

\leq_{SP} CAN HAVE INFINITELY MANY CLASSES

1065

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ABSTRACT. Building off of recent results on Keisler's order, we show that consistently, \leq_{SP} has infinitely many classes. In particular, we define the property of $\leq k$ -type amalgamation for simple theories, for each $2 \leq k < \omega$. If we let $T_{n,k}$ be the theory of the random k -ary, n -clique free random hyper-graph, then $T_{n,k}$ has $\leq k - 1$ -type amalgamation but not $\leq k$ -type amalgamation. We show that consistently, if T has $\leq k$ -type amalgamation then $T_{k+1,k} \not\leq_{\text{SP}} T$, thus producing infinitely many \leq_{SP} -classes. The same construction gives a simplified proof of the theorem from [10] that consistently, the maximal \leq_{SP} -class is exactly the class of non-simple theories. Finally, we show that consistently, if T has $< \aleph_0$ -type amalgamation, then $T \leq_{\text{SP}} T_{\text{rg}}$, the theory of the random graph.

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§ 0. INTRODUCTION

Convention 0.1. T is always a complete theory in a countable language. We will fix a monster model $\mathfrak{C} \models T$ and work within it so $\mathfrak{C} = \mathfrak{C}_T$ but if T is clear from the context we do not mention it.

The first author introduced the following definition in [10], although he had previously investigated the phenomenon in [8] (without giving it a name):

Definition 0.2. Suppose $\lambda \geq \theta$. Define $\text{SP}_T(\lambda, \theta)$ to mean: for every $M \models T$ of size λ , there is a θ -saturated $N \models T$ of size λ extending M .

In this paper, we will restrict to the following special case:

Definition 0.3. 1) Say that (θ, λ) is a nice pair if θ is a regular cardinal and $\lambda \geq \theta$ has $\lambda = \lambda^{\aleph_0}$.

2) Given T_0, T_1 complete first order theories, say that $T_0 \leq_{\text{SP}} T_1$ if whenever (θ, λ) is a nice pair, if $\text{SP}_{T_1}(\lambda, \theta)$ then $\text{SP}_{T_0}(\lambda, \theta)$.

Thus, \leq_{SP} is a pre-ordering of theories which measures how difficult it is to build saturated models.

In [8], the first author proves: the stable theories are the minimal SP-class, and non-simple theories are always maximal. In [10], the first author additionally proves that consistently, non-simple theories are exactly the maximal class.

Recently, there has been substantial progress on Keisler's order \trianglelefteq , another pre-ordering of theories which measures how difficult it is to build saturated models; see for instance [6] and [7] by the first author and Malliaris. In particular, in [7] it is shown that Keisler's order has infinitely many classes, these being separated by certain amalgamation properties. In this paper we use similar ideas to continue investigation of \leq_{SP} .

In §2 we summarize what is already known on \leq_{SP} .

In §3, we introduce several amalgamation-related properties of forcing notions (Definition 2.2), and show that it is preserved under iterations in a suitable sense (Theorem 2.5). In light of this, we define a class of forcing axioms (Definition 2.7); these are closely related to the forcing axiom Ax_{μ_0} , defined by the first author in [9] and used to demonstrate the consistent maximality of non-simple theories under \leq_{SP} in [10]. However, the forcing axioms we develop are designed specifically for what we want and have been simplified somewhat.

In §4, we define and prove some helpful facts about non-forking diagrams of models.

In §5, we introduce, for each $3 \leq k < \omega$, a property of simple theories called $< k$ -type amalgamation (Definition 4.1), and discuss some of its properties. For example, if for $n > k$ we let $T_{n,k}$ be the theory of the k -ary, n -clique free hypergraph, then if $k \geq 3$, $T_{n,k}$ has $< k$ -type amalgamation but not $< k+1$ -type amalgamation. We also show that if T has $< \aleph_0$ -type amalgamation (i.e., $< k$ -type amalgamation for all k), then $\text{SP}_T(\lambda, \theta)$ holds whenever we have that there is some $\theta \leq \mu \leq \lambda$ with $\mu^{<\theta} \leq \lambda$ and $2^\mu \geq \lambda$ (Theorem 4.6). This implies that if the singular cardinals hypothesis holds, then whenever T has $< \aleph_0$ -type amalgamation, then $T \leq_{\text{SP}} T_{\text{rg}}$, where T_{rg} is the theory of the random graph.

In §6, we put everything together to show that consistently, for all $k \geq 3$, if T has the $< k$ -type amalgamation property, then $T_{k,k-1} \not\leq_{\text{SP}} T$ (Theorem 5.4). In particular, for $k < k'$, $T_{k+1,k} \not\leq_{\text{SP}} T_{k'+1,k'}$; this is similar to the situation for Keisler's order in [7].

By a forcing notion, we mean a pre-ordered set (P, \leq^P) such that P has a least element 0^P (pre-order means that \leq^P is transitive); we are using the convention where $p \leq q$ means q is a stronger condition than p . That is, when we force by P we add a generic ideal, rather than a generic filter. Thus, a finite sequence $(p_i : i < k)$ from P is compatible if it has an upper bound in P .

§ 1. BACKGROUND

The following theorem is closely related to the classical Hewitt-Marczewski-Pondiczery theorem of topology; the special case $\theta = \aleph_0$ is implied by Theorem 8 of [1], and the general case is also noted there. It will be central for our investigations.

Theorem 1.1. *Suppose $\theta \leq \mu \leq \lambda$ are infinite cardinals such that θ is regular, $\mu = \mu^{<\theta}$, and $\lambda \leq 2^\mu$. Then there is a sequence $(\mathbf{f}_\gamma : \gamma < \mu)$ from ${}^\lambda\mu$ such that for all partial functions f from λ to μ of cardinality less than θ , there is some $\gamma < \mu$ such that \mathbf{f}_γ extends f . If $\lambda > 2^\mu$ then this fails, in fact, there is no sequence $(\mathbf{f}_\gamma : \gamma < \mu)$ from ${}^\lambda 2$ such that for all partial functions f from λ to 2 of cardinality less than θ , there is some $\gamma < \mu$ such that \mathbf{f}_γ extends f .*

We will also want the following technical device, which will allow us to apply Theorem 1.1 to conclude $\text{SP}_T(\lambda, \theta)$ holds. Here is the idea: suppose $M \models T$ with $|M| \leq \lambda$, and we want to find some θ -saturated $N \succeq M$ with $|N| \leq \lambda$. To do this, we will always first find some $N_0 \succeq M$ with $|N_0| \leq \lambda$ which realizes every type over M of cardinality less than θ , and then we iterate θ -many times. The key step is to find N_0 , and the following definitions capture when this is possible.

Definition 1.2. 1) Suppose T is a simple theory, θ is a regular uncountable cardinal, and $M_* \preceq M \models T$. then let Γ_{M, M_*}^θ be the forcing notion of all partial types $p(x)$ over M of cardinality less than θ which do not fork over M_* , ordered by inclusion, where x is a single variable. Also, if $p_*(x)$ is a complete type over M_* , then let $\Gamma_{M, p_*}^\theta \subseteq \Gamma_{M, M_*}^\theta$ be the set of all $p(x)$ which extend $p_*(x)$.

2) Given (θ, λ) a nice pair and given μ with $\theta \leq \mu \leq \lambda$, define $\text{SP}_T^1(\lambda, \mu, \theta)$ to mean: for every $M \models T$ of size $\leq \lambda$ and for every countable $M_* \preceq M$, there are complete types $p_\gamma(x) : \gamma < \mu$ over M which do not fork over M_* , such that whenever $p(x) \in \Gamma_{M, M_*}^\theta$, then $p(x) \subseteq p_\gamma(x)$ for some $\gamma < \mu$.

3) Given in addition a fixed countable $M_* \models T$ and type $p_*(x)$ over M_* , define $\text{SP}_{T, p_*}^1(\lambda, \mu, \theta)$ similarly: whenever $M \succeq M_*$ has size at most λ , there are complete, non-forking extensions $p_\gamma(x) : \gamma < \mu$ of $p_*(x)$ to M , such that whenever $p(x) \in \Gamma_{M, p_*}^\theta$, then $p(x) \subseteq p_\gamma(x)$ for some $\gamma < \mu$.

Note that if $\mu \geq 2^{\aleph_0}$, then $\text{SP}_T^1(\lambda, \mu, \theta)$ if and only if $\text{SP}_{T, p_*}^1(\lambda, \mu, \theta)$ for every complete type $p_*(x)$ over a countable model M_* (the forward direction is unconditional in μ , but for the reverse direction, we need to concatenate witnesses for each $p_*(x)$, of which there are 2^{\aleph_0} -many). In particular this holds when $\mu = \lambda$, since $\lambda^{\aleph_0} = \lambda$.

The following is an important example. Let T_{rg} be the theory of the random graph, i.e. the model completion of the theory of graphs. T_{rg} admits quantifiers, and given $A \subseteq B$ and $p(x) \in S(B)$, p forks over A if and only if p is realized in $B \setminus A$.

Example 1.3. Suppose (θ, λ) is a nice pair and suppose μ is a cardinal with $\mu = \mu^{<\theta}$ and $\theta \leq \mu \leq \lambda$. Then $\text{SP}_{T_{\text{rg}}}^1(\lambda, \mu, \theta)$ holds if and only if $\lambda \leq 2^\mu$; and this is equivalent to $\text{SP}_{T_{\text{rg}}, p_*}^1(\lambda, \mu, \theta)$ holding for some or any nonalgebraic complete type $p_*(x)$ over a countable model M_* .

Proof. Suppose $M \models T$ has size $\leq \lambda$.

Then the non-algebraic types in $S^1(A)$ correspond naturally to functions from A to 2 , and so this is just a restatement of Theorem 1.1. \square

Theorem 1.4. *Suppose T is a simple theory (in a countable language, as always see 0.1).*

Suppose (θ, λ) is a nice pair:

- (A) *If $\text{SP}_T^1(\lambda, \lambda, \theta)$, then $\text{SP}_T(\lambda, \theta)$.*
- (B) *If $T = T_{rg}$ and $\text{SP}_T(\lambda, \theta)$ then $\text{SP}_T(\lambda, \lambda, \theta)$.*
- (C) *Suppose $p_*(x)$ is a complete type over a countable model $M_* \models T$, and $\text{SP}_{T,p_*}^1(\lambda, \lambda, \theta)$ holds, and $\text{cof}(\lambda) < \theta$. Then for some μ with $\theta \leq \mu < \lambda$, $\text{SP}_{T,p_*}^1(\lambda, \mu, \theta)$ holds.*
- (D) *Suppose $2^{\aleph_0} < \text{cof}(\lambda) < \theta$. Suppose $\text{SP}_T^1(\lambda, \lambda, \theta)$ holds. Then $\text{SP}_T^1(\lambda, \mu, \theta)$ holds for some $\mu < \lambda$.*

Proof. (A) Suppose $M \models T$ has size $\leq \lambda$. Using $\text{SP}_T^1(\lambda, \lambda, \theta)$, we can find $N \succeq M$ of size λ , such that every partial type $p(x)$ over M of cardinality less than θ is realized in N , using every type in M does not fork over some countable submodel of M (we are also using $\lambda = \lambda^{\aleph_0}$, so there are only λ -many countable elementary submodels M_* of M). If we iterate this θ -many times then we will get a θ -saturated model of T .

(B): Suppose $M \models T_{rg}$ has size $\leq \lambda$ and $M_* \preceq M$ is countable. Choose $N \succeq M$, a θ -saturated model of size λ . Let $a_\alpha : \alpha < \lambda$ enumerate N . For each $\alpha < \lambda$ let $p_\alpha(x)$ be the type over M asserting $x \neq a$ for each $a \in M$, and $R(x, a) \in p_\alpha$ if and only if $R(a_\alpha, a)$ holds for each $a \in M$. This is a complete type over M which does not fork over \emptyset . Then $\{p_\alpha(x) : \alpha < \lambda\}$ along with all algebraic types over M_* witness $\text{SP}_{T_{rg}}^1(\lambda, \lambda, \theta)$.

(C): Suppose towards a contradiction that $\text{SP}_{T,p_*}^1(\lambda, \mu, \theta)$ failed for all $\theta \leq \mu < \lambda$. Write $\kappa = \text{cof}(\lambda)$, and let $(\mu_\beta : \beta < \kappa)$ be a cofinal sequence of cardinals in λ with each $\mu_\beta \geq \theta$. For each $\beta < \kappa$, choose $M_\beta \succeq M_*$ with $|M_\beta| \leq \lambda$, witnessing that $\text{SP}_{T,p_*}^1(\lambda, \mu_\beta, \theta)$ fails. We can suppose that $(M_\beta : \beta < \kappa)$ is independent over M_* .

Let $N \models T$ have size $\leq \lambda$ such that each $M_\beta \preceq N$. Then by $\text{SP}_{T,p_*}^1(\lambda, \lambda, \theta)$, we can find $(q_\alpha(x) : \alpha < \lambda)$ such that each $q_\alpha(x)$ extends $p_*(x)$, does not fork over M_* and whenever $q(x) \in \Gamma_{N,p_*}^\theta$, then $q(x) \subseteq q_\alpha(x)$ for some $\alpha < \lambda$.

For each $\beta < \kappa$, we can by hypothesis choose $p_\beta(x) \in \Gamma_{M_\beta,p_*}^\theta$ such that $p_\beta(x) \not\subseteq q_\alpha(x)$ for any $\alpha < \mu_\beta$; note that still $p(x) \supseteq p_*(x)$. By the independence theorem for simple theories, $p(x) := \bigcup_{\beta < \kappa} p_\beta(x)$ does not fork over M_* . Hence $p(x) \subseteq q_\alpha(x)$

for some $\alpha < \lambda$. Choose $\beta < \kappa$ with $\alpha < \mu_\beta$; then this implies that $p_\beta(x) \subseteq q_\alpha(x)$, a contradiction.

(D): Enumerate, up to isomorphism, all types over countable models $(p_\alpha(x, M_\alpha) : \alpha < 2^{\aleph_0})$. For each $\alpha < 2^{\aleph_0}$, $\text{SP}_{T,p_\alpha}^1(\lambda, \lambda, \theta)$ holds, so by (C) there is $\mu_\alpha < \lambda$ such that $\text{SP}_{T,p_\alpha}^1(\lambda, \mu_\alpha, \theta)$ holds. Let μ be the supremum of 2^{\aleph_0} and $\{\mu_\alpha : \alpha < 2^{\aleph_0}\}$; then $\mu < \lambda$ and easily $\text{SP}_T^1(\lambda, \mu, \theta)$ holds. \square

Finally, the following theorem is a collection of most of what has been previously known on \leq_{SP} .

Theorem 1.5. *Suppose T is a complete first order theory in a countable language.*

Suppose (θ, λ) is a nice pair:

- (A) If $\lambda = \lambda^{<\theta}$, then $\text{SP}_T(\lambda, \theta)$ holds; if T is non-simple then the converse is true as well. Thus non-simple theories are all \leq_{SP} -maximal. (See [10] Conclusion 4.6 and Theorem 4.7.)
- (B) T_{rg} is the \leq_{SP} -minimal unstable theory. (Theorem 4.8 of [10].)
- (C) If T is stable, then $\text{SP}_T(\lambda, \theta)$ holds (see [10] Theorem 4.7(2)).
- (D) If λ is a strong limit with $\text{cof}(\lambda) < \theta$ (and as $\lambda = \lambda^{\aleph_0}$, we have $\aleph_0 < \text{cof}(\lambda)$), and if $\text{SP}_T(\lambda, \theta)$ holds, then T is stable. (Theorem 4.7(4) of [10].) Thus the stable theories are exactly the minimal \leq_{SP} -class. Also, under GCH, all unstable theories are maximal.
- (E) If $\theta \leq \mu \leq \lambda$ and $\mu^{<\theta} = \mu$ and $\lambda \leq 2^\mu$, then $\text{SP}_{T_{\text{rg}}}(\lambda, \theta)$ holds. (This is Exercise VIII 4.5 in [8].)
- (F) It is consistent that there exists a nice pair (θ, λ) such that for all simple T , $\text{SP}_T(\theta, \lambda)$ holds. Hence, it is consistent that the non-simple theories are exactly the \leq_{SP} -maximal class. (This is Theorem 4.10 of [10].)

For the reader's convenience, we prove (A) through (E), making use of the language of SP^1 . Theorem (F) will be a special case of our main theorem, namely Theorem 5.4(B).

Proof. (A): By standard arguments, if $\lambda^{<\theta} = \lambda$ then $\text{SP}_T(\lambda, \theta)$ holds. Suppose T is non-simple, and $\text{SP}_T(\lambda, \theta)$ holds, and suppose towards a contradiction that $\lambda^{<\theta} > \lambda$. Choose a formula $\phi(x, y)$ with the tree property (possibly y is a tuple).

Let $\kappa < \theta$ be least such that $\lambda^\kappa > \lambda$. Choose $M \models T$ and $(a_\eta : \eta \in <^\kappa \lambda)$ such that for all $\eta \in <^\kappa \lambda$, $p_\eta(x) := \{\phi(x, a_{\eta \upharpoonright \beta}) : \beta < \kappa\}$ is consistent, and for all $\eta \in <^\kappa \lambda$ and for all $\alpha < \beta < \lambda$, $\phi(x, a_{\eta \upharpoonright \alpha})$ and $\phi(x, a_{\eta \upharpoonright \beta})$ are inconsistent. Note that each $|p_\eta(x)| < \theta$; but clearly if $N \succeq M$ realizes each $p_\eta(x)$ then $|N| \geq \lambda^\kappa > \lambda$.

(B): Suppose T is unstable; we show $T_{\text{rg}} \leq_{\text{SP}} T$. By (A), this is true if T is non-simple, so we can suppose that T is simple, hence has the independence property via some formula $\phi(x, y)$. Now suppose (θ, λ) is a nice pair. By Theorem 1.4(B), it suffices to show that if $\text{SP}_T(\lambda, \theta)$ holds, then $\text{SP}_{T_{\text{rg}}}^1(\lambda, \lambda, \theta)$ holds. Choose some $(a_\alpha : \alpha < \lambda)$ from \mathfrak{C} such that for all $\mathbf{f} : \lambda \rightarrow 2$, $\{\phi(x, a_\alpha)^{\mathbf{f}(\alpha)} : \alpha < \lambda\}$ is consistent. By $\text{SP}_T(\lambda, \theta)$ we can find some θ -saturated $M \prec \mathfrak{C}$ with $|M| \leq \lambda$ and each $a_\alpha \in M$.

Suppose $N \models T_{\text{rg}}$ has cardinality λ , say $N = \{b_\alpha : \alpha < \lambda\}$ without repetitions. For each $c \in M$, let $p_c(x)$ be the complete nonalgebraic type over N , defined by putting $R(x, b_\alpha) \in p_c(x)$ if and only if $M \models \phi(c, a_\alpha)$. Then recalling the proof of 1.3 this witnesses $\text{SP}_{T_{\text{rg}}}^1(\lambda, \lambda, \theta)$ holds (since $|M| \leq \lambda$).

(C): Suppose T is stable. It suffices to show that $\text{SP}_T^1(\lambda, \lambda, \theta)$ holds. But this is clear: given $M \models T$ of size $\leq \lambda$ and $M_* \preceq M$ countable, there are $\leq 2^{\aleph_0} \leq \lambda^{\aleph_0} = \lambda$ many types over M that do not fork over M_* , seeing as types over M_* are stationary.

(D): Suppose towards a contradiction that $\text{SP}_T(\lambda, \theta)$ holds for some unstable T . Then in particular $\text{SP}_{T_{\text{rg}}}(\lambda, \theta)$ holds. Let $p_*(x)$ be a complete non-algebraic type over some countable $M_* \models T_{\text{rg}}$. By Theorem 1.4 we can find $\theta \leq \mu < \lambda$ such that $\text{SP}_{T_{\text{rg}}, p_*}^1(\lambda, \mu, \theta)$ holds. By possibly replacing μ with $\mu^{<\theta}$ we can suppose $\mu = \mu^{<\theta}$. Then this contradicts Example 1.3, since $2^\mu < \lambda$.

(E): By Example 1.3 and Theorem 1.4(A).

(F): See [10] or [3]. □

If the singular cardinals hypothesis holds, then we can say more. Recall that

Definition 1.6. The singular cardinals hypothesis states that if λ is singular and $2^{\text{cof}(\lambda)} < \lambda$, then $\lambda^{\text{cof}(\lambda)} = \lambda^+$. (Note that $2^{\text{cof}(\lambda)} \neq \lambda$ since $\text{cof}(2^\kappa) > \kappa$ for all cardinals κ , by König’s theorem.)

The failure of the singular cardinals hypothesis is a large cardinal axiom; see Chapter 5 of [5].

We want the following simple lemma.

Lemma 1.7. *Suppose the singular cardinals hypothesis holds. Suppose θ is regular, $\lambda \geq \theta$, $\lambda^{<\theta} > \lambda$, and $2^{<\theta} \leq \lambda$. Then for every $\mu < \lambda, \mu^{<\theta} < \lambda$. Further, λ is singular of cofinality $< \theta$.*

Proof. First of all, note that $2^{<\theta} < \lambda$, as otherwise $\lambda^{<\theta} = \lambda$.

Now suppose towards a contradiction there were some $\mu < \lambda$ with $\mu^{<\theta} \geq \lambda$; then necessarily $\mu^{<\theta} > \lambda$, as otherwise again $\lambda^{<\theta} = \lambda$. We can choose μ least with $\mu^{<\theta} > \lambda$. Let $\kappa < \theta$ be least such that $\mu^\kappa > \lambda$.

Note that $2^\kappa < \mu$, as otherwise $2^\kappa = (2^\kappa)^\kappa \geq \mu^\kappa > \lambda$, contradicting $2^{<\theta} < \lambda$. Thus, by a consequence of the singular cardinals hypothesis (Theorem 5.22(ii)(b),(c) of [5]), $\mu^\kappa \leq \mu^+$. But since $\mu < \lambda, \mu^+ \leq \lambda$, so this is a contradiction.

To finish, suppose towards a contradiction that $\text{cof}(\lambda) \geq \theta$. Then $\lambda^{<\theta} = \lambda + \sup\{\mu^{<\theta} : \mu < \lambda\} = \lambda$, a contradiction. \square

Theorem 1.8. *Suppose the singular cardinals hypothesis holds, and suppose (θ, λ) is a nice pair. Then $\text{SP}_T^1(\lambda, \lambda, \theta)$ holds if and only if T is stable, or $\lambda = \lambda^{<\theta}$, or else T is simple and for every complete type $p_*(x)$ over a countable model $M_* \models T$, there is some μ with $\theta \leq \mu < \lambda$ and with $\mu^{<\theta} = \mu$ and $2^\mu \geq \lambda$, such that $\text{SP}_{T,p_*}^1(\lambda, \mu, \theta)$ holds.*

Proof. If T is stable or $\lambda = \lambda^{<\theta}$, then $\text{SP}_T^1(\lambda, \lambda, \theta)$ holds. If T is non-simple and $\lambda < \lambda^{<\theta}$, then $\text{SP}_T^1(\lambda, \lambda, \theta)$ fails by Theorem 1.5(A) and Theorem 1.4(A). Thus we can assume T is unstable, simple (hence has the independence property) and $\lambda < \lambda^{<\theta}$.

It suffices to show that $\text{SP}_T^1(\lambda, \lambda, \theta)$ holds if and only if for every complete type $p_*(x)$ over a countable model M_* , there is some $\theta \leq \mu < \lambda$ with $\mu^{<\theta} = \mu$ and $2^\mu \geq \lambda$, such that $\text{SP}_{T,p_*}^1(\lambda, \mu, \theta)$ holds.

Suppose first $\text{SP}_T^1(\lambda, \lambda, \theta)$ holds, and $p_*(x)$ is given. Since T is unstable with the independence property, $\text{SP}_T^1(\lambda, \lambda, \theta)$ clearly implies that $2^{<\theta} \leq \lambda$. Hence, by Lemma 1.7, λ is singular with $\text{cof}(\lambda) < \theta$, and there are cofinally many $\mu < \lambda$ with $\mu^{<\theta} = \mu$. By Theorem 1.5(D), λ is not a strong limit. Thus by Theorem 1.4(C), we can find $\theta \leq \mu < \lambda$ such that $\mu = \mu^{<\theta}$ and $2^\mu \geq \lambda$ and $\text{SP}_{T,p_*}^1(\lambda, \mu, \theta)$ holds.

Conversely, we have in particular that each $\text{SP}_{T,p_*}^1(\lambda, \lambda, \theta)$ holds; since $\lambda = \lambda^{\aleph_0} \geq 2^{\aleph_0}$ we get that $\text{SP}_T^1(\lambda, \lambda, \theta)$ holds. \square

§ 2. FORCING AXIOMS

In this section, we introduce the forcing axioms which will produce the desired behavior in SP. It is well-known that the countable chain condition is preserved under finite support iterations; we aim to find generalizations to the κ -closed, κ^+ -c.c. context.

Definition 2.1. For a regular cardinal θ and sets X, Y , define $P_{XY\theta}$ to be the forcing notion of all partial functions from X to Y of cardinality less than θ , ordered by inclusion. Note that $P_{XY\theta}$ has the $|Y|^{<\theta}$ -c.c. by the Δ -system lemma and is θ -closed.

Definition 2.2. Suppose P, Q are forcing notions, and suppose $k \geq 3$ is a cardinal (typically finite). Then say that $P \rightarrow_k Q$ if there is a dense subset P_0 of P and a map $F : P_0 \rightarrow Q$ such that for all sequences $(p_i : i < i_*)$ from P_0 with $i_* < k$, if $(F(p_i) : i < i_*)$ is compatible in Q (that is, has a common upper bound), then $(p_i : i < i_*)$ has a least upper bound in P ; we write $F : (P, P_0) \rightarrow_k Q$. Say that $P \rightarrow_k^w Q$ (where w stands for weak) if there is a map $F : P \rightarrow Q$ such that whenever $(p_i : i < i_*)$ is a sequence from P with $i_* < k$, if $(F(p_i) : i < i_*)$ is compatible in Q , then $(p_i : i < i_*)$ is compatible in P .

Suppose P is a forcing notion, $\aleph_0 < \theta \leq \mu$ are cardinals with θ regular, and $3 \leq k \leq \theta$ is a cardinal (often finite). Then say that P has the $(< k, \mu, \theta)$ -amalgamation property if every ascending chain from P of length less than θ has a least upper bound in P , and for some set X , $P \rightarrow_k P_{X\mu\theta}$.

For example, $P_{X\mu\theta}$ has the $(< k, \mu, \theta)$ -amalgamation property. The following lemma sums up several obvious facts.

Lemma 2.3. *Suppose $\aleph_0 < \theta \leq \mu$ are cardinals with $\theta = \text{cf}(\lambda) > \aleph_0$, and $3 \leq k \leq \theta$ is a cardinal.*

- (1) *If $P \rightarrow_k Q$ and $Q \rightarrow_k^w Q'$ then $P \rightarrow_k Q'$.*
- (2) *If P, Q have the $(< k, \mu, \theta)$ -amalgamation property, then P forces that \dot{Q} has the $(< k, |\mu|, \theta)$ -amalgamation property. (We write $|\mu|$ because possibly P collapses μ .) (This is where we use $k \leq \theta$.)*
- (3) *Suppose P has the $(< k, \mu, \theta)$ -amalgamation property for some $k \geq 3$. Then P is θ -closed (hence $< \theta$ -distributive) and $(\mu^{<\theta})^+$ -c.c.*
- (4) *If P is θ -closed and has the least upper bound property, then P has the $(< k, \mu, \theta)$ -amalgamation property if and only if $P \rightarrow_k^w P_{\lambda\mu\theta}$ for some λ .*

We note the following:

Lemma 2.4. *Suppose $\aleph_0 < \theta \leq \mu$ are cardinals with θ regular, and $3 \leq k \leq \theta$. Then P has the $(< k, \mu, \theta)$ -amalgamation property if and only if P has the $(< k, \mu^{<\theta}, \theta)$ -amalgamation property.*

Proof. Define $\mu' = \mu^{<\theta}$, and let λ be a cardinal. It suffices to show there is a cardinal λ' such that $P_{\lambda\mu'\theta} \rightarrow_k^w P_{\lambda'\mu\theta}$, by Lemma 2.3(1). Write $Y' = {}^{<\theta}\mu$; it suffices to find a set X' such that $P_{\lambda Y'\theta} \rightarrow_k^w P_{X'\mu\theta}$.

Let $X' = \lambda \times (\theta + 1)$. Define $F : P_{\lambda Y'\theta} \rightarrow P_{X'\mu\theta}$ as follows. Let $f \in P_{\lambda Y'\theta}$ be given. Let $\text{dom}(F(f)) = \{((\gamma, \delta) : \gamma \in \text{dom}(f) \text{ and either } \delta < \text{dom}(f(\gamma)) \text{ or } \delta = \theta)\}$.

Define $F(f)(\gamma, \delta) = f(\gamma)(\delta)$ if $\delta < \theta$, and otherwise $F(f)(\gamma, \theta) = \text{dom}(f(\gamma))$. Clearly this works. \square

The following is key; it states that the $(< k, \mu, \theta)$ -amalgamation property is preserved under $< \theta$ -support iterations. Note that it follows that the $(< k, \mu, \theta)$ -amalgamation property is preserved under $< \theta$ -support products.

Theorem 2.5. *Suppose θ is a regular uncountable cardinal, $\mu \geq \theta$ and $3 \leq k \leq \theta$. Suppose $(P_\alpha : \alpha \leq \alpha_*)$, $(\dot{Q}_\alpha : \alpha < \alpha_*)$ is a $< \theta$ -support forcing iteration, such that each P_α forces that \dot{Q}_α has the $(< k, |\mu|, \theta)$ -amalgamation property. Then P_{α_*} has the $(< k, \mu, \theta)$ -amalgamation property.*

Proof. Let λ be large enough.

Inductively, choose $(P_\alpha^0 : \alpha \leq \alpha_*, \dot{Q}_\alpha^0 : \alpha < \alpha_*)$ a $< \theta$ -support forcing iteration, and $(\dot{F}_\alpha : \alpha < \alpha_*)$, such that each P_α^0 is dense in P_α (and hence $< \theta$ -distributive), and each P_α forces $\dot{F}_\alpha : (\dot{Q}_\alpha, \dot{Q}_\alpha^0) \rightarrow_k \dot{P}_{\lambda\mu\theta}$. There is a subtlety here: \dot{Q}_α^0 needs to be a P_α^0 -name for \dot{Q}_α , not just a P_α -name. This follows from a general fact that if P is a forcing notion and P_0 is dense in P then any P -name is forced to be equivalent to a P_0 -name; this can be checked by an induction on the foundation rank of P -names.

By revising the choice of \dot{Q}_α^0 and \dot{F}_α , we can suppose \dot{Q}_α^0 contains the minimal element $0^{\dot{Q}_\alpha}$ of \dot{Q}_α and we can suppose \dot{F}_α is forced to take $0^{\dot{Q}_\alpha}$ to the empty function in $\dot{P}_{\lambda\mu\theta}$.

Claim 2.6. *For each $\gamma_* < \theta$, if $(p_\gamma : \gamma < \gamma_*)$ is an ascending chain from P_{α_*} , then it has a least upper bound p in P_{α_*} , such that $\text{supp}(p) \subseteq \bigcup_{\gamma < \gamma_*} \text{supp}(p_\gamma)$.*

Proof. By induction on $\alpha \leq \alpha_*$, we construct $(q_\alpha : \alpha \leq \alpha_*)$ such that each $q_\alpha \in P_\alpha$ with $\text{supp}(q_\alpha) \subseteq \bigcup_{\gamma < \gamma_*} \text{supp}(p_\gamma) \cap \alpha$, and for $\alpha < \beta \leq \alpha_*$, $q_\beta \upharpoonright_\alpha = q_\alpha$, and for each $\alpha \leq \alpha_*$, q_α is a least upper bound of $(p_\gamma \upharpoonright_\alpha : \gamma < \gamma_*)$ in P_α . At limit stages there is nothing to do; so suppose we have defined q_α . If $\alpha \notin \bigcup_{\gamma < \gamma_*} \text{supp}(p_\gamma)$ then

let $q_{\alpha+1} = q_\alpha \frown (0^{\dot{Q}_\alpha})$. Otherwise, since q_α forces that $(p_\gamma(\alpha) : \gamma < \gamma_*)$ is an ascending chain from \dot{Q}_α , we can find \dot{q} , a P_α -name for an element of \dot{Q}_α , such that q_α forces \dot{q} is the least upper bound. Let $q_{\alpha+1} = q_\alpha \frown (\dot{q})$. \square

Now suppose $p \in P_{\alpha_*}^0$. Note that $\text{supp}(p) \in [\alpha_*]^{<\theta}$.

By a similar proof to the claim we can find, for each $n < \omega$, elements $\mathbf{q}_n(p) \in P_{\alpha_*}^0$ with $\mathbf{q}_0(p) = p$, so that for all $n < \omega$:

- $\mathbf{q}_{n+1}(p) \geq \mathbf{q}_n(p)$;
- For all $\alpha < \alpha_*$, $\mathbf{q}_{n+1}(p) \upharpoonright_\alpha$ decides $\dot{F}_\alpha(\mathbf{q}_n(p)(\alpha))$. (This is automatic whenever $\alpha \notin \text{supp}(\mathbf{q}_n)$, since then P forces that $\dot{F}_\alpha(\mathbf{q}_n(p)(\alpha)) = \emptyset$.)

So we can choose $f_{n,\alpha,p} \in P_{\lambda\mu\sigma}$ such that each $\mathbf{q}_{n+1}(p) \upharpoonright_\alpha$ forces that $\dot{F}_\alpha(\mathbf{q}_n(p)(\alpha)) = \check{f}_{n,\alpha,p}$.

Let $\mathbf{q}_\omega(p) \in P$ be the least upper bound of $(\mathbf{q}_n(p) : n < \omega)$, which is possible by the claim. Let $P^0 = \{\mathbf{q}_\omega(p) : p \in P_{\alpha_*}^0\}$. For each $q \in P^0$, choose $\mathbf{p}(q) \in P_{\alpha_*}^0$ such that $q = \mathbf{q}_\omega(\mathbf{p}(q))$. For each $n < \omega$, let $\mathbf{p}_n(q) = \mathbf{q}_n(\mathbf{p}(q))$, and for each $\alpha < \alpha_*$, let $f_{n,\alpha,q} = f_{n,\alpha,\mathbf{p}(q)}$.

Thus we have arranged that for all $q \in P^0$, q is the least upper bound of $(\mathbf{p}_n(q) : n < \omega)$, and for all $n < \omega$ and $\alpha < \alpha_*$, $\mathbf{p}_{n+1}(q) \upharpoonright_\alpha$ forces that $\dot{F}_\alpha(\mathbf{p}_n(q)(\alpha)) = \dot{f}_{n,\alpha}(q)$.

Write $X = \omega \times \alpha_* \times \lambda$. Choose $F : P^0 \rightarrow P_{X\mu\theta}$ so that for all $q, q' \in P^0$, if $F(q)$ and $F(q')$ are compatible, then for all $n < \omega$ and for all $\alpha < \alpha_*$, $f_{n,\alpha,q}$ and $f_{n,\alpha,q'}$ are compatible. For instance, let the domain of $F(q)$ be the set of all (n, α, β) such that β is in the domain of $f_{n,\alpha,q}$, and let $F(q)(n, \alpha, \beta) = f_{n,\alpha,q}(\beta)$.

Now suppose $(q_i : i < i_*)$ is a sequence from P^0 with $i_* < k$, such that $(F(q_i) : i < i_*)$ are compatible. Write $\Gamma = \bigcup_{i < i_*, n < \omega} \text{supp}(\mathbf{p}_n(q_i))$.

By induction on $\alpha \leq \alpha_*$, we construct a least upper bound s_α to $(\mathbf{p}_n(q_i) \upharpoonright_\alpha : i < i_*, n < \omega)$ in P_α , such that $\text{supp}(s_\alpha) \subseteq \Gamma \cap \alpha$, and for $\alpha < \alpha'$, $s_{\alpha'} \upharpoonright_\alpha = s_\alpha$.

Limit stages of the induction are clear. So suppose we have constructed s_α . If $\alpha \notin \Gamma$ clearly we can let $s_{\alpha+1} = s_\alpha \wedge (0^{\dot{Q}_\alpha})$; so suppose instead $\alpha \in \Gamma$. Let $n < \omega$ be given. Then $(f_{n,\alpha,q_i} : i < i_*)$ are compatible, and s_α forces that $\dot{F}_\alpha(\mathbf{p}_n(q_i)(\alpha)) = \dot{f}_{n,\alpha,q_i}$ for each $i < i_*$, since $\mathbf{p}_{n+1}(q_i) \upharpoonright_\alpha$ does. Thus s_α forces that $(\mathbf{p}_n(q_i)(\alpha) : i < i_*)$ has a least upper bound \dot{r}_n . Now s_α forces that $(\dot{r}_n : n < \omega)$ is an ascending chain in \dot{Q}_α , so let \dot{q} be such that s_α forces \dot{q} is a least upper bound to $(\dot{r}_n : n < \omega)$. Let $s_{\alpha+1} = s_\alpha \wedge (\dot{q})$.

Thus the induction goes through, and s_{α_*} is a least upper bound $(q_i : i < i_*)$. \square

The following class of forcing axioms, for $k = 3$, is related to Shelah's $\text{Ax}\mu_0$ from [9] although the formulation is different.

Definition 2.7. Suppose $\aleph_0 < \theta = \theta^{<\theta} \leq \lambda$, and suppose $3 \leq k < \omega$. Then say that $\text{Ax}(< k, \theta, \lambda)$ holds if for every forcing notion P such that $|P| \leq \lambda$ and P has the $(< k, \theta, \theta)$ -amalgamation property, if $(D_\alpha : \alpha < \lambda)$ is a sequence of dense subsets of P , then there is an ideal of P meeting each D_α . (By dense, we mean upwards dense: for every $p \in P$, there is $q \in D_\alpha$ with $q \geq p$.) Say that $\text{Ax}(< k, \theta)$ holds iff $\text{Ax}(< k, \theta, \lambda)$ holds for all $\lambda < 2^\theta$.

By a typical downward Lowenheim-Skolem argument we could drop the condition that $|P| \leq \lambda$ in $\text{Ax}(k, \theta, \lambda)$, but we won't need this. Finally, note that $\text{Ax}(k, \theta, \lambda)$ implies that $2^\theta > \lambda$, since $P_{\theta 2^\theta}$ has the $(< k, \theta, \theta)$ -amalgamation property and there is a family 2^θ dense sets such that no ideal meets them all.

Theorem 2.8. *Suppose $\aleph_0 < \theta$ are cardinals such that θ is regular and $\theta = \theta^{<\theta}$, and suppose $3 \leq k \leq \theta$. Suppose $\kappa \geq \theta$ has $\kappa^{<\kappa} = \kappa$. Then there is a forcing notion P with the $(< k, \theta, \theta)$ -amalgamation property (in particular, θ -closed and θ^+ -c.c.), such that P forces that $\text{Ax}(< k, \theta)$ holds and that $2^\theta = \kappa$. We can arrange $|P| = \kappa$.*

Proof. The proof is very similar to the proof of the consistency of Martin's axiom, see Theorem 16.13 of [5].

Let $(P_\alpha : \alpha \leq \kappa), (\dot{Q}_\alpha : \alpha < \kappa)$ be a $< \theta$ -support iteration, such that (viewing P_α -names as P_β -names in the natural way, for $\alpha \leq \beta < \kappa$):

- Each P_α forces that \dot{Q}_α has the $(< k, \theta, \theta)$ -amalgamation property;
- Whenever $\alpha < \kappa$, and \dot{Q} is a P_α -name such that $|\dot{Q}| < \kappa$ and P_α forces \dot{Q} has the $(< k, \theta, \theta)$ -amalgamation property, then there is some $\beta \geq \alpha$ such that P_β forces that \dot{Q}_β is isomorphic to \dot{Q} ;
- Each $|P_\alpha| \leq \kappa$.

This is possible by the θ^+ -c.c., and using Lemma 2.3(2). The point is that at each stage α , if P_α forces that $|\dot{Q}| = \lambda < \kappa$, then we can choose a P_α -name \dot{Q}' such that P_α -forces $\dot{Q} \cong \dot{Q}'$ and that \dot{Q}' has universe λ ; then there are only $|P_\alpha|^{\theta \cdot \lambda} \leq \kappa$ -many possibilities for \dot{Q}' , up to P_α -equivalence. Thus we can eventually deal with all of them.

Note that by $\kappa = \kappa^{<\kappa}$ we have in particular that P_κ forces $2^\theta \leq \kappa$. Once we verify P_κ forces that $\text{Ax}(< k, \theta, \lambda)$ for all $\lambda < \kappa$, it follows that P_κ forces $2^\theta = \kappa$.

By the θ^+ -c.c., we have that whenever \dot{X} is a P_κ -name for a subset of λ for some $\lambda < \kappa$, then for some $\alpha < \kappa$ and some P_α -name \dot{X}_α we have that \dot{X} is forced to be equal to \dot{X}_α .

Let $\mathbb{V}[G_\kappa]$ be a P_κ -generic extension of \mathbb{V} ; for $\alpha < \kappa$ let G_α be the associated P_α -generic extension of \mathbb{V} . Rephrasing the previous paragraph, we have that whenever $\lambda < \kappa$ and $X \subseteq \lambda$ is in $\mathbb{V}[G_\kappa]$, we have $X \in \mathbb{V}[G_\alpha]$ for some $\alpha < \kappa$.

Let Q be a forcing notion in $\mathbb{V}[G_\kappa]$ with the $(< k, \theta, \theta)$ -amalgamation property, with $|Q| < \kappa$; let $F : (Q, Q_0) \rightarrow_k P_{X\theta\theta}$ witness this, where we can suppose $X = \lambda < \kappa$. Let $\mathcal{D} = \{D_\alpha : \alpha < \lambda'\}$ be a set of dense subsets of Q where $\lambda' < \kappa$. By the preceding, we can find $\alpha < \kappa$ such that $(Q, Q_0, F, \mathcal{D}) \in \mathbb{V}[G_\alpha]$. Thus we can find P_α -names for them, $\dot{Q}, \dot{Q}_0, \dot{F}, \dot{\mathcal{D}}$. We have that P_α forces \dot{Q} has the $(< k, \theta, \theta)$ -amalgamation property. Then we can find some $\beta \geq \alpha$ such that it is forced $\dot{Q} \cong \dot{Q}_\beta$. Then in $\mathbb{V}[G_\kappa]$, if we let H be the $\mathbb{V}[G_\beta]$ -generic subset of Q added by \dot{Q}_β , then this is an ideal of Q meeting each dense set in \mathcal{D} , thus verifying $\text{Ax}(< k, \theta)$. □

We now relate this to model theory.

Definition 2.9. Suppose (θ, λ) is a nice pair, and $\theta \leq \mu \leq \lambda$, and T is simple. Then say that T has $(< k, \lambda, \mu, \theta)$ -type amalgamation if whenever $M \models T$ has size $\leq \lambda$, and whenever $M_* \preceq M$ is countable, then Γ_{M, M_*}^θ has the $(< k, \mu, \theta)$ -amalgamation property, or equivalently, $\Gamma_{M, M_*}^\theta \rightarrow_k^w P_{X\mu\theta}$ for some set X .

Lemma 2.10. *Suppose T fails $(< k, \lambda, \mu, \theta)$ -type amalgamation, and P has the $(< k, \mu, \theta)$ -amalgamation property. Then P forces that \dot{T} fails $(< k, \lambda, |\mu|, \theta)$ -type amalgamation.*

Proof. It suffices to show that if Q is a forcing notion and P forces that $\dot{Q} \rightarrow_k^w \dot{P}_{\dot{X}\mu\theta}$, then $Q \rightarrow_k^w P_{X'\mu\theta}$ for some X' , by Lemma 2.3(4). (We then apply this to $Q = \Gamma_{M, M_*}^\theta$ witnessing the failure of $(< k, \mu, \theta)$ -amalgamation.)

Choose some $F_* : (P, P_0) \rightarrow_k P_{X_*\mu\theta}$, and let \dot{G} be a P -name so that P forces $\dot{g} : \dot{Q} \rightarrow_k^w \dot{P}_{\dot{X}\mu\theta}$. For every $q \in Q$, choose $\mathbf{p}(q) \in P_0$ such that $\mathbf{p}(q)$ decides $\dot{G}(\dot{q})$, say $\mathbf{p}(q)$ forces that $\dot{G}(\dot{q}) = f(q)$. Let X be the disjoint union of X_* and X , and choose $F : Q \rightarrow P_{X\mu\theta}$ so that if $F(q)$ and $F(q')$ are compatible, then $f(q)$ and $f(q')$ are compatible, and $F_*(\mathbf{p}(q))$ and $F_*(\mathbf{p}(q'))$ are compatible.

Suppose $(q_i : i < i_*)$ is a sequence from Q with $(F(q_i) : i < i_*)$ compatible in $P_{X\mu\theta}$. Then $(F_*(\mathbf{p}(q_i)) : i < i_*)$ are all compatible in $P_{X_*\mu\theta}$, so $(\mathbf{p}(q_i) : i < i_*)$ are compatible in P_0 with the least upper bound p . Then p forces each $\dot{F}(\dot{q}_i) = f(q_i)$. But also (by choice of F), $(f(q_i) : i < i_*)$ are compatible in $P_{Y, \mu, \theta}$, so p forces that $(\dot{q}_i : i < i_*)$ is compatible in \dot{Q} , i.e. $(q_i : i < i_*)$ is compatible in Q . □

Theorem 2.11. *Suppose T simple, and $\aleph_0 < \theta = \theta^{<\theta} \leq \lambda = \lambda^{\aleph_0}$, and $Ax(< k, \theta)$ holds. Suppose $2^\theta > \lambda^{<\theta}$, and suppose $3 \leq k \leq \aleph_0$. Then the following are equivalent:*

- (A) T has $(< k, \lambda, \theta, \theta)$ -type amalgamation;
- (B) $SP_T^1(\lambda, \theta, \theta)$ holds.

Proof. (B) implies (A): suppose (B) holds and $M \models T$ has size λ and $M_* \preceq M$ is countable. Let $(p_\alpha(x) : \alpha < \theta)$ be as in the definition of $SP_T^1(\lambda, \theta, \theta)$. Let $X = \{x\}$ be a singleton. Then $\Gamma_{M, M_*}^\theta \rightarrow_k^w P_{X\theta\theta}$, namely send $p(x) \in \Gamma_{M, M_*}^\theta$ to $\{(x, \alpha)\}$ for some α with $p(x) \subseteq p_\alpha(x)$.

(A) implies (B): let $M \models T$ have size at most λ and let $M_* \preceq M$ be countable. Let P be the $< \theta$ -support product of θ -many copies of Γ_{M, M_*}^θ ; then P has the $(< k, \theta, \theta)$ -amalgamation property and $|P| \leq \lambda^{<\theta}$. For each $p(x) \in \Gamma_{M, M_*}^\theta$ let D_p be the dense subset of P consisting of all $f \in P$ such that for some $\gamma \in \text{dom}(f)$, $f(\gamma)$ extends $p(x)$. By $Ax(< k, \lambda^{<\theta}, \theta)$ we can choose an ideal I of P meeting each D_p . This induces a sequence $(p_\gamma(x) : \gamma < \theta)$ of partial types over M that do not fork over M_* , such that for all $p(x) \in \Gamma_{M, M_*}^\theta$ there is $\gamma < \theta$ with $p(x) \subseteq p_\gamma(x)$. To finish, extend each $p_\gamma(x)$ to a complete type over M not forking over M_* . \square

§ 3. NON-FORKING DIAGRAMS

Suppose T is a simple theory in a countable language. We wish to study various type amalgamation properties of T ; in particular we will be looking at systems of types $(p_s(x) : s \in P)$ over a system of models $(M_s : s \in P)$, for some $P \subseteq \mathcal{P}(I)$ closed under subsets. For this to be interesting, we need $(M_s : s \in P)$ to be independent in a suitable sense, which we define in this section.

The following definition is similar to the first author’s definition of independence in [8] in the context of stable theories, see Section XII.2. In fact we are modeling our definition after Fact 2.5 there (we cannot take the definition exactly from [8] because we allow P to contain infinite subsets of I).

Definition 3.1. Let T be simple.

Suppose I is an index set and $P \subseteq \mathcal{P}(I)$ is downward closed. Say that $(A_s : s \in P)$ is a diagram (of subsets of \mathfrak{C}) if each $A_s \subseteq \mathfrak{C}$ and $s \subseteq t$ implies $A_s \subseteq A_t$. Say that $(A_s : s \in P)$ is a non-forking diagram if for all $s_i : i < n, t_j : j < m \in P$, $\bigcup_{i < n} A_{s_i} \perp_{\bigcup_{i,j} A_{s_i \cap t_j}} \bigcup_{j < m} A_{t_j}$. Say that $(A_s : s \in P)$ is a continuous diagram if for every $X \subseteq P$, $\bigcap_{s \in X} A_s = A_{\bigcap X}$. (If X is finite then this is a consequence of non-forking.)

Note that $(A_s : s \in P)$ is continuous if and only if for every $a \in \bigcup_{s \in P} A_s$, there is some least $s \in P$ with $a \in A_s$. Also note that if $(A_s : s \in P)$ is non-forking (continuous) and $Q \subseteq P$ is downward closed then $(A_s : s \in Q)$ is non-forking (continuous).

Lemma 3.2. *Suppose $(A_s : s \in P)$ is a diagram of subsets of \mathfrak{C} . Then the following are equivalent:*

- (A) For all downward-closed subsets $S, T \subseteq P$, $\bigcup_{s \in S} A_s \perp_{\bigcup_{s \in S \cap T} A_s} \bigcup_{t \in T} A_t$.
- (B) $(A_s : s \in P)$ is non-forking.

Proof. (A) implies (B) is trivial.

(B) implies (A): we proceed by induction on κ to show that for all $s_\alpha : \alpha < \kappa, t_\beta : \beta < \kappa, \bigcup_\alpha A_{s_\alpha} \perp_{\bigcup_{\alpha,\beta} A_{s_\alpha \cap t_\beta}} \bigcup_\beta A_{t_\beta}$. This suffices to prove (A) since when $\kappa \geq |S| + |T|$ then s_α, t_β can just enumerate S and T , in which case $s_\alpha \cap t_\beta$ enumerates $S \cap T$. For the induction, when κ is finite use the hypothesis (B), and when κ is infinite use the local character of nonforking. \square

The following lemma is similar to Lemma 2.3 from [8] Section XII.2.

Lemma 3.3. *Suppose $P \subseteq \mathcal{P}(I)$ is downward closed and $(A_s : s \in P)$ is a continuous diagram of subsets of \mathfrak{C} . Suppose there is a well-ordering $<_*$ of $\bigcup_s A_s$ such that for all $a \in \bigcup_s A_s$, a is free from $\{b \in \bigcup_s A_s : b <_* a\}$ over $\{b \in s_a : b <_* a\}$, where s_a is the least element of P with $a \in A_{s_a}$. Then $(A_s : s \in P)$ is non-forking.*

Proof. Let $(a_\alpha : \alpha < \alpha_*)$ be the $<_*$ -increasing enumeration of $\bigcup_s A_s$, and let s_α be the least element of P with $a_\alpha \in A_{s_\alpha}$. For each $\alpha \leq \alpha_*$ and for each $s \in P$ let $A_{s,\alpha} = A_s \cap \{a_\beta : \beta < \alpha\}$. We show by induction on α that $(A_{s,\alpha} : s \in P)$ is non-forking.

Limit stages are clear. So suppose we have shown $(A_{s,\alpha} : s \in P)$ is non-forking. Let $(s_i : i < n), (t_j : j < m) \in P$ be given. We wish to show

$\bigcup_{i < n} A_{s_i, \alpha+1} \downarrow_{\bigcup_{i < n, j < n} A_{s_i \cap t_j, \alpha+1}} \bigcup_{j < m} A_{t_j, \alpha+1}$. Write $A = \bigcup_{i < n} A_{s_i, \alpha}$, write $B = \bigcup_{j < m} A_{t_j, \alpha}$, and write $C = \bigcup_{i, j} A_{s_i \cap t_j, \alpha}$. Define A', B', C' similarly except with $\alpha + 1$ replacing α . We are trying to show $A' \downarrow_{C'} B'$, and by the inductive hypothesis, $A \downarrow_C B$, and we also know $a_\alpha \downarrow_{A_{s_\alpha, \alpha}} AB$.

If $a_\alpha \notin s_i$ and $a_\alpha \notin t_j$ for any i, j then $A' = A, B' = B, C' = C$ and so we are done. If $a_\alpha \in s_i \cap t_j$ and hence $A_{s_\alpha, \alpha} \subseteq C$, then $A' = Aa_\alpha, B' = Ba_\alpha, C' = Ca_\alpha$ and $a_\alpha \downarrow_C AB$ (by monotonicity), so we are done by $A \downarrow_C B$ and transitivity.

Up to symmetry, the final case is $a_\alpha \in s_i$ but is not in any t_j . Then $A_{s_\alpha, \alpha} \subseteq A$ and $A' = Aa_\alpha, B' = B, C' = C$. By monotonicity we have $a_\alpha \downarrow_A AB$, so $a_\alpha A \downarrow_A B$, so by transitivity $a_\alpha A \downarrow_C B$ as desired. \square

Theorem 3.4. *Suppose T is a simple theory in a countable language, and suppose \mathbf{A} is a set of cardinality λ , where $\lambda = \aleph_0$. Then we can find a continuous, non-forking diagram of models $(M_s : s \in [\lambda]^{\leq \aleph_0})$ such that $\mathbf{A} \subseteq \bigcup_s M_s$, and such that for all $S \subseteq \lambda$, $\bigcup_{s \in [S]^{\leq \aleph_0}} M_s$ has size at most $|S| \cdot \aleph_0$.*

Proof. Enumerate $\mathbf{A} = (a_\alpha : \alpha < \lambda)$.

We define $(\text{cl}(\{\alpha\}) : \alpha < \lambda)$ inductively as follows, where each $\text{cl}(\{\alpha\})$ is a countable subset $\alpha + 1$ with $\alpha \in \text{cl}(\{\alpha\})$. Suppose we have defined $(\text{cl}(\{\beta\}) : \beta < \alpha)$. Choose a countable set $\Gamma \subseteq \alpha$ such that $a_\alpha \downarrow_{\{a_\beta : \beta \in \Gamma\}} \{a_\beta : \beta < \alpha\}$; put $\text{cl}(\{\alpha\}) = \{\alpha\} \cup \bigcup_{\beta \in \Gamma} \text{cl}(\{\beta\})$.

Now, for each $s \subseteq \lambda$, let $\text{cl}(s) := \bigcup_{\alpha \in s} \text{cl}(\{\alpha\})$. Say that $A \subseteq \lambda$ is closed if $\text{cl}(A) = A$; this satisfies the usual properties of a set-theoretic closure operation, that is $\text{cl}(A) \supseteq A$, and $A \subseteq B$ implies $\text{cl}(A) \subseteq \text{cl}(B)$, and $\text{cl}^2(A) = \text{cl}(A)$, and cl is finitary: in fact $\text{cl}(A) = \bigcup_{\alpha \in A} \text{cl}(\{\alpha\})$, which is even stronger. Finally, $|\text{cl}(A)| \leq |A| + \aleph_0$.

For each $s \in [\lambda]^{\leq \omega}$, let $A_s = \{a_\alpha : \alpha < \lambda \text{ and } \text{cl}(\{\alpha\}) \subseteq s\}$. Since each $a_\alpha \in A_{\text{cl}(\{\alpha\})}$, we have $\bigcup_s A_s = \mathbf{A}$. Further, $(A_s : s \in [\lambda]^{\leq \omega})$ is clearly a continuous diagram of sets; we claim that $(A_s : s \in [\lambda]^{\leq \omega})$ is a non-forking diagram of sets. But this follows from Lemma 3.3, since each $a_\alpha \downarrow_{A_{\text{cl}(\{\alpha\})} \cap \{a_\beta : \beta < \alpha\}} \{a_\beta : \beta < \alpha\}$.

For each $\alpha \leq \lambda$, and each $u \in [\lambda]^{< \omega}$ let $\mathcal{A}_{\alpha, u} = \{\text{cl}(s \cup u) \cap \alpha : s \in [\alpha]^{< \omega}\}$. We show by induction on $\alpha \leq \lambda$ that for all $u \in [\lambda]^{< \omega}$, $(\mathcal{A}_{\alpha, u}, \subseteq)$ is well-founded. There will be separate step case and limit case.

Suppose we have shown $(\mathcal{A}_{\alpha, u}, \subseteq)$ is well-founded. Then $\mathcal{A}_{\alpha+1, u} = X_0 \cup X_1$ where $X_0 = \{\text{cl}(s \cup u) \cap (\alpha+1) : s \in [\alpha]^{< \omega}\}$ and $X_1 = \{\text{cl}(s \cup \{\alpha\} \cup u) \cap (\alpha+1) : s \in [\alpha]^{< \omega}\}$. It suffices to show each of X_0, X_1 is well-founded under subset. Write $v_0 = u$ and $v_1 = u \cup \{\alpha\}$. Then $X_i = \{\text{cl}(s \cup v_i) \cap (\alpha+1) : s \in [\alpha]^{< \omega}\}$.

By the induction hypothesis, $\mathcal{A}_{\alpha, v_i}$ is well-founded under subset, so it suffices to show that $\text{cl}(s \cup v_i) \cap (\alpha+1) \subseteq \text{cl}(t \cup v_i) \cap (\alpha+1)$ if and only if $\text{cl}(s \cup v_i) \cap \alpha \subseteq \text{cl}(t \cup v_i) \cap \alpha$, for all $s, t \in [\alpha]^{< \omega}$. It suffices to show that $\alpha \in \text{cl}(s \cup v_i)$ if and only if $\alpha \in \text{cl}(t \cup v_i)$, but both are equivalent to $\alpha \in \text{cl}(v_i)$.

Now suppose we have shown $(\mathcal{A}_{\alpha, u}, \subseteq)$ is well-founded for all u and for all $\alpha < \delta$ where δ is a limit. Suppose towards a contradiction $(\mathcal{A}_{\delta, u}, \subseteq)$ were not well-founded, say it had the infinite descending chain $\text{cl}(s_n \cup u) \cap \delta$. Choose $\alpha < \delta$ with $s_0 \in [\alpha]^{< \omega}$. Then we have each $s_n \subseteq \text{cl}(s_0 \cup u)$; thus $s_n \subseteq \alpha \cup \text{cl}(u)$. Put $t_n = s_n \cap \alpha$; then each $\text{cl}(s_n \cup u) \cap \delta = \text{cl}(t_n \cup u) \cap \delta$. But then $(\text{cl}(t_n \cup u) \cap \alpha : n < \omega)$ must be strictly descending, since $(\text{cl}(t_n \cup u) \cap \delta : n < \omega)$ is, and each $\text{cl}(t_n \cup u) \cap (\delta \setminus \alpha) = \text{cl}(u) \cap (\delta \setminus \alpha)$.

Hence $\mathcal{A} := \mathcal{A}_{\lambda, \emptyset} = \{\text{cl}(s) : s \in [\lambda]^{<\aleph_0}\}$ is well-founded under subset. Note that for all $s \in \mathcal{A}$, since $\text{cl}(s) = s$ we have $A_s = \{a_\alpha : \alpha \in s\}$.

Let $<_*$ be a well-order of \mathcal{A} refining \subset . Now by induction on $<_*$, choose countable models $(M_s : s \in \mathcal{A})$ so that $M_s \supseteq A_s$ and $M_s \supseteq M_t$ for $t \subseteq s$ and such that $M_s \downarrow_{A_s \cup \bigcup\{M_t : t \in \mathcal{A}, t \subset s\}} \mathbf{A} \cup \bigcup\{M_t : t \in \mathcal{A}, t <_* s\}$. Finally, given $s \in [\lambda]^{\leq \omega}$, let $M_s := \bigcup\{M_t : t \in \mathcal{A}, t \subseteq s\}$. This is a continuous diagram of models, and for all $S \subseteq \lambda$, $\{t \in \mathcal{A} : t \subseteq S\}$ has size at most $|S| \cdot \aleph_0$, so to finish the proof of the theorem it suffices to show $(M_s : s \in [\lambda]^{\leq \aleph_0})$ is non-forking.

Enumerate $\mathcal{A} = (u_\alpha : \alpha < \alpha_*)$ in $<_*$ -increasing order. For each $\alpha \leq \alpha_*$ let $(B_s^\alpha : s \in [\lambda]^{\leq \aleph_0})$ be the continuous diagram of models defined via $B_s^\alpha = A_s \cup \bigcup\{M_{u_\beta} : u_\beta \subseteq s, \beta < \alpha\}$. So $B_s^0 = A_s$, $B_s^{\alpha_*} = M_s$, and it suffices to show by induction on α that $(B_s^\alpha : s \in [\lambda]^{\leq \aleph_0})$ is nonforking.

The base case and limit cases are clear. So suppose $(B_s^\alpha : s \in [\lambda]^{\leq \aleph_0})$ is nonforking; we try to show $(B_s^{\alpha+1} : s \in [\lambda]^{\leq \aleph_0})$ is nonforking. Let $s_i : i < n, t_j : j < m$ be from $[\lambda]^{\leq \aleph_0}$; we want to show $\bigcup_{i < n} B_{s_i}^{\alpha+1} \downarrow_{\bigcup_{i,j} B_{s_i \cap t_j}^{\alpha+1}} \bigcup_{j < m} B_{t_j}^{\alpha+1}$.

Write $A = \bigcup_{i < n} B_{s_i}^\alpha$, write $B = \bigcup_{j < m} B_{t_j}^\alpha$, and write $C = \bigcup_{i,j} B_{s_i \cap t_j}^\alpha$, and let A', B', C' be the same but with $\alpha + 1$. We know $A \downarrow_C B$ by the inductive hypothesis and we are trying to show $A' \downarrow_{C'} B'$. We also know, by construction of M_{u_α} , that $M_{u_\alpha} \downarrow_{B_{u_\alpha}^\alpha} \bigcup\{B_s^\alpha : s \in [\lambda]^{\leq \aleph_0}\}$.

If u_α is not contained in any s_i or t_j then $A = A', B = B', C = C'$ and we are done. If u_α is contained in some $s_i \cap t_j$ then $A' = AM_{u_\alpha}, B' = BM_{u_\alpha}, C' = CM_{u_\alpha}$, and $M_{u_\alpha} \downarrow_C AB$, so we are done by transitivity. The remaining case (up to symmetry) is that u_α is contained in some s_i but not in any t_j . Then $A' = AM_{u_\alpha}, B' = B, C' = C$, and $M_{u_\alpha} \downarrow_A B$, so we are again done by transitivity. \square

§ 4. AMALGAMATION PROPERTIES

Suppose T is a simple theory in a countable language. We now explain what we mean by T having $< k$ -type amalgamation.

Definition 4.1. Given $\Lambda \subseteq {}^n m$, let P_Λ be the set of all partial functions from n to m which can be extended to an element of Λ ; so P_Λ is a downward-closed subset of $\mathcal{P}(n \times m)$, and Λ is the set of maximal elements of P_Λ .

Suppose $(M_u : u \subseteq n)$ is a non-forking diagram of models. Then by a (Λ, \overline{M}) -array, we mean a non-forking diagram of models $(N_s : s \in P_\Lambda)$, together with maps $(\pi_s : s \in P_\Lambda)$ such that each $\pi_s : M_{\text{dom}(s)} \cong N_s$, and such that $s \subseteq t$ implies $\pi_s \subseteq \pi_t$.

Definition 4.2. Suppose $\Lambda \subseteq {}^n m$. Then T has Λ -type amalgamation if, whenever $(M_u : u \subseteq n)$ is a non-forking diagram of models, and whenever $p(x)$ is a complete type over M_n in a single variable which does not fork over M_0 , and whenever $(N_s, \pi_s : s \in P_\Lambda)$ is a (Λ, \overline{M}) -array, then $\bigcup_{\eta \in \Lambda} \pi_\eta(p(x))$ does not fork over N_0 .

Suppose $3 \leq k \leq \aleph_0$; then say that T has $< k$ -type amalgamation if whenever $|\Lambda| < k$, then T has Λ -type amalgamation.

In the definition of Λ -type amalgamation, it would not matter if we required each M_u to be countable, by a downward Lowenheim-Skolem argument.

Example 4.3. Every simple theory has < 3 -type amalgamation.

Proof. Suppose $\Lambda \subseteq {}^n m$ has $|\Lambda| = 2$ and $(M_u : u \subseteq n)$ is a non-forking diagram of models and $p(x)$ is a complete type over M_n in a single variable which does not fork over M_0 . Suppose $(N_s, \pi_s : s \in P_\Lambda)$ is a (Λ, \overline{M}) -array. Write $\Lambda = \{\eta_0, \eta_1\}$. Write $K_i = \pi_{\eta_i}[M_n]$ for $i < 2$ and let $q_i = \pi_{\eta_i}(p(x))$. By the independence theorem for simple theories, $q_0(x) \cup q_1(x)$ does not fork over $K_0 \cap K_1$. But $K_0 \cap K_1 \subseteq K_0$ and since $q_0(x)$ does not fork over N_0 , also $q_0(x) \cup q_1(x)$ does not fork over N_0 by transitivity. \square

Example 4.4. T_{rg} has $< \aleph_0$ -type amalgamation.

Proof. This follows from the fact that if $(A_s : s \in P)$ is any nonforking diagram of sets and $p_s(x) \in S^1(A_s)$ for each $s \in P$, if each $p_s(x)$ does not fork over A_0 and if $p_s(x) \subseteq p_t(x)$ for $s \subseteq t$, then $\bigcup_s p_s(x)$ is consistent and does not fork over A_0 . \square

Example 4.5. Suppose $\ell > k \geq 2$. Let $T_{\ell,k}$ be the theory of the generic k -ary, ℓ -clique free hypergraph; these examples were introduced by Hrushovski [4], where he proved they have quantifier elimination, and $T_{\ell,k}$ is simple if and only if $k \geq 3$. For $k \geq 3$ and $A \subseteq B, p(x) \in S(B)$, we have that p forks over A if and only if p is realized in $B \setminus A$.

Then: for $k \geq 3$, $T_{\ell,k}$ has $< k$ -type amalgamation but not $< k + 1$ -type amalgamation.

Proof. Let R denote the edge relation of $T_{\ell,k}$.

First we show $T_{\ell,k}$ has $< k$ -type amalgamation. Suppose $\Lambda \subseteq {}^n m$ with $|\Lambda| < k$, and $(M_u : u \subseteq n)$ are given, and suppose $p(x)$ is a complete type over M_n . Suppose towards a contradiction there were a (Λ, \overline{M}) -array $(N_s, \pi_s : s \in P_\Lambda)$ with $\bigcup_{\eta \in \Lambda} \pi_\eta[p(x)]$ forking over N_0 . Then we must have created some ℓ -clique $(x, a_i : <$

$\ell - 1$), where each $a_i \in N_\eta$ for some $\eta \in \Lambda$. That is, $a_i : i < \ell - 1$ is an R -clique and for each $u \in [\ell - 1]^{k-1}$ there is some $\eta \in \Lambda$ such that $\pi_\eta[p(x)]$ implies $R(x, a_i : i \in u)$.

For each $i < \ell - 1$, let $h[\{i\}]$ be the least $s \in P_\Lambda$ with $a_i \in N_s$. For $u \subseteq \ell - 1$ let $h[u] = \bigcup_{i \in u} h[\{i\}]$. The following must hold:

- (I) For every $u \in [\ell - 1]^{k-1}$, $h[u] \in P_\Lambda$, as some $\pi_\eta(p(x))$ implies $R(x, a_i : i \in u)$, and any such η contains $h[u]$;
- (II) $h[\ell - 1] \notin P_\Lambda$, as if $h[\ell - 1] \subseteq \eta$ then $\pi_\eta(p(x))$ would contain $R(x, a_i : i \in u)$ for all $u \in [\ell - 1]^{k-1}$ and so would be inconsistent.

By (II), for each $\eta \in \Lambda$ we must have $h[\ell - 1] \not\subseteq \eta$; thus we can choose $i_\eta < \ell - 1$ such that $h[\{i_\eta\}] \not\subseteq \eta$. Let $u = \{i_\eta : \eta \in \Lambda\} \in [\ell - 1]^{<k}$. Clearly then $h[u] \notin P_\Lambda$, but this contradicts (I).

Now we show that $T_{\ell,k}$ fails $< k + 1$ -type amalgamation. Indeed, let $\Lambda \subseteq {}^k 2$ be the set of all $f : k \rightarrow 2$ for which there is exactly one $i < k$ with $f(i) = 1$; so $|\Lambda| = k$. Also, let $(M_u : u \subseteq k)$ be a non-forking diagram of models so that there are $a_i \in M_{\{i\}}$ for $i < k$ and there are $b_j \in M_0$ for $n < \ell - k - 1$, such that every k -tuple of distinct elements from $(a_i, b_j : i < k, j < \ell - k - 1)$ is in R except for $(a_i : i < k)$. Let $p(x)$ be the partial type over M_k which asserts that $R(x, \bar{a})$ holds for every $k - 1$ -tuple \bar{a} of distinct elements from $(a_i, b_j : i < k, j < \ell - k - 1)$.

It is not hard to find a (Λ, \bar{M}) -array $(N_s, \pi_s : s \in P_\Lambda)$ such that, if we write $\pi_{\{(i,0)\}}(a_i) = c_i$, then $R(c_i : i < k)$ holds; but now we are done, since $\bigcup_{f \in \Lambda} \pi_f[p(x)]$ is inconsistent. □

The following is the key consequence of $< k$ -type amalgamation.

Theorem 4.6. *Suppose T is a simple theory with $< k$ -type amalgamation. Then for all nice pairs (θ, λ) , T has $(< k, \lambda, \theta, \theta)$ -type amalgamation.*

Proof. By Theorem 3.4, it suffices to show that if $(\mathbf{M}_s : s \in [\lambda]^{<\theta})$ is a continuous non-forking diagram of countable models such that each $|\mathbf{M}_s| < \theta$, then writing $\mathbf{M} = \bigcup_s \mathbf{M}_s$, we have that $\Gamma_{\mathbf{M}, \mathbf{M}_0}^\theta \rightarrow_k^w P_{X\theta\theta}$ for some X . Let $<_*$ be a well-ordering of \mathbf{M} .

Given $A \in [\mathbf{M}]^{<\theta}$ let s_A be the \subseteq -minimal $s \in [\lambda]^{<\theta}$ with $A \subseteq M_{s_A}$, possible by continuity.

Let P be the set of all $p(x) \in \Gamma_{\mathbf{M}, \mathbf{M}_0}^\theta$ such that for some $s \in [\lambda]^{<\theta}$, $p(x)$ is a complete type over \mathbf{M}_s ; we write $p(x, \mathbf{M}_s)$ to indicate this. P is dense in $\Gamma_{\mathbf{M}, \mathbf{M}_0}^\theta$, so it suffices to show that $P \rightarrow_k^w P_{\lambda\theta\theta}$ for some λ .

Claim 4.7. *For some set X , we can find $F : P \rightarrow P_{X\theta\theta}$ so that if $F(p(x, \mathbf{M}_s))$ is compatible with $F(q(x, \mathbf{M}_t))$, then:*

- s and t have the same order-type, and if we let $\rho : s \rightarrow t$ be the unique order-preserving bijection, then ρ is the identity on $s \cap t$;
- \mathbf{M}_s and \mathbf{M}_t have the same $<_*$ -order-type, and the unique $<_*$ -preserving bijection from \mathbf{M}_s to \mathbf{M}_t is in fact an isomorphism $\tau : \mathbf{M}_s \cong \mathbf{M}_t$ which is the identity on $\mathbf{M}_{s \cap t}$;
- For each finite $\bar{a} \in \mathbf{M}_s^{<\omega}$, if we write $s' = s_{\bar{a}}$ and if we write $t' = s_{\tau(\bar{a})}$, then: $\rho[s'] = t'$ and $\tau \upharpoonright_{\mathbf{M}_{s'}} : \mathbf{M}_{s'} \cong \mathbf{M}_{t'}$.
- $\tau[p(x)] = q(x)$.

Proof. Given $p(x, \mathbf{M}_s) \in P$ let \mathcal{D}_p consist of the following data:

- The order-type of s , call it γ ; let $\rho_0 : \gamma \rightarrow s$ be the order-preserving bijection
- The $<_*$ -order-type of \mathbf{M}_s , call it δ ; let $\tau_0 : \delta \rightarrow \mathbf{M}_s$ be the order-preserving bijection;
- The structure N with universe δ , such that τ_0 is an isomorphism from N to \mathbf{M}_s ;
- $\tau_0^{-1}(p(x))$;
- The set of all (\bar{a}, α) such that $\bar{a} \in N$ and $\alpha < \gamma$ and $\rho_0(\alpha) \in s_{\tau_0(\bar{a})}$.

Note that there are only $2^{<\theta}$ possibilities for \mathcal{D}_p . Thus, letting X_0 be any set of cardinality θ , by choosing an antichain in $P_{X_0\theta\theta}$ of size $2^{<\theta}$, we can find $F_0 : P \rightarrow P_{X_0\theta\theta}$ so that for all $p, q \in P$, if $F_0(p)$ and $F_0(q)$ are compatible then $\mathcal{D}_p = \mathcal{D}_q$.

Let $F_1 : P \rightarrow P_{\lambda\theta\theta}$ send $p(x, \mathbf{M}_s)$ to the function with domain s sending $\alpha \in s$ to its order-type in s . Let $F_2 : P \rightarrow P_{\mathbf{M}_s\theta\theta}$ send $p(x, \mathbf{M}_s)$ to the function with domain \mathbf{M}_s sending $a \in \mathbf{M}_s$ to its $<_*$ -order-type in \mathbf{M}_s . Let $X = X_0 \cup \lambda \cup \mathbf{M}$ and let $F(p) = F_0(p) \cup F_1(p) \cup F_2(p)$. A straightforward verification shows that this works. \square

Fix F, X as in the claim. Note that it follows that for every $s' \subseteq s$, $\rho \upharpoonright_{\mathbf{M}_{s'}} : \mathbf{M}_{s'} \cong \mathbf{M}_{\rho[s']}$, since $\mathbf{M}_{s'} = \bigcup \{ \mathbf{M}_{s\bar{\pi}} : \bar{\pi} \in (\mathbf{M}_{s'})^{<\omega} \}$ and similarly for $\mathbf{M}_{t'}$.

We claim that F works. So suppose $p_i(x, \mathbf{M}_{s_i}) : i < i_*$ is a sequence from P for $i_* < k$, such that $(F(p_i(x)) : i < i_*)$ is compatible in $P_{x\theta\theta}$.

Let γ_* be the order-type of some or any s_i . Enumerate $s_i = \{s_i(\gamma) : \gamma < \gamma_*\}$ in increasing order, and for $u \subseteq \gamma_*$ let $s_i[u] = \{s_i(\gamma) : \gamma \in u\}$. Let E be the equivalence relation on γ_* defined by: $\gamma E \gamma'$ iff for all $i, i' < k$, $s_i(\gamma) = s_{i'}(\gamma)$ iff $s_i(\gamma') = s_{i'}(\gamma')$. Then E has finitely many classes; enumerate them as $(u_j : j < n)$. Then s_i is the disjoint union of $s_i[u_j]$ for $j < n$. Moreover, $s_i[u_j] \cap s_{i'}[u_{j'}] = \emptyset$ unless $j = j'$; and if $s_i[u_j] \cap s_{i'}[u_j] \neq \emptyset$ then $s_i[u_j] = s_{i'}[u_j]$. For each $j < n$, enumerate $\{s_i[u_j] : i < i_*\} = (Y_{\ell, j} : \ell < m_j)$ without repetitions. Let $m = \max(m_j : j < n)$; and for each $i < i_*$, define $\eta_i \in {}^n m$ via: $\eta_i(j) =$ the unique $\ell < m_i$ with $s_i[u_j] = Y_{\ell, j}$.

Let $\Lambda = \{\eta_i : i < i_*\}$. For each $s \in P_\Lambda$, let $N_s = \mathbf{M}_{t_s}$ where $t_s = \bigcup_{(j, \ell) \in s} Y_{\ell, j}$. Also, define $(M_u : u \subseteq n) := (N_{\eta_0 \upharpoonright u} : u \subseteq n)$. Then the hypotheses on F give commuting isomorphisms $\pi_s : M_{\text{dom}(s)} \cong N_s$ for each $s \in P_\Lambda$, in such a way that $(\bar{N}, \bar{\pi})$ is a (λ, \bar{M}) -array, and each $\pi_{\eta_i}(p_0(x)) = p_i(x)$. It follows by hypothesis on T that $\bigcup_{i < i_*} p_i(x)$ does not fork over N_0 , as desired. \square

Corollary 4.8. *Suppose T is simple, with $< \aleph_0$ -type amalgamation.*

- Suppose θ is a regular uncountable cardinal. Then for any $M \models T$ and any $M_0 \preceq M$ countable, Γ_{M, M_0}^θ has the $(< \aleph_0, \theta, \theta)$ -amalgamation property.
- Suppose (θ, λ) is a nice pair, and suppose that $\theta \leq \mu \leq \lambda$ satisfies $\mu = \mu^{<\theta}$ and $2^\mu \geq \lambda$. Then $SP_T^1(\lambda, \mu, \theta)$ holds.
- If the singular cardinals hypothesis holds, then $T \leq_{SP} T_{rg}$.

Proof. (A) follows immediately from Theorem 4.6.

(B): Suppose $M \models T$ has $|M| \leq \lambda$, and suppose $M_0 \preceq M$ is countable. Choose some $F : \Gamma_{M, M_0}^\theta \rightarrow_{\aleph_0}^w P_{\lambda\theta\theta}$. By Corollary 1.1, we can find $(\mathbf{f}_\gamma : \gamma < \mu)$ such that whenever $f \in P_{\lambda\theta\theta}$ then $f \subseteq \mathbf{f}_\gamma$ for some $\gamma < \mu$; for each $\gamma < \mu$, choose $q_\gamma(x)$, a complete type over M not forking over M_0 , and extending $\bigcup \{p(x) : F(p(x)) \subseteq \mathbf{f}_\gamma\}$. Then clearly $(q_\gamma(x) : \gamma < \mu)$ witnesses $SP_T^1(\lambda, \mu, \theta)$.

(C): Suppose (θ, λ) is a nice pair, and $SP_{T_{rg}}(\lambda, \theta)$ holds; we want to show $SP_T(\lambda, \theta)$ holds. We can suppose $\lambda < \lambda^{<\theta}$. Then by Theorem 1.4(B), $SP_{T_{rg}}^1(\lambda, \lambda, \theta)$ holds. By Theorem 1.8, there is $\theta \leq \mu < \lambda$ with $\mu = \mu^{<\theta}$ and $2^\mu \geq \lambda$. By (B), $SP_T^1(\lambda, \mu, \theta)$ holds, so by Theorem 1.4(A), $SP_T(\lambda, \theta)$ holds, as desired. \square

§ 5. CONCLUSION

We begin to put everything together. We aim to produce a forcing extension in which, whenever T has $< k$ -type amalgamation, then $T_{k,k-1} \not\leq_{SP} T$. We will choose in advance nice pairs (θ_k, λ_k) to witness this. In order to arrange that $SP_T(\lambda_k, \theta_k)$ holds we will use Theorems 2.11 and 4.6. To arrange that $SP_{T_{k,k-1}}(\lambda_k, \theta_k)$ fails, we will use the following.

Theorem 5.1. *Suppose (θ, λ) is a nice pair such that $\theta = \theta^{<\theta}$ and $\lambda > \theta$ is a limit cardinal. Let $3 \leq k < \omega$. Then $P_{\lambda\theta}$ forces that for all $\mu < \lambda$, $\tilde{T}_{k+1,k}$ fails $(< k + 1, \lambda, \mu, \theta)$ -type amalgamation.*

Proof. Fix $\theta \leq \mu < \lambda$, and write $P = P_{[\lambda]^k\theta\theta}$. We show that P forces $\tilde{T}_{k+1,k}$ fails $(< k + 1, \lambda, \mu, \theta)$ -type amalgamation. Since $P \cong P_{\lambda\theta\theta}$, this suffices.

We pass to a P -generic forcing extension $\mathbb{V}[G]$ of \mathbb{V} . Let $R \subseteq [\lambda]^k$ be the set of all v with $\{(v, 0)\} \in G$. Choose $M_0 \preceq M \models T_{k+1,k}$, and $(a_{i,\alpha} : i < k, \alpha < \lambda)$ such that, writing $\bar{a}_s = \{a_{i,\alpha} : (i, \alpha) \in s\}$ for $s \subseteq k \times \lambda$:

- M_0 is countable, and $|M| \leq \lambda$ and each $a_{i,\alpha} \in M \setminus M_0$;
- $a_{i,\alpha} = a_{j,\beta}$ iff $\alpha = \beta$ and $i = j$
- For every $v_* \in [k \times \lambda]^k$, if v_* is not the graph of the increasing enumeration of some $v \in [\lambda]^k$, then $R^M(\bar{a}_{v_*})$ fails. Otherwise, $R^M(\bar{a}_{v_*})$ holds if and only if $v \in R$.

For each $v \in [\lambda]^k$, let $\phi_v(x, \bar{a}_{k \times v})$ be the formula that asserts that $R(x, \bar{a}_u)$ holds for each $u \in [k \times v]^{k-1}$. Note that $\phi_v(x, \bar{a}_{k \times v})$ is consistent exactly when $v \notin R$.

It suffices to show that there is no cardinal λ' and function $F_0 : \Gamma_{M, M_0}^\theta \rightarrow_{k+1}^w P_{\lambda'\mu\theta}$; so suppose towards a contradiction some such F_0 existed. Then we can find $F : [\lambda]^k \setminus R \rightarrow P_{\lambda'\mu\theta}$ such that for all sequences $(w_i : i < k + 1)$ from $[\lambda]^k \setminus R$, if $(F(w_i) : i < k)$ is compatible in P then $\bigwedge_{i < k} \phi_{w_i}(x, \bar{a}_{k \times w_i})$ is consistent. This is all we will need, and so we can replace λ' by λ (since $||[\lambda]^k|| = \lambda$).

Pulling back to \mathbb{V} , we can find $p_* \in P$, and P -names $\dot{R}, \dot{M}, \dot{M}_0, \dot{a}_{i,\alpha}, \dot{F}$, such that p_* forces these behave as above.

Write $X = \lambda \setminus \bigcup \text{dom}(p_*)$; so $|X| = \lambda$.

Suppose $v \in [X]^k$. Choose $p_v \in P$ such that $p_v \geq p_* \cup \{(v, 1)\}$ (so p_v forces $v \notin \dot{R}$), and so that p_v decides $\dot{F}(v)$, say p_v forces that $\dot{F}(v) = f_v \in P_{\lambda\mu\theta}$.

Choose $F_* : [\lambda]^k \rightarrow P_{\lambda\mu\theta}$ so that for all v, v' , if $F_*(v)$ and $F_*(v')$ are compatible, then $p_v, p_{v'}$ are compatible, and $f_v, f_{v'}$ are compatible.

For each $u \in [\lambda]^{k-1}$ let $\mathcal{P}_u = \{F_*(v) : v \in [\lambda]^k, u \subseteq v\}$ and let \mathcal{Q}_u be the set of all $g \in P_{\lambda\mu\theta}$ such that g extends some $f \in \mathcal{P}_u$. Choose a maximal antichain $(g_{u,\alpha} : \alpha < \kappa_u)$ from \mathcal{Q}_u . For each $\alpha < \kappa_u$ choose $w_{u,\alpha} \in \mathcal{P}_u$ such that $g_{u,\alpha}$ extends $F_*(w_{u,\alpha})$.

Since P has the μ^+ -c.c., we have that each $\kappa_u \leq \mu$. For $u \in [\lambda]^{k-1}$ let $S(u) \in [\lambda]^\mu$ be sufficiently large so that each $w_{u,\alpha} \in [S(u)]^k$ and $\text{dom}(p_{w_{u,\alpha}}) \subseteq [S(u)]^k$.

By Theorem 46.1 of [2], using $\lambda \geq \mu^{+\omega}$, we can find some $v \in [\lambda]^k$ such that for all $u \in [v]^{k-1}$, $S(u) \cap v = u$. Enumerate $[v]^{k-1} = (u_i : i < k)$. By induction on $i < k$ we pick $w_i \in \mathcal{P}_{u_i}$ such that $(F_*(v), F_*(w_j) : j \leq i)$ is compatible in $P_{\lambda\mu\theta}$. To see this is possible, suppose we have $w_j : j < i$. Put $f = F_*(v) \cup \bigcup \{F_*(w_j) : j < i\}$. Since f extends $F_*(v)$, we have $f \in \mathcal{Q}_{u_i}$; by maximality of the antichain $(g_{u_i,\alpha} : \alpha < \kappa_{u_i})$,

we must have that f is compatible with some $g_{u_i, \alpha}$, and hence with $F_*(w_{u_i, \alpha})$, so put $w_i = w_{u_i, \alpha}$.

Writing $v_{u_i} := w_i$, we have found $(v_u : u \in [v]^{k-1})$ such that each $u \subseteq v_u \in [S(u)]^k$, and $(F_*(v_u) : u \in [v]^{k-1})$ is compatible. Thus $(p_{v_u} : u \in [v]^{k-1})$ is compatible in P ; write $p = \bigcup_{u \in [v]^{k-1}} p_{v_u}$. Note that $v \notin \text{dom}(p)$, since if $v \in \text{dom}(p_{v_u})$ then $v \in [S(u)]^k$, contradicting that $S(u) \cap v = u$. Thus we can choose $p' \geq p$ in P with $p'(v) = 0$.

Now p' forces that each $\dot{F}(v_u) = \dot{f}_{v_u}$, and $(f_{v_u} : u \in [v]^{k-1})$ is compatible; thus p' forces that $\phi(x) := \bigwedge_{u \in [v]^{k-1}} \phi_{v_u}(x, \vec{a}_{k \times v_u})$ is consistent. But this is impossible, since if we let v_* be the graph of the increasing enumeration of v , then p' forces that $\dot{R}^M(\vec{a}_{v_*})$ holds, and $\phi(x)$ in particular implies that $\dot{R}^M(x, \vec{a}_{u_*})$ holds for all $u_* \in [v_*]^{k-1}$, thus creating a k -clique. \square

Lemma 5.2. *Suppose (λ, θ) is a nice pair and $\ell > k \geq 3$. Then $\text{SP}_{T_{\ell, k}}^1(\lambda, \lambda, \theta)$ if and only if $\text{SP}_{T_{\ell, k}}(\lambda, \theta)$.*

Proof. Forward direction is Theorem 1.4(A). The reverse direction is like Theorem 1.4(B): suppose $M \models T_{\ell, k}$ has $|M| \leq \lambda$ and $M_* \leq M$ is countable. Choose $N \succeq M$, a θ -saturated model of size λ . Let $a_\alpha : \alpha < \lambda$ enumerate N . For each $\alpha < \lambda$ let $p_\alpha(x)$ be the type over M asserting $x \neq a$ for each $a \in M$, and $R(x, \vec{a}) \in p_\alpha$ if and only if $R(a_\alpha, \vec{a})$ holds for each $\vec{a} \in [M]^{k-1}$. This is a complete type over M which does not fork over \emptyset . Then $\{p_\alpha(x) : \alpha < \lambda\}$ along with all realized types over M_* witness $\text{SP}_{T_{\ell, k}}^1(\lambda, \lambda, \theta)$. \square

Lemma 5.3. *Suppose (θ, λ) is a nice pair with $2^{\aleph_0} < \text{cof}(\lambda) < \theta$. Suppose $\theta^{<\theta} = \theta$ and suppose there is $\kappa > \lambda^{<\theta}$ with $\kappa^{<\kappa} = \kappa$. Suppose $k \geq 3$. Then there is a forcing notion P with $|P| = \kappa$ which is θ -closed, θ^+ -c.c. and which forces: (θ, λ) is a nice pair, and for all T with $< k$ -type amalgamation, $\text{SP}_T(\lambda, \theta)$ holds, and $\text{SP}_{T_{k, k-1}}(\lambda, \theta)$ fails.*

Proof. Note that any P which is θ -closed, θ^+ -c.c. will force that (θ, λ) is a nice pair.

Let $P = P_{\lambda\theta\theta}$. By Theorem 2.8 we can choose a P -name \dot{Q} such that P forces \dot{Q} has the $(< k, \theta, \theta)$ -amalgamation property, and forces $\text{Ax}(< k, \theta)$ to hold, and forces $2^\theta = \kappa$. We claim $P * \dot{Q}$ works.

If T has $< k$ -type amalgamation, then by Theorem 4.6 and Theorem 2.11, $P * \dot{Q}$ forces that $\text{SP}_T^1(\lambda, \theta, \theta)$ holds, thus by Theorem 1.4(A), $P * \dot{Q}$ forces $\text{SP}_T(\lambda, \theta)$.

So it suffices to show $P * \dot{Q}$ forces that $\text{SP}_{T_{k, k-1}}(\lambda, \theta)$ fails. If $k = 3$ then this follows from $\lambda < \lambda^{<\theta}$ and the fact that $T_{3, 2}$ is non-simple, by Theorem 1.5(A). So suppose $k \geq 4$.

By Theorem 5.1 and Theorem 2.10, $P * \dot{Q}$ forces that $\text{SP}_{T_{k, k-1}}^1(\lambda, \mu, \theta)$ fails for all $\mu < \lambda$. By Theorem 1.4(D), $P * \dot{Q}$ forces that $\text{SP}_{T_{k, k-1}}^1(\lambda, \lambda, \theta)$ fails; by Lemma 5.2, $P * \dot{Q}$ forces that $\text{SP}_{T_{k, k-1}}(\lambda, \theta)$ fails. \square

Theorem 5.4. *Suppose GCH holds. Then there is a forcing notion P , which forces: for every $k \geq 3$, if T is a simple theory with $< k$ -type amalgamation, then $T_{k, k-1} \not\leq_{\text{SP}} T$. In particular, the non-simple theories are exactly the maximal \leq_{SP} -theories.*

Of course, we can also force to make GCH hold (via a proper-class forcing notion). Thus, this can consistently hold.

Proof. The “in particular” clause follows since simple theories have < 3 -type amalgamation and non-simple theories are maximal by Theorem 1.5(A).

Choose nice pairs $(\theta_n, \lambda_n) : n < \omega$, such that each $\theta_{n+1} > \lambda_n^{++}$ is regular, and each λ_n is singular with $2^{\aleph_0} < \text{cof}(\lambda_n) < \theta_n$ (so each $\lambda_n^{<\theta_n} = \lambda_n^+$).

We will define a full-support forcing iteration $(P_n : k \leq \omega)$, $(\dot{Q}_n : k < \omega)$; for each $n < \omega$, we will have that $|P_n| \leq \lambda_{n-1}^{++}$, and P_n will force that \dot{Q}_n is θ_n -closed and has the θ_n^+ -c.c. Having defined P_n , we will inductively have that $|P_n| < \theta_n$ and is $(2^{\aleph_0})^+$ -closed, and so P_n forces (θ_n, λ_n) is a nice pair with $2^{\aleph_0} < \text{cof}(\lambda) < \theta_n$ and $\theta_n^{<\theta_n} = \theta_n$ and GCH holds above $|P_n|$. Let \dot{Q}_n be as supplied by Lemma 5.3 with $\theta = \theta_n, \lambda = \lambda_n, \kappa = \lambda_n^{++}, k = n + 3$.

Then each P_{n+1} forces that for all T with $< n+3$ -type amalgamation, $\text{SP}_T(\lambda_n, \theta_n)$ holds, and $\text{SP}_{T_{n+3, n+2}}(\lambda_n, \theta_n)$ fails. Since $\dot{Q}_{\geq n+1}$ is forced to be λ_n^{++} -closed, it does not disturb this, so we are done. □

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