(\Delta \tau' \amalg \Delta \tau'') + \tau \otimes \mathbf{1}\right) \\
&= (\mathbf{a}_{\ll} \otimes \mathbf{a}_{\ll})\left((\text{Id} \otimes B_a^+)(\Delta(\tau' \amalg \tau'')) + \tau \otimes \mathbf{1}\right) \\
&= (\mathbf{a}_{\ll} \otimes \mathbf{a}_{\ll})(\Delta \tau).
\end{aligned}$$

Compatibility with units and co-units is immediate, and compatibility with antipodes comes for free due to connectedness of both Hopf algebras. \square

Proposition 7.3. *Let $\gamma \in]0, 1]$, let A be a finite alphabet with d letters, and let \mathbb{X}_{st} be a γ -regular rough path in the classical sense on \mathbb{R}^d . Then its arborified version $\tilde{\mathbb{X}}_{st} := \mathbb{X}_{st} \circ \mathbf{a}_{\ll}$ is a γ -regular planarly branched rough path on \mathbb{R}^d .*

Proof. This is an immediate consequence of Proposition 4.8. \square

Proposition 7.3 calls for the following definition:

Definition 7.4. *A γ -regular planarly branched rough path \mathbb{Z}_{st} on \mathbb{R}^d is geometric if there exists a γ -regular rough path \mathbb{X}_{st} in the classical sense such that \mathbb{Z}_{st} is its arborified version, i.e.*

$$\mathbb{Z}_{st} = \mathbb{X}_{st} \circ \mathbf{a}_{\ll}.$$

Remark 7.5. Any geometric branched rough path is then geometric by definition. The converse is true at the price of inflating the alphabet, as in the branched case [30, Paragraph 4.2]. Indeed, the Hopf algebra $\mathcal{H}_{\text{MKW}}^A$ is, as an algebra, the shuffle algebra of the set of A -decorated planar rooted trees. Any planarly branched rough path is then geometric provided this bigger alphabet is considered.

7.2. Planar contracting arborification. We present a contracting version of planar arborification which has some interest in its own right, although it will not be directly used in the present paper. Suppose that the alphabet A carries an Abelian semigroup structure $(a, b) \mapsto [a + b]$. The quasi-shuffle Hopf algebra is isomorphic to $\mathcal{H}_{\text{III}}^A$ as coalgebra. The quasi-shuffle product is recursively defined by $\emptyset \amalg w = w \amalg \emptyset = w$ for any word $w \in A^*$ and:

$$av \amalg bw = a(v \amalg bw) + b(av \amalg w) + [a + b](v \amalg w)$$

for any letters $a, b \in A$ and words $(v, w) \in A^*$. For example, $a \amalg b = ab + ba + [a + b]$, and $ab \amalg c = abc + acb + cab + [a + c]b + a[b + c]$. It is well-known [32] that the quasi-shuffle product together with deconcatenation give rise to a Hopf algebra $\mathcal{H}_{\text{III}}^A$ isomorphic to the shuffle Hopf algebra $\mathcal{H}_{\text{III}}^A$.

The *planar contracting arborification map* $\mathfrak{a}_{\ll}^c : \mathcal{H}_{\text{MKW}}^A \rightarrow \mathcal{H}_{\text{IH}}^A$ sends any planar decorated forest to the sum of its linear extensions *including contraction terms*. It is defined for any degree n planar A -decorated forest as follows:

$$\mathfrak{a}_{\ll}^c(\tau, \varphi) := \sum_{r \geq 0} \sum_{\alpha: (V(\tau), \ll) \twoheadrightarrow \{1, \dots, n-r\}} \varphi \circ \alpha^{-1}(1) \cdots \varphi \circ \alpha^{-1}(n-r)$$

where the inner sum runs over the increasing surjections from the poset $(V(\tau), \ll)$ onto $\{1, \dots, n-r\}$, i.e., surjective maps α such that $u \ll u' \in V(\tau)$ and $u \neq u'$ implies $\alpha(u) < \alpha(u')$. It can happen that $\alpha^{-1}(j)$ contains several terms: in that case, $\varphi \circ \alpha^{-1}(j)$ is to be understood as the sum in A of the terms $\varphi(u)$, $u \in \alpha^{-1}(j)$. As an example, we have:

$$\mathfrak{a}_{\ll}^c \left(\begin{array}{c} b \\ | \\ \bullet \\ | \\ a \bullet c \end{array} \right) = bac, \quad \mathfrak{a}_{\ll}^c \left(\bullet \begin{array}{c} | \\ \bullet \\ | \\ a \end{array} \right) = bca + cba + [b + c]a.$$

This definition is directly inspired from the contracting arborification map $\mathfrak{a}^c : \mathcal{H}_{\text{BCK}}^A \rightarrow \mathcal{H}_{\text{IH}}^A$ which sends any (non-planar) decorated forest to the sum of its linear extensions including contraction terms [19, 20, 22, 25]. It is defined for any degree n non-planar A -decorated forest by:

$$\mathfrak{a}^c(f, \varphi) := \sum_{r \geq 0} \sum_{\alpha: (V(f), <) \twoheadrightarrow \{1, \dots, n-r\}} \varphi \circ \alpha^{-1}(1) \cdots \varphi \circ \alpha^{-1}(n-r)$$

where the inner sum runs over the increasing surjections from the poset $(V(f), <)$ onto $\{1, \dots, n-r\}$, and is a surjective Hopf algebra morphism from $\mathcal{H}_{\text{BCK}}^A$ onto $\mathcal{H}_{\text{IH}}^A$.

An analogue of Theorem 7.2 holds:

Theorem 7.6.

(1) *The planar contracting arborification map can be recursively defined by $\mathfrak{a}_{\ll}^c(\mathbf{1}) = \mathbf{1}$ and*

$$\mathfrak{a}_{\ll}^c(\tau' \times_a \tau'') = [\mathfrak{a}_{\ll}^c(\tau') \text{IH} \mathfrak{a}_{\ll}^c(\tau'')]a.$$

(2) *The planar contracting arborification map \mathfrak{a}_{\ll}^c is a surjective Hopf algebra morphism, combinatorial if the alphabet A is finite, and the diagram below commutes.*

$$\begin{array}{ccc} \mathcal{H}_{\text{BCK}}^A & \xrightarrow{\Omega} & \mathcal{H}_{\text{MKW}}^A \\ & \searrow \mathfrak{a}^c & \downarrow \mathfrak{a}_{\ll}^c \\ & & \mathcal{H}_{\text{IH}}^A \end{array}$$

Proof. Entirely similar to proof of the analogous results on planar arborification on Paragraph 7.1. Details are left to the reader. \square

8. ROUGH DIFFERENTIAL EQUATIONS ON HOMOGENEOUS SPACES

In this section, we prove the convergence of the formal solutions of the rough differential equation (1) under particular analyticity assumptions.

8.1. Formal solutions of a rough differential equation on a homogeneous space. Let $t \mapsto X_t := (X_t^1, \dots, X_t^d)$ be a differentiable path with values in \mathbb{R}^d . Let $A = \{a_1, \dots, a_d\}$ be an alphabet with d letters. The controlled differential equation we are looking at writes:

$$(54) \quad dY_{st} = \sum_{i=1}^d \#f_i(Y_{st}) dX_t^i$$

with initial condition $Y_{ss} = y$. The unknown is a path $Y_s : t \mapsto Y_{st}$ in a homogeneous space \mathcal{M} , with transitive action $(g, y) \mapsto g.y$ of a Lie group G on it. The elements in $f := \{f_i\}_{i=1}^d$ are smooth maps from \mathcal{M} into the Lie algebra $\mathfrak{g} = \text{Lie}(G)$, which in turn define smooth vector fields $y \mapsto \#f_i(y)$ on \mathcal{M} :

$$(55) \quad \#f_i(y) := \frac{d}{dt} \Big|_{t=0} \exp(t f_i(y)).y \in T_y \mathcal{M}.$$

It has been explained in the Introduction how Equation (54) is lifted to the following differential equation with unknown $\mathbf{Y}_{st} \in C^\infty(\mathcal{M}, \mathcal{U}(\mathfrak{g}))[[h]]$ and step size $h = t - s$:

$$(56) \quad d\mathbf{Y}_{st} = \sum_{i=1}^d \mathbf{Y}_{st} * f_i dX_t^i$$

with initial condition $\mathbf{Y}_{ss} = \mathbf{1}$. Recall that the $*$ product stands for the Grossman–Larson product in the post-associative algebra $C^\infty(\mathcal{M}, \mathcal{U}(\mathfrak{g}))$. The formal solution of (54) is recovered by

$$(57) \quad \psi(Y_{st}) = (\#\mathbf{Y}_{st}.\psi)(y).$$

for any test function $\psi \in C^\infty(\mathcal{M})$. A further step in abstraction leads to the fundamental differential equation in the character group of the Hopf algebra $\mathcal{H}_{\text{MKW}}^A$:

$$(58) \quad d\mathbb{Y}_{st} = \sum_{i=1}^d \mathbb{Y}_{st} * \bullet_i dX_t^i$$

with initial condition $\mathbb{Y}_{ss} = \mathbf{1}$. The $*$ product now stands for the Grossman–Larson product in the completed free post-associative algebra $\widehat{\mathcal{D}}_A$ generated by A . The solution of (56) then is obtained by $\mathbf{Y}_{st} = \mathcal{F}_f(\mathbb{Y}_{st})$, where \mathcal{F}_f is the h -adic completion (with $h = t - s$) of the unique post-associative algebra morphism from \mathcal{D}_A to $C^\infty(\mathcal{M}, \mathcal{U}(\mathfrak{g}))$ which sends \bullet_j to f_j . By using the integral formulation and Picard iteration, the solution of (58) is given by the word series expansion:

$$(59) \quad \mathbb{Y}_{st} = \sum_{\ell \geq 0} \sum_{w = a_{i_1} \cdots a_{i_\ell} \in A^*} \langle \mathbb{X}_{st}, w \rangle \bullet_{i_\ell} * \cdots * \bullet_{i_1}.$$

Theorem 8.1 (Planar arborification-coarborification transform). *The solution of (58) is given by the following expansion indexed by A -decorated planar rooted forests:*

$$(60) \quad \mathbb{Y}_{st} = \sum_{\tau \in F_{\text{pl}}^A} \langle \mathbb{X}_{st} \circ \mathfrak{a}^{\ll}, \tau \rangle \tau.$$

Proof. For any $\bar{\tau} \in F_{\text{pl}}$, let $\mathcal{L}(\bar{\tau})$ be the set of linear extensions of $\bar{\tau}$, i.e., the set of total orders \prec on $V(\bar{\tau})$ compatible with the partial order \ll , i.e., such that $u \ll v \Rightarrow u \prec v$ for any $u, v \in V(\bar{\tau})$. Now let $\tau = (\bar{\tau}, \alpha)$ be an A -decorated forest, and let τ_{\prec} be the word in A^* obtained from $(\bar{\tau}, \alpha)$ by displaying the decorations of the

vertices of $\bar{\tau}$ from left to right according to the total order \prec . We will use the notation $\mathcal{L}(\tau)$ instead of $\mathcal{L}(\bar{\tau})$. It can be easily shown that the planar arborification admits the following explicit expression:

$$\mathbf{a}^{\ll}(\tau) = \sum_{\prec \in \mathcal{L}(\tau)} \tau_{\prec}.$$

The following lemma is easily proven by induction on the length:

Lemma 8.2. *For any word $w = a_{i_1} \cdots a_{i_n} \in A^*$, we have:*

$$(61) \quad \bullet_{i_n} * \cdots * \bullet_{i_1} = \sum_{\bar{\tau} \in T_{\text{pl}}^{[n]}} \sum_{\prec \in \mathcal{L}(\bar{\tau})} (\bar{\tau}, \alpha_{\prec}),$$

where $\alpha_{\prec} : V(\bar{\tau}) \rightarrow A$ is the decoration map which sends the j -th vertex to a_{i_j} according to \prec .

The total number of terms is $n!$. For example we have

$$\bullet_j * \bullet_i = \bullet_j \bullet_i + \begin{array}{c} \bullet_j \\ | \\ \bullet_i \end{array}, \quad \bullet_k * \bullet_j * \bullet_i = \bullet_k \bullet_j \bullet_i + \begin{array}{c} \bullet_k \\ | \\ \bullet_j \bullet_i \end{array} + \begin{array}{c} \bullet_k \\ | \\ \bullet_j \\ | \\ \bullet_i \end{array} + \begin{array}{c} \bullet_k \\ | \\ \bullet_i \end{array} + \begin{array}{c} \bullet_j \\ | \\ \bullet_i \end{array}.$$

We compute, using Lemma 8.2:

$$\begin{aligned} \mathbb{Y}_{st} &= \sum_{\ell \geq 0} \sum_{w = a_{i_1} \cdots a_{i_\ell} \in A^*} \langle \mathbb{X}_{st}, w \rangle \bullet_{i_\ell} * \cdots * \bullet_{i_1} \\ &= \sum_{\ell \geq 0} \sum_{w = a_{i_1} \cdots a_{i_\ell} \in A^*} \langle \mathbb{X}_{st}, w \rangle \sum_{\bar{\tau} \in T_{\text{pl}}^{[n]}} \sum_{\prec \in \mathcal{L}(\bar{\tau})} (\bar{\tau}, \alpha_{\prec}) \\ &= \sum_{\tau \in T_{\text{pl}}^A} \sum_{\prec \in \mathcal{L}(\tau)} \langle \mathbb{X}_{st}, \tau_{\prec} \rangle \tau \\ &= \sum_{\tau \in T_{\text{pl}}^A} \langle \mathbb{X}_{st} \circ \mathbf{a}^{\ll}, \tau \rangle \tau. \end{aligned}$$

□

Theorem 8.1 calls for the following definition.

Definition 8.3. *Let $\gamma \in]0, 1]$ and let $t \mapsto X_t := (X_t^1, \dots, X_t^d)$ be a γ -Hölder continuous path with values in \mathbb{R}^d . A formal solution of Equation (54) driven by X is defined by*

$$(62) \quad Y_{st} = \#\mathcal{F}_f(\mathbb{Y}_{st})(y)$$

where \mathbb{Y}_{st} is given by the expansion

$$(63) \quad \mathbb{Y}_{st} = \sum_{\tau \in F_{\text{pl}}^A} \langle \tilde{\mathbb{X}}_{st}, \tau \rangle \tau$$

where $\tilde{\mathbb{X}}_{st}$ is any γ -regular planarly branched rough path such that $\langle \tilde{\mathbb{X}}_{st}, \bullet_j \rangle = X_t^j - X_s^j$ for any $j \in \{1, \dots, d\}$.

We will freely identify the planarly branched rough path $\tilde{\mathbb{X}}_{st}$ with the expansion \mathbb{Y}_{st} as grouplike elements of the dual of $\mathcal{H}_{\text{MKW}}^A$.

8.2. Cauchy estimates. We borrow material from [20], see also [22, 13], adapting it to general homogeneous spaces. For any compact neighbourhood \mathcal{U} of the origin in \mathbb{C}^n , let $\mathcal{A}_{\mathcal{U}}$ be the subspace of analytic germs defined on \mathcal{U} . We have precisely

$$\mathcal{A}_{\mathcal{U}} = \{\varphi, \|\varphi\|_{\mathcal{U}} < +\infty\},$$

with the norm

$$\|\varphi\|_{\mathcal{U}} := \sup_{y \in \mathcal{U}} |\varphi(y)|$$

making $\mathcal{A}_{\mathcal{U}}$ a Banach space. Now let \mathcal{V} be another compact neighbourhood of the origin such that $\mathcal{V} \subset \mathring{\mathcal{U}}$. We consider the operator norm defined for any linear operator $P : \mathcal{A}_{\mathcal{U}} \rightarrow \mathcal{A}_{\mathcal{U}}$ by

$$\|P\|_{\mathcal{U}, \mathcal{V}} = \sup_{\varphi \in \mathcal{A}_{\mathcal{U}} - \{0\}} \frac{\|P\varphi\|_{\mathcal{V}}}{\|\varphi\|_{\mathcal{U}}}.$$

The two following lemmas are straightforward.

Lemma 8.4. *Let $0 \in \mathcal{V} \subset \mathring{\mathcal{U}} \subset \mathcal{U}$ be two compact neighbourhoods of the origin, and let $f \in \mathcal{A}_{\mathcal{U}}$. Let $P : \mathcal{A}_{\mathcal{U}} \rightarrow \mathcal{A}_{\mathcal{U}}$ be a linear operator. Denoting by $f \cdot : \mathcal{A}_{\mathcal{U}} \rightarrow \mathcal{A}_{\mathcal{U}}$ the pointwise multiplication operator by f , then the following estimate holds:*

$$\|fP\|_{\mathcal{U}, \mathcal{V}} \leq \|f\|_{\mathcal{V}} \|P\|_{\mathcal{U}, \mathcal{V}}.$$

Lemma 8.5. *Let $0 \in \mathcal{V} \subset \mathring{\mathcal{W}} \subset \mathcal{W} \subset \mathring{\mathcal{U}} \subset \mathcal{U}$ be three compact neighbourhoods of the origin, and let $P, Q : \mathcal{A}_{\mathcal{U}} \rightarrow \mathcal{A}_{\mathcal{U}}$ be a two linear operators. Then we have:*

$$\|P \circ Q\|_{\mathcal{U}, \mathcal{V}} \leq \|P\|_{\mathcal{W}, \mathcal{V}} \|Q\|_{\mathcal{U}, \mathcal{W}}.$$

Proposition 8.6. *Let $0 \in \mathcal{V} \subset \mathring{\mathcal{U}} \subset \mathcal{U}$ be two compact neighbourhoods of the origin, and let $r > 0$ be such that the n -fold product of open disks of radius r centered at y is included in \mathcal{U} for any $y \in \mathcal{V}$. Let $f = \sum_{\alpha=1}^n f^{\alpha} \partial_{\alpha}$ be a vector field on \mathcal{U} with analytic coefficients, and let us define*

$$\|f\|_{\mathcal{V}} := \sup_{\alpha=1, \dots, n} \|f^{\alpha}\|_{\mathcal{V}}.$$

Then we have:

$$\|f\|_{\mathcal{U}, \mathcal{V}} \leq \frac{n\|f\|_{\mathcal{V}}}{r}.$$

Proof. This is an immediate application of Lemma 8.4 and the Cauchy estimate for the partial derivation operator ∂_{α} , which is immediately derived from the Cauchy integral formula

$$\varphi(y) = \varphi(y_1, \dots, y_n) = \frac{1}{(2i\pi)} \int_{C_{\alpha}} \frac{\varphi(y_1, \dots, y_{\alpha-1}, \eta_{\alpha}, y_{\alpha+1}, \dots, y_n)}{\eta_{\alpha} - y_{\alpha}} d\eta_{\alpha},$$

valid for any $\varphi \in \mathcal{A}_{\mathcal{U}}$ and for any $y \in \mathcal{V}$, where C_{α} is the circle of radius r in \mathbb{C} centered at y_{α} , counterclockwise oriented. \square

Corollary 8.7. *Let $0 \in \mathcal{V} \subset \mathring{\mathcal{U}} \subset \mathcal{U}$ be two compact neighbourhoods of the origin, and let $r > 0$ be such that the open polydisk of radius r centered at y is included in \mathcal{U} for any $y \in \mathcal{V}$. Let $f = \{f_1, \dots, f_k\}$ be a finite collection of vector fields*

$$f_j = \sum_{\alpha=1}^n f_j^{\alpha} \partial_{\alpha}$$

on \mathcal{U} with analytic coefficients, and let us define

$$\|f\|_{\mathcal{V}} := \sup_{\substack{\alpha=1,\dots,n \\ j=1,\dots,k}} \|f_j^\alpha\|_{\mathcal{V}}.$$

and $\|f\|_{\mathcal{U}}$ similarly. Then we have:

$$\|f_1 \circ \dots \circ f_k\|_{\mathcal{U},\mathcal{V}} \leq \left(\frac{n\|f\|_{\mathcal{U}}}{r}\right)^k k^k.$$

Proof. The case $k = 1$ is covered by Proposition 8.6. For $k \geq 2$, we define intermediate compact neighbourhoods

$$\mathcal{V} = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \dots \subset \mathcal{V}_k = \mathcal{U}$$

as follows: \mathcal{V}_j is the closure of the union of the polydisks of radius r/k centered at any point of \mathcal{V}_{j-1} , for any $j \in \{1, \dots, k-1\}$. The result follows then from Proposition 8.6 and the k -fold iteration of Lemma 8.5 associated with these data, as well as from the obvious inequality $\|f\|_{\mathcal{V}_j} \leq \|f\|_{\mathcal{U}}$ for any $j = 1, \dots, k$. \square

8.3. Convergence of a formal solution. We address the question whether the formal diffeomorphism $\mathbf{Y}_{st} := \#\mathcal{F}_f(\mathbb{Y}_{st})$ converges at least for $|t - s|$ sufficiently small. Any homogeneous space \mathcal{M} under the action of a finite-dimensional Lie group has a canonical analytic structure. We denote by $C^\omega(\mathcal{M}, V)$ the space of weakly analytic maps from \mathcal{M} into a vector space V . We suppose that the data $f = \{f_j\}_{j=1}^d$ are analytic maps from \mathcal{M} to \mathfrak{g} , thus yielding analytic vector fields $\#f_j$ on \mathcal{M} . Choosing $y \in \mathcal{M}$ and two compact chart neighbourhoods \mathcal{U}, \mathcal{V} such that $y \in \mathcal{V} \subset \overset{\circ}{\mathcal{U}}$, we have to prove that the operator norm $\|\mathbf{Y}_{st}\|_{\mathcal{U},\mathcal{V}}$ is finite for small $h = t - s$.

Choosing a basis $(E_\alpha)_{\alpha=1,\dots,N}$ of the Lie algebra \mathfrak{g} , we have:

$$(64) \quad f_j = \sum_{\beta=1}^N \tilde{f}_j^\beta E_\beta, \quad \#E_\beta = \sum_{\alpha=1}^n \varepsilon_\beta^\alpha \partial_\alpha,$$

where the coefficients \tilde{f}_j^β and ε_β^α are analytic on \mathcal{U} , and where

$$\|\tilde{f}\|_{\mathcal{V}} := \sup_{\substack{j=1,\dots,d \\ \beta=1,\dots,N}} \|\tilde{f}_j^\beta\|_{\mathcal{V}}.$$

Theorem 8.8. *There exists a positive constant $C_{\mathcal{U},\mathcal{V}}$ such that for any A -decorated rooted planar forest $\sigma = \sigma_1 \cdots \sigma_k$ with connected components $\sigma_j = B_+^{\alpha_j}(\tau_j)$, the following estimates hold:*

$$(65) \quad \|\tilde{f}_\sigma^\beta\|_{\mathcal{V}} \leq \tau_1! \cdots \tau_k! C_{\mathcal{U},\mathcal{V}}^{|\sigma|-k} \|\tilde{f}\|_{\mathcal{U}}^{|\sigma|},$$

where the coefficients $\tilde{f}_\sigma^\beta \in C^\omega(\mathcal{U})$ are considered with respect to the Poincaré–Birkhoff–Witt basis:

$$\mathcal{F}_\sigma = \sum_{\substack{\beta \in \{1,\dots,N\}^k \\ \beta_1 \leq \dots \leq \beta_k}} \tilde{f}_\sigma^\beta E_\beta,$$

and

$$(66) \quad \|\#\mathcal{F}_\sigma\|_{\mathcal{U},\mathcal{V}} \leq \sigma! C_{\mathcal{U},\mathcal{V}}^{|\sigma|} \|\tilde{f}\|_{\mathcal{U}}^{|\sigma|}.$$

Proof. Let us first treat the case $|\tau| = 1$, i.e., $\tau = \bullet_j$, $j = 1, \dots, d$. Estimate (65) holds by definition of $\|\tilde{f}\|_{\mathcal{V}}$. Applying Proposition 8.6 we have:

$$(67) \quad \|\#E_{\beta}\|_{\mathcal{U}, \mathcal{V}} \leq \frac{n\|\varepsilon\|_{\mathcal{V}}}{r},$$

where $r > 0$ is chosen so that any polydisk of radius r centered at a point of \mathcal{V} is included in \mathcal{U} , and where

$$\|\varepsilon\|_{\mathcal{V}} := \sup_{\substack{\alpha=1, \dots, n \\ \beta=1, \dots, N}} \|\varepsilon_{\beta}^{\alpha}\|_{\mathcal{V}}.$$

Applying Estimate (67) and Lemma 8.4 we get the estimates:

$$(68) \quad \|\#f_j\|_{\mathcal{U}, \mathcal{V}} \leq \frac{nN\|\tilde{f}\|_{\mathcal{V}}\|\varepsilon\|_{\mathcal{V}}}{r}.$$

We introduce the constant

$$(69) \quad C_{\mathcal{U}, \mathcal{V}} := e \frac{nN\|\varepsilon\|_{\mathcal{U}}}{r},$$

so that we immediately get

$$(70) \quad \|\#f_j\|_{\mathcal{U}, \mathcal{V}} \leq C_{\mathcal{U}, \mathcal{V}}\|\tilde{f}\|_{\mathcal{V}} \leq C_{\mathcal{U}, \mathcal{V}}\|\tilde{f}\|_{\mathcal{U}},$$

which is estimate (66). Let us now proceed by induction to the higher degree case. The necessity of the extra Euler prefactor $e = 2, 71828\dots$ in (69) will appear in the proof, as a consequence of the inequality $k^k \leq e^k k!$ coming from Stirling's formula. For any decorated planar forest $\sigma = \sigma_1 \cdots \sigma_k$ with connected components σ_j , we can write its decomposition in the Poincaré–Birkhoff–Witt basis:

$$(71) \quad \mathcal{F}_{\sigma} = \sum_{\substack{\beta \in \{1, \dots, N\}^k \\ \beta_1 \leq \dots \leq \beta_k}} \tilde{f}_{\sigma}^{\beta} E_{\beta}$$

with $\tilde{f}_{\sigma}^{\beta} = \tilde{f}_{\sigma_1}^{\beta_1} \cdots \tilde{f}_{\sigma_k}^{\beta_k}$ and $E_{\beta} = E_{\beta_1} \cdots E_{\beta_k} \in \mathcal{U}(\mathfrak{g})$. Two cases occur for higher-degree forests:

- (1) *First case:* the decorated forest τ is not a tree, i.e., $k \geq 2$. In this case we have, using the induction hypothesis on each connected component,

$$\begin{aligned} \|\tilde{f}_{\sigma}^{\beta}\|_{\mathcal{V}} &\leq \prod_{j=1}^k \|\tilde{f}_{\sigma_j}^{\beta_j}\|_{\mathcal{U}} \\ &\leq \tau_1! \cdots \tau_k! C_{\mathcal{U}, \mathcal{V}}^{|\sigma|-k} \|\tilde{f}\|_{\mathcal{U}}^{|\sigma|}. \end{aligned}$$

From decomposition (71) and Proposition 8.7 we get then:

$$\begin{aligned} \|\#\mathcal{F}_{\sigma}\|_{\mathcal{U}, \mathcal{V}} &\leq \tau_1! \cdots \tau_k! C_{\mathcal{U}, \mathcal{V}}^{|\sigma|-k} \|\tilde{f}\|_{\mathcal{U}}^{|\sigma|} \sum_{\substack{\beta \in \{1, \dots, N\}^k \\ \beta_1 \leq \dots \leq \beta_k}} \|E_{\beta}\|_{\mathcal{U}, \mathcal{V}} \\ &\leq N^k \tau_1! \cdots \tau_k! C_{\mathcal{U}, \mathcal{V}}^{|\sigma|-k} \|\tilde{f}\|_{\mathcal{U}}^{|\sigma|} \left(\frac{n\|\varepsilon\|_{\mathcal{U}}}{r}\right)^k k^k \\ &\leq k! \tau_1! \cdots \tau_k! C_{\mathcal{U}, \mathcal{V}}^{|\sigma|} \|\tilde{f}\|_{\mathcal{U}}^{|\sigma|} \\ &\leq \sigma! C_{\mathcal{U}, \mathcal{V}}^{|\sigma|} \|\tilde{f}\|_{\mathcal{U}}^{|\sigma|}. \end{aligned}$$

The last inequality derives from the recursive formula (52) for the planar factorial.

(2) *Second case:* $k = 1$, i.e., the decorated forest is a tree $\sigma = B_{a_j}^+(\tau)$. From the definition

$$(72) \quad \mathcal{F}_\sigma = \mathcal{F}_\tau \triangleright f_j$$

we get

$$(73) \quad \tilde{f}_\sigma^\beta = (\#\mathcal{F}_\tau) \cdot \tilde{f}_j^\beta$$

for any $\beta \in \{1, \dots, N\}$. Applying the induction hypothesis to τ and Lemma 8.4, we get:

$$\begin{aligned} \|\tilde{f}_\sigma^\beta\|_{\mathcal{U}} &\leq \|\#\mathcal{F}_\tau\|_{\mathcal{U}, \mathcal{V}} \|f_j^\beta\|_{\mathcal{U}} \\ &\leq C_{\mathcal{U}, \mathcal{V}}^{|\tau|} \tau! \|\tilde{f}\|_{\mathcal{U}}^{|\tau|+1} \\ &\leq C_{\mathcal{U}, \mathcal{V}}^{|\sigma|-1} \tau! \|\tilde{f}\|_{\mathcal{U}}^{|\sigma|}. \end{aligned}$$

Finally, from (71) in the special case $k = 1$ we derive:

$$\begin{aligned} \|\#\mathcal{F}_\sigma\|_{\mathcal{U}, \mathcal{V}} &\leq \tau! C_{\mathcal{U}, \mathcal{V}}^{|\sigma|-1} \|\tilde{f}\|_{\mathcal{U}}^{|\sigma|} \sum_{\beta \in \{1, \dots, N\}} \|E_\beta\|_{\mathcal{U}, \mathcal{V}} \\ &\leq N \tau! C_{\mathcal{U}, \mathcal{V}}^{|\sigma|-1} \|\tilde{f}\|_{\mathcal{U}}^{|\sigma|} \left(\frac{n \|\varepsilon\|_{\mathcal{U}}}{r} \right) \\ &\leq \tau! C_{\mathcal{U}, \mathcal{V}}^{|\sigma|} \|\tilde{f}\|_{\mathcal{U}}^{|\sigma|} \\ &\leq \sigma! C_{\mathcal{U}, \mathcal{V}}^{|\sigma|} \|\tilde{f}\|_{\mathcal{U}}^{|\sigma|}. \end{aligned}$$

□

Corollary 8.9. *The series $\|\mathbf{Y}_{st}\|_{\mathcal{U}, \mathcal{V}}$ is dominated by a series of Gevrey type $1 - \gamma$ with respect to the variable $|t - s|^\gamma$.*

Proof. Recall that a power series $\sum_{k \geq 0} b_k x^k$ is of Gevrey type $\beta \geq 0$ if and only if there exists a constant $C > 0$ such that

$$(74) \quad |b_k| \leq C^k (k!)^\beta.$$

The series \mathbf{Y}_{st} is given by $\sum_{k \geq 0} a_k$, with

$$a_k = \sum_{\tau \in F_{\text{pl}, k}^A} \langle \tilde{\mathbb{X}}_{st}, \tau \rangle \#\mathcal{F}_\tau.$$

We compute, using estimates (29), (42), (66), and the majoration of the number of planar A -decorated rooted forests of degree k by (4d)^k:

$$\begin{aligned} \|a_k\|_{\mathcal{U}, \mathcal{V}} &= \left\| \sum_{\tau \in F_{\text{pl}, k}^A} \langle \tilde{\mathbb{X}}_{st}, \tau \rangle \#\mathcal{F}_\tau \right\|_{\mathcal{U}, \mathcal{V}} \\ &\leq \sum_{\tau \in F_{\text{pl}, k}^A} |\langle \tilde{\mathbb{X}}_{st}, \tau \rangle| \cdot \|\#\mathcal{F}_\tau\|_{\mathcal{U}, \mathcal{V}} \\ &\leq \sum_{\tau \in F_{\text{pl}, k}^A} c^{|\tau|} q_\gamma(\tau) \|\#\mathcal{F}_\tau\|_{\mathcal{U}, \mathcal{V}} \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{\tau \in F_{p^1, k}^A} c^{|\tau|} C_\gamma^{|\tau|} \frac{|\tau|!^{1-\gamma}}{\tau!} \tau! |t-s|^{|\tau|} C_{\mathcal{U}, \mathcal{V}}^{|\tau|} \|\tilde{f}\|_{\mathcal{V}}^{|\tau|} \\
&\leq \left(4dcC_\gamma C_{\mathcal{U}, \mathcal{V}} \|\tilde{f}\|_{\mathcal{V}}\right)^k (k!)^{1-\gamma} |t-s|^{\gamma k}.
\end{aligned}$$

□

Corollary 8.10. *In the case when the driving path X is Lipschitz, i.e., if $\gamma = 1$, the norm $\|\mathbf{Y}_{st}\|_{\mathcal{U}, \mathcal{V}}$ is finite for small $h = t - s$.*

APPENDIX: THE SEWING LEMMA

Let S, T be two real numbers with $S < T$. A map Φ from $[S, T] \times [S, T]$ into a vector space B is *additive* if it verifies the chain rule $\Phi(s, t) = \Phi(s, u) + \Phi(u, t)$ for any $s, u, t \in [S, T]$. In that case there obviously exists a map $\varphi : [S, T] \rightarrow B$, unique up to an additive constant, such that $\Phi(s, t) = \varphi(t) - \varphi(s)$. Indeed, choose an arbitrary origin $o \in [S, T]$ and set $\varphi(t) := \Phi(o, t)$.

Loosely speaking, the sewing lemma stipulates that, under an appropriate completeness assumption on the vector space B , a *nearly* additive map $(s, t) \mapsto \mu(s, t)$ is *nearly* given by a difference $\varphi(t) - \varphi(s)$, in the sense that if $\mu(s, t) - \mu(s, u) - \mu(u, t)$ is small, then there is a unique φ , defined up to an additive constant, such that $\mu(s, t) - \varphi(t) + \varphi(s)$ is small. In view of the importance of this result, we give an account of it in the precised version given by Gubinelli, together with a detailed proof adapted from [23]. For the original proof, see [28, Appendix A1].

Proposition 8.11. [28, Proposition 1] *Let μ be a continuous function on a square $[S, T] \times [S, T]$ with values in a Banach space B , and let $\varepsilon > 0$. Suppose that there exist a positive integer n and two collections $a_i, b_i \geq 0$ indexed by $i \in \{1, \dots, n\}$, with $a_i + b_i = 1 + \varepsilon$, such that μ satisfies:*

$$(75) \quad |\mu(s, t) - \mu(s, u) - \mu(u, t)| \leq \sum_{i=1}^n C_i |t-u|^{a_i} |u-s|^{b_i}$$

for any $s, t, u \in [S, T]$ with $s \leq u \leq t$ or $t \leq u \leq s$, where the C_i 's are positive constants. Then there exists a function $\varphi : [S, T] \rightarrow B$, unique up to an additive constant, such that:

$$(76) \quad |\varphi(t) - \varphi(s) - \mu(s, t)| \leq C' |t-s|^{1+\varepsilon},$$

where the best constant in (76) is

$$C' := \frac{1}{2^{1+\varepsilon} - 2} \sum_{i=1}^n C_i.$$

The proof, adapted from reference [23], is based on dyadic decompositions of intervals. A sequence $(\mu_n)_{n \geq 0}$ of continuous maps from $[S, T] \times [S, T]$ into B is defined by $\mu_0 = \mu$ and

$$(77) \quad \mu_n(s, t) := \sum_{i=0}^{2^n-1} \mu(t_i, t_{i+1})$$

with $t_i = s + i(t-s)2^{-n}$. Denoting by C the sum $C_1 + \dots + C_n$, the estimate

$$|\mu_{n+1}(s, t) - \mu_n(s, t)| \leq C 2^{-n\varepsilon-1-\varepsilon} |t-s|^{1+\varepsilon}$$

holds, which is easily seen by applying (75) to each of the 2^n intervals in (77). Hence the sequence $(\mu_n)_{n \geq 0}$ is Cauchy in the complete metric space $\mathcal{C}([S, T]^2, B)$ of continuous maps from $[S, T] \times [S, T]$ into B endowed with the uniform convergence norm:

$$\|f\| := \sup_{(s,t) \in [S,T]^2} \|f(s,t)\|_B,$$

and thus converges uniformly to a limit Φ , which moreover verifies:

$$(78) \quad |\Phi(s,t) - \mu(s,t)| \leq 2^{-1-\varepsilon} C |t-s|^{1+\varepsilon} \sum_{n \geq 0} 2^{-n\varepsilon} = C |t-s|^{1+\varepsilon} \frac{1}{2^{1+\varepsilon} - 2}.$$

Lemma 8.12. *The limit Φ is additive, that is, it satisfies*

$$\Phi(s,t) = \Phi(s,u) + \Phi(u,t)$$

for any $s, u, t \in [S, T]$.

Proof. From $\mu_{n+1}(s,t) = \mu_n(s, (s+t)/2) + \mu_n((s+t)/2, t)$ we get that Φ is *semi-additive*, i.e., it satisfies

$$\Phi(s,t) = \Phi(s, (s+t)/2) + \Phi((s+t)/2, t)$$

for any $s, t \in [S, T]$. Moreover, Φ is the unique semi-additive map satisfying estimates (78). Indeed, if Ψ is another one, then

$$\begin{aligned} |(\Phi - \Psi)(s,t)| &= \left| \sum_{i=0}^{2^n-1} (\Phi - \Psi)(t_{i+1} - t_i) \right| \\ &\leq 2C' \sum_{i=0}^{2^n-1} |t_{i+1} - t_i|^{1+\varepsilon} \\ &\leq 2C' |t-s| 2^{-n\varepsilon} \end{aligned}$$

with $C' = C/(2^{1+\varepsilon} - 2)$. Hence $\Psi = \Phi$ by letting n go to infinity. Now, if r is any positive integer, then the map Ψ_r defined by

$$\Psi_r(s,t) = \sum_{j=0}^{r-1} \Phi(t_j, t_{j+1}),$$

with $t_j = s + j(t-s)/r$, is also semi-additive, hence $\Psi_r = \Phi$. From this we easily get

$$\Phi(s,t) = \Phi(s,u) + \Phi(u,t)$$

for any rational barycenter u of s and t , i.e., such that $u = as + (1-a)t$ with $a \in [0, 1] \cap \mathbb{Q}$. Additivity of Φ follows from continuity. \square

The proof of Proposition 8.11 follows by choosing $\varphi(t) := \Phi(o, t)$ for any fixed but arbitrary $o \in [S, T]$. Uniqueness of φ up to an additive constant follows immediately from the uniqueness of the additive function Φ satisfying estimate (78).

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DEPT. OF MATHEMATICAL SCIENCES, NORWEGIAN UNIVERSITY OF SCIENCE AND TECHNOLOGY (NTNU), 7491 TRONDHEIM, NORWAY.

E-mail address: `charles.curry@ntnu.no`

DEPT. OF MATHEMATICAL SCIENCES, NORWEGIAN UNIVERSITY OF SCIENCE AND TECHNOLOGY (NTNU), 7491 TRONDHEIM, NORWAY.

E-mail address: `kurusch.ebrahimi-fard@ntnu.no`

URL: `https://folk.ntnu.no/kurusche/`

UNIV. BLAISE PASCAL, C.N.R.S.-UMR 6620, 63177 AUBIÈRE, FRANCE

E-mail address: `manchon@math.univ-bpclermont.fr`

URL: `http://math.univ-bpclermont.fr/~manchon/`

DEPT. OF MATHEMATICS, UNIVERSITY OF BERGEN, POSTBOX 7800, N-5020 BERGEN, NORWAY

E-mail address: `hans.munthe-kaas@uib.no`

URL: `http://hans.munthe-kaas.no`