

## ON THE LOCAL CARTESIAN CLOSURE OF EXACT COMPLETIONS

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ABSTRACT. This paper presents a necessary and sufficient condition on a category  $\mathbb{C}$  with weak finite limits for its exact completion  $\mathbb{C}_{\text{ex}}$  to be (locally) cartesian closed. A paper by Carboni and Rosolini already claimed such a characterisation using a different property on  $\mathbb{C}$ , but we shall show that weak finite limits are not enough for their proof to go through. We shall also indicate how to strengthen the hypothesis for that proof to work. It will become clear that, in the case of *ex/lex* completions, their characterisation is still valid and it coincides with the one presented here.

This paper presents a condition on a category  $\mathbb{C}$  with weak finite limits which is equivalent to the (local) cartesian closure of its exact completion  $\mathbb{C}_{\text{ex}}$ . In the next section we recall some background notions and results, besides fixing notation. Section 2 introduces the concepts needed to formulate the condition on  $\mathbb{C}$  and proves the characterisation of cartesian closure in Theorem 2.14. Section 3 does the same for local cartesian closure proving Theorem 3.6. In Section 4 we discuss the differences between our condition and the one given by Carboni and Rosolini in [5], and show that the characterisation of cartesian closure based on the latter requires an additional assumption. Section 5 contains some concluding remarks.

## 1. PRELIMINARIES

1.1. **Exact categories and projective covers.** A category  $\mathbb{E}$  is *exact* [2] if it has finite limits, regular epis are stable under pullback and every equivalence relation  $r = (r_1, r_2): R \hookrightarrow X \times X$  fits in a diagram

$$(1) \quad R \begin{array}{c} \xrightarrow{r_1} \\ \xrightarrow{r_2} \end{array} X \xrightarrow{q} Q$$

which is exact, *i.e.*  $q$  is a coequaliser of  $r_1, r_2$  and  $r_1, r_2$  is a kernel pair of  $q$ . When we know an arrow to be a regular epi, we shall write it with a triangle head as for  $q$  above. For an introduction to exact categories, see [4, Ch. 2] and [9, Ch. A1].

A *quasi limit* of a finite diagram in an exact category  $\mathbb{E}$  is a cone over that diagram such that the unique arrow into the limit cone is a regular epi. Similarly, a diagram as (1) is *quasi exact* if  $qr_1 = qr_2$  and the unique arrow into the kernel of  $q$  is a regular epi. In particular, a quasi exact diagram is a coequaliser.

**Definition 1.1.** Let  $\mathbb{E}$  be an exact category. A *covering square* in  $\mathbb{E}$  is a quasi pullback

$$(2) \quad \begin{array}{ccc} X & \xrightarrow{p} & A \\ \hat{f} \downarrow & & \downarrow f \\ Y & \xrightarrow{q} & B \end{array}$$

where  $q$  (and so  $p$ ) is a regular epi.

An object  $X$  *covers* another object  $A$  if there is a regular epi  $X \rightarrow A$ , and we shall refer to regular epis also as covers. An arrow  $\hat{f}: X \rightarrow Y$  *covers*  $f: A \rightarrow B$  *via*  $p$  *and*  $q$  if they fit in a covering square as (2). We shall just say that  $\hat{f}: X \rightarrow Y$  covers  $f: A \rightarrow B$  when the covers  $p$  and  $q$  are made clear by the context.

Covering squares enjoy the Beck-Chevalley property for subobjects.

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**Lemma 1.2.** *Let  $\mathbb{E}$  be an exact category. For any covering square as in (2), the canonical natural transformation  $\exists_p \hat{f}^* \rightrightarrows f^* \exists_q : \text{Sub}(Y) \rightarrow \text{Sub}(A)$  is invertible.*

An object  $X$  in a category  $\mathbb{E}$  is called (*regular*) *projective* if, for every regular epi  $g: A \twoheadrightarrow B$  and arrow  $f: X \rightarrow B$ , there is a *lift* of  $f$  along  $g$ , *i.e.* an arrow  $f': X \rightarrow A$  such that  $gf' = f$ .

**Definition 1.3.** A *projective cover* of an exact category  $\mathbb{E}$  is a full subcategory  $\mathcal{P}: \mathbb{C} \hookrightarrow \mathbb{E}$  such that

- a) for every object  $X$  in  $\mathbb{C}$ ,  $\mathcal{P}X$  is projective in  $\mathbb{E}$ , and
- b) every object in  $\mathbb{E}$  is covered by an object in  $\mathbb{C}$ , *i.e.* for every  $A$  in  $\mathbb{E}$  there are  $X$  in  $\mathbb{C}$  and a regular epi  $\mathcal{P}X \rightarrow A$ .

$\mathbb{E}$  has *enough projectives* if it has a projective cover.

When an exact category  $\mathbb{E}$  has a projective cover  $\mathcal{P}$ , we shall refer to a regular epi  $\mathcal{P}X \rightarrow A$  as a  *$\mathcal{P}$ -cover of  $A$* . Every arrow  $f: A \rightarrow B$  in  $\mathbb{E}$  can be covered by  $\mathcal{P}\hat{f}$  for some  $\hat{f}: X \rightarrow Y$  in  $\mathbb{C}$ , just take a  $\mathcal{P}$ -cover of a pullback of  $f$  and a  $\mathcal{P}$ -cover of  $Y$ . Abusing terminology, we shall often refer to  $X$  and  $\hat{f}$  as  $\mathcal{P}$ -covers of  $A$  and  $f$ , respectively.

**Example 1.4.** Every category  $\mathbb{E}$  monadic over the category  $\text{Set}$  of sets is exact with enough projectives and the full subcategory on the free algebras is a projective cover [4, Theorem 4.3.5]. This subcategory is equivalent to the Kleisli category via the comparison functor  $\mathcal{K}$ . The counit of the free-forgetful adjunction between  $\mathbb{E}$  and  $\text{Set}$  provides a choice of  $\mathcal{K}$ -covers of objects, and postcomposition with the unit yields a choice of  $\mathcal{K}$ -covers of arrows.

Consider in particular the topos  $\text{Set}^G$  of  $G$ -sets for a group  $G = (G, \cdot, 1_G, (-)^{-1})$ . Recall that objects of the Kleisli category  $\text{Set}_G$  are sets, and arrows  $f: A \twoheadrightarrow B$  in  $\text{Set}_G$  are functions  $f = \langle f_1, f_2 \rangle: A \rightarrow B \times G$  in  $\text{Set}$ . The identity on  $A$  is the unit  $\eta_A$  of the free-forgetful adjunction and the composite  $gf$  of two arrows  $f: A \twoheadrightarrow B$  and  $g: B \twoheadrightarrow C$  in  $\text{Set}_G$  is given by the function  $\mu_C(g \times G)f$ , where  $\mu_C$  is the free action of  $G$  on  $C \times G$ . The comparison functor  $\mathcal{K}: \text{Set}_G \rightarrow \text{Set}^G$  maps an arrow  $f: A \twoheadrightarrow B$  to the morphism of free actions  $\mu_B(f \times G): (A \times G, \mu_A) \rightarrow (B \times G, \mu_B)$ . For a morphism  $g: (A, \alpha) \rightarrow (B, \beta)$ , the choice of  $\mathcal{K}$ -covers given by the free-forgetful adjunction is exhibited by the square

$$(3) \quad \begin{array}{ccc} (A \times G, \mu_A) & \xrightarrow{\alpha} & (A, \alpha) \\ g \times G \downarrow & & \downarrow g \\ (B \times G, \mu_B) & \xrightarrow{\beta} & (B, \beta), \end{array}$$

where the horizontal arrows are counit components and  $g \times G = \mathcal{K}(\eta_B g)$ . In this case the square (3) is not just covering, but an actual pullback.

Let  $\mathcal{P}: \mathbb{C} \hookrightarrow \mathbb{E}$  be a projective cover. The poset of subobjects in  $\mathbb{E}$  of an object  $\mathcal{P}X$  is isomorphic to the poset reflection  $(\mathbb{C}/X)_{\text{po}}$  of  $\mathbb{C}/X$ . More generally, for a tuple  $X_1, \dots, X_n$  of objects in  $\mathbb{C}$ , let  $\mathbb{C}/(X_1, \dots, X_n)$  denote the category of cones over  $X_1, \dots, X_n$ , *i.e.* the comma category  $\Delta \downarrow (X_1, \dots, X_n)$ , where  $\Delta: \mathbb{C} \hookrightarrow \mathbb{C}^n$  is the diagonal functor and  $(X_1, \dots, X_n): \mathbb{1} \rightarrow \mathbb{C}^n$ . Hence, for every  $n > 0$  and every tuple  $X_1, \dots, X_n$  of objects in  $\mathbb{C}$ ,

$$(4) \quad \text{Sub}_{\mathbb{E}}(\mathbb{1}) \cong \mathbb{C}_{\text{po}} \quad \text{and} \quad \text{Sub}_{\mathbb{E}}(\mathcal{P}X_1 \times \dots \times \mathcal{P}X_n) \cong (\mathbb{C}/(X_1, \dots, X_n))_{\text{po}}, \text{ for } n \geq 1.$$

For this and other properties of projective covers that we shall need, we refer to [6] and [14].

The following proposition is a useful characterisation of the full subcategory of projectives among all projective covers of an exact category.

**Proposition 1.5** ([6]). *Let  $\mathcal{P}: \mathbb{C} \hookrightarrow \mathbb{E}$  be a projective cover of an exact category  $\mathbb{E}$ . The following are equivalent.*

1. *Idempotents split in  $\mathbb{C}$ .*
2. *The category  $\mathbb{C}$  is closed under retracts in  $\mathbb{E}$ .*
3. *The category  $\mathbb{C}$  is the full subcategory  $\mathcal{P}(\mathbb{E})$  of  $\mathbb{E}$  on the projective objects.*

**1.2. Weak finite limits.** Weak limits are defined as usual limits but without requiring uniqueness of the induced arrow. More precisely, an object is weakly terminal if every object has an arrow into it, and a *weak limit* of a diagram  $D$  is a cone over  $D$  which is weakly terminal among cones over  $D$ . For example, a weak product of  $X$  and  $Y$  is a span  $X \leftarrow W \rightarrow Y$  such that, for any span  $X \leftarrow Z \rightarrow Y$ , there is a (not necessarily unique) arrow  $Z \rightarrow W$  making the two triangles commute. A weak limit is a limit if and only if its projections are jointly monic. Below we list some examples of weak limits.

**Example 1.6.**

1. Consider the diagram  $X \leftarrow X \times 2 \times Y \rightarrow Y$  in  $\mathbf{Set}$  where  $X$  and  $Y$  are inhabited sets,  $2$  is a set with two elements and the two functions are the product projections. It is clearly a weak product in  $\mathbf{Set}$  with non-jointly monic projections.
2. The Kleisli category  $\mathbf{Set}_G$  from Example 1.4 has weak finite limits. A weak product of two objects  $A$  and  $B$  in  $\mathbf{Set}_G$  is

$$A \xleftarrow{\text{pr}_{1,2}} A \times G \times B \times G \xrightarrow{\text{pr}_{3,4}} B$$

where  $\text{pr}_{i,j} := \langle \text{pr}_i, \text{pr}_j \rangle$ . Since postcomposition in  $\mathbf{Set}_G$  with  $\text{pr}_{1,2}$  and  $\text{pr}_{3,4}$  involves a free action on  $A$  and  $B$ , respectively, if the group  $G$  is non-trivial the projections  $\text{pr}_{1,2}$  and  $\text{pr}_{3,4}$  are not jointly monic in  $\mathbf{Set}_G$ .

3. Homotopy pullbacks in  $\mathbf{Top}$ , the category of spaces and continuous functions, become weak pullbacks when mapped in the category  $\mathbf{Ho}(\mathbf{Top})$  of topological spaces and homotopy classes of continuous maps. In particular, the weak kernel pair in  $\mathbf{Ho}(\mathbf{Top})$  of the homotopy class of the universal cover of the circle  $f: \mathbb{R} \rightarrow S^1$  can be computed as the homotopy pullback

$$\mathbb{R} \times_{S^1}^h \mathbb{R} := \left\{ (x, y, h) \in \mathbb{R} \times \mathbb{R} \times (S^1)^{[0,1]} \mid h(0) = f(x) \text{ and } h(1) = f(y) \right\}$$

together with the two projections onto  $\mathbb{R}$ . Since loops in  $S^1$  with different winding numbers are not homotopic, the two projections cannot be jointly monic in  $\mathbf{Ho}(\mathbf{Top})$ .

Example 2 above is an instance of the following fact.

**Lemma 1.7.** *Let  $\mathcal{P}: \mathbb{C} \hookrightarrow \mathbb{E}$  be a projective cover of an exact category  $\mathbb{E}$ . Let  $\mathcal{D}: \mathbb{X} \rightarrow \mathbb{C}$  be a finite diagram in  $\mathbb{C}$ , then a cone  $(p_x: W \rightarrow \mathcal{D}x)_{x \in \mathbb{X}}$  over  $\mathcal{D}$  is a weak limit in  $\mathbb{C}$  if and only if  $(\mathcal{P}p_x: \mathcal{P}W \rightarrow \mathcal{P}\mathcal{D}x)_{x \in \mathbb{X}}$  is a quasi limit of  $\mathcal{P}\mathcal{D}$  in  $\mathbb{E}$ , i.e. the unique arrow  $p: \mathcal{P}W \rightarrow \lim(\mathcal{P}\mathcal{D})$  over  $\mathcal{P}\mathcal{D}$  is a regular epi. In particular,  $\mathbb{C}$  has finite weak limits.*

*Proof.* If  $(\mathcal{P}p_x)_x$  is a quasi limit, we can lift along  $p$  a universal arrow into the limit as in the proof of Proposition 4 in [6]. The converse is straightforward.  $\square$

We stated the lemma for finite diagrams, but the finiteness assumption is actually irrelevant as soon as the limit exists in  $\mathbb{E}$ .

**Remark 1.8.** Lemma 1.7 amounts to say that a projective cover maps weak limits into quasi limits, and reflects quasi limits into weak limits. A functor  $\mathbb{C} \rightarrow \mathbb{E}$  from a category with weak finite limits into an exact category mapping weak limits into quasi limits is called *left covering* [6, 14]. Left covering functors  $\mathbb{C} \rightarrow \mathbb{E}$  preserve jointly monomorphic families and thus all finite limits that happen to exist in  $\mathbb{C}$ . However, a projective cover (and so a left covering functor) need not preserve weak finite limits.

Let  $\mathbb{C}$  be a category with weak finite limits. Weak pullbacks along an arrow  $f: V \rightarrow X$  in  $\mathbb{C}$  define a functor  $f^{*w}: (\mathbb{C}/X)_{\text{po}} \rightarrow (\mathbb{C}/V)_{\text{po}}$ . In particular, when  $f$  is a projection of a weak product  $Z \leftarrow V \rightarrow X$ , we denote the functor defined by weak pullback along  $f$  as

$$\times_X^w: (\mathbb{C}/Z)_{\text{po}} \rightarrow (\mathbb{C}/(Z, X))_{\text{po}}$$

and call it the *weak product functor*. In the case of a projective cover  $\mathcal{P}: \mathbb{C} \hookrightarrow \mathbb{E}$ , the weak product functor  $\times_X^w$  is isomorphic to the product functor  $(-)\times \mathcal{P}X: \text{Sub}_{\mathbb{E}}(\mathcal{P}Z) \rightarrow \text{Sub}_{\mathbb{E}}(\mathcal{P}Z \times \mathcal{P}X)$  via the isomorphisms in (4).

**1.3. Cartesian closed exact categories.** Let  $\mathbb{E}$  be an exact category. If it is cartesian closed, then not only it has exponentials but all simple products, that is, right adjoints  $\Pi_A$  to pullback along product projections. The simple product functor  $\Pi_A$  restricts to subobjects, endowing the internal logic of  $\mathbb{E}$  with universal quantification  $\forall_A: \text{Sub}(I \times A) \rightarrow \text{Sub}(I)$ .

**Remark 1.9.** When  $\mathbb{E}$  is cartesian closed and has a projective cover  $\mathcal{P}: \mathbb{C} \hookrightarrow \mathbb{E}$ , it follows by (4) that, for every  $Z, X \in \mathbb{C}$ , the weak product functor  $\times_X^w$  has a right adjoint  $\forall_X^w: (\mathbb{C}/(Z, X))_{\text{po}} \rightarrow (\mathbb{C}/Z)_{\text{po}}$ . As it was already observed in [5], the converse is true as well. Indeed, suppose that  $\times_X^w$  is left adjoint for every  $X$  and  $Z$  in  $\mathbb{C}$ , then by (4) there is  $\forall_{\mathcal{P}X}$  right adjoint to  $(-)\times \mathcal{P}X$ . Let  $I$  and  $A$  be objects in  $\mathbb{E}$  and take  $\mathcal{P}$ -covers  $p: \mathcal{P}X \rightarrow A$  and  $q: \mathcal{P}Z \rightarrow I$ , which result in a covering square exhibiting  $\text{pr}_1: \mathcal{P}Z \times \mathcal{P}X \rightarrow \mathcal{P}Z$  as a cover of  $\text{pr}_1: I \times A \rightarrow I$ . It follows that the square of left adjoints below commutes by Lemma 1.2. To obtain a right adjoint  $\forall_A$  to  $(-)\times A$ , we can thus apply Theorem 2 in Section 3.7 of [3] to the diagram below.

$$\begin{array}{ccc} \text{Sub}(I) & \xrightarrow{(-)\times A} & \text{Sub}(I \times A) \\ \Sigma_q \uparrow \dashv \downarrow q^* & & \Sigma_{(q \times p)} \uparrow \dashv \downarrow (q \times p)^* \\ \text{Sub}(\mathcal{P}Z) & \xrightarrow[\forall_{\mathcal{P}X}]{(-)\times \mathcal{P}X} & \text{Sub}(\mathcal{P}Z \times \mathcal{P}X) \end{array}$$

The right adjoints  $\forall_A$  satisfy the Beck-Chevalley condition since their left adjoints do.

**1.4. The exact completion.** Carboni and Vitale have shown that a projective cover  $\mathcal{P}: \mathbb{C} \hookrightarrow \mathbb{E}$  exhibits the exact category  $\mathbb{E}$  as the free exact category over  $\mathbb{C}$  as a category with weak finite limits [6]. An essential step in their proof consists in embedding  $\mathbb{C}$  into an exact category  $\mathbb{C}_{\text{ex}}$ , the *exact completion* of  $\mathbb{C}$ . The embedding  $\Gamma: \mathbb{C} \hookrightarrow \mathbb{C}_{\text{ex}}$  is equivalent to a projective cover of  $\mathbb{C}_{\text{ex}}$ , namely the image of  $\Gamma$ , and it is universal among left covering functors from  $\mathbb{C}$  into exact categories. It follows that every exact category with enough projectives is (equivalent to) the exact completion of any of its projective covers [6, Th. 16]. We shall state and prove the characterisation of cartesian closure for a projective cover  $\mathcal{P}: \mathbb{C} \hookrightarrow \mathbb{E}$  of  $\mathbb{E}$  exact, in order to take advantage of the richer structure of  $\mathbb{E}$ . Here we briefly recall from [6, 14] the construction of  $\mathbb{C}_{\text{ex}}$  from  $\mathbb{C}$  and some properties which we shall need in the next sections.

A *pseudo equivalence relation*  $x_1, x_2: \bar{X} \rightrightarrows X$  in  $\mathbb{C}$  consists of two arrows  $x_1, x_2: \bar{X} \rightarrow X$  together with arrows  $\rho: X \rightarrow \bar{X}$ ,  $\sigma: \bar{X} \rightarrow \bar{X}$  and  $\tau: W \rightarrow \bar{X}$ , where  $\bar{X} \xleftarrow{p_2} W \xrightarrow{p_1} \bar{X}$  is a weak pullback of  $\bar{X} \xrightarrow{x_2} X \xleftarrow{x_1} \bar{X}$ , such that

$$x_1\rho = \text{id}_X, \quad x_2\rho = \text{id}_X, \quad x_1\sigma = x_2, \quad x_2\sigma = x_1, \quad x_1\tau = x_1p_1, \quad \text{and} \quad x_2\tau = x_2p_2.$$

We shall denote a pseudo equivalence relation just by its legs  $x_1, x_2$ . Let  $x_1, x_2: \bar{X} \rightrightarrows X$  and  $y_1, y_2: \bar{Y} \rightrightarrows Y$  be two pseudo equivalence relations in  $\mathbb{C}$  and let  $f: X \rightarrow Y$ . A *tracking of  $f$  from  $x_1, x_2$  to  $y_1, y_2$*  is an arrow  $\bar{f}$  making the diagram

$$(5) \quad \begin{array}{ccc} \bar{X} & \xrightarrow{\bar{f}} & \bar{Y} \\ & \searrow x_2 & \searrow y_2 \\ & X & \xrightarrow{f} Y \\ & \swarrow x_1 & \swarrow y_1 \\ X & \xrightarrow{f} & Y \end{array}$$

commute. In case  $f$  has a tracking as above, we say that it is *extensional from  $x_1, x_2$  to  $y_1, y_2$* . Finally, two arrows  $f, f': X \rightarrow Y$  in  $\mathbb{C}$  are  *$\bar{Y}$ -related* if there is  $h: X \rightarrow \bar{Y}$  making

$$\begin{array}{ccc} & X & \\ f \swarrow & \downarrow h & \searrow f' \\ Y & \xleftarrow{y_1} \bar{Y} \xrightarrow{y_2} & Y \end{array}$$

commute. Sometimes we find it convenient to write a commutative diagram involving pseudo equivalence relations in more compact ways, as in

$$\begin{array}{ccc}
 \bar{X} & \xrightarrow{\bar{f}} & \bar{Y} \\
 \downarrow x_1 & & \downarrow y_1 \\
 X & \xrightarrow{f} & Y \\
 \downarrow x_2 & & \downarrow y_2
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 \bar{U} & \xrightarrow{v_1} & V \\
 \downarrow u_1 & & \downarrow k \\
 U & \xrightarrow{h} & Z \\
 \downarrow u_2 & & 
 \end{array}$$

where the parallel arrows shall always be the legs of a pseudo equivalence relation. When we say that such diagrams commute, we mean that they commute componentwise, that is  $f x_i = y_i \bar{f}$  and  $h u_i = k v_i$  for  $i = 1, 2$ , respectively.

Objects of  $\mathbb{C}_{\text{ex}}$  are pseudo equivalence relations in  $\mathbb{C}$  and arrows of  $\mathbb{C}_{\text{ex}}$  between  $x_1, x_2: \bar{X} \rightrightarrows X$  and  $y_1, y_2: \bar{Y} \rightrightarrows Y$  are equivalence classes  $[f, \bar{f}]$  of extensional arrows  $f$  from  $x_1, x_2$  to  $y_1, y_2$  together with a tracking  $\bar{f}$ , where  $[f, \bar{f}]$  and  $[f', \bar{f}']$  are identified if  $f$  and  $f'$  are  $\bar{Y}$ -related. The embedding  $\Gamma: \mathbb{C} \hookrightarrow \mathbb{C}_{\text{ex}}$  maps an object  $X$  to the free pseudo equivalence relation on  $X$ , namely the pair of identities  $\text{id}_X, \text{id}_X$ , and an arrow  $f: X \rightarrow Y$  to the equivalence class  $[f, f]$  (which consists of  $f$  alone). As for every left covering functor,  $\Gamma$  preserves all finite limits that happen to exist in  $\mathbb{C}$ .

For a projective cover  $\mathcal{P}: \mathbb{C} \hookrightarrow \mathbb{E}$ , the equivalence between  $\mathbb{C}_{\text{ex}}$  and  $\mathbb{E}$  follows from the fact that  $\mathcal{P}$  generates  $\mathbb{E}$  via coequalisers, see [6, Proposition 15]. In particular, for every pseudo equivalence relation  $x_1, x_2: \bar{X} \rightrightarrows X$  in  $\mathbb{C}$  there is  $p: \mathcal{P}\bar{X} \rightarrow A$  making  $\mathcal{P}\bar{X} \rightrightarrows \mathcal{P}X \rightarrow A$  quasi exact and, conversely, for every  $A$  in  $\mathbb{E}$ , there are  $p$  and  $x_1, x_2$  as above. We shall say that  $p$  is a  $\mathcal{P}$ -quotient of  $x_1, x_2$  and that  $x_1, x_2$  is a  $\mathcal{P}$ -kernel of  $p$  if they form a quasi exact diagram. It is then clear that an arrow in  $\mathbb{C}$  is extensional with respect to a pair of pseudo equivalence relations if and only if it induces a (necessarily unique) arrow in  $\mathbb{E}$  between the  $\mathcal{P}$ -quotients of the two relations. We shall spell out a particular case in Lemma 2.8. For details we refer to [14] and, in particular, to the proofs of Theorems 1.5.2 and 1.6.1 therein.

When the projective cover is the embedding  $\Gamma: \mathbb{C} \hookrightarrow \mathbb{C}_{\text{ex}}$ , every object  $(X, \bar{X})$  of  $\mathbb{C}_{\text{ex}}$  has a canonical  $\Gamma$ -cover  $[\text{id}_X, \rho]: \Gamma X \rightarrow (X, \bar{X})$ . If we denote by  $x_1, x_2: \bar{X}' \rightrightarrows X$  its  $\Gamma$ -kernel, then  $\text{id}_X$  gives rise to an isomorphism  $(X, \bar{X}) \cong (X, \bar{X}')$  in  $\mathbb{C}_{\text{ex}}$ .

**Example 1.10.** Here we illustrate the above equivalence in the case of the projective cover  $\mathcal{K}: \text{Set}_G \rightarrow \text{Set}^G$  from Example 1.4. The functor  $\text{Set}^G \rightarrow (\text{Set}_G)_{\text{ex}}$  maps an algebra  $(A, \alpha)$  in  $\text{Set}^G$  to the pseudo equivalence relation

$$(6) \quad A \times G \begin{array}{c} \xrightarrow{\text{id}_{A \times G}} \\ \rightrightarrows_{\eta_A \alpha} \end{array} A$$

in  $\text{Set}_G$ . The morphism  $\alpha$  is a  $\mathcal{K}$ -quotient of it. Moreover, since a square like the one in (3) from Example 1.4 is a pullback, the diagram

$$(7) \quad (A \times G \times G, \mu_{A \times G}) \begin{array}{c} \xrightarrow{\mu_A} \\ \rightrightarrows_{\alpha \times G} \end{array} (A \times G, \mu_A) \xrightarrow{\alpha} (A, \alpha)$$

is exact. It is then easy to see that the  $\mathcal{K}$ -cover  $\eta_B f: A \rightarrow B$  of a morphism  $f: (A, \alpha) \rightarrow (B, \beta)$  is extensional from  $\text{id}_{A \times G}, \eta_A \alpha$  to  $\text{id}_{B \times G}, \eta_B \beta$ , with tracking  $\langle f \times G, 1_G \rangle: A \times G \rightarrow B \times G$ . This gives the action of  $\text{Set}^G \rightarrow (\text{Set}_G)_{\text{ex}}$  on arrows. Conversely, the functor  $(\text{Set}_G)_{\text{ex}} \rightarrow \text{Set}^G$  maps a pseudo equivalence relation as in (6) to its  $\mathcal{K}$ -quotient object  $(A, \alpha)$ . The action on arrows is given by the universal property of the coequaliser (7). It follows that an arrow  $f: A \rightarrow B$  is extensional from  $\text{id}_{A \times G}, \eta_A \alpha$  to  $\text{id}_{B \times G}, \eta_B \beta$  if and only if  $\beta f \alpha = \beta \mu_B(f \times G)$  if and only if  $\beta f$  is a morphism  $(A, \alpha) \rightarrow (B, \beta)$ .

**Remark 1.11.** In light of Proposition 1.5, for every projective cover  $\mathcal{P}: \mathbb{C} \hookrightarrow \mathbb{E}$ , the full subcategory  $\text{P}(\mathbb{E})$  of  $\mathbb{E}$  on the projectives is equivalent to the splitting of idempotents  $\bar{\mathbb{C}}$  of  $\mathbb{C}$ . It follows that two categories with weak finite limits have equivalent exact completions if and only if they have equivalent splitting of idempotents. We shall freely regard the splitting of idempotents  $\bar{\mathbb{C}}$  of  $\mathbb{C}$  as the full subcategory of projectives of  $\mathbb{C}_{\text{ex}}$ .

## 2. CARTESIAN CLOSURE

In this section we present the characterisation of cartesian closure for exact completions of categories with weak finite limits. We begin with some definitions.

**Definition 2.1.** Let  $(g_i: Y \rightarrow X_i)_{i=1}^n$  be a span with  $n$  legs in a category  $\mathbb{C}$  with weak finite limits. A pair of arrows  $y_1, y_2: \bar{Y} \rightarrow Y$  is a *weak kernel pair of the span*  $(g_i)_i$  if it is a weak limit of the diagram

$$\begin{array}{ccc} Y & & Y \\ & \swarrow \quad \searrow & \swarrow \quad \searrow \\ & g_1 \dots g_n & g_1 \dots g_n \\ & \swarrow \quad \searrow & \swarrow \quad \searrow \\ X_1 & \cdots & X_n \end{array}$$

It is well-known that, in a category with pullbacks, a kernel pair of an arrow is an equivalence relation. One proves similarly that, in a category with weak pullbacks, a weak kernel pair of a span is a pseudo equivalence relation. We shall freely regard weak kernel pairs as pseudo equivalence relations.

**Definition 2.2.** Let  $\mathbb{C}$  be a category with weak finite limits. Let  $f: Y \rightarrow X$  be an arrow and  $x_1, x_2: \bar{X} \rightrightarrows X$  a pseudo equivalence relation in  $\mathbb{C}$ . An *extensional image of  $f$  in  $x_1, x_2$*  consists of three arrows  $\bar{y}: \bar{Y} \rightarrow \bar{X}$  and  $y_1, y_2: \bar{Y} \rightarrow Y$  which form a weak limit of

$$\begin{array}{ccc} Y & & \bar{X} & & Y \\ & \swarrow & & \searrow & \downarrow f \\ & f & & x_1 & x_2 & \\ & \downarrow & & & & \downarrow f \\ X & & & & & X \end{array}$$

In a category with weak finite limits  $\mathbb{C}$ , the arrows  $y_1, y_2$ , which are part of an extensional image, are the legs of a pseudo equivalence relation  $y_1, y_2: \bar{Y} \rightrightarrows Y$ . The arrow  $\bar{y}: \bar{Y} \rightarrow \bar{X}$  is a tracking for  $f$  from  $y_1, y_2$  to  $x_1, x_2$ , and  $[f, \bar{y}]$  is monic in  $\mathbb{C}_{\text{ex}}$ . In particular, when  $f$  also has tracking  $\bar{f}$  from a pseudo equivalence relation  $z_1, z_2: \bar{Z} \rightrightarrows Z$  to  $x_1, x_2$ , the arrow  $[f, \bar{y}]$  is the image factorisation of  $[f, \bar{f}]$  in  $\mathbb{C}_{\text{ex}}$  [6]. Note also that a weak kernel pair of  $f$  is the same thing as an extensional image of  $f$  in  $\text{id}_X, \text{id}_X$ . In this case,  $[f, \bar{y}]$  is the image factorisation of  $\Gamma f: \Gamma Y \rightarrow \Gamma X$  and the assignation  $f \mapsto [f, \bar{y}]: (Y, \bar{Y}) \hookrightarrow \Gamma X$  describes the action of the right-to-left part of the isomorphism  $\text{Sub}_{\mathbb{C}_{\text{ex}}}(\Gamma X) \cong (\mathbb{C}/X)_{\text{po}}$ .

**Definition 2.3.** Let  $\mathbb{C}$  be a category with weak finite limits. Let  $Z \xleftarrow{p_1} V \xrightarrow{p_2} X$  be a weak product and let  $y_1, y_2: \bar{Y} \rightrightarrows Y$  be a pseudo equivalence relation in  $\mathbb{C}$ . An arrow  $f: V \rightarrow Y$  *preserves projections with respect to  $y_1, y_2$*  if it is extensional from a weak kernel pair of  $p_1, p_2$  into  $y_1, y_2$ .

Note that this definition does not depend on the particular weak kernel pair of  $p_1, p_2$ . When the pseudo equivalence relation on  $Y$  is clear from the context we just say that  $f$  preserves projections.

**Definition 2.4.** Let  $X$  be an object and  $y_1, y_2: \bar{Y} \rightrightarrows Y$  be a pseudo equivalence relation in a category  $\mathbb{C}$  with weak finite limits. An *extensional exponential of  $y_1, y_2$  and  $X$*  consists of an object  $W$ , a weak product  $W \leftarrow V \rightarrow X$  and an arrow  $e: V \rightarrow Y$  which preserves projections with respect to  $y_1, y_2$  and which is weakly terminal with these properties, that is, for every object  $W'$ , weak product  $W' \leftarrow V' \rightarrow X$  and arrow  $e': V' \rightarrow Y$  preserving projections with respect to  $y_1, y_2$ , there are arrows  $h: W' \rightarrow W$  and  $k: V' \rightarrow V$  making the diagram

$$\begin{array}{ccccc} & & W' & \longleftarrow & V' \\ & & \swarrow & & \searrow \\ & & h & & k \\ & & \swarrow & & \searrow \\ W & \longleftarrow & V & \longrightarrow & X \\ & & \downarrow e & & \downarrow e' \\ & & Y & & \end{array}$$

commute. The arrow  $e$  is called *extensional evaluation*.

A category with weak finite limits *has extensional exponentials* if, for every pseudo equivalence relation  $y_1, y_2$  and for every object  $X$ , there is an extensional exponential of  $y_1, y_2$  and  $X$ .

The name draws from the type of extensional functions in dependent type theory, which is an example of extensional exponential in the category of types described in [8, Section 6] or in [11, Section 7.1]. An example in  $\text{Set}_G$  will be given in 2.10, for the moment we introduce a slight strengthening of the notion of extensional exponential and prove two properties of it.

**Definition 2.5.** Let  $\mathbb{C}$  be a category with weak finite limits. Let  $y_1, y_2: \bar{Y} \rightrightarrows Y$  be a pseudo equivalence relation in  $\mathbb{C}$  and let  $Z \xleftarrow{g_1} Y \xrightarrow{g_2} X$  be a span such that both  $g_1$  and  $g_2$  coequalise  $y_1, y_2$ . An *extensional simple product* of  $g_1$  and  $g_2$  with respect to  $y_1, y_2$  consists of a commutative diagram

$$(8) \quad \begin{array}{ccccc} W & \longleftarrow & V & & \\ w \downarrow & & e \downarrow & \searrow & \\ Z & \xleftarrow{g_1} & Y & \xrightarrow{g_2} & X \end{array}$$

where  $W \leftarrow V \rightarrow X$  is a weak product and  $e$  preserves projections with respect to  $y_1, y_2$ , such that, for every commutative diagram

$$(9) \quad \begin{array}{ccccc} W' & \longleftarrow & V' & & \\ w' \downarrow & & e' \downarrow & \searrow & \\ Z & \xleftarrow{g_1} & Y & \xrightarrow{g_2} & X \end{array}$$

where  $W' \leftarrow V' \rightarrow X$  is a weak product and  $e'$  preserves projections, there are arrows  $h: W' \rightarrow W$  and  $k: V' \rightarrow V$  making

$$(10) \quad \begin{array}{ccccc} & & W' & \longleftarrow & V' \\ & & \swarrow h & & \swarrow k \\ W & \longleftarrow & V & & \\ w \downarrow & & e \downarrow & \searrow & \\ Z & \xleftarrow{g_1} & Y & \xrightarrow{g_2} & X \end{array}$$

commute. The arrow  $e$  in (8) is called *extensional evaluation*.

A category with weak finite limits *has extensional simple products* if, for every pseudo equivalence relation  $y_1, y_2: \bar{Y} \rightrightarrows Y$  and for every span  $Z \xleftarrow{g_1} Y \xrightarrow{g_2} X$  such that  $g_1 y_1 = g_1 y_2$  and  $g_2 y_1 = g_2 y_2$ , there is an extensional simple product of  $g_1$  and  $g_2$  with respect to  $y_1, y_2$ .

**Lemma 2.6.** *Let  $\mathbb{C}$  be a category with weak finite limits. If  $\mathbb{C}$  has extensional simple products, then it has extensional exponentials.*

*Proof.* Let  $X$  be an object and  $y_1, y_2: \bar{Y} \rightrightarrows Y$  be a pseudo equivalence relation in  $\mathbb{C}$ . Take first a weak product  $U$  of  $T, X$  and  $Y$ , where  $T$  is weakly terminal, and then a weak limit  $u_1, u_2: \bar{U} \rightrightarrows U$ ,  $\bar{u}: \bar{U} \rightarrow \bar{Y}$  of the diagram

$$\begin{array}{ccccc} U & & \bar{Y} & & U \\ \downarrow & \searrow & \swarrow & \searrow & \downarrow \\ Y & \longleftarrow & T & \longleftarrow & X & \longrightarrow & Y \end{array}$$

It is easy to check that the pair  $u_1, u_2$  form a pseudo equivalence relation on  $U$ . In fact a weak limit of the above diagram can also be constructed taking first a weak kernel pair of the span  $T \leftarrow U \rightarrow X$ , and then an extensional image in  $y_1, y_2$  (or vice versa, first an extensional image and then a weak kernel pair). Informally, two elements in  $U$  are  $\bar{U}$ -related if their  $T$  and  $X$  components coincide and their  $Y$  components are  $\bar{Y}$ -related. The two product projections in  $T$  and  $X$  coequalise  $u_1, u_2$  by construction and a straightforward computation shows that an extensional simple product of  $T \leftarrow U \rightarrow X$  with respect to  $u_1, u_2$  is an extensional exponential of  $y_1, y_2$  and  $X$ .  $\square$

**Lemma 2.7.** *Let  $\mathbb{C}$  be a category with weak finite limits. If  $\mathbb{C}$  has extensional simple products, then it has right adjoints to weak product functors.*

*Proof.* Consider the weak product functor  $\times_X^w: (\mathbb{C}/Z)_{\text{po}} \rightarrow (\mathbb{C}/(Z, X))_{\text{po}}$  mapping  $f: Y \rightarrow Z$  to the span  $Z \xleftarrow{f p_1} V \xrightarrow{p_2} X$ , where  $Y \xleftarrow{p_1} V \xrightarrow{p_2} X$  is a weak product. We use extensional simple products to define a functor  $\forall_X^w$  going the other way. Let  $Z \xleftarrow{g_1} Y \xrightarrow{g_2} X$  be a span and let  $y_1, y_2: \bar{Y} \rightrightarrows Y$  be its weak kernel pair. Take an extensional simple product of  $g_1, g_2$  with respect to  $y_1, y_2$  as (8) and define  $\forall_X^w[g_1, g_2] := [w]$ . A simple verification shows that the universal property of extensional simple products exhibits the extensional evaluation  $e: V \rightarrow Y$  as the counit of the adjunction  $\times_X^w \dashv \forall_X^w$ : one only needs to observe that, in a commutative diagram as (9) the arrow  $e': V' \rightarrow Y$  always preserves projections with respect to the weak kernel pair of  $g_1, g_2$ .  $\square$

Let us fix for the rest of the section an exact category  $\mathbb{E}$  together with a projective cover  $\mathcal{P}: \mathbb{C} \hookrightarrow \mathbb{E}$  exhibiting  $\mathbb{E}$  as the exact completion of  $\mathbb{C}$ . We can characterise arrows preserving projections as follows.

**Lemma 2.8.** *Let  $Z \leftarrow V \rightarrow X$  be a weak product and  $y_1, y_2: \bar{Y} \rightrightarrows Y$  a pseudo equivalence relation in  $\mathbb{C}$  and let  $q: \mathcal{P}Y \rightarrow B$  be a  $\mathcal{P}$ -quotient of it in  $\mathbb{E}$ . The following are equivalent for an arrow  $f: V \rightarrow Y$  in  $\mathbb{C}$ .*

1. *The arrow  $f$  preserves projections with respect to  $y_1, y_2$ .*
2. *There is a (necessarily unique) arrow  $g: \mathcal{P}Z \times \mathcal{P}X \rightarrow B$  in  $\mathbb{E}$  such that the square*

$$\begin{array}{ccc} \mathcal{P}V & \longrightarrow & \mathcal{P}Z \times \mathcal{P}X \\ \mathcal{P}f \downarrow & & \downarrow g \\ \mathcal{P}Y & \xrightarrow{q} & B \end{array}$$

*commutes.*

*Proof.* Immediate from the definition.  $\square$

One half of the characterisation is straightforward.

**Proposition 2.9.** *If  $\mathbb{E}$  is cartesian closed, then  $\mathbb{C}$  has extensional simple products.*

*Proof.* Let  $\bar{Y} \rightrightarrows Y$  be a pseudo equivalence relation in  $\mathbb{C}$  and let  $q: \mathcal{P}Y \rightarrow B$  be its  $\mathcal{P}$ -quotient in  $\mathbb{E}$ . Any span  $Z \leftarrow Y \rightarrow X$  in  $\mathbb{C}$  whose legs coequalise  $\bar{Y} \rightrightarrows Y$  induces an arrow  $g: B \rightarrow \mathcal{P}Z \times \mathcal{P}X$  in  $\mathbb{E}$ . Let  $w: W \rightarrow Z$  be the reflection in  $\mathbb{C}$  of the composite of the simple product  $\Pi_{\mathcal{P}X} g: \Pi_{\mathcal{P}X} B \rightarrow \mathcal{P}Z$  with a  $\mathcal{P}$ -cover  $p: \mathcal{P}W \rightarrow \Pi_{\mathcal{P}X} B$ . The extensional evaluation  $e: V \rightarrow Y$  is obtained covering the composite  $\text{ev}(p \times \text{id})$ :

$$\begin{array}{ccc} \mathcal{P}V & \longrightarrow & \mathcal{P}W \times \mathcal{P}X \\ \mathcal{P}e \downarrow & & \downarrow \text{ev}(p \times \text{id}) \\ \mathcal{P}Y & \xrightarrow{q} & B. \end{array}$$

The top row exhibits  $V$  as a weak product of  $W$  and  $X$  by Lemma 1.7, Lemma 2.8 ensures that  $e$  preserves projections and commutativity of (8) is immediate. The required universal property follows from that one of  $\Pi_{\mathcal{P}X} f$  using again Lemmas 1.7 and 2.8.  $\square$

We can now provide an example of an extensional exponential in a category with weak finite limits.

**Example 2.10.** Consider again the topos of  $G$ -sets from Example 1.4. Let  $X$  be any set and let  $\text{id}_{B \times G}, \eta_B \beta$  the pseudo equivalence relation in  $\text{Set}_G$  associated to an action  $(B, \beta)$  as in Example 1.10. An exponential in  $\text{Set}^G$  of  $(B, \beta)$  and  $\mathcal{K}X$  consists of the set  $B^{X \times G}$  and the action  $\epsilon: B^{X \times G} \times G \rightarrow B^{X \times G}$  mapping a pair  $(f, g_1)$  to the function  $\epsilon(f, g_1): (x, g_2) \mapsto \beta(f(x, g_2 \cdot g_1^{-1}), g_1)$ . The evaluation  $\text{ev}: B^{X \times G} \times (X \times G) \rightarrow B$  in  $\text{Set}$  is a morphism from  $\epsilon$  to  $\beta$ .

An extensional exponential in  $\text{Set}_G$  of the pseudo equivalence relation  $\text{id}_{B \times G}, \eta_B \beta$  and the set  $X$  is given by the set  $B^{G \times X}$  with extensional evaluation defined by the function

$$\begin{aligned} (B^{X \times G} \times G) \times (X \times G) &\xrightarrow{e} B \times G \\ (f, g_1, x, g_2) &\longmapsto (\epsilon(f, g_1)(x, g_2), 1_G). \end{aligned}$$



We now proceed to prove the converse to Proposition 2.9. We begin with an immediate consequence of Lemma 2.7 and Remark 1.9.

**Lemma 2.11.** *If  $\mathbb{C}$  has extensional simple products, then  $\mathbb{E}$  has right adjoints to inverse images along products projections.*

The validity of following lemma was already observed in [5].

**Lemma 2.12** (Carboni–Rosolini). *The category  $\mathbb{E}$  is cartesian closed if and only if  $\mathcal{P}X$  is exponentiable in  $\mathbb{E}$  for every object  $X$  in  $\mathbb{C}$ .*

*Proof.* One direction is trivial, let us then assume that every object in  $\mathbb{C}$  is exponentiable in  $\mathbb{E}$ . Let  $A$  and  $B$  be two objects in  $\mathbb{E}$  and take a  $\mathcal{P}$ -cover  $p: \mathcal{P}X \rightarrow A$  and a pseudo equivalence relation  $x_1, x_2: \bar{X} \rightrightarrows X$  in  $\mathbb{C}$  such that  $\mathcal{P}\bar{X} \rightrightarrows \mathcal{P}X \rightarrow A$  is quasi exact. Consider the following equaliser

$$E \hookrightarrow B^{\mathcal{P}X} \begin{array}{c} \xrightarrow{B^{\mathcal{P}x_1}} \\ \xrightarrow{B^{\mathcal{P}x_2}} \end{array} B^{\mathcal{P}\bar{X}}.$$

We shall prove that  $E$  is an exponential of  $A$  and  $B$ . The evaluation arrow  $e: E \times A \rightarrow B$  is obtained from the universal property of the coequaliser in the top row below, using commutativity of the solid arrows.

$$\begin{array}{ccccc} E \times \mathcal{P}\bar{X} & \begin{array}{c} \xrightarrow{\text{id} \times \mathcal{P}x_1} \\ \xrightarrow{\text{id} \times \mathcal{P}x_2} \end{array} & E \times \mathcal{P}X & \xrightarrow{\text{id} \times p} & E \times A \\ \text{id} \times \text{id} \downarrow & & \downarrow \text{id} \times \text{id} & & \downarrow e \\ B^{\mathcal{P}X} \times \mathcal{P}\bar{X} & \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} & B^{\mathcal{P}X} \times \mathcal{P}X & & \\ B^{\mathcal{P}x_2} \times \text{id} \downarrow & & \downarrow B^{\mathcal{P}x_1} \times \text{id} & & \downarrow \text{ev} \\ B^{\mathcal{P}\bar{X}} \times \mathcal{P}\bar{X} & \xrightarrow{\quad} & & & B \end{array}$$

Given  $C \in \mathbb{E}$  and  $f: C \times A \rightarrow B$ , there is a unique arrow  $g': C \rightarrow B^{\mathcal{P}X}$  making the obvious triangle commute. This unique arrow factors through  $i: E \hookrightarrow B^{\mathcal{P}X}$  since the left-hand diagram below commutes and  $B^{\mathcal{P}\bar{X}}$  is an exponential. The resulting arrow  $g: C \rightarrow E$  satisfies  $e(g \times A) = f$  because of the commutativity of the right-hand diagram below.

$$\begin{array}{ccc} C \times \mathcal{P}\bar{X} \xrightarrow{g' \times \mathcal{P}\bar{X}} B^{\mathcal{P}X} \times \mathcal{P}\bar{X} \begin{array}{c} \xrightarrow{B^{\mathcal{P}x_1} \times \text{id}} \\ \xrightarrow{B^{\mathcal{P}x_2} \times \text{id}} \end{array} B^{\mathcal{P}\bar{X}} \times \mathcal{P}\bar{X} & & C \times \mathcal{P}X \xrightarrow{C \times p} C \times A \\ \text{id} \times \mathcal{P}x_1 \downarrow \quad \text{id} \times \mathcal{P}x_2 \downarrow & & \downarrow C \times p \\ C \times \mathcal{P}X \xrightarrow{g' \times \mathcal{P}X} B^{\mathcal{P}X} \times \mathcal{P}X & & C \times A \xrightarrow{g \times A} E \times A \\ C \times p \downarrow & & \downarrow g \times A \\ C \times A \xrightarrow{f} B & & B \xrightarrow{e} E \times A \end{array}$$

Uniqueness of  $g$  follows from uniqueness of  $g'$  and monicity of  $i: E \hookrightarrow B^{\mathcal{P}X}$ .  $\square$

Before proving our main theorem we formulate one last lemma which will turn out to be useful in the next section too.

**Lemma 2.13.** *Suppose that  $\mathbb{E}$  has right adjoints to inverse images along product projections. If  $\mathbb{C}$  has extensional exponentials, then  $\mathbb{E}$  is cartesian closed.*

*Proof.* Thanks to Lemma 2.12, it is enough to construct an exponential of  $B$  and  $\mathcal{P}X$  with  $X \in \mathbb{C}$ . Take a  $\mathcal{P}$ -cover  $b: \mathcal{P}Y \rightarrow B$  of  $B$  and a  $\mathcal{P}$ -kernel  $y_1, y_2: \bar{Y} \rightrightarrows Y$  of  $b$  and let  $W$  and  $e: V \rightarrow Y$  be the object and extensional evaluation of an extensional exponential of  $y_1, y_2$  and  $X$ . The arrow  $e$  induces an arrow  $w: \mathcal{P}W \times \mathcal{P}X \rightarrow B$  by Lemma 2.8, and the kernel pair of  $\langle w, \text{pr}_{\mathcal{P}X} \rangle: \mathcal{P}W \times \mathcal{P}X \rightarrow B \times \mathcal{P}X$  factors through  $\mathcal{P}W \times \mathcal{P}W \times \mathcal{P}X$  by construction via an arrow  $k: K \hookrightarrow \mathcal{P}W \times \mathcal{P}W \times \mathcal{P}X$ . Define  $r := \forall_{\mathcal{P}X} k: R \hookrightarrow \mathcal{P}W \times \mathcal{P}W$ , where  $(-) \times \mathcal{P}X \dashv \forall_{\mathcal{P}X}$ . The adjunction relation yields that

$$(11) \quad t = \langle t_1, t_2 \rangle: T \rightarrow \mathcal{P}W \times \mathcal{P}W \text{ factors through } r \text{ if and only if } w(t_1 \times \mathcal{P}X) = w(t_2 \times \mathcal{P}X).$$

Using (11) one proves easily that  $r$  is an equivalence relation and that  $w$  coequalises  $r_1 \times \mathcal{P}X$  and  $r_2 \times \mathcal{P}X$ . It follows that  $r$  has a quotient  $q: \mathcal{P}W \rightarrow E$  and that there is a (unique) arrow  $v: E \times \mathcal{P}X \rightarrow B$  such that  $v(q \times \mathcal{P}X) = w$ .

Consider now an arrow  $f: C \times \mathcal{P}X \rightarrow B$ . Take a  $\mathcal{P}$ -cover  $c: \mathcal{P}Z \rightarrow C$ , a  $\mathcal{P}$ -kernel  $z_1, z_2: \bar{Z} \rightrightarrows Z$  and a  $\mathcal{P}$ -cover  $e': V' \rightarrow Y$  of the composite  $f(c \times \mathcal{P}X): \mathcal{P}Z \times \mathcal{P}X \rightarrow C \times \mathcal{P}X \rightarrow B$ . Lemma 1.7 ensures that  $V'$  with the obvious projections is a weak product of  $Z$  and  $X$  in  $\mathbb{C}$ . The arrow  $e'$  preserves projections by Lemma 2.8, so the universal property of  $e$  yields arrows  $h: Z \rightarrow W$  and  $k: V' \rightarrow V$  such that the diagram in Definition 2.4 commutes. This easily entails that  $w(\mathcal{P}h \times \mathcal{P}X) = f(c \times \mathcal{P}X)$ . Since the diagram below commutes,

$$\begin{array}{ccccc}
\mathcal{P}\bar{Z} \times \mathcal{P}X & \xrightarrow{\mathcal{P}z_2 \times \mathcal{P}X} & \mathcal{P}Z \times \mathcal{P}X & \xrightarrow{\mathcal{P}h \times \mathcal{P}X} & \mathcal{P}W \times \mathcal{P}X \\
\mathcal{P}z_1 \times \mathcal{P}X \downarrow & & \downarrow c \times \mathcal{P}X & & \downarrow w \\
\mathcal{P}Z \times \mathcal{P}X & \xrightarrow{c \times \mathcal{P}X} & C \times \mathcal{P}X & \xrightarrow{f} & B \\
\mathcal{P}h \times \mathcal{P}X \downarrow & & & & \downarrow w \\
\mathcal{P}W \times \mathcal{P}X & \xrightarrow{w} & & & B
\end{array}$$

the arrow  $\langle \mathcal{P}(hz_1), \mathcal{P}(hz_2) \rangle: \mathcal{P}\bar{Z} \rightarrow \mathcal{P}W \times \mathcal{P}W$  factors through  $r$  because of (11). It follows that  $qh: \mathcal{P}Z \rightarrow E$  coequalises  $\mathcal{P}z_1, \mathcal{P}z_2$ , and it thus induces an arrow  $\hat{f}: C \rightarrow E$ . The equation  $v(\hat{f} \times \mathcal{P}X) = f$  follows immediately once we precompose the two sides with the (regular) epi  $c \times \mathcal{P}X$ .

For uniqueness, let  $g: C \rightarrow E$  be such that  $v(g \times \mathcal{P}X) = f$  and denote with  $l: \mathcal{P}Z \rightarrow \mathcal{P}W$  the lift of  $gc: \mathcal{P}Z \rightarrow E$  along  $q: \mathcal{P}W \rightarrow E$ . The equation

$$w(l \times \mathcal{P}X) = v((ql) \times \mathcal{P}X) = v(g \times \mathcal{P}X)(c \times \mathcal{P}X) = f(c \times \mathcal{P}X) = w(\mathcal{P}h \times \mathcal{P}X),$$

entails by (11) that  $\langle l, \mathcal{P}h \rangle: \mathcal{P}Z \rightarrow \mathcal{P}W \times \mathcal{P}W$  factors through  $r$ . Therefore  $g = \hat{f}$  as desired.  $\square$

**Theorem 2.14.** *The category  $\mathbb{E}$  is cartesian closed if and only if  $\mathbb{C}$  has extensional simple products.*

*Proof.* It is now only a matter of putting all the pieces together. The left-to-right direction is Proposition 2.9. For the converse, Lemma 2.11 and Lemma 2.6 provide the hypothesis for Lemma 2.13, which then yields the cartesian closure of  $\mathbb{E}$ .  $\square$

**Remark 2.15.** Inspecting the proof of Lemma 2.7, one sees that only a specific kind of extensional simple product is used. In light of this fact, if we say that a *logical simple product* of  $Z \xleftarrow{g_1} Y \xrightarrow{g_2} X$  is an extensional simple product of  $g_1, g_2$  with respect to a weak kernel pair of  $g_1, g_2$ , then we could weaken the hypothesis in Lemma 2.11 to requiring only logical simple products.

It is not difficult to see that the converse is true as well: if  $\mathbb{E}$  has right adjoints to inverse images along products projections, a logical simple product of  $g_1, g_2$  in  $\mathbb{C}$  is obtained taking  $\mathcal{P}$ -covers of  $\forall_{\mathcal{P}X} b$  and of the counit  $\forall_{\mathcal{P}X} b \times \mathcal{P}X \rightarrow b$ , where  $b: B \hookrightarrow \mathcal{P}Z \times \mathcal{P}X$  is the image factorisation of  $\langle \mathcal{P}g_1, \mathcal{P}g_2 \rangle$  in  $\mathbb{E}$ . In particular, we could replace the hypothesis on  $\mathbb{E}$  in Lemma 2.13 requiring instead that  $\mathbb{C}$  has logical simple products. It follows that existence of extensional simple products is equivalent to existence of extensional exponentials and of logical simple products.

**Remark 2.16.** There is an apparently weaker notion of extensional simple product which in fact turns out to be equivalent to the one we considered in Definition 2.5, in the sense that existence of one implies existence of the other one. Say that a diagram as (8) in Definition 2.5 is a *weakly extensional simple product* if, for every diagram as (9), there are three arrows  $h: W' \rightarrow W$ ,  $k: V' \rightarrow V$  and  $j: V' \rightarrow \bar{Y}$  making the two diagrams below commute.

$$\begin{array}{ccc}
& W' & \longleftarrow V' \\
w' \swarrow & \downarrow h & \searrow k \\
Z & \longleftarrow W & \longleftarrow V & \longrightarrow X
\end{array}
\qquad
\begin{array}{ccc}
V & \longleftarrow k & V' \\
e \downarrow & & \downarrow j \\
Y & \longleftarrow y_1 & \bar{Y} & \xrightarrow{y_2} Y
\end{array}$$

That is, the arrow  $k: V' \rightarrow V$  makes  $e$  and  $e'$  not equal but only  $\bar{Y}$ -related, as witnessed by  $j$ . One can define weakly extensional exponentials similarly.

Of course, an extensional simple product is also a weakly extensional simple product, where the arrow  $V' \rightarrow \bar{Y}$  is the composite of  $e': V' \rightarrow Y$  with reflexivity of  $y_1, y_2$ . The converse is false in

general. Indeed, consider Example 2.10: the function  $e': B^{X \times G} \times G \times X \times G \rightarrow B \times G$  defined by  $f, g_1, x, g_2 \mapsto \langle f(x, g_2 \cdot g_1^{-1}), g_1 \rangle$  gives rise to a weakly extensional exponential in  $\text{Set}_G$  of  $y_1, y_2$  and  $X$ , and it is possible to show that it is not an extensional exponential. Nevertheless, given a weakly extensional simple product of  $g_1, g_2$  with respect to  $y_1, y_2$  we obtain an extensional simple product of  $g_1, g_2$  with respect to  $y_1, y_2$  simply replacing the (weakly) extensional evaluation  $e: V \rightarrow Y$  with the arrow  $f: U \rightarrow Y$  in

$$\begin{array}{ccccc} & & f & & \\ & & \curvearrowright & & \\ U & \longrightarrow & \bar{Y} & \xrightarrow{y_2} & Y \\ \downarrow & & \downarrow y_1 & & \\ V & \xrightarrow{e} & Y & & \end{array}$$

where the square is a weak pullback. It is then clear that  $f$  is an extensional evaluation once we know that  $W \leftarrow U \rightarrow X$  is a weak product and  $f$  preserves projections. To show these two facts we may assume without loss of generality that there is a projective cover  $\mathcal{P}: \mathbb{C} \hookrightarrow \mathbb{E}$  of an exact category  $\mathbb{E}$ . In this setting, the arrow  $f$  fits in a covering square

$$\begin{array}{ccc} \mathcal{P}U & \longrightarrow & \mathcal{P}V \\ \mathcal{P}f \downarrow & & \downarrow q\mathcal{P}e \\ \mathcal{P}Y & \xrightarrow{q} & B, \end{array}$$

where  $q$  is a  $\mathcal{P}$ -quotient of  $y_1, y_2$ . The top cover exhibits  $U$  as a weak product of  $W$  and  $X$  by Lemma 1.7. Since  $e: V \rightarrow Y$  preserves projections, Lemma 2.8 implies that  $q\mathcal{P}e$  factors through  $\mathcal{P}V \rightarrow \mathcal{P}W \times \mathcal{P}X$  via an arrow  $\hat{e}: \mathcal{P}W \times \mathcal{P}X \rightarrow B$ . It follows that the square below commutes

$$\begin{array}{ccc} U & \longrightarrow & \mathcal{P}W \times \mathcal{P}X \\ f \downarrow & & \downarrow \hat{e} \\ Y & \xrightarrow{q} & B \end{array}$$

and an additional application of Lemma 2.8 let us conclude that  $f$  preserves projections.

### 3. LOCAL CARTESIAN CLOSURE

Whenever  $\mathcal{P}: \mathbb{C} \hookrightarrow \mathbb{E}$  is a projective cover and  $X$  is an object in  $\mathbb{C}$ , the induced functor  $\mathcal{P}/_X: \mathbb{C}/X \hookrightarrow \mathbb{E}/\mathcal{P}X$  is a projective cover. Hence we see that  $\mathbb{E}$  is locally cartesian closed if and only if every slice of  $\mathbb{C}$  has extensional simple products: one simply applies Theorem 2.14 to derive the cartesian closure of slices of the form  $\mathbb{E}/\mathcal{P}X$ , and then use descent along a regular epi  $p: \mathcal{P}X \rightarrow A$  to obtain cartesian closure for an arbitrary slice  $\mathbb{E}/A$ . Yet, it would be good to also have a ‘‘global’’ characterisation, *i.e.* as a property of  $\mathbb{C}$  instead of each of its slices, in the same way as existence of dependent products in a category  $\mathbb{E}$  with finite limits is equivalent to existence of exponentials in every slice of  $\mathbb{E}$ . In this section we accomplish this goal.

Let  $X \leftarrow V \rightarrow W$  be a weak pullback of a cospan  $X \rightarrow Z \leftarrow W$  and let  $y_1, y_2: \bar{Y} \rightrightarrows Y$  be a pseudo equivalence relation. Say that an arrow  $V \rightarrow Y$  *preserves (pullback) projections with respect to  $y_1, y_2$*  if it has a tracking from a weak kernel pair of the pullback span  $X \leftarrow V \rightarrow W$  in  $y_1, y_2$ . Note that the forgetful functor  $\mathbb{C}/Z \rightarrow \mathbb{C}$  preserves and reflects pseudo equivalence relations and that an arrow preserves product projections in  $\mathbb{C}/Z$  if and only if it preserves pullback projections in  $\mathbb{C}$ .

**Definition 3.1.** Let  $\mathbb{C}$  be a category with weak finite limits. Let  $y_1, y_2: \bar{Y} \rightrightarrows Y$  be a pseudo equivalence relation and let  $Y \xrightarrow{g} X \xrightarrow{f} Z$  be a pair of arrows such that  $gy_1 = gy_2$ . An *extensional dependent product of  $g$  along  $f$  with respect to  $y_1, y_2$*  consists of a commutative diagram

$$(12) \quad \begin{array}{ccccc} Y & \xleftarrow{e} & V & \longrightarrow & W \\ & \searrow g & \downarrow & & \downarrow w \\ & & X & \xrightarrow{f} & Z \end{array}$$

where the square is a weak pullback and  $e$  preserves projections with respect to  $y_1, y_2$ , and such that for any commutative diagram

$$(13) \quad \begin{array}{ccccc} Y & \xleftarrow{e'} & V' & \longrightarrow & W' \\ & \searrow g & \downarrow & & \downarrow w' \\ & & X & \xrightarrow{f} & Z \end{array}$$

where the square is a weak pullback and  $e'$  preserves projections with respect to  $y_1, y_2$ , there are arrows  $h: W' \rightarrow W$  and  $k: V' \rightarrow V$  making

$$\begin{array}{ccccc} & & V' & \longrightarrow & W' \\ & \swarrow e' & \downarrow k & & \downarrow h \\ Y & \xleftarrow{e} & V & \longrightarrow & W \\ & \searrow g & \downarrow & & \downarrow w \\ & & X & \xrightarrow{f} & Z \end{array}$$

commute. The arrow  $e: V \rightarrow Y$  is called *extensional evaluation*.

A category with weak finite limits *has extensional dependent products* if, for every pseudo equivalence relation  $y_1, y_2: \bar{Y} \rightrightarrows Y$  and arrows  $Y \xrightarrow{g} X \xrightarrow{f} Z$  such that  $gy_1 = gy_2$ , there is an extensional dependent products of  $g$  along  $f$  with respect to  $y_1, y_2$ .

**Lemma 3.2.** *Let  $\mathbb{C}$  be a category with weak finite limits. If every slice of  $\mathbb{C}$  has extensional simple products, then  $\mathbb{C}$  has extensional dependent products.*

*Proof.* This is straightforward: an extensional dependent product of  $Y \xrightarrow{g} X \xrightarrow{f} Z$  with respect to  $y_1, y_2$  is given by an extensional simple product of  $fg, g$  with respect to  $y_1, y_2$  in  $\mathbb{C}/Z$ , one only needs to observe that  $y_1, y_2$  is a pseudo equivalence relation in  $\mathbb{C}/Z$  on  $fg$  because  $gy_1 = gy_2$ .  $\square$

The converse is true as well —it will follow from Theorem 3.6 and Proposition 2.9. But what we need is a bit less.

**Lemma 3.3.** *Let  $\mathbb{C}$  be a category with weak finite limits. If  $\mathbb{C}$  has extensional dependent products, then every slice of  $\mathbb{C}$  has extensional exponentials.*

*Proof.* Let  $x, y \in \mathbb{C}/Z$  and let  $y_1, y_2: \bar{Y} \rightrightarrows Y$  be a pseudo equivalence relation in  $\mathbb{C}/Z$  on  $y: Y \rightarrow Z$ . Take a weak pullback  $X \xleftarrow{f_1} U \xrightarrow{f_2} Y$  of  $x$  and  $y$  and let  $u_1, u_2: \bar{U} \rightrightarrows U$  be the pseudo equivalence relation defined in the weak limit diagram

$$(14) \quad \begin{array}{ccccc} & & \bar{U} & & \\ & \swarrow u_1 & \downarrow & \searrow u_2 & \\ U & & \bar{Y} & & U \\ f_2 \downarrow & \swarrow & \downarrow & \swarrow & \downarrow f_2 \\ Y & & X & & Y \end{array}$$

Clearly,  $f_1$  coequalises  $u_1, u_2$ . Consider then an extensional dependent product of  $U \xrightarrow{f_1} X \xrightarrow{x} Z$  with respect to  $u_1, u_2$  and let  $w: W \rightarrow Z$  and  $e: V \rightarrow U$  be its object in  $\mathbb{C}/Z$  and its extensional evaluation, respectively. We shall prove that  $w$  and  $f_2e: V \rightarrow Y$  form an extensional exponential of  $y_1, y_2$  and  $x$  in  $\mathbb{C}/Z$ .

Suppose then that there are  $w': W' \rightarrow Z$ , a weak pullback  $X \xleftarrow{p'_1} V' \xrightarrow{p'_2} W'$  of  $x$  and  $w'$ , and  $e': V' \rightarrow Y$  which preserves projections with respect to  $y_1, y_2$  and such that  $ye' = xp'_1$ . It follows by the latter equation that there is an arrow  $\hat{e}: V' \rightarrow U$  such that  $f_1\hat{e} = p'_1$  and  $f_2\hat{e} = e'$ . Using the tracking of  $e'$  and the universal property of diagram (14) one sees that  $\hat{e}$  preserves projections with

respect to  $u_1, u_2$ . By the universal property of  $w$  and  $e$  there are dotted arrows below which make the diagram

$$\begin{array}{ccccc}
 & & & V' & \longrightarrow & W' \\
 & & e' & \nearrow & & \nearrow \\
 Y & \xleftarrow{f_2} & U & \xleftarrow{e} & V & \longrightarrow & W \\
 & \searrow y & & \searrow f_1 & \downarrow & & \downarrow w \\
 & & Z & \xleftarrow{x} & X & \xrightarrow{x} & Z \\
 & & & & & & \downarrow w' \\
 & & & & & & W'
 \end{array}$$

commute. We can thus conclude that  $w: W \rightarrow Z$  and  $f_2 e: V \rightarrow Y$  enjoy the universal property of extensional exponentials in  $\mathbb{C}/Z$  as required.  $\square$

**Lemma 3.4.** *Let  $\mathbb{C}$  be a category with weak finite limits. If  $\mathbb{C}$  has extensional dependent products, then it has right adjoints to weak pullback functors.*

*Proof.* The value of the right adjoint along  $f$  at a given arrow  $g$  is defined taking an extensional dependent product of  $g$  along  $f$  with respect to a weak kernel pair of  $g$ . The proof goes, mutatis mutandis, as that one of Lemma 2.7.  $\square$

**Lemma 3.5.** *Let  $\mathcal{P}: \mathbb{C} \hookrightarrow \mathbb{E}$  be a projective cover of an exact category  $\mathbb{E}$ . If  $\mathbb{C}$  has extensional dependent products, then  $\mathbb{E}$  has right adjoints to inverse images along any arrow.*

*Proof.* From Lemma 3.4 and the isomorphisms in (4) we conclude that  $\mathbb{E}$  has right adjoints to functors  $(\mathcal{P}f)^*: \text{Sub}(\mathcal{P}Y) \rightarrow \text{Sub}(\mathcal{P}X)$  with  $f: X \rightarrow Y$  in  $\mathbb{C}$ . The right adjoint to reindexing along an arrow  $g$  in  $\mathbb{E}$  is obtained from the right adjoint to reindexing along a  $\mathcal{P}$ -cover of  $g$  as in Remark 1.9.  $\square$

**Theorem 3.6.** *Let  $\mathcal{P}: \mathbb{C} \hookrightarrow \mathbb{E}$  be a projective cover of an exact category  $\mathbb{E}$ . Then  $\mathbb{E}$  is locally cartesian closed if and only if  $\mathbb{C}$  has extensional dependent products.*

*Proof.* The left-to-right direction follows from Proposition 2.9 and Lemma 3.2. For the converse, let  $Z \in \mathbb{C}$ . Lemma 3.5 entails that  $\mathbb{E}/\mathcal{P}Z$  has right adjoints along product projections, and Lemma 3.3 ensures that  $\mathbb{C}/Z$  has extensional exponentials. The slice  $\mathbb{E}/\mathcal{P}Z$  is then cartesian closed by Lemma 2.13.

For the general case, let  $I$  be an object in  $\mathbb{E}$ ,  $p: \mathcal{P}Z \rightarrow I$  a  $\mathcal{P}$ -cover of  $I$ ,  $z_1, z_2: \bar{Z} \rightrightarrows Z$  a  $\mathcal{P}$ -kernel of  $p$  and  $\bar{p} := p\mathcal{P}z_1 = p\mathcal{P}z_2$ . Given  $a: A \rightarrow I$  and  $b: B \rightarrow I$ , there are quasi exact diagrams  $\mathcal{P}\bar{Z} \times_I A \rightrightarrows \mathcal{P}Z \times_I A \rightarrow A$  and  $\mathcal{P}\bar{Z} \times_I B \rightrightarrows \mathcal{P}Z \times_I B \rightarrow B$  over  $\mathcal{P}\bar{Z} \rightrightarrows \mathcal{P}Z \rightarrow I$ . We can form exponentials  $(p^*b)^{p^*a}: E \rightarrow \mathcal{P}Z$  in  $\mathbb{E}/\mathcal{P}Z$  and  $(\bar{p}^*b)^{\bar{p}^*a}: \bar{E} \rightarrow \mathcal{P}\bar{Z}$  in  $\mathbb{E}/\mathcal{P}\bar{Z}$ . Pasting pullbacks together

$$\begin{array}{ccccccc}
 \bar{E} \times_{\mathcal{P}\bar{Z}} (\mathcal{P}\bar{Z}_i \times_{\mathcal{P}Z} A) & \longrightarrow & \mathcal{P}\bar{Z}_i \times_{\mathcal{P}Z} A & \xrightarrow{\mathcal{P}z_i \times A} & \mathcal{P}Z \times_I A & \longrightarrow & A \\
 \downarrow & & \downarrow \bar{p}^* a & & \downarrow p^* a & & \downarrow a \\
 \bar{E} & \xrightarrow{(\bar{p}^* b)^{\bar{p}^* a}} & \mathcal{P}\bar{Z} & \xrightarrow{\mathcal{P}z_i} & \mathcal{P}Z & \xrightarrow{p} & I
 \end{array}$$

we see that, for  $i = 1, 2$ ,  $\mathcal{P}\bar{Z}_i \times_{\mathcal{P}Z} A \cong \mathcal{P}\bar{Z} \times_I A$ ,  $\bar{E} \times_{\mathcal{P}\bar{Z}} (\mathcal{P}\bar{Z}_i \times_{\mathcal{P}Z} A) \cong \bar{E}_i \times_{\mathcal{P}Z} (\mathcal{P}Z \times_I A) \cong \bar{E} \times_I A$  and, similarly, that  $E \times_{\mathcal{P}Z} (\mathcal{P}Z \times_I A) \cong E \times_I A$ . We can thus apply the universal property of  $(p^*b)^{p^*a}$  to obtain  $e_i: (\mathcal{P}z_i)(\bar{p}^*b)^{\bar{p}^*a} \rightarrow (p^*b)^{p^*a}$  in  $\mathbb{E}/\mathcal{P}Z$  such that the diagram below commutes, for  $i = 1, 2$ .

$$\begin{array}{ccccc}
 \bar{E} \times_I A & \xrightarrow{e_i \times A} & E \times_I A & & \\
 \searrow & \searrow \text{ev} & \searrow \text{ev} & & \\
 & & \mathcal{P}\bar{Z} \times_{\mathcal{P}Z} B & \xrightarrow{\mathcal{P}z_i \times B} & \mathcal{P}Z \times_I B \\
 \searrow & \searrow \bar{p}^* b & \searrow p^* b & & \\
 & & \mathcal{P}\bar{Z} & \xrightarrow{\mathcal{P}z_i} & \mathcal{P}Z
 \end{array}$$

It is possible to show that the image factorisation  $r: R \hookrightarrow E \times E$  of  $(e_1, e_2): \bar{E} \rightarrow E \times E$  is an equivalence relation in  $\mathbb{E}$ . Indeed, in the internal logic of  $\mathbb{E}$ , the exponential object  $E$  is a set of functions with values in  $\mathcal{P}Z \times_I B$ . A pair functions in  $E \times E$  is in  $r$  if and only if their  $B$ -components are equal and their  $Z$ -components are  $\bar{Z}$ -related. Let  $q: E \rightarrow Q$  be a quotient of  $r$  and note that it is also a coequaliser of  $e_1, e_2$ . Let  $Q \rightarrow I$  be the universal arrow induced by  $p(p^*a)^{p^*b}: E \rightarrow \mathcal{P}Z \rightarrow I$ . The evaluation is given by the universal property of the coequaliser

$$\bar{E} \times_I A \begin{array}{c} \xrightarrow{e_1 \times A} \\ \xrightarrow{e_2 \times A} \end{array} E \times_I A \xrightarrow{q \times A} Q \times_I A$$

applied to  $(b^*p)\text{ev}: E \times_I A \rightarrow \mathcal{P}Z \times_I B \rightarrow B$ . The verification of the universal property is straightforward and it is left to the reader.  $\square$

#### 4. CARBONI AND ROSOLINI'S CHARACTERISATION

In this section we investigate the relation between extensional and weak simple products, introduced in [5], and show that the proof of Theorem 2.5 in [5] tacitly uses an additional assumption. In particular, when  $\mathbb{C}$  has finite limits, Carboni and Rosolini's characterisation is still valid and ours reduces to it.

Let us begin recalling few basic facts about internally projective objects. An object  $P$  in a category with binary products is *internally (regular) projective* if, for every object  $C$ , regular epi  $A \rightarrow B$  and arrow  $C \times P \rightarrow B$ , there are an object  $T$  and arrows  $T \rightarrow C$  and  $T \times P \rightarrow A$  such that the following diagram commutes.

$$\begin{array}{ccc} T \times P & \longrightarrow & A \\ \downarrow & & \downarrow \\ C \times P & \longrightarrow & B \end{array}$$

**Remark 4.1.** Let  $\mathcal{P}: \mathbb{C} \hookrightarrow \mathbb{E}$  be a projective cover of an exact category  $\mathbb{E}$ . If  $P$  is internally projective in  $\mathbb{E}$ , we may take  $T$  above to be of the form  $\mathcal{P}X$  for  $X$  in  $\mathbb{C}$ , so that the arrow  $T \rightarrow C$  is a  $\mathcal{P}$ -cover of  $C$ . Furthermore, the following are equivalent for an object  $X$  in  $\mathbb{C}$ :

1.  $\mathcal{P}X$  is internally projective,
2. the functor  $(-) \times \mathcal{P}X: \mathbb{E} \rightarrow \mathbb{E}$  preserves projectives,

and, if  $\mathcal{P}X$  is exponentiable in  $\mathbb{E}$ , we may add the following to the above list of equivalents, because of the adjunction relation:

3. the exponential functor  $(-)^{\mathcal{P}X}$  preserves regular epis,
4. the simple product functor  $\Pi_{\mathcal{P}X}$  preserves regular epis.

In the last two conditions above we may also replace “regular epis” with “ $\mathcal{P}$ -covers”. More precisely, we have the following for  $(-)^{\mathcal{P}X}$  and a similar statement for  $\Pi_{\mathcal{P}X}$ .

**Lemma 4.2.** *Let  $\mathbb{E}$  be an exact category with a projective cover  $\mathcal{P}: \mathbb{C} \hookrightarrow \mathbb{E}$ . Let  $X$  be an object in  $\mathbb{C}$  and suppose that  $\mathcal{P}X$  is exponentiable in  $\mathbb{E}$ . Then the following are equivalent.*

1.  $\mathcal{P}X$  is internally projective.
2. For every  $B$  in  $\mathbb{E}$  and every  $\mathcal{P}$ -cover  $q: \mathcal{P}Y \rightarrow B$ ,  $q^{\mathcal{P}X}: \mathcal{P}Y^{\mathcal{P}X} \rightarrow B^{\mathcal{P}X}$  is regular epic.
3. For every  $B$  in  $\mathbb{E}$  there is a  $\mathcal{P}$ -cover  $q: \mathcal{P}Y \rightarrow B$  such that  $q^{\mathcal{P}X}: \mathcal{P}Y^{\mathcal{P}X} \rightarrow B^{\mathcal{P}X}$  is regular epic.

*Proof.* We only need to prove that (3) implies (1), the other two implications being obvious. Given  $f: A \rightarrow B$ , take a lift  $l: \mathcal{P}Y \rightarrow A$  of  $q: \mathcal{P}Y \rightarrow B$  along  $f$ . It follows that  $f^{\mathcal{P}X}l^{\mathcal{P}X} = q^{\mathcal{P}X}$ , which entails that  $f^{\mathcal{P}X}$  is a regular epi.  $\square$

The relation between projective objects and internally projective objects is clarified by the following equivalences.

**Lemma 4.3.** *Let  $\mathbb{E}$  be an exact category with enough projectives. The following equivalences hold.*

1. Internal projectives are projective if and only if  $\mathbf{1}$  is projective.
2. Projectives are internally projective if and only if projectives are closed under binary products.

In particular, there are exact categories with enough projectives where internal projectives and projectives do not coincide. In the topos of  $G$ -sets the terminal object is internally projective (this is always the case in a category with enough projectives) but not projective. On the contrary, projectives in  $\mathbf{Set}^G$  are internally projective, since binary products of free algebras are projective. The following is an example of an exact category with enough projectives whose projectives are not closed under binary products.

**Example 4.4.** Let  $\mathbf{Set}^M$  be the topos of actions of a monoid  $M$  on sets. Being monadic over  $\mathbf{Set}$ , considerations similar to the topos of  $G$ -sets apply. In particular, we shall use the same notation. One difference that makes this example slightly more involved is that the square (3) in Example 1.4 is not necessarily a pullback.

If the monoid  $M$  contains an idempotent element  $i$  and no invertible elements, then binary products of free algebras are not projective. To show this, it is enough to check that the canonical  $\mathcal{K}$ -cover

$$\begin{array}{ccc} (M \times M \times M, \mu_{M \times M}) & \xrightarrow{\mu \boxtimes \mu} & (M \times M, \mu \boxtimes \mu) \\ (x_1, x_2, x_3) & \longmapsto & (x_1 \cdot x_3, x_2 \cdot x_3) \end{array}$$

does not have a section in  $\mathbf{Set}^M$ , where  $\mu$  is another name for the monoid multiplication. If a function  $s = \langle s_1, s_2, s_3 \rangle: M \times M \rightarrow M \times M \times M$  is a section of  $\mu \boxtimes \mu$  in  $\mathbf{Set}$ , then  $s(1, 1) = (1, 1, 1)$ ,  $s(1, i) = (1, i, 1)$  and  $s(i, 1) = (i, 1, 1)$ . On the other hand,  $s$  is a morphism of algebras if and only if  $s(x_1 \cdot x, x_2 \cdot x) = (s_1(x_1, x_2), s_2(x_1, x_2), s_3(x_1, x_2) \cdot x)$ . It follows that, if  $s$  were section of  $\mu \boxtimes \mu$  in  $\mathbf{Set}^M$ , then it would be  $s(i, i) = s(1 \cdot i, 1 \cdot i) = (1, 1, i)$  and  $s(i, i) = s(1 \cdot i, i \cdot i) = (1, i, i)$ .

We need a few more definitions before discussing Carboni and Rosolini's characterisation. Let us say that an arrow  $f: V \rightarrow Y$  out of a weak product  $Z \xleftarrow{p_1} V \xrightarrow{p_2} X$  is *determined by projections* if it preserves projections with respect to  $\text{id}_Y, \text{id}_V$ , *i.e.* if, for all parallel arrows  $h, k$  into  $V$ ,  $p_1 h = p_1 k$  and  $p_2 h = p_2 k$  imply  $fh = fk$ . Mutatis mutandis, the same definition applies to arrows out of a weak pullback.

**Remark 4.5.** Let  $\mathcal{P}: \mathbb{C} \hookrightarrow \mathbb{E}$  be a projective cover of an exact category  $\mathbb{E}$  and  $Z \leftarrow V \rightarrow X$  a weak product in  $\mathbb{C}$ . By Lemma 2.8, an arrow  $f: V \rightarrow Y$  is determined by projections in  $\mathbb{C}$  if and only if  $\mathcal{P}f$  factors in  $\mathbb{E}$  through the  $\mathcal{P}$ -cover  $\mathcal{P}V \rightarrow \mathcal{P}Z \times \mathcal{P}X$ . It follows that an arrow is determined by projections if and only if it preserves projections with respect to any pseudo equivalence relation on its codomain.

Let  $\mathbb{C}$  be a category with weak finite limits. If  $\mathbb{C}$  has binary products, then for every weak product  $Z \leftarrow V \rightarrow X$  there is an idempotent  $i: V \rightarrow V$  such that  $p_1 i = p_1$  and  $p_2 i = p_2$  and  $i$  is determined by projections. The converse does not hold in general, but we do have the following.

**Lemma 4.6.** *Let  $\mathbb{C}$  be a category with weak finite limits. The following are equivalent for objects  $Z$  and  $X$  in  $\mathbb{C}$ .*

1. *The product  $\Gamma Z \times \Gamma X$  is projective in  $\mathbb{C}_{\text{ex}}$ .*
2. *For every weak product  $Z \xleftarrow{p_1} V \xrightarrow{p_2} X$  in  $\mathbb{C}$  there is an idempotent  $i: V \rightarrow V$  determined by projections and such that  $p_1 i = p_1, p_2 i = p_2$ .*

*Proof.* Let  $Z$  and  $X$  be objects in  $\mathbb{C}$ . Suppose that  $\Gamma Z \times \Gamma X$  is projective in  $\mathbb{C}_{\text{ex}}$ , and consider a weak product  $Z \xleftarrow{p_1} V \xrightarrow{p_2} X$ . The  $\Gamma$ -cover  $\langle \Gamma p_1, \Gamma p_2 \rangle: \Gamma V \rightarrow \Gamma Z \times \Gamma X$  has then a section  $[s, \bar{s}]: \Gamma Z \times \Gamma X \rightarrow \Gamma V$ . By faithfulness of  $\Gamma$ , the composite  $sq$  is an idempotent on  $V$  such that  $p_1 sq = sq$  and  $p_2 sq = p_2$ , and  $sq$  is determined by projections by Remark 4.5. Conversely, let  $[\text{id}_V, \rho_V]: \Gamma V \rightarrow \Gamma Z \times \Gamma X$  be the canonical cover of  $\Gamma Z \times \Gamma X$  and let  $v_1, v_2: \bar{V} \rightrightarrows V$  be its  $\Gamma$ -kernel, so that  $Z \xleftarrow{p_1} V \xrightarrow{p_2} X$  is a weak product in  $\mathbb{C}$ , where  $p_1, p_2$  are the arrows in  $\mathbb{C}$  giving rise to the projections of  $\mathcal{P}Z \times \mathcal{P}X$ , and  $\Gamma Z \times \Gamma X \cong (v_1, v_2: \bar{V} \rightrightarrows V)$  in  $\mathbb{C}_{\text{ex}}$ . Given  $i: V \rightarrow V$  as in 2, there is  $\bar{s}: \bar{V} \rightarrow V$  such that  $iv_1 = \bar{s} = iv_2$ , since  $i$  is determined by projections, and there is  $h: V \rightarrow \bar{V}$  such that  $v_1 h = i$  and  $v_2 h = \text{id}_V$ , since  $p_1 i = p_1$  and  $p_2 i = p_2$ . This amounts to say that  $\bar{s}$  is a tracking for  $i$  from  $\mathcal{P}Z \times \mathcal{P}X$  to  $\Gamma V$  and that  $[i, \bar{s}]$  in  $\mathbb{C}_{\text{ex}}$  is a section of  $[\text{id}_V, \rho_V]$ .  $\square$

**Proposition 4.7.** *Let  $\mathbb{C}$  be a category with weak finite limits. The following are equivalent.*

1.  $\bar{\mathbb{C}}$  has binary products.

2. For every weak product  $Z \xleftarrow{p_1} V \xrightarrow{p_2} X$  in  $\mathbb{C}$  there is an idempotent  $i: V \rightarrow V$  determined by projections and such that  $p_1 i = p_1$ ,  $p_2 i = p_2$ .

*Proof.* The equivalence follows from Lemma 4.6 and Remark 1.11.  $\square$

**Corollary 4.8.** *Let  $\mathbb{C}$  be a category with weak finite limits and suppose that  $\overline{\mathbb{C}}$  has binary products. Then every arrow preserving projections with respect to a pseudo equivalence relation  $y_1, y_2$  is  $\bar{Y}$ -related to an arrow determined by projections.*

*Proof.* Let  $Z \xleftarrow{p_1} V \xrightarrow{p_2} X$  be a weak product,  $i: V \rightarrow V$  the idempotent from Proposition 4.7.2 and let  $f: V \rightarrow Y$  preserve projections with respect to a pseudo equivalence relation  $y_1, y_2: \bar{Y} \rightrightarrows Y$ . The arrow  $f i: V \rightarrow Y$  is determined by projections as  $i$  is. Note that  $i$  is  $\bar{V}$ -related to  $\text{id}_V$ , where  $\bar{V} \rightrightarrows V$  is a weak kernel pair of  $p_1, p_2$ , as in the second part of the proof of Lemma 4.6. It follows that the arrow  $f i$  is  $\bar{Y}$ -related to  $f$  since  $f$  preserves projections.  $\square$

Note that the converse of Corollary 4.8 holds as well: given its conclusion, an idempotent on a weak product  $V$  is obtained as the arrow  $\bar{V}$ -related to the identity on  $V$ , where  $\bar{V} \rightrightarrows V$  is the weak kernel pair of the weak product projections.

**Definition 4.9** ([5], 2.1). A *weak simple product* of a span  $Z \xleftarrow{g_1} Y \xrightarrow{g_2} X$  in a category  $\mathbb{C}$  with weak finite limits is a commutative diagram as (8) where  $W \leftarrow V \rightarrow X$  is a weak product and the arrow  $e: V \rightarrow Y$ , called *weak evaluation*, is determined by projections, such that, for every commutative diagram as (9) where  $W' \leftarrow V' \rightarrow X$  is a weak product and  $e': V' \rightarrow Y$  is determined by projections, there are arrows  $h: W' \rightarrow W$  and  $k: V' \rightarrow V$  making diagram (10) commutative.

Clearly, weak simple products are extensional simple products with respect to free pseudo equivalence relations, *i.e.* those of the form  $\text{id}_Y, \text{id}_Y$ . One can define a *weak exponential* of  $Y$  and  $X$  to be an extensional exponential of  $Y$  and  $X$  with respect to  $\text{id}_Y, \text{id}_Y$ . It consists of an object  $W$  together with a weak product  $W \leftarrow V \rightarrow X$  and an arrow  $V \rightarrow Y$  determined by projections, which is weakly terminal among arrows  $V' \rightarrow Y$  determined by projections, where  $W' \leftarrow V' \rightarrow X$  is a weak product. As it may be expected, in a category with weak finite limits, weak exponentials can be constructed from weak simple products. The relation with extensional simple products is clarified by the following proposition together with Remark 2.16, where the notion of *weakly extensional simple product* is introduced and discussed.

**Proposition 4.10.** *Let  $\mathbb{C}$  be a category with weak finite limits. If  $\overline{\mathbb{C}}$  has binary products, then*

- (\*) *for every span  $Z \xleftarrow{g_1} Y \xrightarrow{g_2} X$ , a weak simple product of  $g_1, g_2$  is a weakly extensional simple product of  $g_1, g_2$  with respect to any pseudo equivalence relation  $y_1, y_2: \bar{Y} \rightrightarrows Y$  such that  $g_1 y_1 = g_1 y_2$  and  $g_2 y_1 = g_2 y_2$ .*

*Conversely, if  $\mathbb{C}$  has weak simple products, then (\*) implies that  $\overline{\mathbb{C}}$  has binary products.*

*Proof.* Suppose that  $\overline{\mathbb{C}}$  has binary products and consider a weak simple product as in diagram (8). The weak evaluation  $e: V \rightarrow Y$  is determined by projections, thus it preserves projections with respect to any pseudo equivalence relation. Suppose a diagram as (9) is given, where  $e': V' \rightarrow Y$  preserves projections with respect to a pseudo equivalence relation  $y_1, y_2$  as in (\*). By Corollary 4.8 there is  $f: V' \rightarrow Y$  determined by projections,  $\bar{Y}$ -related to  $e'$  and, in addition, such that the diagram (9) commutes with  $f$  in place of  $e'$ . It follows that the weak evaluation  $e$  is also a weakly extensional one.

Conversely, let  $Z \xleftarrow{p_1} V \xrightarrow{p_2} X$  be a weak product span and let  $v_1, v_2: \bar{V} \rightrightarrows V$  be the weak kernel pair of  $p_1, p_2$ . Take a weak simple product of  $p_1, p_2$  with object  $w: W \rightarrow Z$ , weak product  $W \xleftarrow{q_1} U \xrightarrow{q_2} X$ , and weak evaluation  $e: U \rightarrow V$ . By (\*) it is a weakly extensional simple product with respect to  $v_1, v_2$  and, since the identity on  $V$  preserves projections with respect to  $v_1, v_2$ , there are arrows  $h: Z \rightarrow W$ ,  $k: V \rightarrow U$  and  $j: V \rightarrow \bar{V}$  such that  $wh = \text{id}_Z$ ,  $q_1 k = h p_1$ ,  $q_2 k = p_2$ ,  $v_1 j = e k$  and  $v_2 j = \text{id}_V$ . It follows that  $p_1 e k = w q_1 k = p_1$ ,  $p_2 e k = p_2$  and, since  $e$  is determined by projections, that the composite  $e k$  is determined by projections and it is an idempotent on  $V$ . The conclusion follows from Proposition 4.7.  $\square$

Carboni and Rosolini's proof in [5] constructs an exponential in the exact completion using weak exponentials and weak simple products instead of extensional ones. Unfortunately two steps of the proof require an additional assumption. These are Lemma 2.6, where a functor is claimed to be



right adjoint, and the last part of the proof of Theorem 2.5, where it is claimed that, for a projective cover  $\mathcal{P}: \mathbb{C} \hookrightarrow \mathbb{E}$ , an exponential  $B^A$  may be obtained as a quotient of an equivalence relation on  $(\mathcal{P}Y)^{\mathcal{P}X}$ , where  $\mathcal{P}Y \rightarrow B$  and  $\mathcal{P}X \rightarrow A$  are  $\mathcal{P}$ -covers. By Lemma 4.2, the latter claim holds if and only if projectives are internally projective, and it turns out that also the first claim is equivalent to projectives being internally projective. This is not always the case when  $\mathbb{C}$  has weak finite limits as we saw, for instance, in Example 4.4.

In order to illustrate the situation for Lemma 2.6 in [5], let us consider a projective cover  $\mathcal{P}: \mathbb{C} \hookrightarrow \mathbb{E}$  of an exact category  $\mathbb{E}$ . If  $\mathbb{C}$  has weak simple products, it is possible to define functors

$$\mathbf{w}_X: \text{Sub}_{\mathbb{E}}(Z \times X) \rightarrow \text{Sub}_{\mathbb{E}}(Z)$$

for any two objects  $Z, X$  in  $\mathbb{C}$  as follows. Given  $a = (a_1, a_2): A \hookrightarrow \mathcal{P}Z \times \mathcal{P}X$ , the subobject  $\mathbf{w}_X(a) \in \text{Sub}_{\mathbb{E}}(Z)$  is the image factorisation of the arrow  $w: W \rightarrow Z$  obtained as a weak simple product of the span  $Z \xleftarrow{g_1} Y \xrightarrow{g_2} X$ , where  $p: \mathcal{P}Y \rightarrow A$  is a  $\mathcal{P}$ -cover and  $g_1$  and  $g_2$  are the reflections in  $\mathbb{C}$  of  $a_1 p$  and  $a_2 p$ , respectively. Note that the definition of  $\mathbf{w}_X(a)$  does not depend (up to isomorphism) on the  $\mathcal{P}$ -cover of  $A$ . The proof of Lemma 2.6 in [5] claims that  $\mathbf{w}_X$  is right adjoint to  $(-)\times \mathcal{P}X$ . However this is true if and only if  $\mathcal{P}X$  is internally projective in  $\mathbb{E}$ . Indeed, suppose for the moment that  $\mathbb{E}$  is cartesian closed. Then for every subobject  $a: A \hookrightarrow \mathcal{P}Z \times \mathcal{P}X$  and  $\mathcal{P}$ -cover  $\mathcal{P}Y \rightarrow A$ , the following commuting diagram

$$(15) \quad \begin{array}{ccc} \prod_{\mathcal{P}X} \mathcal{P}Y & \xrightarrow{\prod_{\mathcal{P}X}(q)} & \forall_X A \\ \downarrow & & \downarrow \forall_{\mathcal{P}X}(a) \\ \mathbf{w}_X A & \xrightarrow{\mathbf{w}_X(a)} & \mathcal{P}Z \end{array}$$

shows that  $\mathbf{w}_X$  coincides with  $\forall_{\mathcal{P}X}$  if and only if the top arrow is a regular epi. Since this argument does not depend on the particular  $\mathcal{P}$ -cover of  $A$ , we conclude that  $\mathbf{w}_X \cong \forall_{\mathcal{P}X}$  if and only if  $\mathcal{P}X$  is internally projective. More generally, the same result can be proven only assuming the existence of weak simple products in  $\mathbb{C}$ , without using the cartesian closure of  $\mathbb{E}$ .

**Proposition 4.11.** *Let  $\mathcal{P}: \mathbb{C} \hookrightarrow \mathbb{E}$  be a projective cover of  $\mathbb{E}$  exact and suppose that  $\mathbb{C}$  has weak simple products. The following are equivalent.*

1. *For every  $Z$  and  $X$  in  $\mathbb{C}$ , there is an adjunction*

$$\text{Sub}(\mathcal{P}Z) \begin{array}{c} \xleftarrow{\mathbf{w}_X} \\ \dashv \\ \xrightarrow{(-)\times \mathcal{P}X} \end{array} \text{Sub}(\mathcal{P}Z \times \mathcal{P}X).$$

2. *For every  $X$  in  $\mathbb{C}$ , the object  $\mathcal{P}X$  is internally projective in  $\mathbb{E}$ .*
3. *Projectives are internally projectives in  $\mathbb{E}$ .*

*Proof.* (1  $\Rightarrow$  2) We shall prove that for every weak product  $Z \xleftarrow{p_1} V \xrightarrow{p_2} X$  there is an idempotent  $i: V \rightarrow V$  such that  $p_1 i = p_1$ ,  $p_2 i = p_2$  and  $i$  is determined by projections. The statement will follow from Proposition 4.7, and the equivalence between conditions 1 and 2 in Remark 4.1. Let  $p = (\mathcal{P}p_1, \mathcal{P}p_2): \mathcal{P}U \rightarrow \mathcal{P}Z \times \mathcal{P}X$  be a  $\mathcal{P}$ -cover and consider the following pair of diagrams, where the left-hand one is a weak simple product of  $p_1, p_2$  in  $\mathbb{C}$  and the right-hand one is an image factorisation in  $\mathbb{E}$ .

$$\begin{array}{ccc} V & \xrightarrow{v_1} & W \\ \swarrow v_2 & & \downarrow w \\ U & \xrightarrow{p_1} & Z \\ \downarrow e & & \downarrow \mathcal{P}w \\ X & \xleftarrow{p_2} & U \end{array} \quad \begin{array}{ccc} \mathcal{P}W & & \\ \downarrow \mathcal{P}w & \searrow q & \\ \mathcal{P}Z & \xleftarrow{\mathbf{w}_X(\text{id}_{Z \times X})} & B \end{array}$$

Since  $\mathcal{P}Z$  is projective, we can lift the unit  $\text{id}_{\mathcal{P}Z} \leq \mathbf{w}_X(\text{id}_{\mathcal{P}Z} \times \mathcal{P}X)$  to a section  $h: Z \rightarrow W$  of  $w$  in  $\mathbb{C}$ . It follows that there is  $k: U \rightarrow V$  in  $\mathbb{C}$  such that  $v_1 k = h p_1$  and  $v_2 k = p_2$ . The composite  $ek: U \rightarrow U$  is easily seen to be idempotent, determined by projections and such that  $p_1 ek = p_1$  and  $p_2 ek = p_2$ .

(2  $\Rightarrow$  3) It follows from the fact that objects in a fixed projective cover are internally projective if and only if all projectives are internally projective.

(3  $\Rightarrow$  1) Observe first that  $\mathbf{w}_X(a) \times \mathcal{P}X \leq a$  always holds for any  $a \in \text{Sub}_{\mathbb{E}}(\mathcal{P}Z \times \mathcal{P}X)$ , therefore we only need to show that  $b \leq \mathbf{w}_X(b \times \mathcal{P}X)$  for every  $b \in \text{Sub}(\mathcal{P}Z)$ . To this aim, let  $p: \mathcal{P}U \rightarrow B \times \mathcal{P}X$  be a  $\mathcal{P}$ -cover and let  $w: W \rightarrow Z$  be a weak simple product of  $Z \xleftarrow{g_1} U \xrightarrow{g_2} X$ , where  $g_1$  and  $g_2$  are the reflections in  $\mathbb{C}$  of  $b\text{pr}_1 p$  and  $\text{pr}_2 p$ , so that  $\mathbf{w}_X(b \times \mathcal{P}X)$  is an image factorisation of  $\mathcal{P}w$ . Since  $\mathcal{P}X$  is internally projective, there are  $y: \mathcal{P}Y \rightarrow B$  and  $u: \mathcal{P}Y \times \mathcal{P}X \rightarrow \mathcal{P}U$  such that  $pu = y \times \mathcal{P}X$ . The composition of any  $\mathcal{P}$ -cover  $\mathcal{P}V \rightarrow \mathcal{P}Y \times \mathcal{P}X$  with  $u$  is an arrow  $V \rightarrow U$  determined by projections by Remark 4.5. It follows that the weak universal property of weak simple products provides an arrow  $Y \rightarrow W$  over  $Z$  which, in turn, induces an arrow  $b \rightarrow \mathbf{w}_X(b \times \mathcal{P}X)$  as required.  $\square$

Proposition 4.11 proves that one of the additional assumptions listed in the remark below must be added to the statement of Lemma 2.6 in [5] in order for its proof to go through.

**Remark 4.12.** Let  $\mathcal{P}: \mathbb{C} \hookrightarrow \mathbb{E}$  be a projective cover of  $\mathbb{E}$  exact and recall that  $\overline{\mathbb{C}}$  denotes the splitting of idempotents of  $\mathbb{C}$  and that  $\text{P}(\mathbb{E})$  denotes the full subcategory of  $\mathbb{E}$  on the projective objects. The following are equivalent by Proposition 4.7 and Lemma 4.3.

1. Projectives in  $\mathbb{E}$  are internally projective.
2.  $\text{P}(\mathbb{E}) \equiv \overline{\mathbb{C}}$  has binary products.
3. For every weak product  $Z \xleftarrow{p_1} V \xrightarrow{p_2} X$  in  $\mathbb{C}$  there is an idempotent  $i: V \rightarrow V$  which is determined by projections and such that  $p_1 i = p_1$  and  $p_2 i = p_2$ .

The following is Theorem 2.5 in [5] with the required assumption. Note that it can be seen as a direct consequence of Proposition 4.10 and Theorem 2.14.

**Theorem 4.13** ([5], 2.5). *Let  $\mathcal{P}: \mathbb{C} \hookrightarrow \mathbb{E}$  be a projective cover of  $\mathbb{E}$  exact. If one of the equivalent conditions from Remark 4.12 holds, then  $\mathbb{E}$  is cartesian closed if and only if  $\mathbb{C}$  has weak simple products.*

**Corollary 4.14.** *Let  $\mathcal{P}: \mathbb{C} \hookrightarrow \mathbb{E}$  be a projective cover of  $\mathbb{E}$  exact and suppose that one of the conditions from Remark 4.12 holds. Then the following are equivalent.*

1.  $\mathbb{C}$  has weak simple products.
2.  $\mathbb{C}$  has extensional simple products.
3.  $\mathbb{E}$  is cartesian closed.

The case of local cartesian closure presents, at this point, no surprise. The conditions in Remark 4.12 take the following form.

**Remark 4.15.** Let  $\mathcal{P}: \mathbb{C} \hookrightarrow \mathbb{E}$  be a projective cover of  $\mathbb{E}$  exact. The following are equivalent.

1. For every  $U$  in  $\mathbb{C}$ , projectives in  $\mathbb{E}/\mathcal{P}U$  are internally projective.
2.  $\text{P}(\mathbb{E}) \equiv \overline{\mathbb{C}}$  has pullbacks.
3. For every weak pullback square in  $\mathbb{C}$

$$\begin{array}{ccc} V & \xrightarrow{p_2} & X \\ p_1 \downarrow & & \downarrow \\ Z & \longrightarrow & U \end{array}$$

there is an idempotent  $i: V \rightarrow V$  which is determined by projections and such that  $p_1 i = p_1$  and  $p_2 i = p_2$ .

We can define a weak dependent product as an extensional dependent product with respect to free pseudo equivalence relations, and a weakly extensional dependent product similarly to the simple version in Remark 2.16. The following is proved as Proposition 4.10.

**Proposition 4.16.** *Let  $\mathbb{C}$  be a category with weak finite limits. If  $\overline{\mathbb{C}}$  has pullbacks, then*

- (\*) *for every pair  $Y \xrightarrow{g} X \xrightarrow{f} Z$ , a weak dependent product of  $f, g$  is a weakly extensional dependent product of  $f, g$  with respect to any pseudo equivalence relation  $y_1, y_2: \bar{Y} \rightrightarrows Y$  such that  $gy_1 = gy_2$ .*

*Conversely, if  $\mathbb{C}$  has weak dependent products, then (\*) implies that  $\overline{\mathbb{C}}$  has pullbacks.*

Theorem 3.3 in [5] is then reformulated as follows, and it can be derived from Proposition 4.16 and Theorem 3.6.

**Theorem 4.17** ([5], 3.3). *Let  $\mathcal{P}: \mathbb{C} \hookrightarrow \mathbb{E}$  be a projective cover of  $\mathbb{E}$  exact. If one of the conditions from Remark 4.15 holds, then  $\mathbb{E}$  is locally cartesian closed if and only if  $\mathbb{C}$  has weak dependent products.*

**Corollary 4.18.** *Let  $\mathcal{P}: \mathbb{C} \hookrightarrow \mathbb{E}$  be a projective cover of  $\mathbb{E}$  exact and suppose that one of the conditions from Remark 4.15 holds. Then the following are equivalent.*

1.  $\mathbb{C}$  has weak dependent products.
2.  $\mathbb{C}$  has extensional dependent products.
3.  $\mathbb{E}$  is locally cartesian closed.

The equivalent conditions in Remarks 4.12 and 4.15 are satisfied, in particular, whenever  $\mathbb{C}$  has finite limits. It follows that an ex/lex completion of  $\mathbb{C}$  is (locally) cartesian closed if and only if  $\mathbb{C}$  has weak simple (resp. dependent) products.

## 5. CONCLUSIONS

It appears from the above discussion that the ultimate reason for the failure of Carboni and Rosolini's characterisation in the case of a category  $\mathbb{C}$  with weak finite limits is related to the absence of certain idempotents on weak products or, equivalently, to the fact that there are arrows preserving projections not related to an arrow determined by projections. It is then clear why their characterisation holds when  $\mathbb{C}$  has finite limits, as in this case the required idempotent on a weak product not only exists but it even splits. Carboni and Rosolini's characterisation then still improves on the non-elementary one presented in [13] which requires  $\mathbb{C}$  to be infinitary lextensive, and it does fulfil their main motivation, that is, providing a common general reason for the local cartesian closure of the effective topos and the category of equiological spaces.

More generally, as the exact completion  $\mathbb{C}_{\text{ex}}$  is determined by the splitting of idempotents  $\overline{\mathbb{C}}$  rather than by  $\mathbb{C}$ , it is no surprise that Carboni and Rosolini's characterisation is still valid when it is  $\overline{\mathbb{C}}$ , rather than  $\mathbb{C}$ , to have limits. This is the case for the topos of  $G$ -sets, where the Kleisli category lacks binary products, but nevertheless exponentials in  $\text{Set}^G$  can be determined by weak simple products in  $\text{Set}_G$ . We mentioned an instance where this is not possible in Example 4.4, but we originally noticed the gap in Carboni and Rosolini's proof when trying to apply the results in [5] to a category of types arising in Martin-Löf type theory, see [8]. It is provable in Martin-Löf type theory that this category has finite limits if and only if the so-called principle of Uniqueness of Identity Proofs holds for all of its objects. But this principle is independent of the theory, and it is not known whether the equivalent conditions in Remark 4.15 are equivalent to it or strictly weaker. In particular, it is not known whether the splitting of idempotents of this category of types has pullbacks.

Together with Erik Palmgren, we presented in [8] a different condition on  $\mathbb{C}$  to derive the local cartesian closure of  $\mathbb{C}_{\text{ex}}$ , inspired by P. Aczel's Fullness Axiom from Constructive Set Theory [1], and we applied it to show when an exact completion produces a model of the Constructive Elementary Theory of the Category of Sets [12], a constructive version of Lawvere's ETCS [10]. This "fullness condition" is rather natural from a type theoretic point of view and, as we argue in [7], from a homotopy theoretic one as it naturally arises, under mild assumptions, in several homotopy categories including the homotopy categories of spaces and CW-complexes. However, it is likely to be stronger than local cartesian closure of  $\mathbb{C}_{\text{ex}}$ . Theorem 3.6 can then be used together with results from [8] to obtain a complete characterisation of models of CETCS in terms of properties of their choice objects.

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