

FURSTENBERG SYSTEMS OF BOUNDED MULTIPLICATIVE FUNCTIONS AND APPLICATIONS

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ABSTRACT. We prove a structural result for measure preserving systems naturally associated with any finite collection of multiplicative functions that take values on the complex unit disc. We show that these systems have no irrational spectrum and their building blocks are Bernoulli systems and infinite-step nilsystems. One consequence of our structural result is that strongly aperiodic multiplicative functions satisfy the logarithmically averaged variant of the disjointness conjecture of Sarnak for a wide class of zero entropy topological dynamical systems, which includes all uniquely ergodic ones. We deduce that aperiodic multiplicative functions with values plus or minus one have super-linear block growth. Another consequence of our structural result is that products of shifts of arbitrary multiplicative functions with values on the unit disc do not correlate with any totally ergodic deterministic sequence of zero mean. Our methodology is based primarily on techniques developed in a previous article of the authors where analogous results were proved for the Möbius and the Liouville function. A new ingredient needed is a result obtained recently by Tao and Teräväinen related to the odd order cases of the Chowla conjecture.

1. INTRODUCTION AND MAIN RESULTS

1.1. **Main results.** A function $f: \mathbb{N} \rightarrow \mathbb{C}$ is called *multiplicative* if

$$f(mn) = f(m)f(n) \quad \text{whenever } (m, n) = 1.$$

Perhaps the most well-known example of a bounded multiplicative function is the Möbius function, which is defined to be 0 on integers divisible by a square, -1 on square-free integers with an odd number of prime factors, and 1 elsewhere. Its non-zero values are expected to fluctuate between -1 and 1 in a random way and many famous conjectures have been formulated based on this belief. One example that has received a lot of attention in recent years is the Möbius disjointness conjecture of Sarnak [27, 28]. It asserts that the Möbius function does not correlate with any bounded deterministic sequence, meaning, any sequence that is produced by a continuous function evaluated along the orbit of a point in a zero entropy topological dynamical system.

In [14] we verified the logarithmically averaged variant of the conjecture of Sarnak for a wide class of deterministic sequences. Our approach was to study measure preserving systems (which we call Furstenberg systems) naturally associated with the Möbius function; in particular, we studied structural properties that allow to deduce disjointness from a wide class of zero entropy systems. Various interesting results, establishing non-correlation of deterministic sequences with the Möbius function and products of its shifts, are natural consequences of these disjointness results.

The main purpose of this article is to extend the approach from [14] in order to cover multiplicative functions that take values on the complex unit disc $\mathbb{U} := \{z \in \mathbb{C} : |z| \leq 1\}$. Our first main result concerns a class of multiplicative functions that are expected to satisfy similar disjointness properties as the Möbius function. These are the strongly aperiodic multiplicative functions (see Definition 2.9), and we verify that they do not

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correlate with a wide class of deterministic sequences. Throughout the paper, we denote by (Y, R) a *topological dynamical system*, meaning, a compact metric space Y together with a continuous homeomorphism $R: Y \rightarrow Y$.

Theorem 1.1. *Let $f: \mathbb{N} \rightarrow \mathbb{U}$ be a strongly aperiodic multiplicative function. Let (Y, R) be a topological dynamical system with zero topological entropy and at most countably many ergodic invariant measures. Then for every $y \in Y$ and every $g \in C(Y)$ we have*

$$\lim_{N \rightarrow \infty} \frac{1}{\log N} \sum_{n=1}^N \frac{g(R^n y) f(n)}{n} = 0.$$

Furthermore, the convergence is uniform in $y \in Y$.

Remarks. • Using rotations on finite cyclic groups, one deduces that non-correlation (using logarithmic averages) of f with all periodic sequences (which implies strong aperiodicity in the real valued case) is a necessary assumption for the conclusion to hold.

• We believe that the countability assumption on the number of ergodic invariant measures of (Y, R) can be dropped. In the case where f is the Möbius function, this is equivalent to the logarithmically averaged variant of the Sarnak conjecture.

An interesting consequence of the previous result is a statement about the block complexity of multiplicative functions $f: \mathbb{N} \rightarrow \mathbb{U}$ that have finite range. In the next statement we denote by $P_f(n)$ the number of patterns of size n that are taken by consecutive values of f (see Section 5.3 for a more formal definition).

Theorem 1.2. *If the multiplicative function $f: \mathbb{N} \rightarrow \mathbb{U}$ has finite range, is strongly aperiodic, and does not converge to zero in logarithmic density, then $\lim_{n \rightarrow \infty} \frac{P_f(n)}{n} = \infty$.*

Remarks. • In fact, we establish a stronger statement, if $a: \mathbb{N} \rightarrow \mathbb{C}$ has finite range and linear block growth, then $\lim_{N \rightarrow \infty} \frac{1}{\log N} \sum_{n=1}^N \frac{a(n) f(n)}{n} = 0$ for every strongly aperiodic multiplicative function $f: \mathbb{N} \rightarrow \mathbb{U}$. Thus, even if we modify the values of f on a set of logarithmic density 0, using values taken from a finite set of complex numbers, the new sequence still has super-linear block growth.

• The assumptions are satisfied if f takes only the values ± 1 and is aperiodic, meaning, it does not correlate with any periodic sequence. Previously, it was not even known that for such multiplicative functions we have $\lim_{n \rightarrow \infty} (P_f(n) - n) = \infty$. On the other hand, a conjecture of Elliott [9, 10] predicts if $f: \mathbb{N} \rightarrow \{-1, 1\}$ is aperiodic, then $P_f(n) = 2^n$ for every $n \in \mathbb{N}$, and if $f: \mathbb{N} \rightarrow \mathbb{U}$ has finite range, is strongly aperiodic, and does not converge to zero in logarithmic density, then $P_f(n)$ grows exponentially.

Henceforth, whenever needed, we assume that a multiplicative function $f: \mathbb{N} \rightarrow \mathbb{U}$ is extended to the integers in an arbitrary way.

In the next result if (Y, R) is a topological dynamical system, we say that a point $y \in Y$ is generic for logarithmic averages for a Borel probability measure ν on Y if for every $g \in C(Y)$ we have $\lim_{N \rightarrow \infty} \frac{1}{\log N} \sum_{n=1}^N \frac{g(R^n y)}{n} = \int g d\nu$. Our methods also allow us to prove non-correlation between products of shifts of arbitrary multiplicative functions with values on the unit disc and totally ergodic deterministic sequences of zero mean.

Theorem 1.3. *Let $f_1, \dots, f_\ell: \mathbb{N} \rightarrow \mathbb{U}$ be multiplicative functions. Let (Y, R) be a topological dynamical system and let $y \in Y$ be generic for logarithmic averages for a measure ν with zero entropy and at most countably many ergodic components, all of which are totally ergodic. Then for every $g \in C(Y)$ that is orthogonal in $L^2(\nu)$ to all R -invariant functions we have*

$$(1) \quad \lim_{N \rightarrow \infty} \frac{1}{\log N} \sum_{n=1}^N \frac{g(R^n y) \prod_{j=1}^{\ell} f_j(n + h_j)}{n} = 0$$

for all $h_1, \dots, h_\ell \in \mathbb{Z}$.

Remarks. • The unweighted version of (1) (take $g := 1$) is expected to hold if the shifts are distinct and at least one of the multiplicative functions is strongly aperiodic. This is the logarithmically averaged variant of a conjecture of Elliott [9, 10] (see [26, Theorem B.1] for a corrected version and the need to assume strong aperiodicity).

• If (Y, R) has zero topological entropy and is uniquely and totally ergodic, then it is easy to deduce from Theorem 1.3 that (1) holds for all $g \in C(Y)$ and $y \in Y$ such that $\lim_{N \rightarrow \infty} \frac{1}{\log N} \sum_{n=1}^N \frac{g(R^n y)}{n} = 0$.

• If (Y, ν, R) is totally ergodic, then using an approximation argument one can also conclude that (1) holds for every $g: Y \rightarrow \mathbb{C}$ that is Riemann integrable with respect to the measure ν and $\int g d\nu = 0$.¹

Theorem 1.3 is new even in the very special case where R is given by an irrational rotation on \mathbb{T} and $g(t) := e^{2\pi it}$, $t \in \mathbb{T}$. In this case we have $g(R^n 0) = e^{2\pi i n \alpha}$, $n \in \mathbb{N}$, for some irrational α , and we get the following result as a consequence:

Corollary 1.4. *Let $f_1, \dots, f_\ell: \mathbb{N} \rightarrow \mathbb{U}$ be multiplicative functions and let $\alpha \in \mathbb{R}$ be irrational. Then*

$$(2) \quad \lim_{N \rightarrow \infty} \frac{1}{\log N} \sum_{n=1}^N \frac{e^{2\pi i n \alpha} \prod_{j=1}^{\ell} f_j(n + h_j)}{n} = 0$$

for all $h_1, \dots, h_\ell \in \mathbb{Z}$.

Remarks. • For $\ell = 1$ the result is the logarithmically averaged variant of a classical result of Daboussi [5, 6, 7]. But even for $\ell = 2$ the result is new.

• More generally, if we apply Theorem 1.3 for R given by appropriate totally ergodic affine transformations on a torus with the Haar measure (as in [16, Section 3.3]), we get that (2) holds with $(e^{2\pi i n \alpha})_{n \in \mathbb{N}}$ replaced by any sequence of the form $(e^{2\pi i P(n)})_{n \in \mathbb{N}}$, where $P \in \mathbb{R}[t]$ has at least one non-constant coefficient irrational. Moreover, one could use as weights zero mean sequences arising from more general totally ergodic nilsystems, giving rise to generalized polynomial sequences. One such example is the sequence $(e^{2\pi i [n\alpha]n\beta})_{n \in \mathbb{N}}$, where $\alpha, \beta \in \mathbb{R}$ are rationally independent. In order to establish this variant, one has to use Theorem 1.3 (see the last remark following this result) for a totally ergodic nilsystem (Y, R) defined on the Heisenberg nilmanifold and an appropriate Riemann integrable function g with respect to the Haar measure on Y with zero integral (see [2, Section 0.14] for details).

A key step in the proof of the previous results is a structural result for measure preserving systems naturally associated with any collection of multiplicative functions that take values on the complex unit disc. We call such systems *Furstenberg systems*, and they are defined as follows: For convenience, let f be a multiplicative function that takes values on a finite subset A of \mathbb{U} and admits correlations for logarithmic averages on a sequence of intervals $\mathbf{N} = ([N_k])_{k \in \mathbb{N}}$ with $N_k \rightarrow \infty$ (see Definition 2.1). Then the Furstenberg system associated with f and \mathbf{N} is defined on the sequence space $X := A^{\mathbb{Z}}$ with the shift transformation, by a measure that assigns to each cylinder set $\{x \in X: x(j) = a_j, j = -m, \dots, m\}$ value equal to the logarithmic density, taken along the sequence \mathbf{N} , of the set $\{n \in \mathbb{N}: f(n + j) = a_j, j = -m, \dots, m\}$, where $a_{-m}, \dots, a_m \in A$ and $m \in \mathbb{N}$. Similarly, one defines Furstenberg systems associated with any finite collection of multiplicative functions $f_1, \dots, f_\ell: \mathbb{N} \rightarrow \mathbb{U}$ and a sequence of intervals \mathbf{N} on which f_1, \dots, f_ℓ admit correlations for logarithmic averages; we call these measure preserving systems *joint Furstenberg systems*. The precise constructions are given in Section 2.3 and are motivated by analogous constructions made by Furstenberg in [16, 17] in order to restate Szemerédi’s theorem on arithmetic progressions in ergodic terms.

¹We say that a function $g: Y \rightarrow \mathbb{C}$ is Riemann integrable with respect to the measure ν if for every $\varepsilon > 0$ there exist $g_1, g_2 \in C(Y)$ such that $g_1(y) \leq g(y) \leq g_2(y)$ for every $y \in Y$ and $\int (g_2 - g_1) d\nu \leq \varepsilon$.

We prove the following structural result for joint Furstenberg systems of multiplicative functions:

Theorem 1.5. *A joint Furstenberg system of the multiplicative functions $f_1, \dots, f_\ell: \mathbb{N} \rightarrow \mathbb{U}$ is a factor of a system that*

- (i) *has no irrational spectrum;*
- (ii) *has ergodic components isomorphic to direct products of infinite-step nilsystems and Bernoulli systems.*

Remarks. • We refer the reader to Section 2 and Appendix A of [14] for the definition of the ergodic notions used in the previous statement.

• The product decomposition depends on the ergodic component, in particular, the infinite-step nilsystems depend on the ergodic component.

See Section 1.3 for more refined conjectural statements regarding the structure of joint Furstenberg systems of multiplicative functions with values on the unit disc.

1.2. Proof strategy. Our general strategy in the proofs of Theorems 1.1-1.3 is similar to the one used in [14] to cover the case of the Möbius and the Liouville functions, but there are also some serious additional difficulties that we have to overcome. Our main focus is to prove the structural result stated in Theorem 1.5; then Theorems 1.1-1.3 are consequences of this result and the deduction is carried out using joining arguments in Section 5 (Theorem 1.1 also uses additional number theory input provided by Theorem 2.10). The first step in the proof of Theorem 1.5 is to apply the identity of Theorem 3.1 which allows to express an arbitrary joint correlation of multiplicative functions as a weighted average of their dilated joint correlations taken over all prime dilates (this step necessitates the use of logarithmic averages). This leads, via the correspondence principle of Furstenberg (see Proposition 2.3), to certain ergodic identities that any joint Furstenberg system (X, μ, T) of these multiplicative functions satisfies.

The next goal is to utilize the ergodic identities in order to prove the structural properties of Theorem 1.5. Unfortunately, the presence of some unwanted weights, which appear because the multiplicative functions are not constant on primes, creates serious problems that do not allow us to continue as in [14], especially when the multiplicative functions take infinitely many distinct values on primes. The way to overcome this obstacle is to first utilize a recent result of Tao and Teräväinen, which proves that joint correlations of multiplicative functions vanish if the product of the multiplicative functions is far from being periodic. This result enables us to obtain a variant of the identity in Theorem 3.1, which has the additional property that all the weights are equal to 1 (see Corollary 3.7). As a consequence, we get an ergodic identity, stated in Theorem 3.8, which allows to show that the system (X, μ, T) is a factor of some system of arithmetic progressions with steps given by all primes in an appropriate congruence class (see Definition 4.1). Finally, this system can be easily linked to a system of arithmetic progressions with prime steps (see Lemma 4.4). The structure of these systems was studied in [14] and they were shown to satisfy the structural properties of Theorem 1.5. Combining the above facts we get a proof of Theorem 1.5.

A simpler and more elementary way to link Furstenberg systems of multiplicative functions to systems of arithmetic progressions with primes steps is explained in Section 4.2; but this simpler approach only works if the range of the multiplicative functions on the primes is a subset of the unit interval or a finite subset of the complex unit disc.

1.3. Further remarks and conjectures. The structural result of Theorem 1.5 is not expected to be optimal and we give below some more refined conjectural structural statements.

In what follows, unless explicitly specified, a Bernoulli system is allowed to be the trivial one point system. Moreover, an ergodic procyclic system (often referred to as

an odometer) is an ergodic inverse limit of periodic systems, or equivalently, an ergodic system (X, μ, T) for which the rational eigenfunctions span a dense subspace of $L^2(\mu)$.

Conjecture 1. *If the multiplicative functions f_1, \dots, f_ℓ take values in $[-1, 1]$ or in a finite subset of \mathbb{U} , then they have a unique joint Furstenberg system,² which is ergodic and isomorphic to the direct product of a procyclic system and a Bernoulli system.*

This generalizes [14, Conjecture 1], which concerned Furstenberg systems of a single multiplicative function $f: \mathbb{N} \rightarrow [-1, 1]$. If we further restrict to the case where f takes values in $\{-1, 1\}$, then we conjectured in [14, Conjecture 2] that f should have a unique Furstenberg system, which is either an ergodic procyclic system or a Bernoulli system. Combining [1, Theorem 1.7] and [7, Theorem 6] we get that the first alternative holds if f is not aperiodic (this happens if and only if $\mathbb{D}(f, \chi) < \infty$ for some Dirichlet character χ , see terminology in Section 2.5). We expect that the second alternative holds exactly when f is aperiodic. This is known to be the case conditionally to the assumption that all Furstenberg systems of f are ergodic [12, Corollary 1.5]. Unconditionally, this is not even known for the Liouville function; it is equivalent to the logarithmically averaged variant of the Chowla conjecture.

Perhaps surprisingly, multiplicative functions with values on the unit circle may have non-ergodic Furstenberg systems, in fact, with uncountably many ergodic components. Consider for instance the multiplicative function $f(n) := n^{it}$, $n \in \mathbb{N}$, for some non-zero $t \in \mathbb{R}$. We claim that it has a unique Furstenberg system (X, μ, T) , which is isomorphic to the system $(\mathbb{T}, m_{\mathbb{T}}, R)$, where R is the identity transformation on \mathbb{T} . Indeed, let $G_0(y) := e^{2\pi iy}$, $y \in \mathbb{T}$, and $X := \mathbb{U}^{\mathbb{Z}}$, T be the shift transformation on X , $F_0(x) := x(0)$, $x \in X$, and $\mu := \lim_{N \rightarrow \infty} \frac{1}{\log N} \sum_{n=1}^N \frac{\delta_{f(n)}}{n}$ (we show below that the weak-star limit exists). We claim that $\Phi: \mathbb{T} \rightarrow X$, defined by $\Phi(y) := (e^{2\pi i n y})_{n \in \mathbb{Z}}$, $y \in \mathbb{T}$, is an isomorphism between the systems $(\mathbb{T}, m_{\mathbb{T}}, R)$ and (X, μ, T) . The map Φ is clearly one to one and satisfies $T \circ \Phi = \Phi \circ R$. It remains to show that $\mu = m_{\mathbb{T}} \circ \Phi^{-1}$. Notice first, that due to the slowly varying nature of n^{it} , for fixed $h \in \mathbb{Z}$ we have $(n+h)^{it} - n^{it} \rightarrow 0$ as $n \rightarrow \infty$. Using this and the fact that $\lim_{N \rightarrow \infty} \frac{1}{\log N} \sum_{n=1}^N \frac{n^{it}}{n} = 0$ for $t \neq 0$, we get that for every $m \in \mathbb{N}$ and $k_{-m}, \dots, k_m \in \mathbb{Z}$, we have

$$\int_X \prod_{j=-m}^m T^{h_j} F_0^{k_j} d\mu = \lim_{N \rightarrow \infty} \frac{1}{\log N} \sum_{n=1}^N \frac{\prod_{j=-m}^m f^{k_j}(n+h_j)}{n} = \int_{\mathbb{T}} \prod_{j=-m}^m T^{h_j} G_0^{k_j} dm_{\mathbb{T}},$$

since the second and third terms are either simultaneously 0 or 1 depending on whether $\sum_{j=-m}^m k_j \neq 0$ or $\sum_{j=-m}^m k_j = 0$. Using this identity and the fact that $G_0 = F_0 \circ \Phi$ we get that $\mu = m_{\mathbb{T}} \circ \Phi^{-1}$, completing the proof that Φ is an isomorphism.

Similarly, if $f(n) := n^{it} \mu(n)$, $n \in \mathbb{N}$, where $t \neq 0$ and μ is the Möbius function, then we expect (but cannot prove) that f has a unique Furstenberg system with uncountably many ergodic components, all of them isomorphic to a direct product of a non-trivial procyclic system and a non-trivial Bernoulli system. It seems likely that a similar structural result holds for general multiplicative functions with values on the unit disc:

Conjecture 2. *A joint Furstenberg system of any multiplicative functions $f_1, \dots, f_\ell: \mathbb{N} \rightarrow \mathbb{U}$ has ergodic components isomorphic to direct products of procyclic systems and Bernoulli systems.*

1.4. Notation. For readers convenience, we gather here some notation that we use frequently throughout the article.

We write \mathbb{T} for the unit circle, which we often identify with \mathbb{R}/\mathbb{Z} , and we write \mathbb{U} for the complex unit disc. We denote by \mathbb{N} the set of positive integers, by \mathbb{P} the set of prime

²This is equivalent to the statement that all sequences of the form $(\prod_{j=1}^m g_j(n+h_j))_{n \in \mathbb{N}}$ have logarithmic averages, where $g_1, \dots, g_m \in \{f_1, \dots, f_\ell, \overline{f_1}, \dots, \overline{f_\ell}\}$ and $m, h_1, \dots, h_m \in \mathbb{N}$ are arbitrary.

numbers, and for $d \in \mathbb{N}$ we denote by \mathbb{P}_d the set $\mathbb{P} \cap (d\mathbb{N} + 1)$. For $N \in \mathbb{N}$ we denote by $[N]$ the set $\{1, \dots, N\}$. Whenever we write \mathbf{N} we mean a sequence of intervals of integers $([N_k])_{k \in \mathbb{N}}$ with $N_k \rightarrow \infty$.

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2. BACKGROUND IN ERGODIC THEORY AND NUMBER THEORY

2.1. Notation regarding averages. If A is a non-empty finite subset of \mathbb{N} we let

$$\mathbb{E}_{n \in A} a(n) := \frac{1}{|A|} \sum_{n \in A} a(n), \quad \mathbb{E}_{n \in A}^{\log} a(n) := \frac{1}{\sum_{n \in A} \frac{1}{n}} \sum_{n \in A} \frac{a(n)}{n}.$$

If A is an infinite subset of \mathbb{N} we let

$$\mathbb{E}_{n \in A} a(n) := \lim_{N \rightarrow \infty} \mathbb{E}_{n \in A \cap [N]} a(n), \quad \mathbb{E}_{n \in A}^{\log} a(n) := \lim_{N \rightarrow \infty} \mathbb{E}_{n \in A \cap [N]}^{\log} a(n)$$

if the limits exist. Let $\mathbf{N} = ([N_k])_{k \in \mathbb{N}}$ be a sequence of intervals with $N_k \rightarrow \infty$. We let

$$\mathbb{E}_{n \in \mathbf{N}} a(n) := \lim_{k \rightarrow \infty} \mathbb{E}_{n \in [N_k]} a(n), \quad \mathbb{E}_{n \in \mathbf{N}}^{\log} a(n) := \lim_{k \rightarrow \infty} \mathbb{E}_{n \in [N_k]}^{\log} a(n)$$

if the limits exist. Using partial summation one sees that if $\mathbb{E}_{n \in \mathbb{N}} a(n) = 0$, then also $\mathbb{E}_{n \in \mathbb{N}}^{\log} a(n) = 0$. The converse does not hold in general, and the forward implication fails if one replaces \mathbb{N} with general sequences $\mathbf{N} = ([N_k])_{k \in \mathbb{N}}$ with $N_k \rightarrow \infty$.

2.2. Measure preserving systems. Throughout the article, we make the standard assumption that all probability spaces (X, \mathcal{X}, μ) considered are Lebesgue, meaning, X can be given the structure of a compact metric space and \mathcal{X} is its Borel σ -algebra. A *measure preserving system*, or simply *a system*, is a quadruple (X, \mathcal{X}, μ, T) where (X, \mathcal{X}, μ) is a probability space and $T: X \rightarrow X$ is an invertible, measurable, measure preserving transformation. We typically omit the σ -algebra \mathcal{X} and write (X, μ, T) . Throughout, for $n \in \mathbb{N}$ we denote by T^n the composition $T \circ \dots \circ T$ (n times) and let $T^{-n} := (T^n)^{-1}$ and $T^0 := \text{id}_X$. Also, for $f \in L^1(\mu)$ and $n \in \mathbb{Z}$ we denote by $T^n f$ the function $f \circ T^n$.

In order to avoid unnecessary repetition, we refer the reader to the article [14] for some other standard notions from ergodic theory. In particular, the reader will find in Section 2 and in Appendix A of [14] the definition of the terms factor, Kronecker factor, isomorphism, inverse limit, spectrum, rational and irrational spectrum, ergodicity, ergodic components, total ergodicity, nilsystem, infinite-step nilsystem, Bernoulli system, joining, and disjoint systems; all these notions are used subsequently.

2.3. Furstenberg systems associated with several sequences. To each finite collection of sequences $a_1, \dots, a_\ell: \mathbb{N} \rightarrow \mathbb{U}$ that are distributed “regularly” along a sequence of intervals, we associate a measure preserving system defined using the joint distribution of the sequences a_1, \dots, a_ℓ . For the purposes of this article, all averages in the definition of joint Furstenberg systems are taken to be logarithmic.

Definition 2.1. Let $\mathbf{N} := ([N_k])_{k \in \mathbb{N}}$ be a sequence of intervals with $N_k \rightarrow \infty$. We say that the sequences $a_1, \dots, a_\ell: \mathbb{Z} \rightarrow \mathbb{U}$ admit *log-correlations on \mathbf{N}* , if the limits

$$(3) \quad \lim_{k \rightarrow \infty} \mathbb{E}_{n \in [N_k]}^{\log} \prod_{j=1}^m b_j(n + h_j)$$

exist for all $m \in \mathbb{N}$, all $h_1, \dots, h_m \in \mathbb{Z}$, and all $b_1, \dots, b_m \in \{a_1, \dots, a_\ell, \overline{a_1}, \dots, \overline{a_\ell}\}$.

Remarks. • Given $a_1, \dots, a_\ell: \mathbb{Z} \rightarrow \mathbb{U}$, using a diagonal argument, we get that every sequence of intervals $\mathbf{N} = ([N_k])_{k \in \mathbb{N}}$ has a subsequence $\mathbf{N}' = ([N'_k])_{k \in \mathbb{N}}$, such that the sequences a_1, \dots, a_ℓ admit log-correlations on \mathbf{N}' .

• If the sequences a_1, \dots, a_ℓ are only defined on \mathbb{N} , then we extend them in an arbitrary way to \mathbb{Z} and give analogous definitions. Then all the limits in (3) do not depend on the choice of the extension.

Definition 2.2. Let (X, T) be a topological dynamical system. We say that the collection of functions $F_1, \dots, F_\ell \in C(X)$ is *T-generating* if the functions $T^n F_1, \dots, T^n F_\ell$, $n \in \mathbb{Z}$, separate points of X .

Remark. By the Stone-Weierstrass theorem, the functions $F_1, \dots, F_\ell \in C(X)$ are *T-generating* if and only if the T, T^{-1} -invariant subalgebra generated by F_1, \dots, F_ℓ and $\overline{F_1}, \dots, \overline{F_\ell}$ is dense in $C(X)$ with the uniform topology.

We use the following variant of the correspondence principle of Furstenberg [16, 17] that applies to finite collections of bounded sequences of complex numbers:

Proposition 2.3. *Let $a_1, \dots, a_\ell: \mathbb{Z} \rightarrow \mathbb{U}$ be sequences that admit log-correlations on $\mathbf{N} := ([N_k])_{k \in \mathbb{N}}$. Then there exist a topological dynamical system (X, T) , a T -invariant Borel probability measure μ , and a T -generating collection of functions $F_{0,1}, \dots, F_{0,\ell} \in C(X)$, such that*

$$(4) \quad \mathbb{E}_{n \in \mathbf{N}}^{\log} \prod_{j=1}^m b_j(n + h_j) = \int_X \prod_{j=1}^m T^{h_j} F_j d\mu$$

for all $m \in \mathbb{N}$, all $h_1, \dots, h_m \in \mathbb{Z}$, and all $b_j, \dots, b_m \in \{a_1, \dots, a_\ell, \overline{a_1}, \dots, \overline{a_\ell}\}$, where for $j = 1, \dots, m$, if the sequence b_j is a_k or $\overline{a_k}$ for some $k \in \{1, \dots, \ell\}$, then F_j is $F_{0,k}$ or $\overline{F_{0,k}}$ respectively.

Remark. In the arguments that follow we often use the explicit choice of X and T made in the proof below, namely, we take $X = (\mathbb{U}^\ell)^\mathbb{Z}$ and let T be the shift transformation on X . We also often assume that the functions $F_{0,1}, \dots, F_{0,\ell}$ are defined by (5) below.

Proof. Let $X := (\mathbb{U}^\ell)^\mathbb{Z}$ and T be the shift transformation on X , namely, T maps an element $((x_1(n), \dots, x_\ell(n)))_{n \in \mathbb{Z}}$ of X to $((x_1(n+1), \dots, x_\ell(n+1)))_{n \in \mathbb{Z}}$. For $j = 0, \dots, \ell$ we define the functions $F_{0,j} \in C(X)$ as follows

$$(5) \quad F_{0,j}(x) := x_j(0), \quad \text{for } x = ((x_1(n), \dots, x_\ell(n)))_{n \in \mathbb{Z}} \in X.$$

Finally, the measure μ is defined to be the weak-star limit of the sequence of measures $\mathbb{E}_{n \in [N_k]}^{\log} \delta_{T^n a}$, $k \in \mathbb{N}$, where $a := ((a_1(n), \dots, a_\ell(n)))_{n \in \mathbb{Z}} \in X$. Then μ is T -invariant and we have $F_{0,j}(T^n a) = a_j(n)$, $n \in \mathbb{Z}$, for $j = 1, \dots, \ell$. It follows that (4) holds and the proof is complete. \square

Definition 2.4. Let $a_1, \dots, a_\ell: \mathbb{Z} \rightarrow \mathbb{U}$ be sequences that admit log-correlations on $\mathbf{N} := (N_k)_{k \in \mathbb{N}}$. We call the system (or the measure μ) defined in Proposition 2.3 the *joint Furstenberg system (or measure) associated with a_1, \dots, a_ℓ and \mathbf{N}* .

Remarks. • Given $a_1, \dots, a_\ell: \mathbb{Z} \rightarrow \mathbb{U}$ and \mathbf{N} , the measure μ is uniquely determined by (4) since this identity determines the values of $\int f d\mu$ for all real valued $f \in C(X)$.

• If two or more sequences coincide, say for example that $a_m = \dots = a_\ell$ for some $m \in \{1, \dots, \ell - 1\}$, then it is not hard to see that the joint Furstenberg system associated with a_1, \dots, a_ℓ and \mathbf{N} is isomorphic with the one associated with a_1, \dots, a_m and \mathbf{N} .

• A collection of sequences $a_1, \dots, a_\ell: \mathbb{Z} \rightarrow \mathbb{U}$ may have several non-isomorphic joint Furstenberg systems depending on which sequence of intervals \mathbf{N} we use in the evaluation of their joint correlations. When we write that a joint Furstenberg measure or system of the sequences a_1, \dots, a_ℓ has a certain property, we mean that any of these measures or systems has the asserted property.

2.4. Convergence results. Henceforth, we use the following notation:

Definition 2.5. If $d \in \mathbb{N}$ we let $\mathbb{P}_d := \mathbb{P} \cap (d\mathbb{N} + 1)$.

We will use the following convergence result that was proved in [32] and also in [15] conditional to some conjectures obtained later in [21, 22]:

Theorem 2.6. *Let (X, μ, T) be a system and $d \in \mathbb{N}$. Then for every $\ell \in \mathbb{N}$ and $F_1, \dots, F_\ell \in L^\infty(\mu)$ the following limit exists in $L^2(\mu)$*

$$\mathbb{E}_{p \in \mathbb{P}_d} \prod_{j=1}^{\ell} T^{pj} F_j.$$

Remark. Convergence is proved in [32] and [15] for $d = 1$. The more general statement follows by using the $d = 1$ case for product systems of the form $T \times R$ acting on $X \times \mathbb{Z}/d\mathbb{Z}$ with the product measure, where R is the translation by 1 on $\mathbb{Z}/d\mathbb{Z}$, and for the functions $F_1 \otimes \mathbf{1}_{d\mathbb{Z}+1}, F_2, \dots, F_\ell$; of course, one also uses the fact that the relative density of the set \mathbb{P}_d in the primes exists.

We will make use of the following consequence of Theorem 2.6:

Proposition 2.7. *Suppose that the sequences $a_1, \dots, a_\ell: \mathbb{Z} \rightarrow \mathbb{U}$ admit log-correlations on the sequence of intervals \mathbf{N} . Then for every $d \in \mathbb{N}$ the limit*

$$\mathbb{E}_{p \in \mathbb{P}_d} \left(\mathbb{E}_{n \in \mathbf{N}}^{\log} \prod_{j=1}^m b_j(n + ph_j) \right)$$

exists for all $m \in \mathbb{N}$, all $h_1, \dots, h_m \in \mathbb{Z}$, and all $b_1, \dots, b_m \in \{a_1, \dots, a_\ell, \overline{a_1}, \dots, \overline{a_\ell}\}$.

Proof. Let (X, \mathcal{X}, μ, T) be the joint Furstenberg system associated with a_1, \dots, a_ℓ and \mathbf{N} , and let also $F_{0,1}, \dots, F_{0,\ell} \in L^\infty(\mu)$ be as in Proposition 2.3. Using Theorem 2.6 we get that the limit

$$\mathbb{E}_{p \in \mathbb{P}_d} \int_X \prod_{j=1}^m T^{ph_j} F_j d\mu$$

exists for all $m \in \mathbb{N}$, all $h_1, \dots, h_m \in \mathbb{Z}$, and all $F_1, \dots, F_m \in L^\infty(\mu)$. Combining this with identity (4) we get the asserted convergence. \square

2.5. Aperiodic and strongly aperiodic multiplicative functions. We denote by \mathcal{M} the set of all multiplicative functions $f: \mathbb{N} \rightarrow \mathbb{U}$, where \mathbb{U} is the complex unit disc. A *Dirichlet character* is a periodic completely multiplicative function χ with $\chi(1) = 1$. We say that $f \in \mathcal{M}$ is *aperiodic* (or *non-pretentious* using terminology from [18]) if it averages to 0 on every infinite arithmetic progression, meaning, if

$$\mathbb{E}_{n \in \mathbb{N}} f(an + b) = 0, \quad \text{for all } a, b \in \mathbb{N}.$$

This is equivalent to asserting that $\mathbb{E}_{n \in \mathbb{N}} f(n) d(n) = 0$ for every periodic sequence $d: \mathbb{N} \rightarrow \mathbb{C}$, or that $\mathbb{E}_{n \in \mathbb{N}} f(n) \chi(n) = 0$ for every Dirichlet character χ . In order to give easier to verify necessary conditions for aperiodicity, we use a notion of distance between two multiplicative functions defined as in [18]:

Definition 2.8. We let $\mathbb{D}: \mathcal{M} \times \mathcal{M} \rightarrow [0, \infty]$ be given by

$$\mathbb{D}(f, g)^2 := \sum_{p \in \mathbb{P}} \frac{1}{p} (1 - \operatorname{Re}(f(p)\overline{g(p)}))$$

where $\operatorname{Re}(z)$ denotes the real part of a complex number z .

It is shown in [8, Theorem 1] that $f \in \mathcal{M}$ is aperiodic if and only if $\mathbb{D}(f, \chi \cdot n^{it}) = \infty$ for every $t \in \mathbb{R}$ and every Dirichlet character χ . Moreover, if f takes real values, then f is aperiodic if and only if $\mathbb{D}(f, \chi) = \infty$ for every Dirichlet character χ . In particular, the Möbius and the Liouville functions are aperiodic.

For our purposes we also need a notion introduced in [26] that is somewhat stronger than aperiodicity.

Definition 2.9. Let $\mathbb{D}: \mathcal{M} \times \mathcal{M} \times \mathbb{N} \rightarrow [0, \infty]$ be given by

$$\mathbb{D}(f, g; N)^2 := \sum_{p \in \mathbb{P} \cap [N]} \frac{1}{p} (1 - \operatorname{Re}(f(p)\overline{g(p)}))$$

and $M: \mathcal{M} \times \mathbb{N} \rightarrow [0, \infty)$ be given by

$$M(f; N) := \min_{|t| \leq N} \mathbb{D}(f, n^{it}; N)^2.$$

The multiplicative function $f \in \mathcal{M}$ is *strongly aperiodic* if $M(f \cdot \chi; N) \rightarrow \infty$ as $N \rightarrow \infty$ for every Dirichlet character χ .

Note that strong aperiodicity implies aperiodicity. The converse is not in general true (see [26, Theorem B.1]), but it is true for real valued multiplicative functions (see [26, Appendix C]). In particular, the Möbius and the Liouville functions are strongly aperiodic. Furthermore, if $f \in \mathcal{M}$ is aperiodic and $f(p)$ is a d -th root of unity for all $p \in \mathbb{P}$, then f is strongly aperiodic (see [13, Proposition 6.1]). In particular, if $f(p)$ is a nontrivial d -th root of unity for all $p \in \mathbb{P}$, then f is strongly aperiodic (see [13, Corollary 6.2]).

The hypothesis of strong aperiodicity is useful for our purposes because it enables us to use the following result of Tao [30, Corollary 1.5]:

Theorem 2.10. *Let $f \in \mathcal{M}$ be a strongly aperiodic multiplicative function. Then we have*

$$\mathbb{E}_{n \in \mathbb{N}}^{\log} f(n) \overline{f(n+h)} = 0$$

for every $h \in \mathbb{N}$.

Remark. By adjusting the example in [26, Theorem B.1], it follows that strong aperiodicity cannot be replaced by aperiodicity; in particular, there exist an aperiodic multiplicative function $f \in \mathcal{M}$, a positive constant c , and a sequence of intervals $\mathbf{N} := ([N_k])_{k \in \mathbb{N}}$ with $N_k \rightarrow \infty$, such that

$$|\mathbb{E}_{n \in \mathbf{N}}^{\log} f(n) \cdot \overline{f(n+h)}| \geq c, \quad \text{for every } h \in \mathbb{N}.$$

3. CORRELATION IDENTITIES AND ERGODIC CONSEQUENCES

3.1. Correlation identities. If $a: \mathbb{P} \rightarrow \mathbb{U}$ is a sequence and A is a non-empty finite or infinite subset of the primes we define $\mathbb{E}_{p \in A}^{\log}$ as in Section 2.1. The following identity of Tao and Teräväinen from [31, Theorem 3.6] is key for our purposes:

Theorem 3.1. *Suppose that the multiplicative functions $f_1, \dots, f_\ell: \mathbb{Z} \rightarrow \mathbb{U}$ admit log-correlations on the sequence of intervals \mathbf{N} . Then we have*

$$(6) \quad \mathbb{E}_{p \in \mathbb{P}}^{\log} \left| c_{p,m} \mathbb{E}_{n \in \mathbf{N}}^{\log} \prod_{j=1}^m g_j(n+h_j) - \mathbb{E}_{n \in \mathbf{N}}^{\log} \prod_{j=1}^m g_j(n+ph_j) \right| = 0$$

for all $m \in \mathbb{N}$, all $h_1, \dots, h_m \in \mathbb{Z}$, and all $g_1, \dots, g_m \in \{f_1, \dots, f_\ell, \overline{f_1}, \dots, \overline{f_\ell}\}$, where $c_{p,m} := \prod_{j=1}^m g_j(p)$, $p \in \mathbb{P}$.

Remarks. • A variant of this result is also implicit in the article of Tao [30] and was also used in [14] for $f_1 = \cdots = f_\ell = \mu$ or λ using a different averaging scheme. The current version is more suitable for our purposes.

• In [31] the result is proved for a class of generalized limit functionals in place of $\mathbb{E}_{n \in \mathbf{N}}^{\log}$. Assuming that $\mathbf{N} = ([N_k])_{k \in \mathbb{N}}$, the asserted version follows if one uses a generalized limit functional of the form $\lim_{k \rightarrow \infty} \mathbb{E}_{n \in [N_k]}^{\log}$ since it coincides with the standard limit $\lim_{k \rightarrow \infty} \mathbb{E}_{n \in [N_k]}^{\log} = \mathbb{E}_{n \in \mathbf{N}}^{\log}$ whenever this limit exists.

For the record, we mention the following identity for general sequences which follows from the proof of [31, Theorem 3.6] without any essential change; Theorem 3.1 is an easy consequence of this identity:

Theorem 3.2. *Let \mathbf{N} be a sequence of intervals, $a_1, \dots, a_\ell: \mathbb{Z} \rightarrow \mathbb{U}$ be sequences, and $h_1, \dots, h_\ell \in \mathbb{Z}$. Then, assuming that for every $p \in \mathbb{P}$ the limits $\mathbb{E}_{n \in \mathbf{N}}^{\log}$ below exist, we have the identity*

$$\mathbb{E}_{p \in \mathbb{P}}^{\log} \left| \mathbb{E}_{n \in \mathbf{N}}^{\log} \prod_{j=1}^{\ell} a_j(pn + ph_j) - \mathbb{E}_{n \in \mathbf{N}}^{\log} \prod_{j=1}^{\ell} a_j(n + ph_j) \right| = 0.$$

3.2. A consequence of the correlation identities. We are going to combine Theorem 3.1 with Theorem 3.5 stated below in order to prove a variant of the identity (6) in which the weights $c_{p,m}$ are all equal to 1. For convenience we introduce the following notation:

Definition 3.3. Let $a, b: \mathbb{P} \rightarrow \mathbb{U}$ be sequences. We write $a \sim b$ if

$$\mathbb{E}_{p \in \mathbb{P}}^{\log} (1 - \operatorname{Re}(a(p) \cdot \overline{b(p)})) = 0.$$

Remarks. • If we restrict to sequences that take values on the unit circle, then \sim is an equivalence relation and $a \sim b$ is equivalent to $\mathbb{E}_{p \in \mathbb{P}}^{\log} |a(p) - b(p)| = 0$.

• Using terminology from [31] we have that two multiplicative functions $f, g: \mathbb{Z} \rightarrow \mathbb{U}$ satisfy $f \sim g$ exactly when “ f weakly pretends to be g ”.

We will use the following basic properties:

Lemma 3.4. *If $a, b, c, d: \mathbb{P} \rightarrow \mathbb{U}$ are sequences, then the following properties hold:*

- (i) *If $a \sim b$, then $\bar{a} \sim \bar{b}$.*
- (ii) *If $a \sim b$ and $b \sim c$, then $a \sim c$.*
- (iii) *If $a \sim b$ and $c \sim d$, then $ac \sim bd$.*
- (iv) *If $a \sim b$, then $\mathbb{E}_{p \in \mathbb{P}}^{\log} |a(p) - b(p)| = 0$.*

Proof. Property (i) is obvious. Properties (ii) and (iii) follow from the estimate

$$(7) \quad 1 - \operatorname{Re}(uv) \leq 2(1 - \operatorname{Re}(u) + 1 - \operatorname{Re}(v))$$

which holds for all $u, v \in \mathbb{U}$. One way to prove this, is to first consider the case where $|u| = |v| = 1$; in this case the estimate is equivalent to $|u - v|^2 \leq 2(|1 - u|^2 + |1 - v|^2)$, which follows from the Cauchy Schwarz inequality. One then deduces from this the general case by expressing arbitrary $u, v \in \mathbb{U}$ as a convex combination of two elements on the unit circle and taking advantage of the linearity features of (7). Property (iv) follows from the estimate

$$|u - v|^2 \leq 2(1 - \operatorname{Re}(u\bar{v}))$$

which holds for all $u, v \in \mathbb{U}$. □

We will use the next result of Tao and Teräväinen [31, Theorem 1.2]:

Theorem 3.5. *Let $f_1, \dots, f_\ell: \mathbb{Z} \rightarrow \mathbb{U}$ be multiplicative functions. Suppose that for every Dirichlet character χ we have $f_1 \cdots f_\ell \approx \chi$. Then*

$$\mathbb{E}_{n \in \mathbb{N}}^{\log} \prod_{j=1}^{\ell} f_j(n + h_j) = 0$$

for all $h_1, \dots, h_\ell \in \mathbb{Z}$.

Remarks. • The use of logarithmic averages is essential for the statement to hold. For example, take $\ell = 1$ and let $f_1(n) := n^{it}$, $n \in \mathbb{N}$, for some non-zero $t \in \mathbb{R}$. Then $f_1 \approx \chi$ for every Dirichlet character χ but the limit $\lim_{N \rightarrow \infty} \mathbb{E}_{n \in [N]} n^{it}$ does not exist (since $\mathbb{E}_{n \in [N]} n^{it} = \frac{N^{it}}{it+1} + o(1)$). On the other hand we have that $\mathbb{E}_{n \in \mathbb{N}}^{\log} n^{it} = 0$.

• The proof of Theorem 3.5 depends crucially on deep results from ergodic theory such as [23, 24, 25] and analytic number theory [19, 20].

The next result is a key ingredient in our argument (recall that $\mathbb{P}_d = \mathbb{P} \cap (d\mathbb{N} + 1)$):

Proposition 3.6. *Let $f_1, \dots, f_\ell: \mathbb{Z} \rightarrow \mathbb{U}$ be multiplicative functions. There exists $d \in \mathbb{N}$ such that the following holds: If f_1, \dots, f_ℓ admit log-correlations on the sequence of intervals \mathbb{N} , then*

$$(8) \quad \mathbb{E}_{p \in \mathbb{P}_d}^{\log} \left| \mathbb{E}_{n \in \mathbb{N}}^{\log} \prod_{j=1}^m g_j(n + h_j) - \mathbb{E}_{n \in \mathbb{N}}^{\log} \prod_{j=1}^m g_j(n + ph_j) \right| = 0$$

for all $m \in \mathbb{N}$, all $h_1, \dots, h_m \in \mathbb{Z}$, and all $g_1, \dots, g_m \in \{f_1, \dots, f_\ell, \overline{f_1}, \dots, \overline{f_\ell}\}$.

Proof. Suppose first that for some $j \in \{1, \dots, \ell\}$ we have $\mathbb{E}_{n \in \mathbb{N}}^{\log} |f_j(n)| = 0$. Then whenever one of the functions g_1, \dots, g_m is equal to f_j or $\overline{f_j}$, all logarithmic averages in (8) vanish and the identity holds trivially for $d = 1$. Thus, without loss of generality, we can assume that for $j = 1, \dots, \ell$ we have $\mathbb{E}_{n \in \mathbb{N}}^{\log} |f_j(n)| > 0$ (note that by our assumptions the average exists). Using Theorem 3.1 for $\ell = 1$, $g_1 = f_j$, and $h_1 = 0$, we deduce that $\mathbb{E}_{p \in \mathbb{P}}^{\log} (1 - |f_j(p)|) = 0$.³ Hence, we can work under the assumption that

$$(9) \quad |f_j| \sim 1, \quad \text{for } j = 1, \dots, \ell.$$

Next, for $k \in \mathbb{N}$ and $j = 1, \dots, \ell$, we denote by f_j^{-k} the function $\overline{f_j^k}$ and let $K = K_{f_1, \dots, f_\ell}$ be the subset of \mathbb{Z}^ℓ defined as follows

$$K := \left\{ (k_1, \dots, k_\ell) \in \mathbb{Z}^\ell : \prod_{j=1}^{\ell} f_j^{k_j} \sim \chi \text{ for some Dirichlet character } \chi \right\}.$$

Using (9) and properties (i)-(iii) of Lemma 3.4, and since products and complex conjugates of Dirichlet characters are Dirichlet characters, we get that K is a subgroup of \mathbb{Z}^ℓ . Since every subgroup of \mathbb{Z}^ℓ is finitely generated, K is finitely generated. We let $F_K = F_{K, f_1, \dots, f_\ell}$ be the following set of multiplicative functions

$$F_K := \left\{ \prod_{j=1}^{\ell} f_j^{k_j} : (k_1, \dots, k_\ell) \in K \right\}.$$

We have that F_K is finitely generated under multiplication. Let $\{f_{0,1}, \dots, f_{0,r}\}$, for some $r \in \mathbb{N}$, be a set of generators for F_K . Then for $j = 1, \dots, r$ there exist Dirichlet characters χ_j such that $f_{0,j} \sim \chi_j$. If $d \in \mathbb{N}$ is a common period of all these Dirichlet characters, then for $j = 1, \dots, r$ we have $\chi_j(dn + 1) = 1$ for every $n \in \mathbb{N}$. Let $f \in F_K$, then $f = \prod_{j=1}^r f_{0,j}^{k_j}$ for some $k_1, \dots, k_r \in \mathbb{Z}$. Since $f_{0,j} \sim \chi_j$ for $j = 1, \dots, r$, we get from property (iii) of

³We note that the $\ell = 1$ case of Theorem 3.1 admits a simple elementary proof via the Turán-Kubilius inequality.

Lemma 3.4 that $f \sim \prod_{j=1}^r \chi_j^{k_j}$, and since $\chi_j(p) = 1$ for all $j \in \{1, \dots, r\}$ and all $p \in \mathbb{P}_d$, we deduce from property (iv) of Lemma 3.4 that $\mathbb{E}_{p \in \mathbb{P}_d}^{\log} |f(p) - 1| = 0$ (we also used that \mathbb{P}_d has positive relative density in \mathbb{P}). Hence,

$$(10) \quad \mathbb{E}_{p \in \mathbb{P}_d}^{\log} |f(p) - 1| = 0, \quad \text{for every } f \in F_K.$$

We now show that (8) holds. Let $\tilde{g} := \prod_{j=1}^m g_j$. Since $g_j \in \{f_1, \dots, f_\ell, \overline{f_1}, \dots, \overline{f_\ell}\}$ for $j = 1, \dots, m$, using (9) and properties (ii) and (iii) of Lemma 3.4, we get that $\tilde{g} \sim \prod_{j=1}^\ell f_j^{k_j}$ for some $k_1, \dots, k_\ell \in \mathbb{Z}$, where we continue to use the notation f^k for $\overline{f^{-k}}$ if k is a negative integer. We consider two cases. If $\tilde{g} \notin F_K$, then $\tilde{g} \approx \chi$ for all Dirichlet characters χ , in which case (8) holds because by Theorem 3.5 we have $\mathbb{E}_{n \in \mathbb{N}}^{\log} \prod_{j=1}^m g_j(n + h_j) = 0$ for all $h_1, \dots, h_m \in \mathbb{Z}$. On the other hand, if $\tilde{g} \in F_K$, we see that (8) holds by combining Theorem 3.1 (with \mathbb{P}_d in place of \mathbb{P}) and (10). This completes the proof. \square

We are going to use the following consequence of Proposition 3.6 which is better suited for our purposes:

Corollary 3.7. *Let $f_1, \dots, f_\ell: \mathbb{Z} \rightarrow \mathbb{U}$ be multiplicative functions. There exists $d \in \mathbb{N}$ such that the following holds: If f_1, \dots, f_ℓ admit log-correlations on the sequence of intervals \mathbb{N} , then the limit on the right hand side below exists and we have*

$$(11) \quad \mathbb{E}_{n \in \mathbb{N}}^{\log} \prod_{j=1}^m g_j(n + h_j) = \mathbb{E}_{p \in \mathbb{P}_d} \mathbb{E}_{n \in \mathbb{N}}^{\log} \prod_{j=1}^m g_j(n + ph_j)$$

for all $m \in \mathbb{N}$, all $h_1, \dots, h_m \in \mathbb{Z}$, and all $g_1, \dots, g_m \in \{f_1, \dots, f_\ell, \overline{f_1}, \dots, \overline{f_\ell}\}$.

Proof. This follows immediately from Proposition 3.6 and the fact that by Proposition 2.7 the limit $\mathbb{E}_{p \in \mathbb{P}_d}$ on the right hand side exists and as a consequence the limit $\mathbb{E}_{p \in \mathbb{P}_d}^*$ is equal to the limit $\mathbb{E}_{p \in \mathbb{P}_d}$. \square

3.3. An ergodic consequence. Using Proposition 2.3 we deduce from Corollary 3.7 the following ergodic result concerning joint Furstenberg systems of multiplicative functions:

Theorem 3.8. *Let $f_1, \dots, f_\ell: \mathbb{Z} \rightarrow \mathbb{U}$ be multiplicative functions. There exists $d \in \mathbb{N}$ such that the following holds: If (X, μ, T) is a joint Furstenberg system of f_1, \dots, f_ℓ , and if $F_{0,1}, \dots, F_{0,\ell}$ are as in Proposition 2.3, then we have*

$$(12) \quad \int_X \prod_{j=1}^m T^{h_j} F_j d\mu = \mathbb{E}_{p \in \mathbb{P}_d} \int_X \prod_{j=1}^m T^{ph_j} F_j d\mu$$

for all $m \in \mathbb{N}$, all $h_1, \dots, h_m \in \mathbb{Z}$, and all $F_1, \dots, F_m \in \{F_{0,1}, \dots, F_{0,\ell}, \overline{F_{0,1}}, \dots, \overline{F_{0,\ell}}\}$.

4. THE STRUCTURE OF FURSTENBERG SYSTEMS OF MULTIPLICATIVE FUNCTIONS

The goal of this section is to prove Theorem 1.5. In the next section we use this structural result to prove Theorems 1.1-1.3.

4.1. Proof of Theorem 1.5. Given a system (X, μ, T) and $d \in \mathbb{N}$ we define the system of arithmetic progressions with steps in \mathbb{P}_d as follows:

Definition 4.1. Let (X, μ, T) be a system and let $X^{\mathbb{Z}}$ be endowed with the product σ -algebra. For $d \in \mathbb{N}$ we write $\tilde{\mu}_d$ for the probability measure on $X^{\mathbb{Z}}$ defined as follows: For every $m \in \mathbb{N}$ and all $F_{-m}, \dots, F_m \in L^\infty(\mu)$, we let

$$(13) \quad \int_{X^{\mathbb{Z}}} \prod_{j=-m}^m F_j(x_j) d\tilde{\mu}_d(\underline{x}) := \mathbb{E}_{p \in \mathbb{P}_d} \int_X \prod_{j=-m}^m T^{pj} F_j d\mu,$$

where $\underline{x} := (x_j)_{j \in \mathbb{Z}}$ and the limit on the right hand side exists by Theorem 2.6. The measure $\tilde{\mu}_d$ is invariant under the shift transformation S on $X^{\mathbb{Z}}$ and induces a system $(X^{\mathbb{Z}}, \tilde{\mu}_d, S)$, which we call the *system of arithmetic progressions with steps in \mathbb{P}_d* associated with the system (X, μ, T) .

Remark. For $d = 1$ the system $(X^{\mathbb{Z}}, \tilde{\mu}_1, S)$ coincides with the system of arithmetic progressions with prime steps introduced in [14, Definition 3.8].

The relevance of the systems $(X^{\mathbb{Z}}, \tilde{\mu}_d, S)$ to our problem is demonstrated by the following result:

Proposition 4.2. *Let $f_1, \dots, f_\ell: \mathbb{Z} \rightarrow \mathbb{U}$ be multiplicative functions. Then there exists $d \in \mathbb{N}$ such that any joint Furstenberg system (X, μ, T) of the multiplicative functions f_1, \dots, f_ℓ is a factor of the system $(X^{\mathbb{Z}}, \tilde{\mu}_d, S)$.*

Proof. We can assume that the joint Furstenberg system is defined on the space $X := (\mathbb{U}^\ell)^{\mathbb{Z}}$ and T is the shift transformation on X . We denote elements of X by $x = (x_1(k), \dots, x_\ell(k))_{k \in \mathbb{Z}}$, where $x_1(k), \dots, x_\ell(k) \in \mathbb{U}$ for $k \in \mathbb{Z}$, and elements of $X^{\mathbb{Z}}$ with $\underline{x} = (x_n)_{n \in \mathbb{Z}}$, where $x_n \in X$ for $n \in \mathbb{Z}$. Hence, $\underline{x} = (x_{n,1}(k), \dots, x_{n,\ell}(k))_{k,n \in \mathbb{Z}}$ can be identified with $(x_{n,1}, \dots, x_{n,\ell})_{n \in \mathbb{Z}}$, where $x_{n,j} = (x_{n,j}(k))_{k \in \mathbb{Z}}$ for $j = 1, \dots, \ell$.

We define the map $\pi: X^{\mathbb{Z}} \rightarrow X$ as follows: For $\underline{x} = (x_{n,1}, \dots, x_{n,\ell})_{n \in \mathbb{Z}} \in X^{\mathbb{Z}}$ let

$$(\pi(\underline{x}))(n) := (x_{n,1}(0), \dots, x_{n,\ell}(0)) = (F_{0,1}(x_{n,1}), \dots, F_{0,\ell}(x_{n,\ell})), \quad n \in \mathbb{Z},$$

where

$$F_{h,j}(x) := x_j(h), \quad x \in X, \quad h \in \mathbb{Z}, \quad j \in \{1, \dots, \ell\}.$$

For $n \in \mathbb{Z}$ we have

$$\begin{aligned} (\pi(S\underline{x}))(n) &= (F_{0,1}((S\underline{x})_n), \dots, F_{0,\ell}((S\underline{x})_n)) = (F_{0,1}(x_{n+1,1}), \dots, F_{0,\ell}(x_{n+1,\ell})) = \\ &= (\pi(\underline{x}))(n+1) = (T\pi(\underline{x}))(n). \end{aligned}$$

Thus

$$\pi \circ S = T \circ \pi.$$

Next, we claim that $\tilde{\mu}_d \circ \pi^{-1} = \mu$. Indeed, for all $m \in \mathbb{N}$, all $h_1, \dots, h_m \in \mathbb{Z}$, and all $k_1, \dots, k_m \in \{\pm 1, \dots, \pm \ell\}$, using identity (12) in Theorem 3.8 and the definition of $\tilde{\mu}_d$ given in (13), we have

$$\begin{aligned} \int_X \prod_{j=1}^m F_{h_j, k_j}(x) d\mu(x) &= \int_X \prod_{j=1}^m F_{0, k_j}(T^{h_j} x) d\mu(x) = \mathbb{E}_{p \in \mathbb{P}_d} \int_X \prod_{j=1}^m F_{0, k_j}(T^{ph_j} x) d\mu(x) = \\ &= \int_{X^{\mathbb{Z}}} \prod_{j=1}^m F_{0, k_j}(x_{h_j}) d\tilde{\mu}_d(\underline{x}) = \int_{X^{\mathbb{Z}}} \prod_{j=1}^m (F_{h_j, k_j} \circ \pi)(\underline{x}) d\tilde{\mu}_d(\underline{x}), \end{aligned}$$

where we let $F_{h, -k} := \overline{F_{h, k}}$ for $h \in \mathbb{Z}$ and $k \in \{1, \dots, \ell\}$. Since the algebra generated by the functions $F_{h,1}, \dots, F_{h,\ell}, \overline{F_{h,1}}, \dots, \overline{F_{h,\ell}}$, $h \in \mathbb{Z}$, is dense in $C(X)$ with the uniform topology, the claim follows.

Therefore, $\pi: (X^{\mathbb{Z}}, \tilde{\mu}_d, S) \rightarrow (X, \mu, T)$ is a factor map and the proof is complete. \square

Our next task is to obtain structural results for the systems $(X^{\mathbb{Z}}, \tilde{\mu}_d, S)$. This crucially depends on the following result from [14] which deals with the case where $d = 1$:

Theorem 4.3. *Let (X, μ, T) be a system. Then the system $(X^{\mathbb{Z}}, \tilde{\mu}_1, S)$*

- (i) *has no irrational spectrum;*
- (ii) *has ergodic components isomorphic to direct products of infinite-step nilsystems and Bernoulli systems.*

Remark. The infinite-step nilsystems and the Bernoulli systems are allowed to be trivial.

The proof of Theorem 4.3 uses some deep ergodic machinery such as the main result from [23] (or [33]) regarding characteristic factors of Furstenberg averages, results about arithmetic progressions on nilmanifolds, and properties of partially strongly stationary systems. It also uses indirectly (via the use of variants of limit formulas obtained in [15]) some deep number theoretic input such as the Gowers uniformity of the W -tricked von Mangoldt function from [20, 21, 22]. Luckily, we do not have to modify the argument from [14] in order to get a similar result for the measures $\tilde{\mu}_d$; instead, we make use of the following simple observation, which allows us to use Theorem 4.3 as a “black box”:

Lemma 4.4. *Let (X, μ, T) be a system and $\tilde{\mu}_d, d \in \mathbb{N}$, be the measures on $X^{\mathbb{Z}}$ defined by (13). Then $\tilde{\mu}_d \leq \phi(d)\tilde{\mu}_1$ for every $d \in \mathbb{N}$.*

Proof. It suffices to show that for all $m \in \mathbb{N}$ and all non-negative $F_{-m}, \dots, F_m \in L^\infty(\mu)$ we have

$$\int_{X^{\mathbb{Z}}} \prod_{j=-m}^m F_j(x_j) d\tilde{\mu}_d(\underline{x}) \leq \phi(d) \int_{X^{\mathbb{Z}}} \prod_{j=-m}^m F_j(x_j) d\tilde{\mu}_1(\underline{x}).$$

This follows immediately from (13), the fact that the relative density $d_{\mathbb{P}}(\mathbb{P}_d)$ of the set \mathbb{P}_d in the primes is $1/\phi(d)$, and the estimate

$$\mathbb{E}_{p \in \mathbb{P}_d} a(p) \leq (d_{\mathbb{P}}(\mathbb{P}_d))^{-1} \mathbb{E}_{p \in \mathbb{P}} a(p)$$

which holds for all sequences $a: \mathbb{P} \rightarrow \mathbb{R}^+$ assuming that the limits on the left and right hand side exist. \square

Combining Theorem 4.3 and Lemma 4.4 we deduce the following:

Theorem 4.5. *Let (X, μ, T) be a system. Then for every $d \in \mathbb{N}$ the system $(X^{\mathbb{Z}}, \tilde{\mu}_d, S)$*

- (i) *has no irrational spectrum;*
- (ii) *has ergodic components isomorphic to direct products of infinite-step nilsystems and Bernoulli systems.*

Proof. By Lemma 4.4 we have $\tilde{\mu}_d \leq \phi(d)\tilde{\mu}_1$, hence the measure $\tilde{\mu}_d$ is absolutely continuous with respect to the measure $\tilde{\mu}_1$. This implies that the spectrum of the system $(X^{\mathbb{Z}}, \tilde{\mu}_d, S)$ is a subset of the spectrum of the system $(X^{\mathbb{Z}}, \tilde{\mu}_1, S)$, so the former has no irrational spectrum since the same holds for the latter by Theorem 4.3. Furthermore, if $\tilde{\mu}_1 = \int_{\Omega} \tilde{\mu}_{1,\omega} dP(\omega)$ is the ergodic decomposition of the measure $\tilde{\mu}_1$, then the ergodic decomposition of the measure $\tilde{\mu}_d$ is $\tilde{\mu}_d = \int_{\Omega} \tilde{\mu}_{1,\omega} dP_d(\omega)$ for some probability measure P_d that is absolutely continuous with respect to P . This implies that property (ii) holds for the ergodic components of the measure $\tilde{\mu}_d$ since it holds for the ergodic components of the measure $\tilde{\mu}_1$ by Theorem 4.3. \square

Theorem 1.5 now follows by combining Proposition 4.2 and Theorem 4.5.

4.2. An alternative proof of Theorem 1.5 for some special cases. In some interesting special cases we can prove Theorem 1.5 (and hence Theorems 1.1-1.3) using an alternative approach that avoids the use of Theorem 3.5. We present the details below, let us emphasize though, that this alternative approach breaks down when $f_j(\mathbb{P})$ is an infinite subset of the unit circle for some $j \in \{1, \dots, \ell\}$, and we do not see how to avoid the use of Theorem 3.5 in order to cover such cases.

4.2.1. The case where $f_1(\mathbb{P}), \dots, f_\ell(\mathbb{P})$ are finite subsets of \mathbb{T} . Suppose first that the multiplicative functions f_1, \dots, f_ℓ are such that $f_j(\mathbb{P})$ is a finite subset of \mathbb{T} for $j = 1, \dots, \ell$. Let (X, μ, T) be a joint Furstenberg system associated with these multiplicative functions and a sequence of intervals \mathbf{N} . Then there exist $c_1, \dots, c_\ell \in \mathbb{U}$, a subset A of \mathbb{P} , and a sequence of intervals $\mathbf{M} = ([M_k])_{k \in \mathbb{N}}$ with $M_k \rightarrow \infty$, such that

- (i) $f_j(p) = c_j, j = 1 \dots, \ell$, for all $p \in A$;

- (ii) $d_{\mathbf{M},\mathbb{P}}^{\log}(A) := \mathbb{E}_{\mathbf{M},p \in \mathbb{P}}^{\log} \mathbf{1}_A(p)$ exists and is positive;
- (iii) the averages

$$\mathbb{E}_{\mathbf{M},p \in A}^{\log} \int_X \prod_{j=-m}^m T^{pj} F_j d\mu$$

exist for all $m \in \mathbb{N}$ and $F_{-m}, \dots, F_m \in L^\infty(\mu)$,

where we used the notation

$$\mathbb{E}_{\mathbf{M},p \in A}^{\log} a(p) := \lim_{k \rightarrow \infty} \mathbb{E}_{p \in A \cap [M_k]}^{\log} a(p)$$

for $a: \mathbb{P} \rightarrow \mathbb{C}$, if the limit exists. Using Theorem 3.1 and property (i), we get as in the proof of Proposition 4.2, using the factor map $\pi: X^{\mathbb{Z}} \rightarrow X$ defined by $(\pi(\underline{x}))(n) := (\overline{c_1} x_{n,1}(0), \dots, \overline{c_\ell} x_{n,\ell}(0))$, that the system (X, μ, T) is a factor of the system $(X^{\mathbb{Z}}, \mu^*, S)$ where the measure μ^* is defined as follows: For every $m \in \mathbb{N}$ and all $F_{-m}, \dots, F_m \in L^\infty(\mu)$, we let

$$(14) \quad \int_{X^{\mathbb{Z}}} \prod_{j=-m}^m F_j(x_j) d\mu^*(\underline{x}) := \mathbb{E}_{\mathbf{M},p \in A}^* \int_X \prod_{j=-m}^m T^{pj} F_j d\mu,$$

where the limit on the right hand side exists by property (iii). Since $d_{\mathbf{M},\mathbb{P}}^*(A) > 0$ (by property (ii)), we get that for every sequence $a: \mathbb{P} \rightarrow \mathbb{R}^+$ for which the limits below exist that

$$\mathbb{E}_{\mathbf{M},p \in A}^{\log} a(p) \leq C \lim_{k \rightarrow \infty} \mathbb{E}_{p \in \mathbb{P} \cap [M_k]}^{\log} a(p) = C \mathbb{E}_{p \in \mathbb{P}} a(p)$$

where $C := (d_{\mathbf{M},\mathbb{P}}^{\log}(A))^{-1}$ (the last identity holds because $\lim_{m \rightarrow \infty} \mathbb{E}_{p \in \mathbb{P} \cap [M_k]}^{\log} a(p) = \mathbb{E}_{p \in \mathbb{P}} a(p)$ since the last limit is assumed to exist). Using this estimate in the case where $a(p) := \int_X \prod_{j=-m}^m T^{pj} F_j d\mu$, $p \in \mathbb{P}$, is non-negative, we get that the measures $\tilde{\mu}_1$ and μ^* , defined by (13) and (14) respectively, satisfy the estimate

$$\int_{X^{\mathbb{Z}}} \prod_{j=-m}^m F_j(x_j) d\mu^*(\underline{x}) \leq C \mathbb{E}_{p \in \mathbb{P}} \int_X \prod_{j=-m}^m T^{pj} F_j d\mu = C \int_{X^{\mathbb{Z}}} \prod_{j=-m}^m F_j(x_j) d\tilde{\mu}_1(\underline{x})$$

for all $m \in \mathbb{N}$ and all non-negative $F_{-m}, \dots, F_m \in L^\infty(\mu)$. Hence, $\mu^* \leq C \tilde{\mu}_1$, and as in the proof of Theorem 4.5 we conclude that the system $(X^{\mathbb{Z}}, \mu^*, S)$ satisfies properties (i) and (ii) of Theorem 4.5. Since (X, μ, T) is a factor of the system $(X^{\mathbb{Z}}, \mu^*, S)$, it also satisfies these two properties.

4.2.2. The case of real valued multiplicative functions. Let (X, μ, T) be a joint Furstenberg system associated with the multiplicative functions $f_1, \dots, f_\ell: \mathbb{Z} \rightarrow [-1, 1]$ and a sequence of intervals \mathbf{N} . Suppose first that for some $j \in \{1, \dots, \ell\}$ we have $\mathbb{E}_{n \in \mathbf{N}} |f_j(n)| = 0$, say for $j = \ell$. Then all correlations involving the function f_ℓ are trivial. As a consequence, the joint Furstenberg system associated with the functions f_1, \dots, f_ℓ and \mathbf{N} is isomorphic (in the measure theoretic sense) to the joint Furstenberg system associated with the functions $f_1, \dots, f_{\ell-1}$ and \mathbf{N} . Hence, it suffices to prove Theorem 1.5 in the case where $\mathbb{E}_{n \in \mathbf{N}} |f_j(n)| \neq 0$ for $j = 1, \dots, \ell$. As in the first part of the proof of Proposition 3.6 we get that $\mathbb{E}_{p \in \mathbb{P}}^{\log} (1 - |f_j(p)|) = 0$ for $j = 1, \dots, \ell$. Hence, $f_j \sim f'_j$ for some $f'_j: \mathbb{P} \rightarrow \{-1, 1\}$ for $j = 1, \dots, \ell$. As a consequence, in the identity of Theorem 3.1 we can replace the weights $c_{p,m} = \prod_{j=1}^m g_j(p)$ with the weights $c'_{p,m} := \prod_{j=1}^m g'_j(p)$, where for $j = 1, \dots, m$, if $g_j(p)$ is $f_k(p)$ or $\overline{f_k(p)}$ for some $k \in \{1, \dots, \ell\}$, then $g'_j(p)$ is $f'_k(p)$ or $\overline{f'_k(p)}$ respectively. Using this new identity, we deduce Theorem 1.5 as in the case treated above where $f_j(\mathbb{P})$ is finite for $j = 1, \dots, \ell$.

5. PROOF OF THEOREMS 1.1-1.3

We will use the following disjointness result, proved in [14, Corollary 3.13]:

Proposition 5.1. *Let (X, μ, T) be a system with ergodic components isomorphic to direct products of infinite-step nilsystems and Bernoulli systems. Let (Y, ν, R) be a zero entropy system with at most countably many ergodic components.*

- (i) *If the two systems have disjoint irrational spectrum, then for every joining σ of the two systems and function $F \in L^\infty(\mu)$ that is orthogonal to $\mathcal{K}_{\text{rat}}(T)$, we have*

$$\int_{X \times Y} F(x) G(y) d\sigma(x, y) = 0$$

for every $G \in L^\infty(\nu)$.

- (ii) *If the two systems have no common eigenvalue except 1, then for every joining σ of the two systems and function $G \in L^\infty(\nu)$ that is orthogonal in $L^2(\nu)$ to all R -invariant functions, we have*

$$\int_{X \times Y} F(x) G(y) d\sigma(x, y) = 0$$

for every $F \in L^\infty(\mu)$.

5.1. Proof of Theorem 1.1. We follow the argument used in [14, Section 3.9]. Arguing by contradiction, suppose that under the assumptions of Theorem 1.1 we do not have uniform convergence to 0 of the related averages. Then there exist a strongly aperiodic multiplicative function $f: \mathbb{N} \rightarrow \mathbb{U}$, which we extend to \mathbb{Z} in an arbitrary way, a topological dynamical system (Y, R) , positive integers $N_k \rightarrow \infty$, points $y_k \in Y, k \in \mathbb{N}$, and a function $g_0 \in C(Y)$ such that the averages

$$\mathbb{E}_{n \in [N_k]}^{\log} g_0(R^n y_k) f(n)$$

converge to a non-zero number as $k \rightarrow \infty$. After passing to a subsequence which we denote again by $([N_k])_{k \in \mathbb{N}}$, we can further assume that the averages $\mathbb{E}_{n \in [N_k]}^{\log} \delta_{R^n y_k}$ converge (as $k \rightarrow \infty$) weak-star to an R -invariant probability measure ν and the limit

$$(15) \quad \lim_{k \rightarrow \infty} \mathbb{E}_{n \in [N_k]}^{\log} g(R^n y_k) \prod_{j=1}^m f_j(n + h_j)$$

exists for all $m \in \mathbb{N}, h_1, \dots, h_m \in \mathbb{Z}, f_1, \dots, f_m \in \{f, \overline{f}\}$, and $g \in C(Y)$. Note that, by our assumptions, the system (Y, ν, R) has zero entropy and at most countably many ergodic components.

Let $X := \mathbb{U}^{\mathbb{Z}}, T: X \rightarrow X$ be the shift transformation, and $x_0 \in X$ be defined by

$$x_0(n) := f(n), \quad n \in \mathbb{Z}.$$

Then the convergence (15) implies that the limit

$$\lim_{k \rightarrow \infty} \mathbb{E}_{n \in [N_k]}^{\log} g(R^n y_k) \left(\prod_{j=1}^m G_{h_j} \right) (T^n x_0)$$

exists for all $m \in \mathbb{N}, h_1, \dots, h_m \in \mathbb{Z}, g \in C(Y)$, and $G_h \in \{F_h, \overline{F_h}\}, h \in \mathbb{Z}$, where $F_h(x) = x(h), x \in X, h \in \mathbb{Z}$. Since the algebra generated by the functions $F_h, \overline{F_h}, h \in \mathbb{Z}$, is dense in $C(X)$ with the uniform topology, we deduce that the sequence of measures

$$\mathbb{E}_{n \in [N_k]}^{\log} \delta_{(T^n x_0, R^n y_k)}, \quad k \in \mathbb{N},$$

converges weak-star to some probability measure σ on $X \times Y$ that satisfies

$$(16) \quad \lim_{k \rightarrow \infty} \mathbb{E}_{n \in [N_k]}^{\log} g(R^n y_k) \prod_{j=1}^m f_j(n + h_j) = \int_{X \times Y} \prod_{j=1}^m G_{h_j}(x) g(y) d\sigma(x, y)$$

for all $m \in \mathbb{N}$, $h_1, \dots, h_m \in \mathbb{Z}$, $f_1, \dots, f_m \in \{f, \overline{f}\}$, and $g \in C(Y)$, where G_h is F_h or $\overline{F_h}$ according to whether f_j is f or \overline{f} . By construction, σ is invariant under $T \times R$.

The projection of σ on Y is the weak-star limit of the sequence of measures $\mathbb{E}_{n \in [N_k]}^{\log} \delta_{R^n y_k}$, $k \in \mathbb{N}$, which is the measure ν , and thus the corresponding measure preserving system has zero entropy and at most countably many ergodic components.

The projection of σ on X is the weak-star limit of the sequence of measures $\mathbb{E}_{n \in [N_k]}^{\log} \delta_{T^n x_0}$, $k \in \mathbb{N}$. It is thus a T -invariant measure μ which is the Furstenberg measure associated with f and $\mathbf{N} = ([N_k])_{k \in \mathbb{N}}$ by Proposition 2.3. Hence, σ is a joining of the systems (X, μ, T) and (Y, ν, R) .

By the $\ell = 1$ case of Proposition 4.2 and its proof, there exists $d \in \mathbb{N}$ such that (X, μ, T) is a factor of the system $(X^{\mathbb{Z}}, \tilde{\mu}_d, S)$, with factor map $\pi: X^{\mathbb{Z}} \rightarrow X$ given by

$$(\pi(\underline{x}))(k) := x_k(0), \quad \underline{x} = (x_n)_{n \in \mathbb{Z}} \in X^{\mathbb{Z}}, \quad k \in \mathbb{Z}.$$

We define the joining $\tilde{\sigma}$ of the systems $(X^{\mathbb{Z}}, \tilde{\mu}_d, S)$ and (Y, ν, R) by

$$\int_{X^{\mathbb{Z}} \times Y} H(\underline{x}) \cdot g(y) d\tilde{\sigma}(\underline{x}, y) := \int_{X \times Y} \mathbb{E}_{\tilde{\mu}_d}(H | X)(x) \cdot g(y) d\sigma(x, y)$$

for every $H \in L^\infty(\tilde{\mu}_d)$ and $g \in L^\infty(\nu)$, where $\mathbb{E}_{\tilde{\mu}_d}(H | X)$ in $L^1(\nu)$ is determined by the property $\int_A \mathbb{E}_{\tilde{\mu}_d}(H | X) d\mu = \int_{\pi^{-1}(A)} H d\tilde{\mu}_d$ for every $A \in \mathcal{X}$.

We show now that the systems $(X^{\mathbb{Z}}, \tilde{\mu}_d, S)$ and (Y, ν, R) verify the assumptions of part (i) of Proposition 5.1. By Theorem 4.5, the system $(X^{\mathbb{Z}}, \tilde{\mu}_d, S)$ has no irrational spectrum and its ergodic components are isomorphic to direct products of infinite-step nilsystems and Bernoulli systems. We show next that the function $F'_0 := F_0 \circ \pi$ is orthogonal to the rational Kronecker factor of the system $(X^{\mathbb{Z}}, \tilde{\mu}_d, S)$, in fact, we establish the stronger property that F'_0 is orthogonal to the Kronecker factor of this system. By a well known consequence of the spectral theorem for unitary operators, this is equivalent to establishing that

$$(17) \quad \mathbb{E}_{n \in \mathbb{N}} \left| \int_{X^{\mathbb{Z}}} F'_0 \cdot S^n \overline{F'_0} d\tilde{\mu}_d \right| = 0.$$

By the Definition 4.1 of the measure $\tilde{\mu}_d$ and since for $n \in \mathbb{N}$ we have $F'_0(\underline{x}) \overline{F'_0}(S^n \underline{x}) = F_0(x_0) \overline{F_0}(x_n)$, we get for every $n \in \mathbb{N}$ that

$$\int_{X^{\mathbb{Z}}} F'_0 \cdot S^n \overline{F'_0} d\tilde{\mu}_d = \mathbb{E}_{p \in \mathbb{P}_d} \int_X F_0 \cdot T^{pn} \overline{F_0} d\mu.$$

By (4), for every $h \in \mathbb{N}$ we have

$$\int_X F_0 \cdot T^h \overline{F_0} d\mu = \mathbb{E}_{n \in \mathbb{N}}^{\log} f(n) \overline{f(n+h)} = 0$$

where the vanishing of the average follows from Theorem 2.10 and our assumption that f is strongly aperiodic. Combining the above identities we get (17).

By part (i) of Proposition 5.1, we have

$$0 = \int_{X^{\mathbb{Z}} \times Y} F'_0(\underline{x}) \cdot g_0(y) d\tilde{\sigma}(\underline{x}, y) = \int_{X \times Y} F_0(x) \cdot g_0(y) d\sigma(x, y) = \lim_{k \rightarrow \infty} \mathbb{E}_{n \in [N_k]}^{\log} g_0(R^n y_k) f(n)$$

where the last identity follows by (16). This contradicts our initial assumption that $\lim_{k \rightarrow \infty} \mathbb{E}_{n \in [N_k]}^{\log} g_0(R^n y_k) f(n) \neq 0$ and completes the proof. \square

5.2. Proof of Theorem 1.3.

We follow the argument used in [14, Section 3.11]. Arguing by contradiction, suppose that the conclusion of Theorem 1.3 fails. Then there exist $\ell \in \mathbb{N}$, multiplicative functions $f_1, \dots, f_\ell: \mathbb{N} \rightarrow \mathbb{U}$, which we extend to \mathbb{Z} in an arbitrary way, a topological dynamical system (Y, R) , a point $y_0 \in Y$ that is generic for a measure ν such that the system (Y, ν, R) has zero entropy and at most countably

many ergodic components all of which are totally ergodic, and a function $g_0 \in C(Y)$ orthogonal in $L^2(\nu)$ to all R -invariant functions, such that for some $h_{0,1}, \dots, h_{0,\ell} \in \mathbb{Z}$ the identity (1) fails, namely, the averages

$$(18) \quad \mathbb{E}_{n \in [N]}^{\log} g_0(R^n y_0) \prod_{j=1}^{\ell} f_j(n + h_{0,j})$$

do not converge to 0 as $N \rightarrow \infty$.

Let $X := (\mathbb{U}^\ell)^\mathbb{Z}$, $T: X \rightarrow X$ be the shift transformation, and $x_0 \in X$ be defined by

$$x_0(n) := (f_1(n), \dots, f_\ell(n)), \quad n \in \mathbb{Z}.$$

If $x = (x_1(n), \dots, x_\ell(n))_{n \in \mathbb{Z}} \in X$, where $x_j(n) \in U$ for $j = 1, \dots, \ell$, $n \in \mathbb{Z}$, we let

$$F_{h,j}(x) := x_j(h), \quad h \in \mathbb{Z}, \quad j \in \{1, \dots, \ell\}.$$

As in the proof of Theorem 1.1 in the previous subsection, we define a sequence of intervals $\mathbf{N} = (N_k)_{k \in \mathbb{N}}$, with $N_k \rightarrow \infty$, such that the averages (18), taken along \mathbf{N} , converge to some non-zero number, and a measure σ on $X \times Y$ which is the weak-star limit of the sequence of measures

$$\mathbb{E}_{n \in [N_k]}^{\log} \delta_{(T^n x_0, R^n y_0)}, \quad k \in \mathbb{N}.$$

In particular, the identity

$$(19) \quad \mathbb{E}_{n \in \mathbf{N}}^{\log} g(R^n y_0) \prod_{j=1}^{\ell} f_j(n + h_j) = \int_{X \times Y} \prod_{j=1}^{\ell} F_{h_j,j}(x) g(y) d\sigma(x, y)$$

holds for all $h_1, \dots, h_\ell \in \mathbb{Z}$, and $g \in C(Y)$.

By construction, σ is invariant under $T \times R$. By assumption and the definition of genericity, the projection of σ on Y is the measure ν , and thus the system (Y, ν, R) has zero entropy, at most countably many ergodic components, and no rational eigenvalue except 1. Moreover, the projection of σ on X is the weak-star limit of the sequence of measures $\mathbb{E}_{n \in [N_k]}^{\log} \delta_{T^n x_0}$, $k \in \mathbb{N}$. It is thus a T -invariant measure μ which is the joint Furstenberg measure associated with the multiplicative functions f_1, \dots, f_ℓ and \mathbf{N} by Proposition 2.3. Hence, by Proposition 4.2, for some $d \in \mathbb{N}$ the system (X, μ, T) is a factor of the system $(X^\mathbb{Z}, \tilde{\mu}_d, S)$. By Theorem 1.5, the system $(X^\mathbb{Z}, \tilde{\mu}_d, S)$ has no irrational spectrum and its ergodic components are isomorphic to direct products of infinite-step nilsystems and Bernoulli systems.

From the previous discussion it follows that the function g_0 and the systems $(X^\mathbb{Z}, \tilde{\mu}_d, S)$ and (Y, ν, R) satisfy the hypothesis of the second part of Proposition 5.1. Hence, for every joining $\tilde{\sigma}$ of these systems and $\tilde{f} \in L^\infty(\tilde{\mu}_d)$, we have $\int \tilde{f}(\underline{x}) g_0(y) d\tilde{\sigma}(\underline{x}, y) = 0$. Since σ is a joining of the systems (X, μ, T) and (Y, ν, R) , and the system (X, μ, T) is a factor of $(X^\mathbb{Z}, \tilde{\mu}_d, S)$, the measure σ can be lifted to a joining $\tilde{\sigma}$ of $(X^\mathbb{Z}, \tilde{\mu}_d, S)$ and (Y, ν, R) . It follows that for every $f \in L^\infty(\mu)$ we have $\int f(x) g_0(y) d\sigma(x, y) = 0$. We deduce that

$$\mathbb{E}_{n \in \mathbf{N}}^{\log} g_0(R^n y_0) \prod_{j=1}^{\ell} f_j(n + h_{0,j}) = \int_{X \times Y} \prod_{j=1}^{\ell} F_{h_{0,j},j}(x) \cdot g_0(y) d\sigma(x, y) = 0.$$

This contradicts our assumption that $\mathbb{E}_{n \in \mathbf{N}}^{\log} g_0(R^n y_0) \prod_{j=1}^{\ell} f_j(n + h_{0,j}) \neq 0$ and completes the proof of Theorem 1.3.

5.3. Block complexity and proof of Theorem 1.2. We start with some definitions. Let A be a non-empty finite set. The set A is endowed with the discrete topology and $A^\mathbb{Z}$ with the product topology and with the shift T . For $n \in \mathbb{N}$, a *word of length n* is a sequence $u = u_1 \dots u_n$ of n letters where $u_1, \dots, u_n \in A$, and we write $[u] = \{x \in$

$A^{\mathbb{Z}}: x_1 \dots x_n = u_1 \dots u_n$. A *subshift* is a closed non-empty T -invariant subset X of $A^{\mathbb{Z}}$. It is *transitive* if it has at least one dense orbit under T .

Let (X, T) be a transitive subshift that is equal to the closed orbit of some point $\omega \in A^{\mathbb{Z}}$. For every $n \in \mathbb{N}$ we let $L_n(X)$ denote the set of words u of length n such that $[u] \cap X \neq \emptyset$. Then $L_n(X)$ is also the set of words of length n that occur (as consecutive values) in ω . Note that the set $L(X) := \bigcup_{n \in \mathbb{N}} L_n(X)$ determines X . The *block complexity* of X or of ω is defined by $p_X(n) = |L_n(X)|$ for $n \in \mathbb{N}$. We say that the subshift (X, T) (or the sequence ω) *has linear block growth* if $\liminf_{n \rightarrow \infty} p_X(n)/n < \infty$. We are going to use the following consequence of a result from [4] (or [11, Theorem 7.3.7]), that was obtained in [14, Section 7.1]:

Proposition 5.2. *Let (X, T) be a transitive subshift with linear block growth. Then (X, T) admits only finitely many ergodic invariant measures.*

This result was proved in [3] under the stronger hypothesis that (X, T) is minimal.

Proof of Theorem 1.2. We argue as in [14, Section 7.2] where a similar result was proved for the Liouville function. Let A be the range of f , which we have assumed to be a finite subset of \mathbb{U} . Suppose that f has linear block growth. We extend f to a two sided sequence, which we denote by $y_0 \in A^{\mathbb{Z}}$, by letting $y_0(n) := 1$ for non-positive $n \in \mathbb{Z}$; then the extended sequence still has linear block growth. Let Y be the closed orbit of y_0 in $A^{\mathbb{Z}}$ and let R be the shift on Y . Then (Y, R) is a transitive subshift, and since it has linear block growth it has zero topological entropy. Moreover, by Proposition 5.2 this system admits only finitely many ergodic invariant measures. Note that for every $n \in \mathbb{N}$ we have $f(n) = F_0(R^n y_0)$, where $F_0: A^{\mathbb{Z}} \rightarrow \mathbb{U}$ is the map defined by $F_0(y) := y(0)$ for $y = (y(n))_{n \in \mathbb{Z}} \in Y$. By Theorem 1.1 we get

$$0 = \mathbb{E}_{n \in \mathbb{N}}^{\log} \overline{F_0}(R^n y_0) f(n) = \mathbb{E}_{n \in \mathbb{N}}^{\log} |f(n)|^2 \neq 0,$$

where we used our assumption that f does not converge to zero in logarithmic density. We have thus established a contradiction and the proof is complete. \square

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