ON GROUP RINGS

GUY RENAULT

Note by M. Guy Renault,
presented by M. Jean Leray.

Abstract. We characterize the rings A and groups G for which the group rings $A[G]$ are local, semi-local, or left perfect [14]. The recent work of M. P. Malliavin [13] and J. L. Pascaud permits the completion of results of [14] on self-injective group rings.

A designates a ring with identity but which is not necessarily commutative, and G is a group. The fields involved are not necessarily commutative. For an exposition on group rings, consult J. Lambek [12] and P. Ribenboim [15].

1. Local group rings

We generalize a result of T. Gulliksen-P. Ribenboim-T. M. Viswanathan [8, p. 153] obtained for the class of commutative group rings.

Theorem 1. *Let A be a ring and G a group $\neq e$ such that the group ring $A[G]$ is local. We then have the following properties:*

(a) A is a local ring whose maximal left ideal will be denoted by M.
(b) The field $K \neq A/M$ has characteristic $p \neq 0$.
(c) G is a p-group.

If, additionally, G is locally finite, these conditions are sufficient for $A[G]$ to be local.

The ring A is isomorphic to a quotient ring of $A[G]$, hence (a). For the same reason $K[G]$ is a local ring. If H is a subgroup of G, then $K[H]$ is local. Indeed, let R (resp. R') be the radical of $K[G]$ (resp. $K[H]$). It follows from a result of Connell [5, p. 665] that $K[H] \cap R \subset R'$; since R is is the fundamental ideal of $K[G]$, $K[H] \cap R$ is the fundamental ideal of $K[H]$: this is a maximal left ideal which is equal to R' and $K[H]$ is local. Let $x \neq e$ be an element of G, H_0 the subgroup generated by x. $K[H_0]$ is a local ring and consequently the element $e + x - x^2$ is invertible. It is easy to see that this last condition implies the finiteness of H_0. Let q be the order of x. If q is invertible in K, the element $e - q^{-i} \sum_{i=0}^{q-1} x_i$ would

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be a nontrivial idempotent of $K[G]$ which is not possible. We deduce immediately properties (b) and (c).

Let A be a ring, G a locally finite group satisfying the conditions of the theorem. Since G is locally finite, $MA[G]$ is contained in the radical of $A[G]$ [5, p. 665] and it is sufficient to demonstrate that the ring $K[G]$ is local, which easily results from the following property that is well-known when the field is commutative. Let K be a (not necessarily commutative) field of characteristic $p \neq 0$, and G a finite p-group. Then $K[G]$ is a local ring whose radical is a nil ideal.

Remark. Let G be the infinite p-group generated by three elements that is described in [10], and let k be the field of p elements. $k[G]$ is a local ring although G is not locally finite.

In what follows, A and G are commutative. The result of [8, p. 153] can also be generalized in the following way:

Proposition 2. The following conditions are equivalent:

(1) $A[G]$ is a semi-local ring.
(2) (a) A is a semi-local ring with radical R;
  (b) G is finite or G is infinite and in this case A/R is a ring of characteristic $p \neq 0$, $G = G_p \times G_0$ where G_p is an infinite p-group, and where G_0 is a finite group whose order is not divisible by p.

The proof of this theorem is not difficult. For the implication (1) $\implies$ (2) consult [3].

2. Left perfect group rings [1]

Let's recall that if A is left perfect, the finitely generated sub-modules of any right A-module satisfy the descending chain condition [2]. The result that follows was also obtained by Sheila Woods [16] by completely different methods.

Theorem 3. *Let A be a ring and G be a group. The following are equivalent:*

(1) $A[G]$ is left perfect.
(2) (a) A is left perfect.
  (b) G is finite.

(2) $\implies$ (1): For the finitely generated right ideals of $A[G]$, which are finitely generated right A-modules, we verify the descending chain condition [2].

(1) $\implies$ (2): Let R be the radical of A. The rings A, $(A/R)[G]$, which are quotient rings of $A[G]$, are left perfect and it is sufficient to study the case when A is a simple ring with center k. $A[G]$ is a free $k[G]$ -module , Lemma 12 of [15, p. 150] and the results of [2] show that $k[G]$ is left perfect. Suppose G is infinite: then $k[G]$ is not semiprimary and it results in the following consequences: the characteristic of k is $p > 0$ and there is a normal subgroup of H_1 of G whose order is divisible by p [12, p. 162], G/H_1 is infinite and $k[G/H_1]$ is left perfect. There exists a normal sub-group H_2 of G containing H_1 such that p divides the order of H_2/H_1. Evidentially, then, there is an increasing sequence of normal subgroups (H_n) of G of order $p^{s(n)}q_n$, p not dividing q_n, such that $s(n) > s(n-1)$. The Sylow theorems permit the construction of an infinite strictly increasing sequence of finite p-groups whose union is an infinite p-group G_0. $k[G]$ is a free $k[G_0]$ -module by Lemma 12 of [15, p. 150] and the results of [2] show that $k[G_0]$ is left perfect. $k[G_0]$ is a local

ring whose radical is the fundamental ideal $\omega(G_0)$; the right socle of $k[G_0]$ is not zero since $k[G_0]$ is left perfect and G_0 is finite [15, p. 137], which contradicts the hypothesis made on G.

As a special case, we obtain the characterization of Artinian group rings [I. G. Connell [5]].

3. Self-injective group rings

Theorem 4. *Let A be a ring and G be a group. The following conditions are equivalent:*

(1) The ring $A[G]$ is left self-injective.
(2) (a) A is left self-injective;
(b) G is a finite group.

(2) $\implies$ (1): This is a result of I. G. Connell [5].

(1) $\implies$ (2): Following [5] we know that A is left self-injective. Let H be a finitely generated subgroup of G, and $\omega(H)$ be the right ideal of $A[G]$ generated by the elements $1-h$, $h \in H$. According to [11] we know the left annihilator of $\omega(H)$ is different from (0), so H is finite [12], which proves that G is locally finite.

Suppose that G is an infinite group; following [9], G contains an infinite Abelian subgroup G_1. $A[G]$ which is a free $A[G_1]$ -module, is an injective $A[G_1]$ -module [4, p. 123], in particular $A[G_1]$ is left self-injective. If H_1 is an infinite subgroup of G_1, $A[G_1]$ is an injective $A[H_1]$ -module, but as $A[H_1]$ is not a quasi-Frobenius ring (See Theorem 3), this implies according to C. Faith [6], that the index of H_1 in G_1 is finite. We deduce that the socle of G_1 is of finite length and G_1 is an Artinian Abelian group [7]. It is easy to see the problem is reduced to the case when G_1 is quasi-cyclic p-group. A contradiction results from the following proposition [cf. also [13]].

Proposition 5. (Pascaud). Let A be a ring and G be the quasi-cyclic p-group defined by generators x_i and relations $x_i = x_{i+1}^p$. Then $A[G]$ is not left self-injective.

$A[G]$ is a free left A-module and we give $B = Hom_A(A[G], A[G])$ a left $A[G]$ -module structure by defining $x \in A[G]$, $f \in B$, $(x \cdot f)(y) = f(yx)$ for $y \in A[G]$.

$A[G]$ embeds into B in the following way: to each $x = \sum_{g_i} a(g_i)g_i$ we associate the endomorphism $\bar{x}$: defined by $\bar{x}(g_i) = a(g_i^{-1})$.

We denote by G_i the group generated by x_i and we consider the elements f, f_i of B defined by:

$$f(g) = \begin{cases} 1 & \text{if } g = x_{2k}^l x_{2k+1} \text{ for some } k,\, l \\ 0 & \text{otherwise} \end{cases}$$

$$f_i(g) = \begin{cases} 1 & \text{if } g = x_{2k}^l x_{2k+1} \text{ for some } k,\, l \text{ with } k \le i \\ 0 & \text{otherwise} \end{cases}$$

For all i, f_i is an element of $A[G]$ and f is an element of B that does not belong to $A[G]$.

Lemma. *(1) Let a, b be two elements of $A[G_i]$, x an element of G, $x \notin G_i$. The relation $a = bx$ implies $a = b = 0$. (2) If g is an element of G not belonging to G_{2i+2}, then $(1 - x_{2i+2}) \cdot f(g) = 0$.*

The proposition will result from the fact that $A[G] + A[G]f$ is an essential extension of $A[G]$. Let a, b two elements of $A[G_{2i}]$ with $a + bf \neq 0$ If $g \notin G_{2i+2}$, the support of bg does not meet G_{2i+2} and according to the Lemma $(1-x_{2i+2})bf(g) = 0$ and consequently

$$y = (1 - x_{2i+2})(a + bf) = (1 - x_{2i+2})(a + bf_i)$$

belongs to $A[G]$. If $y = 0$, according to the lemma we have $a + bf_i = 0$, from which it follows that $a + bf = b(f - f_i)$. Let n_0 be the smallest integer $\geq i + 1$ such that we have $b(f_n - f_i) \neq 0$; showing, as before, that

$$(1 - x_{2n_0+2})(a + bf) = (1 - x_{2n_0+2})b(f_{n_0} - f_i)$$

which is an element $\neq 0$ in $A[G]$ according to property (1) of the Lemma.

89 avenue du Recteur-Pineau
86-Poitiers, Vienne

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Jean-Yves CHAPRON
Directeur des publications
Académie des sciences
Institut de France
23 quai de Conti
75006 Paris

The following changes were made to the original text owing to the high likelihood that they were typographical mistakes:

(1) Page 1 third line of the introductory paragraph: "Lambek" was formerly "Lambeck".
(2) Page 2 third line of intro to Section 2: "Woods" was formerly "Wood".
(3) Page 2 line -2: The $k[G]$ at the beginning of the sentence was formerly $K[G]$.
(4) Page 3 third line of Proposition 5: $x = \sum_{g_i} a(g_i)g_i$ was formerly $x \sum_{g_i} a(g_i)g_i$.
(5) Page 3 first case in definition of f: the x_{2k+1} was formerly X_{2k+1}.
(6) Page 4 line 2: $a + bf \neq 0$ was formerly $a = bf \neq 0$