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Solving generalized Abel's integral equations of the first and second kinds via Taylor-collocation method

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Abstract

The aim of this paper is to present an efficient numerical procedure to approximate the generalized Abel's integral equations of the first and second kinds. For this reason, the Taylor polynomials and the collocation method are applied. Also, the error analysis of presented method is illustrated. Several examples are approximated and the numerical results show the accuracy and efficiency of this method.

keywords: Generalized Abel's integral equation, Collocation method, Taylor polynomials.

1 Introduction

The great mathematician Niels Abel, gave the initiative of integral equations in 1823 in his study of mathematical physics [13, 32, 33, 34]. Zeilon in 1924, studied the generalized Abel's integral equation on a finite segment [38].

The Abel's integral equations are the singular form of Volterra integral equations. The singular integral equations are the important and applicable kinds of integral equations which solved by many authors [9, 10, 11, 24, 28, 34]. These integral equations have many applications in various areas such as simultaneous dual relations [29], stellar winds [18], water wave [4], spectroscopic data [3] and the others [2, 13, 19].

In this paper, we investigate the following generalized Abel's integral equations of the first kind

$$\int_a^x \frac{\Phi(t)}{(\phi(x)-\phi(t))^\alpha} dt = g(x), \qquad 0<\alpha<1, \tag{1}$$

and the second kind

$$\Phi(x) = g(x) + \int_a^x \frac{\Phi(t)}{(\phi(x)-\phi(t))^\alpha} dt, \qquad 0<\alpha<1, \tag{2}$$

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where a is a given real value, $g(x)$ and $\phi(x)$ are known functions and $\Phi(x)$ is an unknown function that $\phi(t)$ is strictly monotonically increasing and differentiable function in some interval $a < t < b$, and $\phi'(t) \neq 0$ for every t in the interval.

Abel's integral equations (1) and (2) have applications in experimental physics such as plasma diagnostics, physical electronics, nuclear physics, optics and astrophysics [18, 19]. Also, Baker [1], Wazwaz [32, 33] and Delves [5] studied the numerical treatment of singular integral equations of the first and second kinds. In recent years, many authors solved the Abel's integral equations of the first and second kinds by using different applicable methods [6, 12, 15, 16, 24, 27, 37]. The properties of Abel's integral equations can be found in [20, 25, 28, 33].

One of powerful and efficient methods to estimate the integral and differential equations is the collocation method [7, 8, 21]. This method is one of the expansion methods which is used with different basis functions. In this paper, the collocation method and the Taylor polynomials are applied to estimate the generalized Abel's integral equations of the first and second kinds.

Kanwal and Liu in [17] presented the Taylor expansion approach for solving integral equations. It was applied for solving the Volterra- Fredholm integral equations [30, 31, 35], system of integral equations [22], Volterra integral equations [26], integro-differential equations [23, 36] and the others [14, 17].

This paper is organized as follows: At first, in Section 2, the Taylor polynomials and the collocation method are combined to estimate the generalized form of Abel's integral equations of the first and second kinds. In Section 3, the error analysis of presented method is illustrated and in sequel, in Section 4, several applicable examples are approximated. Also, in this section the numerical results and the absolute errors for different values of x and n are tabulated. Finally, Section 5 is conclusion.

2 Main Idea

In order to estimate the Abel's integral equations of the first and second kinds, the Taylor polynomials of degree n at $x = z$ is introduced as follows

$$\Phi_n(x) = \sum_{j=0}^{n} \frac{1}{j!} \Phi^{(j)}(z)(x - z)^j, \qquad a \leq x, z \leq b, \tag{3}$$

where the coefficients $\Phi^{(j)}(z), j = 0, 1, ..., n$ should be determined.

To approximate the Abel's integral equation of the first kind (1), we rewrite it in the form

$$\sum_{j=0}^{n} \frac{1}{j!} \int_a^x \frac{\Phi^{(j)}(z)(t - z)^j}{\left[\phi(x) - \phi(t)\right]^{\alpha}} dt = g(x), \qquad 0 < \alpha < 1 \ \ and \ \ a \leq x, z \leq b. \tag{4}$$

In order to estimate the integral equation (4), the collocation points

$$x_i = a + \left(\frac{b - a}{n}\right) i, \qquad i = 0, 1, 2, ..., n, \tag{5}$$

are substitute into Eq. (4) as follows

$$\sum_{j=0}^{n} \frac{1}{j!} \int_a^{x_i} \frac{(t-z)^j}{\big[\phi(x_i)-\phi(t)\big]^\alpha} dt \ \Phi^{(j)}(z) = g(x_i), \qquad 0<\alpha<1. \tag{6}$$

Now, we can write Eq. (6) in the form

$$AX = G, \tag{7}$$

where

$$A = \begin{bmatrix} \frac{1}{0!}\int_a^{x_0} \frac{(t-z)^0}{\big[\phi(x_0)-\phi(t)\big]^\alpha}dt & \frac{1}{1!}\int_a^{x_0} \frac{(t-z)^1}{\big[\phi(x_0)-\phi(t)\big]^\alpha}dt & \cdots & \frac{1}{n!}\int_a^{x_0} \frac{(t-z)^n}{\big[\phi(x_0)-\phi(t)\big]^\alpha}dt \\ \frac{1}{0!}\int_a^{x_1} \frac{(t-z)^0}{\big[\phi(x_1)-\phi(t)\big]^\alpha}dt & \frac{1}{1!}\int_a^{x_1} \frac{(t-z)^1}{\big[\phi(x_1)-\phi(t)\big]^\alpha}dt & \cdots & \frac{1}{n!}\int_a^{x_1} \frac{(t-z)^n}{\big[\phi(x_1)-\phi(t)\big]^\alpha}dt \\ \vdots & \vdots & \cdots & \vdots \\ \frac{1}{0!}\int_a^{x_n} \frac{(t-z)^0}{\big[\phi(x_n)-\phi(t)\big]^\alpha}dt & \frac{1}{1!}\int_a^{x_n} \frac{(t-z)^1}{\big[\phi(x_n)-\phi(t)\big]^\alpha}dt & \cdots & \frac{1}{n!}\int_a^{x_n} \frac{(t-z)^n}{\big[\phi(x_n)-\phi(t)\big]^\alpha}dt \end{bmatrix}_{(n+1)(n+1)},$$

$$X = \begin{bmatrix} \Phi^{(0)}(z) \\ \Phi^{(1)}(z) \\ \vdots \\ \Phi^{(n)}(z) \end{bmatrix}_{(n+1)\times 1}, \quad G = \begin{bmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ g(x_n) \end{bmatrix}_{(n+1)\times 1}.$$

By solving the system of equations (7) and substitute $\Phi^{(j)}(z), j = 0,1,...,n$ in Eq. (3) the approximate solution of Abel's integral equation of the first kind (1) can be obtained.

In order to estimate the second kind form of Abel's integral equation (2), by substitute Eq. (3) into Eq. (2) we have

$$\sum_{j=0}^{n} \frac{1}{j!}\Phi^{(j)}(z)(x-z)^j - \int_a^x \frac{\sum_{j=0}^{n} \frac{1}{j!}\Phi^{(j)}(z)(t-z)^j}{(\phi(x)-\phi(t))^\alpha} dt = g(x), \qquad 0<\alpha<1, \tag{8}$$

and by substitute collocation points (5) into Eq.(8) the following equation is obtained

$$\sum_{j=0}^{n} \frac{1}{j!}\left[(x_i-z)^j - \int_a^{x_i} \frac{(t-z)^j}{(\phi(x_i)-\phi(t))^\alpha} dt\right]\Phi^{(j)}(c) = g(x_i), \qquad 0<\alpha<1. \tag{9}$$

Finally, we rewrite Eq. (9) in the matrix form $(B-A)X=G$ where matrices A,X,G were presented and

$$B=\begin{bmatrix} \frac{1}{0!}(x_0-z)^0 & \frac{1}{1!}(x_0-z)^1 & \cdots & \frac{1}{n!}(x_0-z)^n \\ \frac{1}{0!}(x_1-z)^0 & \frac{1}{1!}(x_1-z)^1 & \cdots & \frac{1}{n!}(x_1-z)^n \\ \vdots & \vdots & \cdots & \vdots \\ \frac{1}{0!}(x_n-z)^0 & \frac{1}{1!}(x_n-z)^1 & \cdots & \frac{1}{n!}(x_n-z)^n \end{bmatrix}_{(n+1)(n+1)}.$$

By solving the obtained system and determine the unknowns X and substitute into Eq. (3), the approximate solution of Eq. (2) can be obtained.

3 Error analysis

In this section, the error analysis theorem of this method is presented.

Theorem 1: Let $\Phi(x)$ is an exact solution of Eqs. (1) and (2), $\Phi_n(x)$ is an approximate solution which is obtained from n-th order Taylor-collocation method and $S_n(x)=\sum_{i=0}^{n}\frac{\widetilde{\Phi}^{(i)}(z)(x-z)^i}{i!}$ is the Taylor polynomial of degree n at $x=z$ then

$$\|\Phi(x)-\Phi_n(x)\|_\infty \le \frac{M}{(i+1)!}\widetilde{\Phi}^{(i+1)}(\xi)+C\max|e_i(z)|, \quad \xi\in[a,b],$$

where $M=\max|(x-z)^{i+1}|$,$C=\|\ell\|_\infty$, $e_i(z)=\widetilde{\Phi}^{(i)}(z)-\Phi^{(i)}(z)$.

Proof: Assume that $R_n(x)$ is the reminder term of n-th order Taylor polynomial S_n at $x=z$ which is given by

$$R_n(x)=\Phi(x)-S_n(x)=\frac{\widetilde{\Phi}^{(i+1)}(\xi)}{(i+1)!}(x-z)^{i+1}, \xi\in[a,b].$$

Thus

$$|R_n(x)|=|\Phi(x)-S_n(x)|\le\frac{\widetilde{\Phi}^{(i+1)}(\xi)}{(i+1)!}.\max|(x-z)^{i+1}|=\frac{M}{(i+1)!}\widetilde{\Phi}^{(i+1)}(\xi), \quad \xi\in[a,b]. \quad (10)$$

Now, one can easily write that

$$\|\Phi(x)-\Phi_n(x)\|_\infty\le\|\Phi(x)-S_n(x)\|_\infty+\|S_n(x)-\Phi_n(x)\|_\infty\le\|R_n(x)\|_\infty+\|S_n(x)-\Phi_n(x)\|_\infty. \quad (11)$$

On the other hand we have

$$|S_n(x)-\Phi_n(x)|=\left|\sum_{i=0}^{n}\left(\widetilde{\Phi}^{(i)}(z)-\Phi^{(i)}(z)\right).\frac{(x-z)^i}{i!}\right|\le|E_i\ell|\le\|E_i\|_\infty.\|\ell\|_\infty\le C\|E_i\|_\infty, \quad (12)$$

where

$$E_i = (e_0(z), e_1(z), ..., e_i(z), ..., e_n(z)), \quad \ell = (\ell_0(z), \ell_1(z), ..., \ell_i(z), ..., \ell_n(z)),$$

and

$$e_i(z) = \widetilde{\Phi}^{(i)}(z) - \Phi^{(i)}(z), \quad \ell_i(z) = \frac{(x-z)^i}{i!}.$$

By substitute Eqs. (10) and (12) into Eq. (11), the bound of absolute error function can be obtained as follows

$$\|\Phi(x) - \Phi_n(x)\|_\infty \leq \frac{M}{(i+1)!}\widetilde{\Phi}^{(i+1)}(\xi) + C\max|e_i(z)|, \quad \xi \in [a,b].\blacksquare$$

4 Numerical illustrations

In this section, several examples of generalized Abel's integral equations of the first and second kinds are solved by using presented method [24, 32, 33]. Also, the absolute error for different values of x and the maximum error are presented in some tables. The Mathematica 8 was applied to numerical calculations.

Example 1: Consider the following Abel's integral equation of the first kind [32, 33]

$$\frac{2}{3}\pi x^3 = \int_0^x \frac{\Phi(t)}{\sqrt{x^2-t^2}}dt, \tag{13}$$

where the exact solution is $\Phi(x) = \pi x^3$. For solving this integral equation by using the Taylor polynomials the following algorithm is presented.

Algorithm

1- Substitute the Taylor polynomial (3) in Eq. (13) for arbitrary value $n = 5$.

$$\frac{2}{3}\pi x^3 = \sum_{j=0}^{5} \frac{1}{j!}\int_0^x \frac{\Phi^{(j)}(z)(t-z)^j}{\sqrt{x^2-t^2}}dt. \tag{14}$$

2- Substitute the collocation point x_i from Eq. (5) in Eq. (14) as

$$\frac{2}{3}\pi x_i^3 = \sum_{j=0}^{5} \frac{1}{j!}\int_0^{x_i} \frac{\Phi^{(j)}(z)(t-z)^j}{\sqrt{x_i^2-t^2}}dt, \qquad 0 \leq i \leq n. \tag{15}$$

3- Construct the matrix form $AX = G$ where

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1.5708 & 0.2 & 0.015708 & 0.000888889 & 0.0000392699 & 1.42222\times10^{-6} \\ 1.5708 & 0.4 & 0.0628319 & 0.00711111 & 0.000628319 & 0.0000455111 \\ 1.5708 & 0.6 & 0.141372 & 0.024 & 0.00318086 & 0.0003456 \\ 1.5708 & 0.8 & 0.251327 & 0.0568889 & 0.0100531 & 0.00145636 \\ 1.5708 & 1 & 0.392699 & 0.111111 & 0.0245437 & 0.00444444 \end{bmatrix},$$

and

$$G = \begin{bmatrix} 0 \\ 0.0167552 \\ 0.134041 \\ 0.452389 \\ 1.07233 \\ 2.0944 \end{bmatrix}.$$

4- Determine the unknowns X and calculate the approximate solution $\Phi_n(x)$

$$\Phi_5(x) = 0.000211964 - 0.00380121x + 0.0198716x^2 + 3.09737x^3 + 0.0441592x^4 - 0.0162574x^5.$$

The absolute error for $n = 5, 7, 9$ and different values of x is shown in Table 1 and the maximum errors are demonstrate in Table 2.

Table 1: Numerical results of Example 1 for $n = 5, 7, 9$.

x	exact	$e_5 =\mid \Phi(x) - \Phi_5(x) \mid$	$e_7 =\mid \Phi(x) - \Phi_7(x) \mid$	$e_9 =\mid \Phi(x) - \Phi_9(x) \mid$
0.0	0	0.000211964	1.17775×10^{-8}	1.37683×10^{-13}
0.2	0.0251327	0.0000417215	8.60494×10^{-10}	1.50574×10^{-15}
0.4	0.201062	4.84617×10^{-6}	1.33074×10^{-10}	7.49401×10^{-16}
0.6	0.678584	7.69255×10^{-6}	7.99231×10^{-11}	0
0.8	1.6085	8.44368×10^{-6}	2.03972×10^{-10}	4.44089×10^{-16}
1.0	3.14159	0.0000361083	1.60657×10^{-9}	9.32587×10^{-15}

Table 2: The maximum errors of Example 1.

$n = 5$	$n = 7$	$n = 9$
0.000211964	1.17775×10^{-8}	1.37683×10^{-13}

Example 2: Consider the generalized Abel's integral equation of the first kind [32]:

$$\frac{4}{3} \sin^{\frac{3}{4}}(x) = \int_0^x \frac{\Phi(t)}{(\sin(x) - \sin(t))^{\frac{1}{4}}} dt, \tag{16}$$

with the exact solution $\Phi(x) = \cos x$. By using the presented method for $n = 9$ the approximate solution is obtained as follows

$$\begin{aligned} \Phi_9(x) = 0.9999999996930083 + 1.3682136756898444 \times 10^{-8} x - 0.5000001933589282x^2 \\ + 1.3397660985165968 \times 10^{-6} x^3 + 0.041661365627426505x^4 + \cdots \ . \end{aligned} \tag{17}$$

Comparison between exact and approximate solutions for different values of x and $n = 5, 7, 9$ are presented in Table 3 and the maximum errors are demonstrate in Table 4.
Example 3: In this example, the following first kind Abel's integral equation [32] is considered:

$$\frac{6}{5}(e^x - 1)^{\frac{5}{6}} = \int_0^x \frac{\Phi(t)}{(e^x - e^t)^{\frac{1}{6}}} dt, \tag{18}$$

Table 3: Numerical results of Example 2 for $n = 5, 7, 9$.

x	exact	$e_5 =\mid \Phi(x) - \Phi_5(x) \mid$	$e_7 =\mid \Phi(x) - \Phi_7(x) \mid$	$e_9 =\mid \Phi(x) - \Phi_9(x) \mid$
0.0	1	0.0000590935	1.82477×10^{-7}	3.06992×10^{-10}
0.2	0.980067	0.0000122884	7.76028×10^{-9}	3.44569×10^{-12}
0.4	0.921061	3.76895×10^{-6}	2.92998×10^{-9}	4.82949×10^{-13}
0.6	0.825336	3.85864×10^{-6}	3.08999×10^{-9}	9.76565×10^{-13}
0.8	0.696707	6.42239×10^{-6}	3.32207×10^{-9}	7.80758×10^{-12}
1.0	0.540302	0.000028944	7.77184×10^{-8}	1.45912×10^{-10}

Table 4: The maximum errors of Example 2.

$n = 5$	$n = 7$	$n = 9$
0.0000590935	1.82477×10^{-7}	3.06992×10^{-10}

with exact solution $\Phi(x) = e^x$. By using presented method, we have following approximate solution

$$\Phi_9(x) = 1.000000012857142 + 0.9999998135066253x + 0.5000012303151726x^2 + \cdots . \quad (19)$$

In Table 5 and 6, the absolute errors and maximum values of absolute error are presented for $n = 5, 7, 9$.

Table 5: Numerical results of Example 3 for $n = 5, 7, 9$.

x	exact	$e_5 =\mid \Phi(x) - \Phi_5(x) \mid$	$e_7 =\mid \Phi(x) - \Phi_7(x) \mid$	$e_9 =\mid \Phi(x) - \Phi_9(x) \mid$
0.0	1	0.00001384	2.06582×10^{-7}	1.28571×10^{-8}
0.2	1.2214	0.0000122884	4.8359×10^{-9}	5.60856×10^{-10}
0.4	1.49182	5.25742×10^{-6}	3.48525×10^{-9}	5.87789×10^{-10}
0.6	1.82212	5.33178×10^{-6}	4.64466×10^{-9}	2.37805×10^{-10}
0.8	2.22554	9.82404×10^{-6}	3.06887×10^{-9}	3.93271×10^{-10}
1.0	2.71828	0.0000471546	1.27079×10^{-7}	1.29719×10^{-9}

Example 4: In this example, the second kind Abel's integral equation [32, 24] is considered as follows

$$\Phi(x) = x^2 + \frac{16}{15}x^{\frac{5}{2}} - \int_0^x \frac{\Phi(t)}{(x-t)^{\frac{1}{2}}}dt,$$

with the exact solution $\Phi(x) = x^2$. Presented method for solving this example is exact for $n = 5$.
Example 5: Let us consider the following integral equation [32]:

$$\Phi(x) = 1 - 2x - \frac{32}{21}x^{\frac{7}{4}} + \frac{4}{3}x^{\frac{3}{4}} - \int_0^x \frac{\Phi(t)}{(x-t)^{\frac{1}{4}}}dt,$$

with the exact solution $\Phi(x) = 1-2x$. In this example, the Taylor-collocation method for $n = 3$ is exact.

Table 6: The maximum errors of Example 3.

$n = 5$	$n = 7$	$n = 9$
0.0000471546	2.06582×10^{-7}	1.28571×10^{-8}

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5 Conclusions

In this paper, Taylor polynomials to estimate the generalized form of Abel's integral equations of the first and second kinds were used. Also, bound of absolute error function was illustrated. Several applicable examples were solved by using Taylor-collocation method. Presented tables and obtained numerical results show the efficiency and accuracy of method.

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