MAXIMAL OPERATORS ASSOCIATED WITH BILINEAR MULTIPLIERS OF LIMITED DECAY

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ABSTRACT. Results analogous to those proved by Rubio de Francia [28] are obtained for a class of maximal functions formed by dilations of bilinear multiplier operators of limited decay. We focus our attention to $L^2 \times L^2 \to L^1$ estimates. We discuss two applications: the boundedness of the bilinear maximal Bochner-Riesz operator and of the bilinear spherical maximal operator. For the latter we improve the known results in [1] by reducing the dimension restriction from $n \geq 8$ to $n \geq 4$.

1. INTRODUCTION

Coifman and Meyer [6, 8, 7] initiated the study of bilinear singular integrals and set the cornerstone of a theory that has recently flourished in view of the breakthrough results in [24, 25] and of the foundational work in [20, 23]. The study of multipliers of limited decay in the bilinear setting, such as of Mihlin-Hörmander type, was initiated in [31] and pursued further in [10, 18, 19, 27] and other works. Many of these results have found weighted extensions in terms of the natural multilinear weights introduced in [26]. Meanwhile, the simple characterization of multipliers bounded on L^2 does not have a bilinear analogue; see [2] and [16].

In this work we investigate the $L^2 \times L^2 \to L^1$ boundedness of maximal operators related to bilinear multipliers with limited decay. This line of investigation was motivated by the study of the bilinear spherical maximal operator introduced in [11] and further studied in [1]; another bilinear version of the spherical maximal operator is studied in [21].

The spherical maximal operator was shown to be L^p bounded by Stein [29] in dimensions $n \geq 3$ (see also [30, Chapter XI]) but its planar version ($n = 2$) was completed by Bourgain [4]. Rubio de Francia [28] introduced a different approach to study this operator in dimensions $n \geq 3$ and proved

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the following theorem concerning general maximal functions that include the spherical maximal operator.

Theorem ([28, Theorem B]). *Let s be an integer with $s > n/2$, let $a > 1/2$, and suppose that m is a function of class $\mathcal{C}^{s+1}(\mathbb{R}^n)$ that satisfies*

$$|D^\alpha m(\xi)| \le C|\xi|^{-a} \qquad \text{for all } |\alpha| \le s+1.$$

Then, $T_m^*(f) = \sup_{t>0} |(m(t\cdot)\widehat{f})^\vee|$ is bounded on $L^p(\mathbb{R}^n)$ for

$$q_a = \frac{2n}{n+2a-1} < p < \frac{2n-2}{n-2a} = r_a$$

(with the understanding that $q_a = 1$ if $a > (n+1)/2$ and $r_a = \infty$ if $a \ge n/2$).

Here $\widehat{f}$ is the Fourier transform of f given by $\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}dx$.

In this paper we are concerned with maximal operators formed by dilations of bilinear multiplier operators of the form

$$T_m(f,g)(x) = \sup_{t>0}\Big|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} m(t\xi,t\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta)}d\xi d\eta\Big|$$

for all Schwartz functions f and g on $\mathbb{R}^n$. Our main result is the following theorem, which presents a bilinear analogue of the aforementioned result of Rubio de Francia.

Theorem 1.1. *Let $a > \frac{n}{2}+1$. Suppose that $m(\xi,\eta) \in C^\infty(\mathbb{R}^{2n})$ satisfies*

$$|\partial^\beta m| \le C_\beta |(\xi,\eta)|^{-a}$$

for all $|\beta| \le [\frac{n}{2}]+2$, where $[\frac{n}{2}]$ is the integer part of $\frac{n}{2}$. Define

$$S_t(f,g)(x) = \int_{\mathbb{R}^{2n}} m(t\xi,t\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta)}d\xi d\eta.$$

Then the bilinear maximal operator defined by

$$\mathcal{M}(f,g) = \sup_{t>0}|S_t(f,g)| \tag{1}$$

is bounded from $L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$.

In studying linear and bilinear spherical maximal operators, we often decompose the multiplier $m = \sum_{j=0}^\infty m_j$ with $m_j = m\psi_j$ for smooth bumps ψ_j supported in annuli $|(\xi,\eta)| \approx 2^j$, $j \ge 1$ and ψ_0 supported in a neighborhood of the origin. We recall the Sobolev space L^r_s of all functions g with $\|(I-\Delta)^{s/2}g\|_{L^r} < \infty$, where Δ is the usual Laplaciand and $s > 0$.

Motivated by Hörmander type conditions, we obtain Theorem 1.1 as a consequence of the following more general result, which is the main contribution of this paper.

Theorem 1.2. *Let $\lambda > 1$, $1 < r \leq 4$, $s > \frac{2n}{r} + 1, j \geq 1$. Suppose that for each $j \in \mathbb{N}$, $M_j(\xi, \eta)$ is a multiplier supported in*

$$\{(\xi, \eta) \in \mathbb{R}^{2n} :\ 2^{j-1} \leq |(\xi, \eta)| \leq 2^{j+1}\},$$

that satisfies

$$\|M_j\|_{L^r_s(\mathbb{R}^{2n})} \leq A 2^{-\lambda j}. \tag{2}$$

Let

$$S_t(f,g)(x) = \int_{\mathbb{R}^{2n}} \sum_{j \geq 0} M_j(t\xi, t\eta) \widehat{f}(\xi) \widehat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta.$$

Then the maximal operator

$$T(f,g) = \sup_{t>0} |S_t(f,g)|$$

is bounded from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ with bound a constant multiple of A.

We prove Theorem 1.2 in Section 3. Below we derive Theorem 1.1.

Proof of Theorem 1.1 assuming Theorem 1.2. We fix a smooth function $\widehat{\varphi}$ supported in $B(0,2)$ whose value is 1 in the unit ball, and define

$$\widehat{\psi}(\cdot) = \widehat{\varphi}(2^{-1} \cdot) - \widehat{\varphi}(\cdot)$$

and

$$m_j(\xi, \eta) = m(\xi, \eta) \widehat{\psi}(2^{-j}(\xi, \eta))$$

for $j \geq 1$, and $m_0 = m - \sum_{j \geq 1} m_j$.

Then m_0 is a compactly supported smooth function, so the corresponding bilinear maximal operator

$$T^*_{m_0}(f,g)(x) = \sup_{t>0} \Big| \int_{\mathbb{R}^{2n}} m_0(t\xi, t\eta) \widehat{f}(\xi) \widehat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta \Big|$$

is bounded by $CM(f)M(g)$, where M is the Hardy-Littlewood maximal function. So T_0 is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for all $1 < p_1, p_2 < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$.

Let $r = 4$, then $\|m_j\|_{L^4_s} \leq C 2^{-ja} 2^{jn/2}$. Hence m_j satisfies conditions of Theorem 1.2 with the decay $\lambda = a - \frac{n}{2} > 1$. Theorem 1.2 then implies that the bilinear maximal operator (1) is bounded from $L^2 \times L^2$ to L^1. $\square$

As an application of Theorem 1.1, we improve the known results concerning the boundedness of the bilinear spherical maximal operator. It was shown in [1] that this operator is bounded from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ for $n \geq 8$. Here we reduce the dimension restriction to $n \geq 4$.

Theorem 1.3. *Let* $m_\alpha(\xi,\eta) = \frac{J_{n+\alpha-1}(2\pi|\xi,\eta|)}{|(\xi,\eta)|^{n+\alpha-1}}$ *for* $\alpha \in \mathbb{R}$*, then the bilinear maximal operator* M_α *defined by*

$$M_\alpha(f,g) = \sup_{t>0} \left| \iint_{\mathbb{R}^{2n}} m_\alpha(t\xi,t\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta)}\,d\xi d\eta \right|$$

is bounded from $L^2 \times L^2$ *to* L^1 *when* $n > 3-2\alpha$*.*

In particular, for $\alpha = 0$*, the bilinear spherical maximal operator*

$$\mathcal{M}_0(f,g) = \sup_{t>0} \left| \int_{\mathbb{S}^{2n-1}} f(x-t\theta)g(x-t\phi)d\sigma(\theta,\phi) \right| \tag{3}$$

is bounded from $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ *to* $L^1(\mathbb{R})$ *when* $n \geq 4$*.*

Proof. The function m_α satisfies the conditions of Theorem 1.1 with $a = n+\alpha-\frac{1}{2}$. Hence when $n > 3-2\alpha$, we obtain the $L^2 \times L^2 \to L^1$ boundedness of M_α. Now recall that the $(2n-1)$-dimensional surface measure σ satisfies $\widehat{d\sigma} = m_0$, i.e., m_α with $\alpha = 0$. Hence the bilinear spherical maximal operator (3) is bounded from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ when $n \geq 4$. □

It was pointed out in [1] that $L^2 \times L^2 \to L^1$ boundedness fails in dimension $n = 1$. As of this writing, we are uncertain about the behavior of this operator in dimensions $n = 2,3$.

We discuss another application of Theorem 1.2 concerning the bilinear maximal Bochner-Riesz means in Section 4.

2. Wavelet decomposition

We use the wavelet decomposition of multipliers as in [15]. So we need to introduce the tensor type wavelets due to [9], and the exact form we use here can be found in [32].

Lemma 2.1 ([32, Section 1.7.3])**.** *For any fixed* $k \in \mathbb{N}$ *there exist real compactly supported functions* $\psi_F, \psi_M \in \mathcal{C}^k(\mathbb{R})$*, which satisfy that* $\|\psi_F\|_{L^2(\mathbb{R})} = \|\psi_M\|_{L^2(\mathbb{R})} = 1$ *and* $\int_{\mathbb{R}} x^\alpha \psi_M(x)dx = 0$ *for* $0 \leq \alpha \leq k$*, such that, if* Ψ^G *is defined by*

$$\Psi^G(\vec{x}) = \psi_{G_1}(x_1)\cdots\psi_{G_{2n}}(x_{2n})$$

for $G = (G_1,\dots,G_{2n})$ *in the set*

$$\mathcal{I} := \left\{(G_1,\dots,G_{2n}) :\ G_i \in \{F,M\}\right\},$$

then the family of functions

$$\bigcup_{\vec{\mu}\in\mathbb{Z}^{2n}} \left[\left\{\Psi^{(F,\dots,F)}(\vec{x}-\vec{\mu})\right\} \cup \bigcup_{\gamma=0}^{\infty} \left\{ 2^{\gamma n}\Psi^G(2^\gamma\vec{x}-\vec{\mu}) :\ G \in \mathcal{I}\setminus\{(F,\dots,F)\} \right\} \right]$$

forms an orthonormal basis of $L^2(\mathbb{R}^{2n})$, where $\vec{x} = (x_1, \ldots, x_{2n})$.

For simplicity, we use often below $\omega(\xi,\eta) = \omega_{k,l}(\xi,\eta)$ to denote the wavelet $2^{\gamma n}\Psi^G(2^\gamma(\xi,\eta) - (k,l))$ when the dilation factor γ is fixed. Moreover we may write $\omega_{k,l}(\xi,\eta) = \omega_{1,k}(\xi)\omega_{2,l}(\eta)$, where

$$\omega_{1,k}(\xi) = 2^{\gamma n/2}\psi_{G_1}(2^\gamma\xi_1 - k_1)\cdots\psi_{G_n}(2^\gamma\xi_n - k_n) \tag{4}$$

and $\omega_{2,l}(\eta)$ is defined in an obvious similar way. For a good function m, we denote by $a_{k,l}$ as the inner product $\langle m, \omega_{k,l}\rangle$ of m and $\omega_{k,l}$.

Let $F^s_{r,q}(\mathbb{R}^{2n})$ and $f^s_{r,q}$ be the Triebel-Lizorkin spaces of functions and sequences, respectively; see [13, Sections 2.2 and 2.3]. To characterize general function spaces, we need the following lemma.

Lemma 2.2 ([32, Theorem 1.64]). *Let $0 < r < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$, and for $\gamma \in \mathbb{N}$ and $\vec{\mu} \in \mathbb{N}^{2n}$ let $\chi_{\gamma\vec{\mu}}$ be the characteristic function of the cube $Q_{\gamma\vec{\mu}}$ centered at $2^{-\gamma}\vec{\mu}$ with length $2^{1-\gamma}$. For a sequence $\eta = \{\eta^{\gamma,G}_{\vec{\mu}}\}$ define the norm*

$$\|\eta|f^s_{r,q}\| = \Big\|\big(\sum_{\gamma,G,\vec{\mu}} 2^{\gamma sq}|\eta^{\gamma,G}_{\vec{\mu}}\chi_{\gamma\vec{\mu}}(\cdot)|^q\big)^{1/q}\Big\|_{L^r(\mathbb{R}^{2n})}.$$

Let $\mathbb{N} \ni k > \max\{s, \frac{4n}{\min(r,q)} + n - s\}$. Let $\Psi^{\gamma,G}_{\vec{\mu}}$ be the $2n$-dimensional Daubechies wavelet with smoothness k as in Lemma 2.1. Let $m \in \mathcal{S}'(\mathbb{R}^{2n})$. Then $m \in F^s_{r,q}(\mathbb{R}^{2n})$ if and only if it can be represented as

$$m = \sum_{\gamma,G,\vec{\mu}} \eta^{\gamma,G}_{\vec{\mu}} 2^{-\gamma n}\Psi^{\gamma,G}_{\vec{\mu}}$$

with $\|\eta|f^s_{rq}\| < \infty$ with unconditional convergence in $\mathcal{S}'(\mathbb{R}^n)$. Furthermore this representation is unique,

$$\eta^{\gamma,G}_{\vec{\mu}} = 2^{\gamma n}\langle m, \Psi^{\gamma,G}_{\vec{\mu}}\rangle,$$

and

$$I : m \to \big\{2^{\gamma n}\langle m, \Psi^{\gamma,G}_{\vec{\mu}}\rangle\big\}$$

is an isomorphism from $F^s_{r,q}(\mathbb{R}^{2n})$ onto $f^s_{r,q}$.

We now return the multipliers considering their wavelet decompositions. Before doing so, we make some comments. The functions ψ_F and ψ_M have compact supports, and all elements in a fixed level, i.e., of the same dilation factor γ, in the basis come from translations of finitely many products, so their supports have finite overlaps. Consequently we can classify the elements in the basis into finitely many classes so that all elements in the same level in each class have distant supports, which means that if ω and ω' are in the same class with the same dilation parameter γ, then $5\text{ supp }\omega \cap 5\text{ supp }\omega' = \emptyset$, where $5\text{ supp }\omega = B(c_0, 5d)$ with c_0 inside the support of ω and d the diameter of the support of ω. So, from now on, we will assume that the supports of ω's related to a given dilation factor γ are far disjoint.

For the multiplier M_j in Theorem 1.2, we have a wavelet decomposition using Lemma 2.2, i.e.

$$M_j = \sum a_\omega \omega, \tag{5}$$

where the summation is over all $\omega = \Psi_{\vec{\mu}}^{\gamma,G}$ in the orthonormal basis described in Lemma 2.1, the order of cancellations of ψ_M is $M = 4n+6$, and $a_\omega = \langle M_j, \omega\rangle$.

Concerning the size of $a_\omega = a_{k,l}$, we have the following estimate.

Corollary 2.3. *The coefficient a_ω in* (5) *related to ω with dilation γ is bounded by $C2^{-j\lambda}2^{-(s+n-\frac{2n}{r})\gamma}$.*

Proof. Since $F^s_{r,2}(\mathbb{R}^{2n}) = L^r_s(\mathbb{R}^{2n})$, we have

$$\left\| \Big(\sum_{k,l} 2^{2\gamma s} |2^{\gamma n} a_{k,l} \chi_{Q_{\gamma,k,l}}|^2 \Big)^{1/2} \right\|_{L^r} \le C \|M_j\|_{L^r_s},$$

by Lemma 2.2, where $Q_{\gamma,k,l}$ is the cube centered at $2^{-\gamma}(k,l)$ with length $2^{1-\gamma}$. Take just one term on the left hand side, and notice that $|Q_{\gamma,k,l}| \sim 2^{-2n\gamma}$, then

$$|a_{k,l}| \le CA2^{-j\lambda}2^{-(s+n)\gamma}2^{2n\gamma/r} \le CA2^{-j\lambda}2^{-\gamma(s+n-\frac{2n}{r})}.$$

□

With the wavelet decompositions in hand, we are able to prove Theorem 1.2. The proof is inspired by [14] and the square function technique (see [5] and [28]). We control

$$T_j = \sup_{t>0} \left| \iint_{\mathbb{R}^{2n}} M_j(t\xi, t\eta) \widehat{f}(\xi) \widehat{g}(\eta) e^{2\pi i x\cdot(\xi+\eta)} d\xi d\eta \right| \tag{6}$$

by two integrals with the diagonal and the off-diagonal parts. For the diagonal part we have just one term, which can be handled using product wavelets. For the off-diagonal parts we introduce two square operators with each one bounded by a product of the Hardy-Littlewood maximal function and a linear operator bounded on $L^2(\mathbb{R}^n)$.

We need to decompose M_j further. Take N to be a fixed large enough number so that $N/10$ is greater than d, the diameters of the support of ω with dilation factor $\gamma = 0$. We write $\omega(\xi,\eta) = \omega_{\vec{\mu}}(\xi,\eta) = \omega_{1,k}(\xi)\omega_{2,l}(\eta)$, where $\vec{\mu} = (k,l)$ with $k,l \in \mathbb{Z}^n$, and denote the corresponding coefficient $\langle \omega_{k,l}, M_j \rangle$ by $a_{k,l}$. We define

$$M_j^1 = \sum_{\gamma \geq 0} M_{j,\gamma}^1 = \sum_{\gamma} \sum_{|k| \geq N} \sum_{|l| \geq N} a_{k,l} \omega_{1,k} \omega_{2,l} \tag{7}$$

$$M_j^2 = \sum_{\gamma \geq 0} M_{j,\gamma}^2 = \sum_{\gamma} \sum_{k} \sum_{|l| \leq N} a_{k,l} \omega_{1,k} \omega_{2,l} \tag{8}$$

$$M_j^3 = \sum_{\gamma \geq 0} M_{j,\gamma}^3 = \sum_{\gamma} \sum_{|k| \leq N} \sum_{|l| \geq N} a_{k,l} \omega_{1,k} \omega_{2,l}. \tag{9}$$

Here M_j^1 is the diagonal part such that the support of each level is away from both ξ and η axes, M_j^2 is the off-diagonal part with each level's support near the ξ axis, and the support of each level of M_j^3 is near the η axis.

Remark 1. This decomposition is more delicate than that in [1]. and allows us to handle more singular operators. Actually for each fix γ, the supports of the wavelets in M_j^2 related to γ are contained in $\{(\xi,\eta) : |\eta| \leq Nd2^{-\gamma}\}$, while the corresponding part in [1] is contained in $\{(\xi,\eta) : |\eta| \leq 2^{j\varepsilon}\}$.

Corresponding to $M_{j,\gamma}^i$, $i = 1,2,3$, we define

$$B_{j,\gamma,t}^i(f,g)(x) = \iint_{\mathbb{R}^{2n}} M_{j,\gamma}^i(t\xi, t\eta) \widehat{f}(\xi) \widehat{g}(\eta) e^{2\pi i x \cdot (\xi+\eta)} d\xi d\eta.$$

We can define $B_{j,\gamma,t}$ in a similar way and $B_{j,\gamma,t}(f,g)(x) = \sum_{i=1}^3 B_{j,\gamma,t}^i(f,g)(x)$. Moreover we can define $T_{j,\gamma}^i$ in the way similar to (6) so that $T_j = \sum_{i=1}^3 \sum_\gamma T_{j,\gamma}^i$.

3. Proof of Theorem 1.2

For $f,g \in \mathcal{S}(\mathbb{R}^n)$, using the fundamental theorem of Calculus, we rewrite

$$\begin{aligned} &B_{j,\gamma,t}^1(f,g)(x) \\ &= \iint_{\mathbb{R}^{2n}} M_{j,\gamma}^1(t\xi, t\eta) \widehat{f}(\xi) \widehat{g}(\eta) e^{2\pi i x \cdot (\xi+\eta)} d\xi d\eta \\ &= \int_0^t \iint_{\mathbb{R}^{2n}} (s\xi, s\eta) \cdot \nabla M_{j,\gamma}^1(s\xi, s\eta) \widehat{f}(\xi) \widehat{g}(\eta) e^{2\pi i x \cdot (\xi+\eta)} d\xi d\eta \frac{ds}{s}, \end{aligned}$$

where the existence of $\nabla M_{j,\gamma}^1$ is guaranteed by that all components in $M_{j,\gamma}^1$ are contained in the same level.

Define the operator related to $(s\xi, s\eta) \cdot \nabla M_{j,\gamma}^1(s\xi, s\eta)$ as

$$\tilde{B}_{j,\gamma,s}^1(f,g)(x) = \iint_{\mathbb{R}^{2n}} (s\xi, s\eta) \cdot \nabla M_{j,\gamma}^1(s\xi, s\eta) \widehat{f}(\xi) \widehat{g}(\eta) e^{2\pi i x \cdot (\xi+\eta)} d\xi d\eta.$$

Then we have the pointwise estimate

$$T^1_{j,\gamma}(f,g)(x) = \sup_{t>0} |B^1_{j,\gamma,t}(f,g)(x)| \le \int_0^\infty |\widetilde{B}^1_{j,\gamma,t}(f,g)(x)| \frac{dt}{t} \tag{10}$$

We now turn to the study of the boundedness of $\widetilde{B}^1_{j,\gamma,t}$. The basic idea is the observation that when $r \in (1,4)$, [14, Remark 2] shows that whenever σ is supported in $B(0,R)$ we have

$$\|T_\sigma(f,g)\|_{L^2\times L^2\to L^1} \le C\|\sigma\|_{L^r_s}$$

with C independent of R.

To make this argument rigorous, in the case $t=1$, we have the following estimate, whose proof can be found in the Appendix (Section 5).

Proposition 3.1. *Let $E = \{\xi \in \mathbb{R}^n : C2^{-\gamma} \le |\xi| \le 2^j\}$. Then we have*

$$\|\widetilde{B}^1_{j,\gamma,1}(f,g)\|_{L^1} \le CAC(j,\gamma)\|\widehat{f}\chi_E\|_{L^2}\|\widehat{g}\chi_E\|_{L^2}, \tag{11}$$

with

$$\begin{aligned} C(j,\gamma) &= n(j+\gamma)2^{-j(\lambda-1)}2^{\gamma(1+\frac{n}{2}-s)} \quad \text{when } r=4 \\ C(j,\gamma) &= 2^{-j(\lambda-1)}2^{\gamma(1+\frac{2n}{r}-s)} \quad \text{when } 1<r<4. \end{aligned}$$

In both cases we have good decay in j.

Corollary 3.2. *For the diagonal part we have*

$$\|T^1_{j,\gamma}(f,g)\|_{L^1} \le CAC(j,\gamma)(j+\gamma)\|f\|_{L^2}\|g\|_{L^2}.$$

Proof. From (10) we know that

$$\|T^1_{j,\gamma}(f,g)\|_{L^1} \le \int_0^\infty \|\widetilde{B}^1_{j,\gamma,t}(f,g)\|_{L^1}\frac{dt}{t} = \int_0^\infty \|\widetilde{B}^1_{j,\gamma,1}(f_t,g_t)\|_{L^1}\frac{dt}{t},$$

where $\widehat{f_t}(\xi) = t^{-n/2}\widehat{f}(\xi/t)$, and $\widehat{g_t}(\xi) = t^{-n/2}\widehat{g}(\xi/t)$. Applying Proposition 3.1, the last integral is dominated by

$$\begin{aligned} &CA\int_0^\infty C(j,\gamma)\|\widehat{f_t}\chi_E\|_{L^2}\|\widehat{g_t}\chi_E\|_{L^2}\frac{dt}{t} \\ \le &CC(j,\gamma)\Big(\int_{\mathbb{R}^n}\int_0^\infty |\widehat{f_t}\chi_E|^2\frac{dt}{t}d\xi\Big)^{1/2}\Big(\int_{\mathbb{R}^n}\int_0^\infty |\widehat{g_t}\chi_E|^2\frac{dt}{t}d\xi\Big)^{1/2}. \end{aligned}$$

The double integral involving f_t is bounded by $\int |\widehat{f}(\xi)|^2 \int_{C2^{-\gamma}/|\xi|}^{2^j/|\xi|}\frac{dt}{t}d\xi$, which is less than $C(j+\gamma)\|f\|^2_{L^2}$. Hence the last expression is controlled by $CAC(j,\gamma)(j+\gamma)\|f\|_{L^2}\|g\|_{L^2}$. □

We next deal with the off-diagonal parts. More specifically, we consider $B^2_{j,\gamma,t}$, since the analysis of $B^3_{j,\gamma,t}$ is similar in view of symmetry. Recall that

$$B^2_{j,\gamma,t} = \iint_{\mathbb{R}^{2n}} M^2_{j,\gamma}(t\xi,t\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta)}d\xi d\eta.$$

We denote $(\xi,\eta)\cdot(\nabla M^2_{j,\gamma})(\xi,\eta)$ by $\widetilde{M}^2_{j,\gamma}(\xi,\eta)$. Then similar to $B^2_{j,t}(f,g)(x)$ we define

$$\widetilde{B}^2_{j,\gamma,t}(f,g)(x) = \iint_{\mathbb{R}^{2n}} \widetilde{M}^2_{j,\gamma}(t\xi,t\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta)}d\xi d\eta.$$

With these notations, by the fundamental theorem of Calculus, we have

$$\begin{aligned}(B^2_{j,\gamma,t}(f,g)(x))^2 &= 2\int_0^t B^2_{j,\gamma,s}(f,g)(x)s\frac{dB^2_{j,\gamma,s}(f,g)(x)}{ds}\frac{ds}{s}\\ &\le 2\int_0^\infty |B^2_{j,\gamma,s}(f,g)(x)||\widetilde{B}^2_{j,\gamma,s}(f,g)(x)|\frac{ds}{s}\\ &\le 2G_{j,\gamma}(f,g)(x)\widetilde{G}_{j,\gamma}(f,g)(x),\end{aligned}$$

where we set

$$G_{j,\gamma}(f,g)(x) = \left(\int_0^\infty |B^2_{j,\gamma,s}(f,g)(x)|^2\frac{ds}{s}\right)^{1/2}$$
$$\widetilde{G}_{j,\gamma}(f,g)(x) = \left(\int_0^\infty |\widetilde{B}^2_{j,\gamma,s}(f,g)(x)|^2\frac{ds}{s}\right)^{1/2}.$$

These g-functions are bounded from $L^2\times L^2$ to L^1 with good decay in j. Indeed, we have the following.

Lemma 3.3. *For any $\varepsilon>0$ there exists a constant C_ε independent of j such that for all $f,g\in\mathcal{S}(\mathbb{R}^n)$,*

$$\|G_{j,\gamma}(f,g)\|_{L^1}\le C_\varepsilon A2^{-j\lambda}2^{(n+1)\gamma}\|f\|_{L^2}\|g\|_{L^2}$$

and

$$\|\widetilde{G}_{j,\gamma}(f,g)\|_{L^1}\le C_\varepsilon A2^{-j(\lambda-1)}2^{(n+1)\gamma}\|f\|_{L^2}\|g\|_{L^2}.$$

The proof of this lemma is inspired by [15].

Proof. We will focus on $\widetilde{G}_{j,\gamma}$ first. For $\widetilde{G}_{j,\gamma}$ we need to consider two typical cases, the derivative falling on ξ and the derivative falling on η.

Let us consider the multiplier

$$\xi_1\partial_{\xi_1}M^2_{j,\gamma} = \sum_{k,l} a_{k,l}v_k(\xi)\omega_{2,l}(\eta)$$

with $v_k(\xi)=\xi_1\partial_{\xi_1}\omega_{1,k}(\xi)$. Using (4) we observe that

$$|v_k(\xi)|\le C2^j2^{\gamma n/2}2^\gamma = C2^j2^{\gamma(n+2)/2}.$$

The g-function related to $\xi_1\partial_{\xi_1}M^2_{j,\gamma}$ is denoted by $\widetilde{G}^1_{j,\gamma}(f,g)$. By the definition (8), for a fixed γ at most N of $\omega_{2,l}$ are involved, so we can consider a single fixed l. Observe that

$$\begin{aligned}&\iint_{\mathbb{R}^{2n}}\sum_k a_{k,l}v_k(\xi)\omega_{2,l}(\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi ix\cdot(\xi,\eta)}d\xi d\eta\\ =&\|a\|_{\ell^\infty}2^{\gamma(n+2)/2}2^j\left(\int_{\mathbb{R}^n}\omega_{2,l}(\eta)\widehat{g}(\eta)e^{2\pi ix\cdot\eta}d\eta\right)\\ &\qquad\qquad\int_{\mathbb{R}^n}\frac{\sum_k a_{k,l}v_k(\xi)}{\|a\|_\infty 2^{\gamma(n+2)/2}2^j}\widehat{f}(\xi)e^{2\pi ix\cdot\xi}d\xi.\end{aligned}$$

By $|v_k|\le C2^{\gamma(n+2)/2}2^j$, $|a_{k,l}|\le\|a\|_{\ell^\infty}$, and the disjointness of the supports of v_k, we know that $\sigma(\xi):=(\sum_k a_{k,l}v_k(\xi))/(\|a\|_{\ell^\infty}2^{\gamma(n+2)/2}2^j)$ is a compactly supported bounded function. Hence the bilinear operator related to the multiplier $\sum_k a_{k,l}v_k(\xi)\omega_{2,l}(\eta)$ is pointwise bounded by

$$C\|a\|_{\ell^\infty}2^{\gamma(n+1)}2^jM(g)(x)T_\sigma(f)(x), \tag{12}$$

where $T_\sigma(f)$ satisfies that $\|T_\sigma(f)\|_{L^2}\le C\|\widehat{f}\chi_F\|_{L^2}$ with

$$F=\{\xi\in\mathbb{R}^n:2^{j-1}\le|\xi|\le 2^{j+1}\}.$$

The operator $\widetilde{G}^1_{j,\gamma}$ is then bounded from $L^2\times L^2$ to L^1. Indeed we can estimate it by a standard dilation argument as follows. Setting $\widehat{f_s}(\xi)=s^{-n/2}\widehat{f}(\xi/s)$, and $\widehat{g}_s(\xi)=s^{-n/2}\widehat{g}(\xi/s)$, we have

$$\begin{aligned}&\int_{\mathbb{R}^n}\widetilde{G}^1_{j,\gamma}(f,g)(x)dx\\ =&\int_{\mathbb{R}^n}\left[\int_0^\infty\Big|\iint_{\mathbb{R}^{2n}}\sum_{k,l}a_{k,l}v_k(s\xi)\omega_{2,l}(s\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi ix\cdot(\xi,\eta)}d\xi d\eta\Big|^2\frac{ds}{s}\right]^{\frac12}dx\\ =&\int_{\mathbb{R}^n}\left[\int_0^\infty\Big|\iint_{\mathbb{R}^{2n}}\sum_{k,l}a_{k,l}v_k(\xi)\omega_{2,l}(\eta)\widehat{f_s}(\xi)\widehat{g}_s(\eta)e^{2\pi i\frac{x}{s}\cdot(\xi,\eta)}\frac{d\xi d\eta}{s^n}\Big|^2\frac{ds}{s}\right]^{\frac12}dx.\end{aligned}$$

Since there are only finitely many l in the sum above, we can use the pointwise estimate (12) to estimate the last displayed expression by

$$\begin{aligned}&C\|a\|_{\ell^\infty}2^j2^{(n+1)\gamma}\int_{\mathbb{R}^n}\left(\int_0^\infty\Big|s^{-n/2}M(g)(x)T_\sigma(f_s)(s^{-1}x)\Big|^2\frac{ds}{s}\right)^{1/2}dx\\ \le&C\|a\|_{\ell^\infty}2^j2^{(n+1)\gamma}\|M(g)\|_{L^2}\left(\int_0^\infty\int_{\mathbb{R}^n}s^{-n}|\widehat{f}(\xi/s)|^2\chi_F(\xi)d\xi\frac{ds}{s}\right)^{\frac12}\\ \le&C\|a\|_{\ell^\infty}2^j2^{(n+1)\gamma}\|g\|_{L^2}\left(\int_{\mathbb{R}^n}|\widehat{f}(\xi)|^2\int_{(2^{j-1})/|\xi|}^{(2^{j+1})/|\xi|}\frac{ds}{s}d\xi\right)^{\frac12}.\end{aligned}$$

The integral with respect to s is $\log \frac{2^{j+1}}{2^{j-1}} \le C$. This, combined with the bound of $\|a\|_{\ell^\infty} \le CA2^{-j\lambda}2^{-(s+n-\frac{2n}{r})\gamma}$ obtained in Corollary 2.3, shows that the last displayed expression is smaller than

$$(13)\quad C\|a\|_{\ell^\infty}2^j 2^{(n+1)\gamma}\|g\|_{L^2}\|f\|_{L^2} \le CA2^{-j(\lambda-1)}2^{-\gamma(s-\frac{2n}{r}-1)}\|g\|_{L^2}\|f\|_{L^2}.$$

When the derivative falls on η, for example we have differentiation with respect to η_1, using the notation $v_l(\eta) = \eta_1\partial_{\eta_1}\omega_{2,l}(\eta)$ we have a similar representation

$$\iint_{\mathbb{R}^{2n}} \sum_k a_{k,l}\omega_{1,k}(\xi)v_l(\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i x\cdot(\xi,\eta)}d\xi d\eta$$
$$=\|a\|_{\ell^\infty}2^{\gamma n/2}\left(\int_{\mathbb{R}^n} v_l(\eta)\widehat{g}(\eta)e^{2\pi i x\cdot\eta}d\eta\right)\int_{\mathbb{R}^n}\frac{\sum_k a_{k,l}\omega_{1,k}(\xi)}{\|a\|_{\ell^\infty}2^{\gamma n/2}}\widehat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi.$$

The integral in the parenthesis in the last line is dominated by $2^{\gamma n/2}M(g)(x)$ as both $\partial_1(\omega_{2,l})^\vee(x)e^{2\pi i x\cdot l}$ and $(\omega_{2,l})^\vee(x)l_1e^{2\pi i x\cdot l}$ are Schwartz functions, and the number of the second type of functions is finite because $|l| \le N$. The bilinear operator related to the multiplier $\sum_k a_{k,l}\omega_{1,k}(\xi)v_l(\eta)$ is therefore bounded by

$$C\|a\|_{\ell^\infty}2^{\gamma n}M(g)(x)T_{\sigma'}(f)(x),$$

where $T_{\sigma'}$ satisfies the same property as T_σ. For the L^1 norm of the g-function $\widetilde{G}^2_{j,\gamma}$ related to the multiplier $\sum_k a_{k,l}\omega_{1,k}(\xi)v_l(\eta)$ we apply an argument similar to that used for $\|\widetilde{G}^1_{j,\gamma}\|_{L^1}$. We obtain

$$(14)\qquad \|\widetilde{G}^2_{j,\gamma}\|_{L^1} \le CA2^{-j\lambda}2^{-\gamma(s-\frac{2n}{r})}\|g\|_{L^2}\|f\|_{L^2}.$$

This estimate and (13) show that

$$\|\widetilde{G}_{j,\gamma}(f,g)\|_{L^1} \le CA2^{-j(\lambda-1)}2^{-\gamma(s-\frac{2n}{r}-1)}\|f\|_{L^2}\|g\|_{L^2}.$$

For $G_{j,\gamma}(f,g)$ an analogous, but simpler argument, applied to the standard representation $\sum a_{k,l}\omega_{1,k}(\xi)\omega_{2,l}(\eta)$ yields

$$\|G_{j,\gamma}(f,g)\|_{L^1} \le CA2^{-j\lambda}2^{-\gamma(s-\frac{2n}{r}-1)}\|f\|_{L^2}\|g\|_{L^2}.$$

The additional decay of 2^{-j} comes from the fact that in the multiplier of $B^2_{j,\gamma,s}$ we miss the term (ξ,η), which is controlled by 2^j. □

Corollary 3.4. *For the off-diagonal part the estimate below holds:*

$$(15)\qquad \|T^2_{j,\gamma}(f,g)\|_{L^1} \le CA2^{-j(\lambda-1/2)}2^{-\gamma(s-\frac{2n}{r}-1)}\|f\|_{L^2}\|g\|_{L^2}.$$

Proof. By the calculation before Lemma 3.3 we have the pointwise control

$$T^2_{j,\gamma}(f,g)(x) \le \sqrt{2}(G_{j,\gamma}(f,g)(x)\widetilde{G}_{j,\gamma}(f,g)(x))^{1/2},$$

which, combined with Lemma 3.3, implies that

$$\begin{aligned}\|T^2_j(f,g)\|_{L^1} \le & \big\|\sqrt{2}\big(G_{j,\gamma}(f,g)\widetilde{G}_{j,\gamma}(f,g)\big)^{1/2}\big\|_{L^1}\\ \le & C(\|G_{j,\gamma}(f,g)\|_{L^1}\|\widetilde{G}_{j,\gamma}(f,g)\|_{L^1})^{1/2}\\ \le & CA\Big(2^{-j\lambda}2^{-j(\lambda-1)}2^{-2\gamma(s-\frac{2n}{r}-1)}\|f\|^2_{L^2}\|g\|^2_{L^2}\Big)^{1/2}\\ = & CA2^{-j(\lambda-1/2)}2^{-\gamma(s-\frac{2n}{r}-1)}\|f\|_{L^2}\|g\|_{L^2}.\end{aligned}$$

In this case we have nice decay in j for $T^2_{j,\gamma}$ since $\lambda > 1 > 1/2$. □

We collect the known results to finish the proof of Theorem 1.2.

Proof of Theorem 1.2. We observe that

$$T(f,g)(x) \le \sum_{j=0}^{\infty}\sum_{\gamma}|T_{j,\gamma}(f,g)(x)|.$$

It is straightforward to verify that

$$\sum_{\gamma} C(j,\gamma)(j+\gamma) \le C_\varepsilon 2^{-j(\lambda-1-\varepsilon)} \le C_\varepsilon 2^{-j(\lambda-1)/2},$$

if we choose ε small enough. So we obtain

$$\sum_j\sum_\gamma \|T^1_{j,\gamma}(f,g)\|_{L^1} \le \sum_j CA2^{-j\frac{\lambda-1}{2}}\|f\|_{L^2}\|g\|_{L^2} \le CA\|f\|_{L^2}\|g\|_{L^2}. \tag{16}$$

This concludes the argument of the diagonal part. A similar argument using (15) show the boundedness of the off-diagonal part. Hence we deduce the conclusion of Theorem 1.2. □

4. Applications to bilinear maximal Bochner-Riesz

Theorem 1.2 can also be used to study the boundedness of the maximal bilinear Bochner-Riesz means. These are the means

$$A^\lambda_t(f,g)(x) = \iint_{\mathbb{R}^{2n}} \widehat{f}(\xi)\widehat{g}(\eta)\big(1-|t\xi|^2-|t\eta|^2\big)^\lambda_+ e^{2\pi i x\cdot(\xi+\eta)}d\xi d\eta, \tag{17}$$

which coincide with $B^\lambda_{1/t}(f\otimes g)(x,x)$ with $B^\lambda_{1/t}$ the linear Bochner-Riesz operator on $\mathbb{R}^{2n}$ and $x \in \mathbb{R}^n$. For test functions we should have $A^\lambda_t(f,g) \to fg$ as $t \to 0$ in the L^p or in the pointwise sense. [17, 3, 22] have proved positive results for $\lambda = 0$ and $\lambda > 0$ respectively, concerning their L^p convergence.

In this section, we are concerned with the pointwise convergence of the means (17), in particular with the boundedness of the maximal bilinear

Bochner-Riesz operator, which of course implies the boundedness of the bilinear Bochner-Riesz operators in the same range.

The bilinear maximal Bochner-Riesz operator for $\lambda > 0$ is defined as

$$T_*^\lambda(f,g)(x) = \sup_{t>0}\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} m^\lambda(t\xi,t\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta)}d\xi d\eta\right|, \tag{18}$$

where $m^\lambda(\xi,\eta) = (1-|\xi|^2-|\eta|^2)_+^\lambda$, which is equal to $(1-(|\xi|^2+|\eta|^2))^\lambda$ when $|(\xi,\eta)| \le 1$ and 0 when $|(\xi,\eta)| > 1$.

Our main theorem concerning the boundedness of bilinear maximal Bochner-Riesz means is as follows:

Theorem 4.1. *When $\lambda > \frac{2n+3}{4}$, for T_*^λ in* (18) *we have that*

$$\|T_*^\lambda(f,g)\|_{L^1} \le C\|f\|_{L^2}\|g\|_{L^2}.$$

We fix a nonnegative smooth function $\varphi(s)$ supported in $[-\frac{3}{4},\frac{3}{4}]$ and a smooth function ψ supported in $[\frac{1}{8},\frac{5}{8}]$ such that $\sum_{j=0}^\infty \psi_j(1-s) = 1$ for $s \in [0,1)$, where $\psi_j(s) = \psi(2^j s)$ for $j \ge 1$ and $\psi_0 = \varphi$.

We decompose the multiplier $m(\xi,\eta) = (1-(|\xi|^2+|\eta|^2))_+^\lambda$ smoothly as $m = \sum_{j\ge 0} m_j$, where

$$m_j(\xi,\eta) = m(\xi,\eta)\psi_j(|(\xi,\eta)|)$$

is supported in an annulus of the form

$$\{(\xi,\eta) \in \mathbb{R}^{2n} : 1-2^{-j} \le |(\xi,\eta)| \le 1-2^{-j-2}\}$$

for $j \ge 1$ and m_0 is supported in a ball of radius $3/4$ centered at the origin. If

$$T_j(f,g)(x) = \sup_{t>0}\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \widehat{f}(\xi)\widehat{g}(\eta)m_j(t\xi,t\eta)e^{2\pi i x\cdot(\xi+\eta)}d\xi d\eta\right|, \tag{19}$$

then

$$T_*^\lambda(f,g)(x) \le \sum_{j=0}^\infty T_j(f,g)(x).$$

The following are straighforward facts about T_*^λ and T_j. Let $\|T\|_{X\times Y\to Z}$ denote the norm of T from $X \times Y$ to Z.

Proposition 4.2. *Assume $1 < p_1, p_2 \le \infty$, and $1/p = 1/p_1 + 1/p_2$. Then for $\lambda > n-1/2$, there exists a finite constant $C = C(p_1,p_2)$ such that $\|T_*\|_{L^{p_1}\times L^{p_2}\to L^p} \le C$. For any fixed j, there exists a finite constant $C_j(p_1,p_2)$ such that $\|T_j\|_{L^{p_1}\times L^{p_2}\to L^p} \le C_j(p_1,p_2)$.*

Proof. Let us consider the kernel $K(y,z) = m^\vee(y,z)$ of A_1^λ defined in (17), which satisfies that $|K(y,z)| \le C(1+|y|+|z|)^{-(n+\lambda+1/2)}$ (see, for example, [12]), hence for $\lambda > n-1/2$, we have

$$\begin{aligned} |A_t(f,g)(x)| = & \left| \int_{\mathbb{R}^{2n}} t^{-2n} K\big(\frac{x-y}{t}, \frac{x-z}{t}\big) f(y) g(z) dydz \right| \\ \le & \ C(\varphi_t * |f|)(x)(\varphi_t * |g|)(x) \\ \le & \ CM(f)(x)M(g)(x), \end{aligned}$$

where M is the Hardy-Littlewood maximal function, and $\varphi_t(y) = t^{-n}\varphi(y/t)$ with $\varphi(y) = (1+|y|)^{-(n+\lambda+1/2)/2}$, which is integrable when $\lambda > n-1/2$. Then $T_*(f,g)(x) \le CM(f)(x)M(g)(x)$, which implies that $\|T_*(f,g)\|_{L^p} \le C(p_1,p_2)\|f\|_{L^{p_1}}\|g\|_{L^{p_2}}$ for $1 < p_1, p_2 \le \infty$ with $1/p = 1/p_1 + 1/p_2$ in view of the boundedness of the Hardy-Littlewood maximal function.

We observe that each m_j is smooth and compactly supported, hence for each j a similar argument yields $\|T_j\|_{L^{p_1}\times L^{p_2}\to L^p} \le C_j(p_1,p_2) < \infty$. □

With the aid of the preceding decomposition and the boundedness of T_j, the study of the boundedness of T_* is reduced to the decay of C_j in j.

We now go back to the multipliers and will apply Theorem 1.2. For this purpose we should study kinds of norms of m_j.

Lemma 4.3. *There exists a constant C such that*

$$\|m_j\|_{L^2} \le C2^{-j(\lambda+1/2)},$$

and for any multiindex α*,*

$$\|\partial^\alpha m_j\|_{L^\infty} \le C_\alpha 2^{-j(\lambda-|\alpha|)}. \tag{20}$$

Proof. A change of variables using polar coordinates implies that

$$\begin{aligned} \|m_j\|_{L^2} =& \Big(\int_{\mathbb{R}^{2n}} |m_j(\xi,\eta)|^2 d\xi d\eta \Big)^{1/2} \\ \le & C\Big(\int_{1-2^{-j}}^{1-2^{-j-2}} (1-r^2)^{2\lambda} r^{2n-1} dr \Big)^{1/2} \\ \le & C(2^{-2j\lambda}2^{-j})^{1/2} \\ =& C2^{-j(\lambda+1/2)} \end{aligned}$$

To estimate the α-th derivatives, we use the Leibniz's rule to write

$$\partial^\alpha m_j(\xi,\eta) = \sum_{\alpha_1+\alpha_2=\alpha} C_{\alpha_1} \partial^{\alpha_1} m(\xi,\eta) \partial^{\alpha_2} \psi_j(|(\xi,\eta)|).$$

Noticing that $|\partial^{\alpha_1} m(\xi,\eta)| \le C2^{-j(\lambda-|\alpha_1|)}$ and $\partial^{\alpha_2}\psi_j(|(\xi,\eta)|) \le C2^{j|\alpha_2|}$, we derive the bound $\partial^\alpha m_j(\xi,\eta)$ by $C2^{-j(\lambda-|\alpha|)}$. □

The multiplier m_j is not supported in the annulus of radius 2^j and one can verify that its Sobolev norm is not as good as would wish. Actually the norm increases as the number of derivatives is large. So a dilation is necessary to apply Theorem 1.2.

Let us define $M_j(\xi,\eta) = m_j(2^{-j}\xi, 2^{-j}\eta)$, which is supported in the annulus $\{(\xi,\eta)\in\mathbb{R}^{2n} : 2^j - 1 \le |(\xi,\eta)| \le 2^j - 1/4\}$, whose width is $3/4$. Based on Lemma 4.3, we have the following corollary.

Corollary 4.4. *The multipliers $M_j(\xi,\eta) = m_j(2^{-j}\xi, 2^{-j}\eta)$ satisfy*

$$\|\partial^\alpha M_j\|_{L^\infty} \le C2^{-j\lambda} \qquad \text{for all multiindex } \alpha,$$

$$\nabla m_j(\xi,\eta) = 2^j(\nabla M_j)(2^j\xi, 2^j\eta),$$

and

$$\|M_j\|_{L^r_s} \le C2^{-j\lambda}2^{j(2n-1)/r},$$

where $\|M_j\|_{L^r_s} = \|(I-\Delta)^{s/2}M_j\|_{L^r}$ is the Sobolev norm of M_j.

Proof. We have

$$|\partial^\alpha M_j| \le 2^{-j|\alpha|}|(\partial^\alpha m_j)(2^{-j}\xi, 2^{-j}\eta)| \le C2^{-j|\alpha| - j(\lambda-|\alpha|)} = C2^{-j\lambda},$$

using (20). The verification of the last identity is straightforward once we notice that M_j is supported in the annulus

$$\{(\xi,\eta) : 2^j - 4 \le |(\xi,\eta)| \le 2^j - 1\}$$

whose volume is about $2^{j(2n-1)}$. □

Proof of Theorem 4.1. It is easy to verify that T_j in (19) stays the same if we replace m_j by M_j. We apply Theorem 1.2 to M_j with $r = 4$, then it follows from this and Proposition 4.2 that T_* is bounded from $L^2 \times L^2$ to L^1 when $\lambda > \frac{2n+3}{4}$. □

Using complex interpolation between Theorem 4.1 and Proposition 19 we can obtain a larger range of boundedness, which we will not pursue here.

As a corollary of Theorem 4.1, we obtain the pointwise convergence, as $t \to 0$, of the operator $A_t^\lambda(f,g)(x)$, which we denote by $A_t(f,g)(x)$ as well.

Proposition 4.5. *Suppose $\lambda > \frac{2n+3}{4}$, then for $f \in L^2$ and $g \in L^2$ we have*

$$\lim_{t\to 0} A_t(f,g)(x) \to f(x)g(x) \qquad a.e.. \tag{21}$$

The proof of this proposition is similar to the linear case, but we sketch it here for completeness.

Proof. It is easy to establish (21) when both f and g are Schwartz functions. To prove (21) for $f \in L^2$ and $g \in L^2$ it suffices to show that for any given $\delta > 0$ the set $E_{f,g}(\delta) = \{y \in \mathbb{R}^n : O_{f,g}(y) > \delta\}$ has measure 0, where

$$O_{f,g}(y) = \limsup_{\theta \to 0} \limsup_{\varepsilon \to 0} \left| A_\theta(f,g)(y) - A_\varepsilon(f,g)(y) \right|.$$

For any positive number η smaller than $\|f\|_{L^2}$, $\|g\|_{L^2}$, there exist Schwartz functions $f_1 = f - a$ and $g_1 = g - b$ such that both $\|a\|_{L^2}$, and $\|b\|_{L^2}$ are bounded by η. We observe that

$$|E_{f,g}(\delta)| \le |E_{f_1,g_1}(\delta/4)| + |E_{a,g_1}(\delta/4)| + |E_{f_1,b}(\delta/4)| + |E_{a,b}(\delta/4)|.$$

Notice that $|E_{f_1,g_1}(\delta/4)| = 0$ since (21) is valid for f_1, g_1. To control the remaining three terms, we observe that, for instance,

$$\begin{aligned} |E_{a,g_1}(\delta/4)| \le\ & |\{y :\ 2T_*(a,g_1)(y) > \delta/4\}| \\ \le\ & C\frac{\|a\|_{L^2}\|g_1\|_{L^2}}{\delta} \\ \le\ & C\frac{\eta\|g\|_{L^2}}{\delta}, \end{aligned}$$

where the last term goes to 0 as $\eta \to 0$ since g and δ are fixed. □

5. Appendix: Proof of Proposition 3.1

The proof of this proposition is essentially contained in [14, Lemma 6], but for the sake of completeness we include it, ignoring some routine calculations that can be found in [14].

Proof of Proposition 3.1. Notice that in the support of $\nabla M^1_{j,\gamma}(\xi,\eta)$, we have $\xi \in E$ and $\eta \in E$, hence we may alway assume that $\widehat{f} = \widehat{f}\chi E$ and $\widehat{g} = \widehat{g}\chi_E$. In other words, it suffices to establish (11) without χ_E.

It suffices to consider, for example, the typical term $\xi_1 \partial_{\xi_1} M^1_{j,\gamma}(\xi,\eta)$, which is $\sum_k \sum_l a_{k,l} \partial_{\xi_1} \omega_{1,k}(\xi)\xi_1 \omega_{2,l}(\eta)$ for allowed k, l in $M^1_{j,\gamma}$. We rewrite this as

$$2^j 2^\gamma \sum_k \sum_l b_{k,l} \tilde{\omega}_{1,k}(\xi) \tilde{\omega}_{2,l}(\eta), \tag{22}$$

where $\tilde{\omega}_{1,k}(\xi) = 2^{-j}\partial_{\xi_1}\omega_{1,k}(\xi)\xi_1 / \|\partial_{\xi_1}\omega_{1,k}(\xi)\|_{L^r}$, $\tilde{\omega}_{2,l} = \omega_{2,l}/\|\omega_{2,l}\|_{L^r}$, and $b_{k,l} = 2^{-\gamma} a_{k,l} \|\partial_{\xi_1}\omega_{1,k}(\xi)\|_{L^r} \|\omega_{2,l}\|_{L^r}$.

We need some estimates of $\tilde{\omega}_{1,k}$ which will be useful later. The function $\partial_{\xi_1}\omega_{1,k}(\xi)$ is of the form $2^\gamma 2^{\gamma n/2} \varphi(2^\gamma \xi)$ for a compactly supported smooth function φ, hence $\|\partial_{\xi_1}\omega_{1,k}(\xi)\|_{L^r} \approx 2^{\gamma(1+\frac{n}{2}-\frac{n}{r})}$. This implies that $\|\tilde{\omega}_{1,k}\|_{L^\infty} \le C 2^{\gamma n/r}$ since $|\xi_1| \le C 2^j$.

We have

$$\left\|\left(\sum_{k,l}2^{\gamma s}|2^{\gamma n}a_{k,l}\chi_{Q_{\gamma,k,l}}|^2\right)^{1/2}\right\|_{L^r}\le C\|M_j\|_{L^r_s},$$

by Lemma 2.2, where $Q_{\gamma,k,l}$ is the cube centered at $2^{-\gamma}(k,l)$ with length $2^{1-\gamma}$. This leads to

$$\left\|\left(\sum_{k,l}2^{\gamma s}|2^{-j}2^{-\gamma}a_{k,l}\partial_{\xi_1}\omega_{1,k}(\xi)\xi_1\omega_{2,l}(\eta)|^2\right)^{1/2}\right\|_{L^r}\le C\|M_j\|_{L^r_s}.$$

Recall that $\|M_j\|_{L^r_s}\le C2^{-j\lambda}$. Then using the disjointness of supports of $\omega_{k,l}$ we obtain further that

$$B=(\sum|b_{k,l}|^r)^{1/r}\le C2^{-j\lambda}2^{-s\gamma}.$$

Each ω in level γ is of the form $\omega=\omega_k\omega_l$ with $\vec{\mu}=(k,l)$, where k and l both range over index sets of cardinality at most $C2^{jn}2^{\gamma n}$. Moreover we denote by b_{kl} the coefficient b_ω, and we define a bilinear multiplier

$$\varsigma_\gamma=\sum_{k\in U_1}\tilde{\omega}_k\sum_{l\in U_2}b_{kl}\tilde{\omega}_l.$$

Let A be a number between $\|b\|_\infty$ and $B=\|b\|_r$. Related to $\tau\ge 0$ we define $U_\tau=\{(k,l):2^{-\tau-1}A\le|b_{k,l}|\le 2^{-\tau}A\}$. Denote by $col_k=\{(k,l)\in U_\tau: k \text{ fixed}\}$. Define

$$U_\tau^1=\{(k,l)\in U_\tau:\#col_k\ge N_1\},$$

where N_1 is a to be determined number. So U_τ^1 is a union of long columns. We denote by $P_1U_\tau^1=\{k:\exists\, l\ s.t.\ (k,l)\in U_\tau^1\}$, the projection of U_τ^1 onto the k-axis. Then the number of columns is $\#P_1U_\tau^1\le B^r(2^{-\tau}A)^{-r}N_1^{-1}:=N_2$.

Let U_τ^2 be the complement of U_τ^1 in U_τ. Associated to U_τ^i we can define a bilinear multiplier $\varsigma_\tau^i=2^j2^\gamma\sum_{(k,l)\in U_\tau^i}b_{k,l}\tilde{\omega}_{k,l}$, and a bilinear operator $T_{\varsigma_\tau^i}$. A well-known argument (see, for instance, [15] or [14]) shows that

$$\|T_{\varsigma_\tau^1}(f,g)\|_{L^1}\le C2^j2^\gamma N_2^{1/2}2^{2\gamma n/r}2^{-\tau}A\|f\|_{L^2}\|g\|_{L^2}$$

and

$$\|T_{\varsigma_\tau^2}(f,g)\|_{L^1}\le C2^j2^\gamma N_1^{1/2}2^{2\gamma n/r}2^{-\tau}A\|f\|_{L^2}\|g\|_{L^2}.$$

Identifying N_1 and N_2, and taking $A=B$ in our situation, we obtain that $N_1=N_2=C2^{\tau r/2}$, which implies that the $\|T_{\varsigma_\tau^i}\|_{L^2\times L^2\to L^1}$ is bounded by $C2^{-j(\lambda-1)}2^{-\gamma(s-\frac{2n}{r}-1)}2^{-\tau(1-\frac{r}{4})}$.

Summing over τ, we obtain the claimed bound for $r<4$.

For the case $r=4$, we may assume that $\tau\le\tau_m=2(j+\gamma)n/4$ since $N_2=2^{\tau r/2}\le 2^{(j+\gamma)n}$ with $r=4$. Actually we define

$$U_{\tau_m}=\{(k,l):|b_{k,l}|\le 2^{-\tau_m}A\}.$$

Then the previous argument gives the bound $C(j+\gamma)n2^{-j(\lambda-1)}2^{-\gamma(s-\frac{n}{2}-1)}$ when $r=4$. □

A lemma concerning the decay of the coefficients related to the orthonormal basis in Lemma 2.1 is given below.

Lemma 5.1 ([15]). *Suppose $\varsigma(\xi,\eta)$ defined on $\mathbb{R}^{2n}$ satisfies that there exists a constant C_M such that $\|\partial^\alpha(\varsigma(\xi,\eta))\|_{L^\infty}\le C_M$ for each multiindex $|\alpha|\le M$, where M is the number of vanishing moments of ψ_M. Then for any nonnegative integer $\gamma\in\mathbb{N}_0=\{n\in\mathbb{Z}:n\ge 0\}$ we have*

$$|\langle\Psi^{\gamma,G}_{\vec{\mu}},\varsigma\rangle|\le CC_M2^{-(M+n)\gamma}. \tag{23}$$

This lemma can be proved by applying Appendix B.2 in [13], and we delete the details which can be found in [15].

By this lemma we have a better decay in j for $b_{k,l}$ compared with Corollary 2.3, namely $|b_{k,l}|\le C2^{-ja}2^{-\gamma(s+n)}$, using $|\partial^\beta m|\le C|(\xi,\eta)|^{-a}$ in Theorem 1.1 if we assume s number of derivatives. It is natural to conjecture that this better decay in j can lower the restriction on a. This, unfortunately, is not true.

As we did before, setting $N_1=N_2$ implies that $N_1=2^{\tau r/2}$. An important observation is that $|b_{k,l}|\ll B$. Actually the smallest τ such that $2^{-\tau}B\sim\|b_{k,l}\|_{\ell^\infty}\le C2^{-ja}2^{-\gamma(s+n)}$ is $\tau_0=\frac{2nj}{r}+n\gamma$, which means that the summation in τ starts from τ_0 other than 0.

Another observation is that N_2 related to τ_0 is $2^{\tau_0 r/2}\sim 2^{nj+n\gamma r/2}$, which is smaller than $2^{\tau_m r/2}\sim 2^{nj+n\gamma}$ when $r>2$. So for $r\in(2,4)$, we take $N_2=2^{nj+n\gamma}$. And the summation in τ consists just one term τ_0.

By the calculation in the proof of Proposition 3.1 the norm of $T_{\varsigma_\gamma}\sum_\tau T_{\varsigma_\tau^i}$, which consists of one term with $\tau=\tau_0$, is bounded by a constant multiple of $2^{-j(a-n/2-1)}2^{-\gamma(s-n/2-1)}$. This provides no new information except for a bound independent of r, which is natural since there is no r in the conditions of Theorem 1.1.

So we still have the restriction $a>\frac{n}{2}+1$. It is also easy to verify that when $r=4$ the bound for $T_{\varsigma_\tau^i}$ does not change, so in this case we need $a>\frac{n}{2}+1$ as well.

Remark 2. We use mainly the case $r=4$ in applying Proposition 3.1, while a smaller A, which reduces the number of τ's involved, does not change the exponential decay in j at all.

Remark 3. Lemma 5.1 implies also a better decay of the off-diagonal part in j, namely $2^{-j(a-1/2)}$, which, however, is useless for us due to the restriction of the diagonal part.

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