THE BÉZOUT EQUATION ON THE RIGHT HALF PLANE IN A WIENER SPACE SETTING

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ABSTRACT. This paper deals with the Bézout equation $G(s)X(s) = I_m$, $\Re s \geq 0$, in the Wiener space of analytic matrix-valued functions on the right half plane. In particular, G is an $m \times p$ matrix-valued analytic Wiener function, where $p \geq m$, and the solution X is required to be an analytic Wiener function of size $p \times m$. The set of all solutions is described explicitly in terms of a $p \times p$ matrix-valued analytic Wiener function Y, which has an inverse in the analytic Wiener space, and an associated inner function Θ defined by Y and the value of G at infinity. Among the solutions, one is identified that minimizes the H^2-norm. A Wiener space version of Tolokonnikov's lemma plays an important role in the proofs. The results presented are natural analogs of those obtained for the discrete case in [11].

1. Introduction and main results

In this paper we deal with the Bézout equation $G(s)X(s) = I_m$ on the closed right half plane $\Re s \geq 0$, assuming that the given function G is of the form

$$G(s) = D + \int_0^\infty e^{-st} g(t)\, dt \quad (\Re s \geq 0),$$

$$\text{where} \quad g \in L^1_{m\times p}(\mathbb{R}_+) \cap L^2_{m\times p}(\mathbb{R}_+). \tag{1.1}$$

In particular, G belongs to the analytic Wiener space $\mathcal{W}_+^{m\times p}$. We are interested in solutions $X \in \mathcal{W}_+^{p\times m}$, that is,

$$X(s) = D_X + \int_0^\infty e^{-st} x(t)\, dt \quad (\Re s \geq 0), \quad \text{where} \quad x \in L^1_{m\times p}(\mathbb{R}_+). \tag{1.2}$$

Throughout $p \geq m$. We refer to the final paragraph of this introduction for a further explanation of the notation.

With G given by (1.1) we associate the Wiener-Hopf operator T_G mapping $L^2_p(\mathbb{R}_+)$ into $L^2_m(\mathbb{R}_+)$ which is defined by

$$(T_G h)(t) = Dh(t) + \int_0^\infty g(t-\tau)h(\tau)d\tau, \quad t \geq 0 \quad (h \in L^2_m(\mathbb{R}_+)). \tag{1.3}$$

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For X as in (1.2) we define the Wiener-Hopf operator T_X mapping $L^2_m(\mathbb{R}_+)$ into $L^2_p(\mathbb{R}_+)$ in a similar way, replacing D by D_X and g by x. If the Bézout equation

$$G(s)X(s) = I_m, \quad \Re s \geq 0. \tag{1.4}$$

has a solution X as in (1.2), then (using the analyticity of G and X) the theory of Wiener-Hopf operators (see [7, Section XII.2] or [2, Section 9]) tells us that $T_G T_X = T_{GX} = I$, where I stands for the identity operator on $L^2_m(\mathbb{R}_+)$. Thus for the Bézout equation (1.4) to be solvable the operator T_G must be surjective or, equivalently, $T_G T_G^*$ must be strictly positive. We shall see that this condition is also sufficient.

To state our main results, we assume that $T_G T_G^*$ is strictly positive. Then $D = G(\infty)$ is surjective, and hence DD^* is strictly positive too. We introduce two matrices D^+ and E, of sizes $p \times m$ and $p \times (p-m)$, respectively, and a $p \times p$ matrix function Y in $\mathcal{W}_+^{p\times p}$, as follows:

(i) $D^+ = D^*(DD^*)^{-1}$, where $D = G(\infty)$;
(ii) E is an isometry mapping $\mathbb{C}^{p-m}$ into $\mathbb{C}^p$ such that $\operatorname{Im} E = \operatorname{Ker} D$;
(iii) Y is the $p \times p$ matrix function given by

$$Y(s) = I_p - \int_0^\infty e^{-st} y(t)dt, \ \Re s \geq 0, \ \text{where } y = T_G^*(T_G T_G^*)^{-1} g. \tag{1.5}$$

From the definitions of D^+ and E it follows that the $p \times p$ matrix $\begin{bmatrix} D^+ & E \end{bmatrix}$ is non-singular. In fact

$$\begin{bmatrix} D \\ E^* \end{bmatrix} \begin{bmatrix} D^+ & E \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & I_{p-m} \end{bmatrix}. \tag{1.6}$$

As we shall see (Proposition 2.1 in Section 2 below), the fact that the given function $g \in L^1_{m\times p}(\mathbb{R}_+) \cap L^2_{m\times p}(\mathbb{R}_+)$ implies that a similar result holds true for y. In particular, $Y \in \mathcal{W}_+^{p\times p}$. In what follows Ξ and Θ are the functions defined by

$$\Xi(s) = \Big(I_p - \int_0^\infty e^{-st} y(t)dt\Big) D^+ = Y(s)D^+, \ \Re s > 0; \tag{1.7}$$

$$\Theta(s) = \Big(I_p - \int_0^\infty e^{-st} y(t)dt\Big) E = Y(s)E, , \ \Re s > 0. \tag{1.8}$$

Since $Y \in \mathcal{W}_+^{p\times p}$, we have $\Xi \in \mathcal{W}_+^{p\times m}$ and $\Theta \in \mathcal{W}_+^{p\times(p-m)}$. Finally, recall that a function Ω in the analytic Wiener space $\mathcal{W}_+^{k\times r}$ is inner whenever $\Omega(s)$ is an isometry for each $s \in i\mathbb{R}$. We now state our main results.

Theorem 1.1. *Let G be the $m \times p$ matrix-valued function given by (1.1). Then the equation $G(s)X(s) = I_m$, $\Re s > 0$, has a solution $X \in \mathcal{W}_+^{p\times m}$ if and only if T_G is right invertible. In that case the function Ξ defined by (1.7) is a particular solution and the set of all solutions $X \in \mathcal{W}_+^{p\times m}$ is given by*

$$X(s) = \Xi(s) + \Theta(s)Z(s), \quad \Re s > 0, \tag{1.9}$$

where Ξ and Θ are defined by (1.7) and (1.8), respectively, and the free parameter Z is an arbitrary function in $\mathcal{W}_+^{(p-m)\times m}$. Moreover, the function Θ belongs to $\mathcal{W}_+^{p\times(p-m)}$ and is inner. Furthermore, the solution Ξ is the minimal H^2 solution *in the following sense*

$$\|X(\cdot)u\|^2_{H^2_p} = \|\Xi(\cdot)u\|^2_{H^2_p} + \|Z(\cdot)u\|^2_{H^2_{p-m}},$$

$$\text{where } u \in \mathbb{C}^m \text{ and } Z \in \widetilde{\mathcal{W}}_+^{(p-m)\times m}. \tag{1.10}$$

In the above theorem, for any positive integer k, $H^2_k = H^2_k(i\mathbb{R})$ is the Hardy space of $\mathbb{C}^k$-valued functions on the right half plane given by $H^2_k(i\mathbb{R}) = JL^2_k(\mathbb{R}_+)$, where J is the unitary operator defined by

$$J = \frac{1}{\sqrt{2\pi}}\mathcal{F} : L^2_k(\mathbb{R}) \to L^2_k(i\mathbb{R}) \tag{1.11}$$

with $\mathcal{F}$ being the Fourier transform mapping $L^2_k(\mathbb{R})$ onto $L^2_k(i\mathbb{R})$. Moreover, $Z \in \widetilde{\mathcal{W}}_+^{(p-m)\times m}$ means that

$$Z(s) = D_Z + \int_0^\infty e^{-st} z(t)\,dt \quad (\Re s \geq 0), \quad \text{where}$$

$$D_Z \text{ is a } (p-m)\times p \text{ matrix and } z \in L^1_{(p-m)\times p}(\mathbb{R}_+) \cap L^2_{(p-m)\times p}(\mathbb{R}_+).$$

See the final part of this introduction for further information about the used notation, in particular, see (1.14) for the definition of the Fourier transform $\mathcal{F}$.

The second theorem is a variant of the Tolokonnikov lemma [20] in the present setting. The result emphasizes the central role of the function Y.

Theorem 1.2. *Assume $T_G T_G^*$ is strictly positive, and let Y be the matrix function defined by* (1.5)*. Then Y belongs to the Wiener space $\mathcal{W}_+^{p\times p}$, $\det Y(s) \neq 0$ whenever $\Re s \geq 0$, and hence Y is invertible in $\mathcal{W}_+^{p\times p}$. Furthermore, the $p \times p$ matrix function*

$$\begin{bmatrix} G(s) \\ E^*Y(s)^{-1} \end{bmatrix}, \quad \Re s \geq 0, \tag{1.12}$$

is invertible in the Wiener algebra $\mathcal{W}_+^{p\times p}$ and its inverse is given by

$$\begin{bmatrix} G(s) \\ E^*Y(s)^{-1} \end{bmatrix}^{-1} = Y(s)\begin{bmatrix} D^+ & E \end{bmatrix} = \begin{bmatrix} \Xi(s) & \Theta(s) \end{bmatrix}, \quad \Re s \geq 0. \tag{1.13}$$

The literature on the Bézout equation and the related corona problem is extensive, starting with Carleson's corona theorem [3] (for the case when $m = 1$) and Fuhrmann's extension to the matrix-valued case [6], both in a H^∞ setting. The topic has beautiful connections with operator theory (see the books [14], [16], [17], [18], and the more recent papers [21], [22], [23]). Rational matrix equations of the form (1.4) play an important role in solving systems and control theory problems, in particularly, in problems involving coprime factorization, see, e.g., [24, Section 4.1], [10, Section A.2], [25, Chapter 21]. For more recent work see [12] and [13], and [15, page 3] where it is proved that the scalar analytic Wiener algebra is a pre-Bézout ring. For matrix polynomials, the equation (1.4) is closely related to the Sylvester resultant; see, e.g., Section 3 in [9] and the references in that paper.

The present paper is inspired by [5] and [11]. The paper [5] deals with equation (1.4) assuming the matrix function G to be a stable rational matrix function, and the solutions are required to be stable rational matrix functions as well. The comment in the final paragraph of [5, Section 2] was the starting point for our analysis. The paper [11] deals with the discrete case (when the right half plane is replaced by the open unit disc). Theorems 1.1 and 1.2 are the continuous analogue of Theorem

1.1 in [11]. The absence of an explicit formula for the function Y^{-1} in the present setting makes the proofs more complicated than those in [11].

The paper consists of five sections, including the present introduction and an appendix. Section 2, which deals with the right invertibility of the operator T_G, has an auxiliary character. Theorem 1.2 is proved in Section 3, and Theorem 1.1 in Section 4. The Appendix, Section A, contains a number of auxiliary results involving the Lebesgue space $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and its vector-valued counterpart, which are collected together simply for the convenience of the reader and contains no significantly new material.

Notation and terminology. We conclude this section with some notation and terminology. Throughout, a linear map $A : \mathbb{C}^r \to \mathbb{C}^k$ is identified with the $k \times r$ matrix of A relative to the standard orthonormal bases in $\mathbb{C}^r$ and $\mathbb{C}^k$. The space of all $k \times r$ matrices with entries in $L^1(\mathbb{R})$ will be denoted by $L^1_{k\times r}(\mathbb{R})$. As usual $\widehat{f}$ denotes the Fourier transform of $f \in L^1_{k\times r}(\mathbb{R})$, that is,

$$\widehat{f}(s) = (\mathcal{F}f)(s) = \int_{-\infty}^{\infty} e^{-st} f(t)dt, \quad s \in i\mathbb{R}. \tag{1.14}$$

Note that $\widehat{f}$ is continuous on the extended imaginary axis $i\mathbb{R} \cup \{\pm\, i\infty\}$, and is zero at $\pm\, i\infty$ by the Riemann-Lebesgue lemma. By $\mathcal{W}^{k\times r}$ we denote the *Wiener space* consisting of all $k \times r$ matrix functions F on the imaginary axis of the form

$$F(s) = D_F + \widehat{f}(s), \ s \in i\mathbb{R}, \text{where } f \in L^1_{k\times r}(\mathbb{R}) \text{ and } D_F \text{ is a constant matrix.} \tag{1.15}$$

Since $\widehat{f}$ is continuous on the extended imaginary axis and is zero at $\pm i\infty$, the function F given by (1.15) is also continuous on the extended imaginary axis and the constant matrix D_F is equal to the value of F at infinity. We write $\mathcal{W}^{k\times r}_+$ for the space of all F of the form (1.15) with the additional property that f has its support in $\mathbb{R}_+ = [0, \infty)$, that is, f is equal to zero on $(-\infty, 0)$. Any function $F \in \mathcal{W}^{k\times r}_+$ is analytic and bounded on the open right half plane. Thus any $F \in \mathcal{W}^{k\times r}_+$ is a matrix-valued H^∞ function. Finally, by $\mathcal{W}^{k\times r}_{-,0}$ we denote the Wiener space consisting of all F of the form (1.15) with the additional property that $D_F = 0$ and f has its support in $(-\infty, 0]$. Thus we have the following direct sum decomposition:

$$\mathcal{W}^{k\times r} = \mathcal{W}^{k\times r}_+ \dot{+} \mathcal{W}^{k\times r}_{-,0}. \tag{1.16}$$

We write $F \in \widetilde{\mathcal{W}}^{k\times r}_+$ if the function f in (1.15) belongs to $L^1_{k\times r}(\mathbb{R}_+) \cap L^2_{k\times r}(\mathbb{R}_+)$. Similarly, $F \in \widetilde{\mathcal{W}}^{k\times r}_{-,0}$ if $f \in L^1_{k\times r}(\mathbb{R}_-) \cap L^2_{k\times r}(\mathbb{R}_-)$ and $D_F = 0$.

Let $F \in \mathcal{W}^{k\times r}$ be given by (1.15). With F we associate the Wiener-Hopf operator T_F mapping $L^2_r(\mathbb{R}_+)$ into $L^2_k(\mathbb{R}_+)$. This operator (see [7, Section XII.2]) is defined by

$$(T_F h)(t) = D_F h(t) + \int_0^\infty f(t-\tau)h(\tau)d\tau, \quad t \geq 0 \quad (h \in L^2_r(\mathbb{R}_+)). \tag{1.17}$$

The orthogonal complement of $H^2_k(i\mathbb{R}) = JL^2_k(\mathbb{R}_+)$, with J as in (1.11), in $L^2_k(i\mathbb{R})$ will be denoted by $K^2_k(i\mathbb{R})$. If $F \in \widetilde{\mathcal{W}}^{k\times r}_+$, then for each $u \in \mathbb{C}^r$ the function $F(\cdot)u$ belongs to $H^2_k(i\mathbb{R})$. Similarly, $F(\cdot)u$ belongs to $K^2_k(i\mathbb{R})$ if $F \in \widetilde{\mathcal{W}}^{k\times r}_{-,0}$.

Finally, for $f \in L^1_{k\times r}(\mathbb{R})$ and $g \in L^1_{r\times m}(\mathbb{R})$ the *convolution product* $f \star g$ is the function in $L^1_{k\times m}(\mathbb{R})$, see [19, Section 7.13], given by

$$(f \star g)(t) = \int_{-\infty}^{\infty} f(t-\tau)g(\tau)\,d\tau \quad \text{a.e. on } \mathbb{R}. \tag{1.18}$$

2. Right invertibility of T_G

In this section $G \in \mathcal{W}_+^{m\times p}$, where G is given by (1.1) and $p \geq m$. We already know that the Bézout equation (1.4) having a solution X in $\mathcal{W}_+^{p\times m}$ implies that T_G is right invertible or, equivalently, $T_G T_G^*$ is strictly positive; see the paragraph containing formula (1.4).

In this section we present an auxiliary result that will be used to prove our main theorems. For this purpose we need the $m \times m$ matrix-valued function R on the imaginary axis defined by $R(s) = G(s)G(s)^*$, $s \in i\mathbb{R}$. It follows that $R \in \mathcal{W}^{m\times m}$. By T_R we denote the corresponding Wiener-Hopf operator acting on $L^2_m(\mathbb{R}_+)$. Thus

$$(T_R f)(t) = DD^* f(t) + \int_0^\infty r(t-\tau)f(\tau)\,d\tau, \quad 0 \leq t < \infty,$$

$$\text{with } r(t) = Dg^*(t) + g(t)D^* + \int_{-\infty}^{\infty} g(t-\tau)g^*(\tau)\,d\tau, \quad t \in \mathbb{R}.$$

Here $g^*(t) = g(-t)^*$ for $t \in \mathbb{R}$. It is well-known (see, e.g., formula (24) in Section XII.2 of [7]) that

$$T_R = T_G T_G^* + H_G H_G^*. \tag{2.1}$$

Here H_G is the Hankel operator mapping $L^2_p(\mathbb{R}_+)$ into $L^2_m(\mathbb{R}_+)$ defined by G, that is,

$$(H_G f)(t) = \int_0^\infty g(t+\tau)f(\tau)d\tau, \quad f \in L^2_p(\mathbb{R}_+). \tag{2.2}$$

We shall prove the following proposition. For the case when G is a rational matrix function, the first part (of the "if and only if" part) of the proposition is covered by Lemma 2.3 in [5]. The proof given in [5] can also be used in the present setting. For the sake of completeness we include a proof of the first part.

Proposition 2.1. *Let G be given by* (1.1)*. Then the operator T_G is right invertible if and only if T_R and $I - H_G^* T_R^{-1} H_G$ are both invertible operators. In that case the inverse of $T_G T_G^*$ is given by*

$$(T_G T_G^*)^{-1} = T_R^{-1} + T_R^{-1} H_G (I - H_G^* T_R^{-1} H_G)^{-1} H_G^* T_R^{-1}. \tag{2.3}$$

Furthermore,

- (a) *$(T_G T_G^*)^{-1}$ maps $L^1_m(\mathbb{R}_+) \cap L^2_m(\mathbb{R}_+)$ in a one-to-one way onto itself;*
- (b) *the function y defined by $y = T_G^*(T_G T_G^*)^{-1} g$ belongs to $L^1_{p\times p}(\mathbb{R}_+) \cap L^2_{p\times p}(\mathbb{R}_+)$, in particular, the function Y given by* (1.5) *is in $\widetilde{\mathcal{W}}_+^{p\times p}$.*

Proof. We split the proof into four parts. In the first part we assume that T_G is right invertible, and we show that T_R and $I - H_G^* T_R^{-1} H_G$ are both invertible operators and that the inverse of $T_G T_G^*$ is given by (2.3). The second part deals with the reverse implication. Items (a) and (b) are proved in the last two parts.

Part 1. Assume T_G is right invertible. Then the operator $T_G T_G^*$ is strictly positive. According to (2.1) we have $T_R \geq T_G T_G^*$, and hence T_R is also strictly positive.

In particular, T_R is invertible. Rewriting (2.1) as $T_GT_G^* = T_R - H_GH_G^*$, and multiplying the latter identity from the left and from the right by $T_R^{-1/2}$ shows that

$$T_R^{-1/2}T_GT_G^*T_R^{-1/2} = I - T_R^{-1/2}H_GH_G^*T_R^{-1/2}. \tag{2.4}$$

Hence $I - T_R^{-1/2}H_GH_G^*T_R^{-1/2}$ is strictly positive which shows that $H_G^*T_R^{-1/2}$ is a strict contraction. But then $H_G^*T_R^{-1}H_G = \big(H_G^*T_R^{-1/2}\big)\big(H_G^*T_R^{-1/2}\big)^*$ is also a strict contraction, and thus the operator $I - H_G^*T_R^{-1}H_G$ is strictly positive. In particular, $I - H_G^*T_R^{-1}H_G$ is invertible. Finally, since $T_GT_G^* = T_R - H_GH_G^*$, a usual Schur complement type of argument (see, e.g., Section 2.2 in [1]), including the well-known inversion formula

$$(A - BC)^{-1} = A^{-1} + A^{-1}B(I - CA^{-1}B)^{-1}CA^{-1},$$

then shows that $(T_GT_G^*)^{-1}$ is given by (2.3).

Part 2. In this part we assume that T_R and $I - H_G^*T_R^{-1}H_G$ are both invertible operators, and we show that T_G is right invertible. According to (2.1) the operator T_R is positive. Since we assume T_R to be invertible, we conclude that T_R is strictly positive. Rewriting (2.1) as $T_GT_G^* = T_R - H_GH_G^*$, and multiplying the latter identity from the left and from the right by $T_R^{-1/2}$ we obtain the identity (2.4). Hence $I - T_R^{-1/2}H_GH_G^*T_R^{-1/2}$ is positive which shows that $H_G^*T_R^{-1/2}$ is a contraction. But then $H_G^*T_R^{-1}H_G = \big(H_G^*T_R^{-1/2}\big)\big(H_G^*T_R^{-1/2}\big)^*$ is also a contraction, and thus the operator $I - H_G^*T_R^{-1}H_G$ is positive. By assumption $I - H_G^*T_R^{-1}H_G$ is invertible. It follows that $I - H_G^*T_R^{-1}H_G$ is strictly positive, and hence $T_R^{-1/2}H_G$ is a strict contraction. But then the same holds true for $T_R^{-1/2}H_GH_G^*T_R^{-1/2}$. This implies that $I - T_R^{-1/2}H_GH_G^*T_R^{-1/2}$ is strictly positive, and (2.4) shows that T_G is right invertible.

Part 3. In this part we prove item (a). Observe that $g \in L^1_{m\times p}(\mathbb{R})$ implies that T_G maps $L^1_p(\mathbb{R}_+) \cap L^2_p(\mathbb{R}_+)$ into $L^1_m(\mathbb{R}_+) \cap L^2_m(\mathbb{R}_+)$. Since $g^* \in L^1_{p\times m}(\mathbb{R})$ and

$$(T_G^*f)(t) = D^*f(t) + \int_0^\infty g^*(t-\tau)f(\tau)\,d\tau, \quad 0 \le t < \infty,$$

the operator T_G^* maps $L^1_m(\mathbb{R}_+) \cap L^2_m(\mathbb{R}_+)$ into $L^1_p(\mathbb{R}_+) \cap L^2_p(\mathbb{R}_+)$. Thus $T_GT_G^*$ maps $L^1_m(\mathbb{R}_+) \cap L^2_m(\mathbb{R}_+)$ into itself. We have to show that the same holds true for its inverse. To do this we apply Lemmas A.3 and A.4.

Lemma A.3 tells us that T_R^{-1} maps $L^1_m(\mathbb{R}_+) \cap L^2_m(\mathbb{R}_+)$ in a one-to-one way onto itself. This allows us to apply Lemma A.4 with

$$Q = T_R^{-1}, \quad H = H_G \quad \text{and} \quad \tilde{H} = H_G^*.$$

Recall that H_G is a Hankel operator, see (2.2), and H_G^* is also a Hankel operator, in fact

$$(H_G^*f)(t) = \int_0^\infty g(t+\tau)^*f(\tau)\,d\tau, \quad 0 \le t < \infty.$$

Since $I - H_G^*T_R^{-1}H_G$ is invertible, Lemma A.4 then shows that $I - H_G^*T_R^{-1}H_G$ maps $L^1_p(\mathbb{R}_+) \cap L^2_p(\mathbb{R}_+)$ in a one-to-one way onto itself, and hence the same holds true for its inverse $(I - H_G^*T_R^{-1}H_G)^{-1}$. To complete the proof of item (a) note that $H_G^*T_R^{-1}$ maps $L^1_m(\mathbb{R}_+) \cap L^2_m(\mathbb{R}_+)$ into $L^1_p(\mathbb{R}_+) \cap L^2_p(\mathbb{R}_+)$, and $T_R^{-1}H_G$ maps $L^1_p(\mathbb{R}_+) \cap L^2_p(\mathbb{R}_+)$ into $L^1_m(\mathbb{R}_+) \cap L^2_m(\mathbb{R}_+)$. But then (2.3) shows that $(T_GT_G^*)^{-1}$ maps $L^1_m(\mathbb{R}_+) \cap$

$L^2_m(\mathbb{R}_+)$ into itself. To see that $(T_G T_G^*)^{-1}$ is one-to-one on $L^1_m(\mathbb{R}_+) \cap L^2_m(\mathbb{R}_+)$ and maps $L^1_m(\mathbb{R}_+) \cap L^2_m(\mathbb{R}_+)$ onto itself, one can follow the same argumentation as in the last part of the proof of Lemma A.3.

PART 4. In this part we prove item (b). Since g belongs to $L^1_{m\times p}(\mathbb{R}_+) \cap L^2_{m\times p}(\mathbb{R}_+)$, item (a) tells us that $f := (T_G T_G^*)^{-1} g$ also belongs to $L^1_{m\times p}(\mathbb{R}_+) \cap L^2_{m\times p}(\mathbb{R}_+)$. We already have seen (in the first paragraph of the previous part) that T_G^* maps $L^1_m(\mathbb{R}_+) \cap L^2_m(\mathbb{R}_+)$ into $L^1_p(\mathbb{R}_+) \cap L^2_p(\mathbb{R}_+)$. It follows that $y = T_G^* f$ belongs to $L^1_{p\times p}(\mathbb{R}_+) \cap L^2_{p\times p}(\mathbb{R}_+)$, as desired. □

3. The functions Y and Θ, and proof of Theorem 1.2

We begin with three lemmas involving the functions Y and Θ defined by (1.5) and (1.8), respectively. From Proposition 2.1, item (b), and (1.8) we know that $Y \in \mathcal{W}_+^{p\times p}$ and $\Theta \in \mathcal{W}_+^{p\times (p-m)}$; see also the paragraph preceding Theorem 1.1.

Lemma 3.1. *Assume that T_G is right invertible, and let $Y \in \mathcal{W}_+^{p\times p}$ be the function defined by* (1.5)*. Then*

$$G(s)Y(s) = D, \qquad \Re s > 0. \tag{3.1}$$

Proof. To prove (3.1) note that $T_G y = T_G T_G^* (T_G T_G^*)^{-1} g = g$. Since the functions g and y both have their support in $\mathbb{R}_+$, the identity $T_G y = g$ can be rewritten as $Dy + g \star y = g$, where $\star$ is the convolution product of matrix-valued functions with entries in $L^1(\mathbb{R})$; see (1.18). Thus

$$Dy(t) + (g \star y)(t) = Dy(t) + \int_{-\infty}^{\infty} g(t-\tau) y(\tau) dt = g(t), \quad t \in \mathbb{R}. \tag{3.2}$$

Next use that the Fourier transform of a convolution product is just the product of the Fourier transforms of the functions in the convolution product. Thus taking Fourier transforms in (3.2) yields $D\widehat{y} + \widehat{g}\widehat{y} = \widehat{g}$. The latter identity can be rewritten as $G\widehat{y} = \widehat{g}$. Hence, using the definition of Y in (1.5), we obtain

$$G(s)Y(s) = G(s)\Big(I_p - \widehat{y}(s)\Big) = G(s) - \widehat{g}(s) = D.$$

This proves (3.1). □

Lemma 3.2. *Assume that T_G is right invertible. Then the function Θ defined by* (1.8) *belongs to $\mathcal{W}_+^{p\times (p-m)}$ and is an inner function, that is, $\Theta(s)$ is an isometry for each $s \in i\mathbb{R}$ and at infinity.*

Proof. We already know that $\Theta \in \mathcal{W}_+^{p\times (p-m)}$. To prove that Θ is inner, let $y = T_G^*(T_G T_G^*)^{-1} g$ as in (1.5), and put $f = (T_G T_G^*)^{-1} g$. Thus $f \in L^1_{m\times m}(\mathbb{R}_+)$, by Proposition 2.1 (b), and $y = T_G^* f$. The latter can be rewritten as

$$y(t) = D^* f(t) + \int_0^{\infty} g^*(t-\tau) f(\tau)\, d\tau, \quad t \geq 0.$$

Note that $g^*(t) = g(-t)^*$, and hence g^* has its support in $(-\infty, 0]$. Therefore

$$y(t) = D^* f(t) + \int_{-\infty}^{\infty} g^*(t-\tau) f(\tau)\, d\tau, \quad t \in \mathbb{R}. \tag{3.3}$$

Put

$$\rho(t) = \begin{cases} 0 & \text{when } t \geq 0, \\ (g^* \star f)(t) & \text{when } t < 0. \end{cases} \tag{3.4}$$

Using the definition of the convolution product $\star$, see (1.18), we can rewrite (3.3) as

$$y(t) = D^* f(t) + (g^* \star f)(t) - \rho(t) \quad t \in \mathbb{R}.$$

Taking Fourier transforms we obtain

$$\widehat{y}(s) = D^* \widehat{f}(s) + \widehat{g^*}(s)\widehat{f}(s) - \widehat{\rho}(s) = G(s)^* \widehat{f}(s) - \widehat{\rho}(s), \quad s \in i\mathbb{R}.$$

Hence, we have

$$Y(s) = I - G(s)^* \widehat{f}(s) + \widehat{\rho}(s), \quad s \in i\mathbb{R}. \tag{3.5}$$

Now let us compute $\Theta(s)^*\Theta(s) = E^*Y(s)^*Y(s)E$ for $s \in i\mathbb{R}$. We have

$$\begin{aligned} E^*Y(s)^*Y(s)E = \\ &= E^*Y(s)^*E - E^*Y(s)^*G(s)^*\widehat{f}(s)E + E^*Y(s)^*\widehat{\rho}(s)E \\ &= E^*Y(s)^*E + E^*Y(s)^*\widehat{\rho}(s)E \\ &\qquad\qquad (\text{because } G(s)Y(s)E = 0 \text{ by (3.1) and } DE = 0) \\ &= E^*E - E^*\widehat{y}(s)^*E + E^*Y(s)^*\widehat{\rho}(s)E \\ &= I_{p-m} - \Omega(s). \end{aligned}$$

Here $\Omega(s) = E^*\widehat{y}(s)^*E - E^*Y(s)^*\widehat{\rho}(s)E$. Note that the functions $\widehat{y}(\cdot)^*$ and $Y(\cdot)^*\widehat{\rho}(\cdot)$ belong to $\mathcal{W}_{-,0}^{p\times p}$, and thus Ω belongs to $\mathcal{W}_{-,0}^{(p-m)\times(p-m)}$. On the other hand, the function $E^*Y(\cdot)^*Y(\cdot)E$ is hermitian on the imaginary axis, and hence the same is true for Ω. But for any positive integer k we have

$$\mathcal{W}_{-,0}^{k\times k} \cap (\mathcal{W}_{-,0}^{k\times k})^* = \{0\}.$$

Thus Ω is identically zero, and thus $\Theta(s)^*\Theta(s) = E^*Y(s)^*Y(s)E = I_{p-m}$ for any $s \in i\mathbb{R}$. Moreover, $\Theta(\infty)^*\Theta(\infty) = E^*E = I$. This proves that Θ is inner. □

Lemma 3.3. *Assume that T_G is right invertible, and let $Y \in \mathcal{W}_+^{p\times p}$ be the function defined by* (1.5). *Then Y is invertible in $\mathcal{W}_+^{p\times p}$.*

Proof. Fix $s \in i\mathbb{R}$, and assume $u \in \mathbb{C}^p$ such that $Y(s)u = 0$. Then $G(s)Y(s) = D$ implies that $Du = 0$. By definition of E, $u = Ev$ for some $v \in \mathbb{C}^{p-m}$. Next use $\Theta(s) = Y(s)E$. It follows that $\Theta(s)v = Y(s)Ev = Y(s)u = 0$. However, $\Theta(s)$ is an isometry, by Lemma 3.2. So $v = 0$, and hence $u = 0$. We see that $\det Y(s) \neq 0$. Also $Y(\infty) = I_p$. We conclude that T_Y is a Fredholm operator; see [7, Theorem XII.3.1].

Next we prove that $\operatorname{Ker} T_Y = \{0\}$. Take $h \in \operatorname{Ker} T_Y$. Then $T_Y h = 0$, and hence $Y(s)\widehat{h}(s) = 0$ for each $s \in i\mathbb{R}$. But $\det Y(s) \neq 0$ for each $s \in i\mathbb{R}$. Hence $\widehat{h} = 0$, and therefore $h = 0$.

We want to prove that T_Y is invertible. Given the results of the preceding first two paragraphs it suffices to show that $\operatorname{ind} T_Y = 0$. This will be done in the next step by an approximation argument, using the fact, from [5], that we know the result is true for rational matrix functions.

Let g be as in (1.1). Note that g is the limit in L^1 of a sequence $g_1, g_2, \ldots$ such that $G_n(s) = D + \widehat{g_n}(s)$ is a stable rational matrix function; cf., Part (v) on page 229 of [7]. Since T_G is right invertible, T_{G_n} will also be right invertible for n sufficiently large. In fact, $T_{G_n}T_{G_n}^* \to T_G T_G^*$ in operator norm. Put $y_n = T_{G_n}^*(T_{G_n}T_{G_n}^*)^{-1}g_n$. Then $y_n \to y$ in the L^1-norm. Put $Y_n(s) = I - \widehat{y_n}(s)$. Then $T_{Y_n} \to T_Y$ in operator norm. For n sufficiently large the operator T_{Y_n} is invertible (see the paragraph preceding Theorem 1.2 in [5] and formula (2.17) in [5]). In particular, the Fredholm index of T_{Y_n} is zero. But $\operatorname{ind} T_Y = \lim_{n\to\infty} \operatorname{ind} T_{Y_n} = 0$. Thus T_Y is invertible, and hence Y is invertible in $\mathcal{W}_+^{p\times p}$. □

Proof of Theorem 1.2. From Lemma 3.3 we know that $Y \in \mathcal{W}_+^{p\times p}$ and that Y is invertible in $\mathcal{W}_+^{p\times p}$. Thus we only have to prove the second part of the theorem. Since Y is invertible in $\mathcal{W}_+^{p\times p}$, the $p \times p$ matrix function given by (1.12) is well-defined and belongs to $\mathcal{W}_+^{p\times p}$. Furthermore, from (1.6) we know that the $p \times p$ matrix $\begin{bmatrix} D^+ & E \end{bmatrix}$ is invertible. Hence the function defined by the right hand side of (1.13) belongs to $\mathcal{W}_+^{p\times p}$ and is invertible in $\mathcal{W}_+^{p\times p}$. Using (3.1) and the identity (1.6) we see that

$$\begin{aligned}\begin{bmatrix} G(s) \\ E^*Y(s)^{-1} \end{bmatrix} Y(s) \begin{bmatrix} D^+ & E \end{bmatrix} &= \begin{bmatrix} G(s)Y(s) \\ E^* \end{bmatrix} \begin{bmatrix} D^+ & E \end{bmatrix} \\ &= \begin{bmatrix} D \\ E^* \end{bmatrix} \begin{bmatrix} D^+ & E \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & I_{p-m} \end{bmatrix}, \quad \Re s \geq 0.\end{aligned}$$

This proves the first identity (1.13). The second identity is an immediate consequence of the definitions of Ξ and Θ in (1.7) and (1.8), respectively. □

4. Proof of Theorem 1.1

We begin with a lemma concerning the functions Ξ and Θ.

Lemma 4.1. *Assume that T_G is right invertible, and let Ξ and Θ be the functions defined by* (1.7) *and* (1.8)*, respectively. Then*

$$\operatorname{Ker} T_G = T_\Theta L^2_{p-m}(\mathbb{R}_+) \quad \textit{and} \quad \Theta^*\Xi \in \widetilde{\mathcal{W}}_{0,-}^{(p-m)\times m}. \tag{4.1}$$

Proof. We split the proof into two parts.

Part 1. In this part we prove the inclusion of (4.1). Take $s \in i\mathbb{R}$. Note that in Proposition 3.2 it was shown for $s \in i\mathbb{R}$ that

$$Y(s) = I - G(s)^*\widehat{f}(s) + \widehat{\rho}(s),$$

where $f = (T_G T_G^*)^{-1} g$ and ρ is defined by (3.4); see (3.5). From (1.7) and (1.8) we then see that

$$\Theta(s)^*\Xi(s) = E^*Y(s)^*Y(s)D^+ = E^*Y(s)^*\Big(I - G(s)^*\widehat{f}(s) + \widehat{\rho}(s)\Big)D^+.$$

Now use that $G(s)Y(s)E = DE = 0$, and hence $E^*Y(s)^*G(s)^* = 0$. The latter identity and the fact that $E^*D^+ = 0$ and $Y = I - \widehat{y}$ imply that

$$\begin{aligned}\Theta(s)^*\Xi(s) &= -E^*\widehat{y}(s)^*D^+ + E^*\widehat{\rho}(s)D^+ - E^*\widehat{y}(s)^*\widehat{\rho}(s)D^+ \\ &= -A(s) + B(s) - C(s).\end{aligned} \tag{4.2}$$

From item (b) in Proposition 4.1 we know that $y \in L^1_{p\times p}(\mathbb{R}_+) \cap L^2_{p\times p}(\mathbb{R}_+)$, and thus $\widehat{y} \in \widetilde{\mathcal{W}}_+^{p\times p}$ and $\widehat{y}(\infty) = 0$, that is, $\widehat{y} \in \widetilde{\mathcal{W}}_{0,+}^{p\times p}$. It follows that

$$A(\cdot) := E^*\widehat{y}(\cdot)^*D^+ \in \widetilde{\mathcal{W}}_{0,-}^{(p-m)\times p}. \tag{4.3}$$

Recall that ρ is given by (3.4) with $f = (T_G T_G^*)^{-1} g$. Since the function g belongs to $L^1_{m\times p}(\mathbb{R}_+) \cap L^2_{m\times p}(\mathbb{R}_+)$, item (b) in Proposition 2.1 tells us that the same holds true for f. It follows that $g^* \star f \in L^1_{p\times p}(\mathbb{R}) \cap L^2_{p\times p}(\mathbb{R})$. The latter implies that $\rho \in L^1_{p\times p}(\mathbb{R}_-) \cap L^2_{p\times p}(\mathbb{R}_-)$. We conclude that

$$B(\cdot) := E^* \widehat{\rho}(\cdot) D^+ \in \widetilde{\mathcal{W}}^{(p-m)\times p}_{0,-}. \tag{4.4}$$

Finally, note that $y^*(t) = y(-t)^*$ for $t \in \mathbb{R}$ and $\big(\widehat{y}(s)\big)^* = \widehat{y^*}(s)$ for $s \in i\mathbb{R}$. Thus

$$\widehat{y}(s)^* \widehat{\rho}(s) = \Big(\widehat{y^* \star \rho}\Big)(s), \quad s \in i\mathbb{R},$$

and

$$(y^* \star \rho)(t) = \int_{-\infty}^{\infty} y^*(t-\tau)\rho(\tau)\, d\tau = \int_{-\infty}^{0} y^*(t-\tau)\rho(\tau)\, d\tau.$$

Since both y^* and ρ belong to $L^1_{p\times p}(\mathbb{R}_-) \cap L^2_{p\times p}(\mathbb{R}_-)$, it is well known (see, e.g., Section 2 in [4]) that the same holds true for $y^* \star \rho$. But then

$$C(\cdot) := E^* \widehat{y}(\cdot)^* \widehat{\rho}(\cdot) D^+ \in \widetilde{\mathcal{W}}^{(p-m)\times p}_{0,-}. \tag{4.5}$$

From (4.3), (4.4), (4.5) and (4.2) it follows that $\Theta^*\Xi \in \widetilde{\mathcal{W}}^{(p-m)\times m}_{0,-}$.

Part 2. In this part we prove the identity of (4.1). Using (3.1) we see that

$$G(s)\Theta(s) = G(s)Y(s)E = DE = 0, \quad s \in i\mathbb{R}.$$

This implies that $T_G T_\Theta = 0$, and hence $\operatorname{Im} T_\Theta \subset \operatorname{Ker} T_G$. To prove the reverse inclusion, take $h \in \operatorname{Ker} T_G$. Thus $h \in L^2_p(\mathbb{R}_+)$ and $T_G h = 0$. It follows that $G(s)\widehat{h}(s) = 0$ for $\Re s > 0$. Put $H(s) = \widehat{h}(s)$. Then $H(\cdot)$ belongs to $H^2_m(i\mathbb{R})$. Next we apply Theorem 1.2. Using the identities in (1.13) we see that

$$\begin{aligned} H(s) &= \begin{bmatrix} \Xi(s) & \Theta(s) \end{bmatrix} \begin{bmatrix} G(s) \\ E^*Y(s)^{-1} \end{bmatrix} H(s) \\ &= \begin{bmatrix} \Xi(s) & \Theta(s) \end{bmatrix} \begin{bmatrix} 0 \\ E^*Y(s)^{-1}H(s) \end{bmatrix} = \Theta(s)E^*Y(s)^{-1}H(s). \end{aligned}$$

Hence $\widehat{h}(s) = \Theta(s)\Psi(s)$, where $\Psi(s) = E^*Y(s)^{-1}\widehat{h}(s)$. Since $h \in L^2_p(\mathbb{R}_+)$ and $Y(\cdot)^{-1}$ is a matrix function with H^∞ entries, we conclude that $\Psi \in H^2_{p-m}$, and hence $\Psi = \widehat{u}$ for some $u \in L^2_{p-m}(\mathbb{R}_+)$. The identity $\widehat{h}(s) = \Theta(s)\Psi(s)$ then yields $\widehat{h}(s) = \Theta(s)\widehat{u}(s)$. This shows that $h = T_\Theta u$, and hence $\operatorname{Ker} T_G \subset \operatorname{Im} T_\Theta$. □

Proof of Theorem 1.1. From Lemma 3.2 we know that Θ is an inner function in $\mathcal{W}^{(p-m)\times m}_+$. The proof of the other statements is split into three parts.

Part 1. In this part we show that the equation $G(s)X(s) = I_m$, $\Re s > 0$, has a solution $X \in \mathcal{W}^{p\times m}_+$ if and only if T_G is right invertible. Furthermore, we show that in that case the function Ξ defined by (1.7) is a particular solution. From the one but last sentence of the paragraph containing (1.4) we know that it suffices to prove the "if part" only. Therefore, in what follows we assume that T_G is right invertible. Since $\Xi(s) = Y(s)D^+$ and $Y \in \mathcal{W}^{p\times p}_+$, we have $\Xi \in \mathcal{W}^{p\times m}_+$. Moreover, using the identity (4.1) we have

$$G(s)\Xi(s) = G(s)Y(s)D^+ = DD^+ = I_m.$$

Thus Ξ is a particular solution.

Part 2. This second part deals with the description of all in solutions in $\mathcal{W}_+^{p\times m}$. Let Z be an arbitrary function in $\mathcal{W}_+^{p\times m}$, and let $X \in \mathcal{W}_+^{p\times m}$ be defined by (1.9). Then

$$G(s)X(s) = G(s)\Xi(s) + G(s)\Theta(s)Z(s) = I_m + G(s)\Theta(s)Z(s), \quad \Re s \geq 0.$$

Recall that $G(s)\Theta(s) = G(s)Y(s)E = DE = 0$. Thus $G(s)X(s) = I_m$, $\Re s \geq 0$, and thus X is a solution.

To prove the converse implication, let $X \in \mathcal{W}_+^{p\times m}$ be a solution of the equation $G(s)X(s) = I_m$. Put $H = X - \Xi$. Then $H \in \mathcal{W}_+^{p\times m}$ and $G(s)H(s) = 0$. Using the identities in (1.13), we obtain

$$\begin{aligned} H(s) &= \begin{bmatrix}\Xi(s) & \Theta(s)\end{bmatrix} \begin{bmatrix} G(s) \\ E^*Y(s)^{-1}\end{bmatrix} H(s) \\ &= \begin{bmatrix}\Xi(s) & \Theta(s)\end{bmatrix} \begin{bmatrix} 0 \\ E^*Y(s)^{-1}H(s)\end{bmatrix} = \Theta(s)E^*Y(s)^{-1}H(s). \end{aligned}$$

Thus $H(s) = \Theta(s)Z(s)$, where $Z(s) = E^*Y(s)^{-1}H(s)$. Since Y is invertible in $\mathcal{W}_+^{p\times p}$, the function $Y(\cdot)^{-1}$ is in $\mathcal{W}_+^{p\times p}$. Together with the fact that $H \in \mathcal{W}_+^{p\times m}$, this yields $Z \in \mathcal{W}_+^{(p-m)\times m}$. It follows X has the desired representation (1.9).

Part 3. In this part we prove the identity (1.10). Assume $Z \in \widetilde{\mathcal{W}}^{(p-m)\times m}$, and let X be the function defined by (1.9). Fix $u \in \mathbb{C}^m$. Then $Z(\cdot)u \in H_m^2(i\mathbb{R})$, and

$$\Theta(\cdot)Z(\cdot)u = M_\Theta Z(\cdot)u \in H_{p-m}^2(i\mathbb{R}). \tag{4.6}$$

Here M_Θ is the operator of multiplication by $\Theta(\cdot)$ mapping $H_m^2(i\mathbb{R})$ into $H_{p-m}^2(i\mathbb{R})$. Furthermore, since M_Θ is an isometry, we also see that

$$\|Z(\cdot)u\| = \|\Theta(\cdot)Z(\cdot)u\|. \tag{4.7}$$

The fact that $y \in L_{p\times p}^1(\mathbb{R}_+) \cap L_{p\times p}^2(\mathbb{R}_+)$ implies that $Y \in \widetilde{\mathcal{W}}_+^{p\times p}$. But then $\Xi(s) = Y(s)D^+$ yields $\Xi(\cdot)u \in H_m^2(i\mathbb{R})$. Using the identity (1.9) we conclude that $\Xi(\cdot)u$ also belongs to $H_m^2(i\mathbb{R})$. It follows that all norms in (1.10) are well defined, and in order to prove the identity (1.10) it suffices that to show that in $H_m^2(i\mathbb{R})$ the function $\Theta(\cdot)Z(\cdot)u$ is orthogonal to the function $\Xi(\cdot)v$ for any $v \in \mathbb{C}^m$. The latter fact follows from the inclusion in the second part of (4.1). Indeed, this inclusion tells us that $M_\Theta^*\Xi(\cdot)v = 0$, and hence

$$\begin{aligned} \langle \Xi(\cdot)v, \Theta(\cdot)Z(\cdot)u\rangle_{H_m^2(i\mathbb{R})} &= \langle \Xi(\cdot)v, M_\Theta Z(\cdot)u\rangle_{H_m^2(i\mathbb{R})} \\ &= \langle M_\Theta^*\Xi(\cdot)v, Z(\cdot)u\rangle_{H_m^2(i\mathbb{R})} = 0. \end{aligned}$$

This completes the proof. □

Appendix A. The Lebesgue space $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$

The material in this section is standard and is presented for the convenience of the reader. Throughout we deal with the Lebesgue spaces of complex-valued functions on the real line $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$, their vector-valued counterparts $L_m^1(\mathbb{R})$ and $L_m^2(\mathbb{R})$, and the intersection of the latter two spaces: $L_m^1(\mathbb{R}) \cap L_m^2(\mathbb{R})$. The norms on these spaces are given by

$$\begin{aligned}
\|f\|_1 &= \int_{-\infty}^{\infty} |f(t)|\, dt \qquad \text{for } f \in L^1(\mathbb{R}),\\
\|f\|_2 &= \Big(\int_{-\infty}^{\infty} |f(t)|^2\, dt\Big)^{1/2} \qquad \text{for } f \in L^2(\mathbb{R}),\\
\|f\|_1 &= \Big(\sum_{i=1}^{m} \|f_i\|_1^2\Big)^{1/2} \qquad \text{for } f = (f_1, \ldots, f_m)^\top \in L^1_m(\mathbb{R}),\\
\|f\|_2 &= \Big(\sum_{i=1}^{m} \|f_i\|_2^2\Big)^{1/2} \qquad \text{for } f = (f_1, \ldots, f_m)^\top \in L^2_m(\mathbb{R}),\\
\|f\|_0 &= \max\{\|f\|_1, \|f\|_2\} \qquad \text{for } f \in L^1_m(\mathbb{R}) \cap L^2_m(\mathbb{R}).
\end{aligned}$$

Let $k \in L^1_{m\times p}(\mathbb{R})$. Thus k is an $m \times p$ matrix function of which the (i, j)-th entry $k_{ij} \in L^1(\mathbb{R})$. For each $\varphi \in L^1_p(\mathbb{R})$ and $\psi \in L^2_p(\mathbb{R})$ the convolution products $k \star \varphi$ and $k \star \psi$, see (1.18), are well defined, $k \star \varphi$ belongs to $L^1_m(\mathbb{R})$ and $k \star \psi$ belongs to $L^2_m(\mathbb{R})$. In particular, if $f \in L^1_p(\mathbb{R}) \cap L^2_p(\mathbb{R})$, then $k \star f$ belongs to $L^1_m(\mathbb{R}) \cap L^2_m(\mathbb{R})$. It follows that for a given $k \in L^1_{m\times p}(\mathbb{R})$ the convolution product induces linear maps from the space $L^1_p(\mathbb{R})$ into $L^1_m(\mathbb{R})$, from the space $L^2_p(\mathbb{R})$ into $L^2_m(\mathbb{R})$, and from the space $L^1_p(\mathbb{R}) \cap L^2_p(\mathbb{R})$ into $L^1_m(\mathbb{R}) \cap L^2_m(\mathbb{R})$. The resulting operators will be denoted by K_1, K_2 and K_0, respectively. The proof of the following lemma is standard (see, e.g., page 216 in [6]) and therefore it is omitted.

Lemma A.1. *The operators K_1, K_2 and K_0 are bounded linear operators, and*

$$\text{(A1)} \qquad \|K_\nu\| \leq \kappa \ \ (\nu = 1, 2, 0), \ \ \textit{where} \ \ \kappa = \Big(\sum_{i=1}^{m}\sum_{j=1}^{p} \|k_{ij}\|_1^2\Big)^{1/2}.$$

With $k \in L^1_{m\times p}(\mathbb{R}_+)$ we also associate the Wiener-Hopf operator W and the Hankel operator H defined by

$$\begin{aligned}
(Wf)(t) &= \int_0^\infty k(t-\tau) f(\tau)\, d\tau, \quad 0 \leq t < \infty,\\
(Hf)(t) &= \int_0^\infty k(t+\tau) f(\tau)\, d\tau, \quad 0 \leq t < \infty.
\end{aligned}$$

Using the classical relation between the convolution operator defined by k and the operators W and H (see, e.g., Section XII.2 in [7]) it is easy to see that W and H map the space $L^1_p(\mathbb{R}_+)$ into $L^1_m(\mathbb{R}_+)$, the space $L^2_p(\mathbb{R}_+)$ into $L^2_m(\mathbb{R}_+)$, and the space $L^1_p(\mathbb{R}_+) \cap L^2_p(\mathbb{R}_+)$ into $L^1_m(\mathbb{R}_+) \cap L^2_m(\mathbb{R}_+)$. We denote the resulting operators by W_1, W_2, W_0, and H_1, H_2, H_0, respectively. Lemma A.1 shows that these operators are bounded and

$$\text{(A2)} \qquad \|W_\nu\| \leq \kappa \text{ and } \|H_\nu\| \leq \kappa \ \ (\nu = 1, 2, 0), \text{ where } \kappa = \Big(\sum_{i=1}^{m}\sum_{j=1}^{p} \|k_{ij}\|_1^2\Big)^{1/2}.$$

Furthermore, using the line of reasoning in Lemma XX.2.4 in [7], we have the following corollary.

Corollary A.2. *The Hankel operators H_1, H_2, and H_0 are the limit in operator norm of finite rank operators, and hence compact.*

Next we present an auxiliary result that is used in the proof of Proposition 2.1. Put

$$R(s) = D_R + \int_{-\infty}^{\infty} e^{-st} r(t)\, dt \quad \text{where} \quad r \in L^1_{m\times m}(\mathbb{R}). \tag{A3}$$

By T_R we denote the Wiener-Hopf operator on $L^2_m(\mathbb{R}_+)$ defined by R, that is,

$$(T_R f)(t) = D_R f(t) + \int_0^{\infty} r(t-\tau) f(\tau)\, d\tau, \quad 0 \le t < \infty. \tag{A4}$$

As we know from the first paragraph of this section, the fact that $r \in L^1_{m\times m}(\mathbb{R})$ implies that T_R maps $L^1_m(\mathbb{R}_+) \cap L^2_m(\mathbb{R}_+)$ into itself.

Lemma A.3. *If T_R is invertible as an operator on $L^2_m(\mathbb{R}_+)$, then T_R^{-1} maps the space $L^1_m(\mathbb{R}_+) \cap L^2_m(\mathbb{R}_+)$ in a one-to-one way onto itself.*

Proof. Since T_R is invertible, R admits a canonical factorization (see Section XXX.10 in [8]), and hence we can write $T_R^{-1} = LU$, where L and U are Wiener-Hopf operators on $L^2_m(\mathbb{R}_+)$,

$$(Lf)(t) = D_L f(t) + \int_0^{t} \ell(t-\tau) f(t)\, dt, \quad 0 \le t < \infty, \tag{A5}$$

$$(Uf)(t) = D_U f(t) + \int_t^{\infty} u(t-\tau) f(t)\, dt, \quad 0 \le t < \infty. \tag{A6}$$

Here ℓ and u both belong to $L^1_{m\times m}(\mathbb{R})$, with support of ℓ in $\mathbb{R}_+$ and support of u in $\mathbb{R}_-$. The fact that both ℓ and u belong to $L^1_{m\times m}(\mathbb{R})$ implies that both L and U map $L^1_m(\mathbb{R}_+) \cap L^2_m(\mathbb{R}_+)$ into itself. Hence T_R^{-1} has the same property. Since T_R^{-1} is one-to-one on $L^2_m(\mathbb{R}_+)$, it is also one-to-one on $L^1_m(\mathbb{R}_+) \cap L^2_m(\mathbb{R}_+)$. For $f \in L^1_m(\mathbb{R}_+) \cap L^2_m(\mathbb{R}_+)$, we have $g = T_R f \in L^1_m(\mathbb{R}_+) \cap L^2_m(\mathbb{R}_+)$ and $f = T_R^{-1} T_R f = T_R^{-1} g$. This shows that T_R^{-1} maps $L^1_m(\mathbb{R}_+) \cap L^2_m(\mathbb{R}_+)$ onto $L^1_m(\mathbb{R}_+) \cap L^2_m(\mathbb{R}_+)$. □

Lemma A.4. *Let $k \in L^1_{p\times m}(\mathbb{R}_+)$ and $\tilde{k} \in L^1_{m\times p}(\mathbb{R}_+)$, and let H and $\tilde{H}$ be the corresponding Hankel operators acting from $L^2_m(\mathbb{R}_+)$ into $L^2_p(\mathbb{R}_+)$ and from $L^2_p(\mathbb{R}_+)$ into $L^2_m(\mathbb{R}_+)$, respectively. Let Q be any operator on $L^2_m(\mathbb{R}_+)$ mapping $L^1_p(\mathbb{R}_+) \cap L^2_m(\mathbb{R}_+)$ into itself, and assume that the restricted operator Q_0 acting on $L^1_m(\mathbb{R}_+) \cap L^2_m(\mathbb{R}_+)$ is bounded. If the operator $I - \tilde{H}QH$ is invertible on $L^2_m(\mathbb{R}_+)$, then $I - \tilde{H}QH$ maps the space $L^1_m(\mathbb{R}_+) \cap L^2_m(\mathbb{R}_+)$ in a one-to-one way onto itself.*

Proof. We know that H maps $L^1_p(\mathbb{R}_+) \cap L^p_2(\mathbb{R}_+)$ into $L^1_m(\mathbb{R}_+) \cap L^2_m(\mathbb{R}_+)$. Furthermore the same holds true for $\tilde{H}$ with the role of p and m interchanged. Hence our hypothesis on Q implies that $I - \tilde{H}QH$ maps the space $L^1_m(\mathbb{R}_+) \cap L^2_m(\mathbb{R}_+)$ into itself. Let M_0 be the corresponding restricted operator. We have to prove that M_0 is invertible. Note that Corollary A.2 implies that M_0 is equal to the identity operator minus a compact operator, and hence M_0 is a Fredholm operator of index zero. Therefore, in order to prove that M_0 is invertible, it suffices to show that $\operatorname{Ker} M_0$ consists of the zero element only. Assume not. Then there exists a non-zero f in $L^1_m(\mathbb{R}_+) \cap L^2_m(\mathbb{R}_+)$ such that $M_0 f = 0$. The fact that f belongs to $L^1_m(\mathbb{R}_+) \cap L^2_m(\mathbb{R}_+) \subset L^2_m(\mathbb{R}_+)$ shows that $0 = M_0 f = (I - \tilde{H}QH)f$. But $I - \tilde{H}QH$ is assumed to be invertible. Hence f must be zero. Thus M_0 is invertible. □

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