

# THE DISCRETE TWOFOLD ELLIS-GOHBERG INVERSE PROBLEM

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ABSTRACT. In this paper a twofold inverse problem for orthogonal matrix functions in the Wiener class is considered. The scalar-valued version of this problem was solved by Ellis and Gohberg in 1992. Under reasonable conditions, the problem is reduced to an invertibility condition on an operator that is defined using the Hankel and Toeplitz operators associated to the Wiener class functions that comprise the data set of the inverse problem. It is also shown that in this case the solution is unique. Special attention is given to the case that the Hankel operator of the solution is a strict contraction and the case where the functions are matrix polynomials.

## 1. INTRODUCTION

To state our main problem we need some notation and terminology about Wiener class functions. Throughout  $\mathcal{W}^{n \times m}$  denotes the space of  $n \times m$  matrix functions with entries in the Wiener algebra on the unit circle. Thus a matrix function  $\varphi$  belongs to  $\mathcal{W}^{n \times m}$  if and only if  $\varphi$  is continuous on the unit circle and its Fourier coefficients  $\dots \varphi_{-1}, \varphi_0, \varphi_1, \dots$  are absolutely summable. We set

$$\begin{aligned}\mathcal{W}_+^{n \times m} &= \{\varphi \in \mathcal{W}^{n \times m} \mid \varphi_j = 0, \quad \text{for } j = -1, -2, \dots\}, \\ \mathcal{W}_-^{n \times m} &= \{\varphi \in \mathcal{W}^{n \times m} \mid \varphi_j = 0, \quad \text{for } j = 1, 2, \dots\}, \\ \mathcal{W}_d^{n \times m} &= \{\varphi \in \mathcal{W}^{n \times m} \mid \varphi_j = 0, \quad \text{for } j \neq 0\}, \\ \mathcal{W}_{+,0}^{n \times m} &= \{\varphi \in \mathcal{W}^{n \times m} \mid \varphi_j = 0, \quad \text{for } j = 0, -1, -2, \dots\}, \\ \mathcal{W}_{-,0}^{n \times m} &= \{\varphi \in \mathcal{W}^{n \times m} \mid \varphi_j = 0, \quad \text{for } j = 0, 1, 2, \dots\}.\end{aligned}$$

Given  $\varphi \in \mathcal{W}^{n \times m}$  the function  $\varphi^*$  is defined by  $\varphi^*(\zeta) = \varphi(\zeta)^*$  for each  $\zeta \in \mathbb{T}$ . Thus the  $j$ -th Fourier coefficient of  $\varphi^*$  is given by  $(\varphi^*)_j = (\varphi_{-j})^*$ . The map  $\varphi \mapsto \varphi^*$  defines an involution which transforms  $\mathcal{W}^{n \times m}$  into  $\mathcal{W}^{m \times n}$ ,  $\mathcal{W}_+^{n \times m}$  into  $\mathcal{W}_-^{m \times n}$ ,  $\mathcal{W}_{-,0}^{n \times m}$  into  $\mathcal{W}_{+,0}^{m \times n}$ , etc.

The data of the inverse problem we shall be dealing with consist of four functions, namely

$$(1.1) \quad \alpha \in \mathcal{W}_+^{p \times p}, \quad \beta \in \mathcal{W}_+^{p \times q}, \quad \gamma \in \mathcal{W}_-^{q \times p}, \quad \delta \in \mathcal{W}_-^{q \times q},$$

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and we are interested in finding  $g \in \mathcal{W}_+^{p \times q}$  such that

$$(1.2) \quad \alpha + g\gamma - e_p \in \mathcal{W}_{-,0}^{p \times p} \quad \text{and} \quad g^* \alpha + \gamma \in \mathcal{W}_{+,0}^{q \times p};$$

$$(1.3) \quad \delta + g^* \beta - e_q \in \mathcal{W}_{+,0}^{q \times q} \quad \text{and} \quad g\delta + \beta \in \mathcal{W}_{-,0}^{p \times q}.$$

Here  $e_p$  and  $e_q$  denote the functions identically equal to the identity matrices  $I_p$  and  $I_q$ , respectively. If  $g$  has these properties, we refer to  $g$  as a *solution to the twofold EG inverse problem associated with the data set*  $\{\alpha, \beta, \gamma, \delta\}$ . If a solution exists, then we know from Theorem 1.2 in [11] that necessarily the following identities hold:

$$(1.4) \quad \alpha^* \alpha - \gamma^* \gamma = a_0, \quad \delta^* \delta - \beta^* \beta = d_0, \quad \alpha^* \beta = \gamma^* \delta.$$

Here  $a_0$  and  $d_0$  are the zero-th Fourier coefficient of  $\alpha$  and  $\delta$ , respectively, and we identify the matrices with  $a_0$  and  $d_0$  with the matrix functions on  $\mathbb{T}$  that are identically equal to  $a_0$  and  $d_0$ , respectively. Our main problem is to find additional conditions that guarantee the existence of a solution and to obtain explicit formulas for a solution.

The EG inverse problem related to (1.2) only and using  $\alpha$  and  $\gamma$  only has been treated in [12]. Here we deal with the inverse problem (1.2) and (1.3) together, and for that reason we refer to the problem as a twofold EG inverse problem. The acronym EG stands for R. Ellis and I. Gohberg, the authors of [2], where the inverse problem is solved for the scalar case, see [2, Section 4].

Given a data set  $\{\alpha, \beta, \gamma, \delta\}$  and assuming both matrices  $a_0$  and  $d_0$  are invertible, our main theorem (Theorem 4.1) gives necessary and sufficient conditions in order that the twofold EG inverse problem associated with the given data set has a solution. Furthermore, we show that the solution is unique and we give an explicit formula for the solution in terms of the given data. The results obtained can be seen as an addition to Chapter 11 in the Ellis and Gohberg book [3]. For some more insight in the role of the matrices  $a_0$  and  $d_0$  in (1.4) we refer to Section A.

To understand better the origin of the problem and to prove our main results we shall restate the twofold EG inverse problem as an operator problem. This requires some further notation and terminology. For any positive integer  $n$  we denote by  $\ell_+^2(\mathbb{C}^n)$  and  $\ell_-^2(\mathbb{C}^n)$  the Hilbert spaces

$$(1.5) \quad \ell_+^2(\mathbb{C}^n) = \left\{ \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{bmatrix} \mid \sum_{j=0}^{\infty} \|x_j\|^2 < \infty \right\}, \ell_-^2(\mathbb{C}^n) = \left\{ \begin{bmatrix} \vdots \\ x_{-2} \\ x_{-1} \\ x_0 \end{bmatrix} \mid \sum_{j=0}^{\infty} \|x_{-j}\|^2 < \infty \right\}.$$

We shall also need the corresponding  $\ell^1$ -spaces which appear when the superscripts 2 in (1.5) are replaced by 1. Since an absolutely summable sequence is square summable,  $\ell_{\pm}^1(\mathbb{C}^n) \subset \ell_{\pm}^2(\mathbb{C}^n)$ . In the sequel the one column matrices of the type appearing in (1.5) will be denoted by

$$\begin{bmatrix} x_0 & x_1 & x_2 & \cdots \end{bmatrix}^{\top} \quad \text{and} \quad \begin{bmatrix} \cdots & x_{-2} & x_{-1} & x_0 \end{bmatrix}^{\top}, \quad \text{respectively,}$$

with the  $\top$ -superscript indicating the block transpose. We will also use this notation when the entries are matrices. Finally, let  $h$  and  $k$  be the linear maps defined by

$$h = [h_0 \ h_1 \ h_2 \ \cdots]^\top : \mathbb{C}^m \rightarrow \ell_+^1(\mathbb{C}^n),$$

$$k = [\cdots \ k_{-2} \ k_{-1} \ k_0]^\top : \mathbb{C}^m \rightarrow \ell_-^1(\mathbb{C}^n).$$

With these linear maps we associate the functions  $\mathcal{F}h$  and  $\mathcal{F}k$  which are given by

$$(\mathcal{F}h)(\lambda) = \sum_{\nu=0}^{\infty} \lambda^\nu h_\nu \quad (|\lambda| \leq 1), \quad (\mathcal{F}k)(\lambda) = \sum_{\nu=0}^{\infty} \lambda^{-\nu} k_{-\nu} \quad (|\lambda| \geq 1).$$

Since in both cases the sequence of coefficients are summable in norm, the function  $\mathcal{F}h$  belongs to  $\mathcal{W}_+^{n \times m}$  and  $\mathcal{F}k$  belongs to  $\mathcal{W}_-^{n \times m}$ . We shall refer to  $\mathcal{F}h$  and  $\mathcal{F}k$  as the *inverse Fourier transforms* of  $h$  and  $k$ , respectively.

**The twofold EG inverse problem as an operator problem.** Let  $\alpha, \beta, \gamma, \delta$  be the functions appearing in (1.1). With these functions we associate the linear maps:

$$(1.6) \quad a = [a_0 \ a_1 \ a_2 \ \cdots]^\top : \mathbb{C}^p \rightarrow \ell_+^1(\mathbb{C}^p),$$

$$(1.7) \quad b = [b_0 \ b_1 \ b_2 \ \cdots]^\top : \mathbb{C}^q \rightarrow \ell_+^1(\mathbb{C}^q),$$

$$(1.8) \quad c = [\cdots \ c_{-2} \ c_{-1} \ c_0]^\top : \mathbb{C}^p \rightarrow \ell_-^1(\mathbb{C}^q).$$

$$(1.9) \quad d = [\cdots \ d_{-2} \ d_{-1} \ d_0]^\top : \mathbb{C}^q \rightarrow \ell_-^1(\mathbb{C}^q).$$

Here for each  $j$  the matrices  $a_j, b_j, c_j, d_j$  denote the  $j$ -th Fourier coefficients of the functions  $\alpha, \beta, \gamma, \delta$ , respectively. Thus these linear maps are uniquely determined by the functions  $\alpha, \beta, \gamma, \delta$  via the following identities

$$(1.10) \quad \alpha = \mathcal{F}a, \quad \beta = \mathcal{F}b, \quad \gamma = \mathcal{F}c, \quad \delta = \mathcal{F}d.$$

Conversely, if  $a, b, c, d$  are linear maps as in (1.6) – (1.9), then the functions  $\alpha, \beta, \gamma, \delta$  defined by (1.10) satisfy the inclusions listed in (1.1).

Next, let  $g$  be any function in  $\mathcal{W}_+^{p \times q}$ , say  $g(\zeta) = \sum_{\nu=0}^{\infty} \zeta^\nu g_\nu$ ,  $\zeta \in \mathbb{T}$ . With  $g$  we associate the Hankel operator  $G$  defined by

$$(1.11) \quad G = \begin{bmatrix} \cdots & g_2 & g_1 & g_0 \\ \cdots & g_3 & g_2 & g_1 \\ \cdots & g_4 & g_3 & g_2 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} : \ell_-^2(\mathbb{C}^q) \rightarrow \ell_+^2(\mathbb{C}^p).$$

Using the linear maps  $a, b, c, d$  defined by (1.6) – (1.9) it is straightforward to check that

$$(1.12) \quad (1.2) \iff \begin{bmatrix} I & G \\ G^* & I \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} \varepsilon_{+,p} \\ 0 \end{bmatrix},$$

$$(1.13) \quad (1.3) \iff \begin{bmatrix} I & G \\ G^* & I \end{bmatrix} \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ \varepsilon_{-,q} \end{bmatrix}.$$

Here  $I$  stands for the identity operator on  $\ell_-^2(\mathbb{C}^q)$  or  $\ell_+^2(\mathbb{C}^p)$ , and

$$\varepsilon_{+,p} = [I_p \ 0 \ 0 \ \cdots]^\top : \mathbb{C}^p \rightarrow \ell_+^1(\mathbb{C}^p),$$

$$\varepsilon_{-,q} = [\cdots \ 0 \ 0 \ I_q]^\top : \mathbb{C}^q \rightarrow \ell_-^1(\mathbb{C}^q).$$

Thus, given the data set  $\{\alpha, \beta, \gamma, \delta\}$  and the associate linear maps  $a, b, c, d$ , the twofold EG inverse problem is equivalent to the problem of finding a function  $g \in \mathcal{W}_+^{p \times q}$ ,  $g(\zeta) = \sum_{\nu=0}^{\infty} \zeta^\nu g_\nu$ , such that for the Hankel operator  $G$  defined by  $g$  in (1.11) the following two identities are satisfied:

$$(1.14) \quad \begin{bmatrix} I & G \\ G^* & I \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} \varepsilon_{+,p} \\ 0 \end{bmatrix}, \quad \begin{bmatrix} I & G \\ G^* & I \end{bmatrix} \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ \varepsilon_{-,q} \end{bmatrix}.$$

In this operator setting the twofold EG inverse problem appears as an infinite dimensional analogue of the classical inverse problem for an  $n \times n$  Hermitian block Toeplitz matrix  $T = [t_{i-j}]_{i,j=0}^n$ , where  $t_k = t_k^*$ ,  $0 \leq k \leq n$ , are  $p \times p$  matrices. For the latter problem the data consist of two matrix polynomials,  $x(\lambda) = \sum_{\nu=0}^n \lambda^\nu x_\nu$  and  $z(\lambda) = \sum_{\nu=0}^n \lambda^{-\nu} z_{-\nu}$ , with the coefficients being  $p \times p$  matrices, and the problem is to find  $p \times p$  matrices  $t_0, t_1, \dots, t_n$  such that

$$T \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} I_p \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{and} \quad T \begin{bmatrix} z_n \\ \vdots \\ z_{-1} \\ z_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_p \end{bmatrix}.$$

The solution of this block Toeplitz matrix inverse problem is due to Gohberg-Heinig [7]; see also Theorem 3.4 in [8].

The twofold EG inverse problem is also closely related to an inversion theorem for the operator  $\Omega$  defined by the  $2 \times 2$  operator matrix appearing in (1.14). Thus

$$(1.15) \quad \Omega = \begin{bmatrix} I & G \\ G^* & I \end{bmatrix}, \quad \text{where } G \text{ is given by (1.11) and } g \in \mathcal{W}_+^{p \times q}.$$

In fact, the operator  $\Omega$  is invertible whenever there exist linear maps  $a, b, c, d$  as in (1.6) – (1.9) such that the identities in (1.14) are satisfied. The latter result is given by Theorem 3.1 in [4], and without the formula for the inverse of  $\Omega$  it can also be found in Section 11 of the Ellis-Gohberg book [3]. The inversion theorem also appears in Section 5 of [1] in a more general non-symmetric setting (see also [10, Theorem 1.1]). We will give a direct proof of this inversion theorem in Section 6. The result itself, see Theorem 3.1 in Section 3, plays an important role in the proof of our main theorem (Theorem 4.1).

**Contents.** The paper consists of ten sections including the present introduction and an appendix. In Section 2 we review a number of standard facts about Laurent, Toeplitz and Hankel operators that are used throughout the paper. In this section we also reformulate the inclusions in (1.2) and (1.3) in operator language. In Section 3 we state the inversion theorem, and in Section 4 we present the solution of the inverse problem. In Section 4 we also consider the special case when  $\det \alpha$  and  $\det \delta$  have no zeros in  $|\lambda| \leq 1$  and  $|\lambda| \geq 1$ , respectively. In Section 5 we prove a number of basic identities that will play a fundamental role in proving Theorem 3.1 in Section 6 and Theorem 4.1 in Section 7. In deriving these basic identities we only use that the data  $\{\alpha, \beta, \gamma, \delta\}$  satisfy the three conditions in (1.4) and that the matrices  $a_0$  and  $d_0$  are invertible. In Section 8 we prove Theorem 4.4 and in Section 9 we state and prove the solution to the EG inverse problem for the case when the functions  $\alpha$  and  $\beta$  are polynomials in  $\lambda$  and the functions  $\gamma$  and  $\delta$  are polynomials in  $\lambda^{-1}$ . In the final section, the appendix, we review (in a somewhat more general setting, allowing the matrices to be non-square) some known results on properties of the

matrices  $a_0$  and  $d_0$  in the identities in (1.4), and present their proofs for the sake of completeness.

## 2. PRELIMINARIES ON HANKEL AND TOEPLITZ OPERATORS

We shall need some standard facts involving Laurent, Toeplitz, and Hankel operators (see, e.g., the first three sections of [6, Chapter XXIII]). Fix a  $\rho \in \mathcal{W}^{n \times m}$ , where  $\rho(\zeta) = \sum_{\nu=-\infty}^{\infty} r_\nu \zeta^\nu$ ,  $\zeta \in \mathbb{T}$ . With  $\rho$  we associate an operator  $L_\rho$  which is given by the following  $2 \times 2$  operator matrix representation:

$$(2.1) \quad L_\rho = \begin{bmatrix} T_{-, \rho} & S_{-, n}^* H_{-, \rho} \\ S_{+, n}^* H_{+, \rho} & T_{+, \rho} \end{bmatrix} : \begin{bmatrix} \ell_-^2(\mathbb{C}^m) \\ \ell_+^2(\mathbb{C}^m) \end{bmatrix} \rightarrow \begin{bmatrix} \ell_-^2(\mathbb{C}^n) \\ \ell_+^2(\mathbb{C}^n) \end{bmatrix}.$$

Here  $T_{+, \rho}$  and  $T_{-, \rho}$  are the block Toeplitz operators defined by

$$(2.2) \quad T_{+, \rho} = \begin{bmatrix} r_0 & r_{-1} & r_{-2} & \cdots \\ r_1 & r_0 & r_{-1} & \cdots \\ r_2 & r_1 & r_0 & \cdots \\ \vdots & \vdots & & \ddots \end{bmatrix} : \ell_+^2(\mathbb{C}^m) \rightarrow \ell_+^2(\mathbb{C}^n),$$

$$(2.3) \quad T_{-, \rho} = \begin{bmatrix} \cdots & \vdots & \vdots & \vdots \\ \cdots & r_0 & r_{-1} & r_{-2} \\ \cdots & r_1 & r_0 & r_{-1} \\ \cdots & r_2 & r_1 & r_0 \end{bmatrix} : \ell_-^2(\mathbb{C}^m) \rightarrow \ell_-^2(\mathbb{C}^n),$$

and  $H_{+, \rho}$  and  $H_{-, \rho}$  are the block Hankel operators defined by

$$(2.4) \quad H_{+, \rho} = \begin{bmatrix} \cdots & r_2 & r_1 & r_0 \\ \cdots & r_3 & r_2 & r_1 \\ \cdots & r_4 & r_3 & r_2 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} : \ell_-^2(\mathbb{C}^m) \rightarrow \ell_+^2(\mathbb{C}^n),$$

$$(2.5) \quad H_{-, \rho} = \begin{bmatrix} \vdots & \vdots & \vdots \\ r_{-2} & r_{-3} & r_{-4} & \cdots \\ r_{-1} & r_{-2} & r_{-3} & \cdots \\ r_0 & r_{-1} & r_{-2} & \cdots \end{bmatrix} : \ell_+^2(\mathbb{C}^m) \rightarrow \ell_-^2(\mathbb{C}^n).$$

Furthermore,  $S_{-, n}$  and  $S_{+, n}$  are the block forward shifts on  $\ell_-^2(\mathbb{C}^n)$  and  $\ell_+^2(\mathbb{C}^n)$ , respectively, that is,

$$(2.6) \quad S_{-, n} [\cdots \ x_{-2} \ x_{-1} \ x_0]^\top = [\cdots \ x_{-1} \ x_0 \ 0]^\top,$$

$$(2.7) \quad S_{+, n} [y_0 \ y_1 \ y_2 \ \cdots]^\top = [0 \ y_0 \ y_1 \ \cdots]^\top,$$

and  $S_{-, n}^*$  and  $S_{+, n}^*$  are the adjoints of these operators. It follows that

$$(2.8) \quad S_{-, n}^* H_{-, \rho} = \begin{bmatrix} \vdots & \vdots & \vdots \\ r_{-3} & r_{-4} & r_{-5} & \cdots \\ r_{-2} & r_{-3} & r_{-4} & \cdots \\ r_{-1} & r_{-2} & r_{-3} & \cdots \end{bmatrix}, \quad S_{+, n}^* H_{+, \rho} = \begin{bmatrix} \cdots & r_3 & r_2 & r_1 \\ \cdots & r_4 & r_3 & r_2 \\ \cdots & r_5 & r_4 & r_3 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

Moreover, we have

$$(2.9) \quad S_{-, n}^* H_{-, \rho} = H_{-, \rho} S_{+, m} \quad \text{and} \quad S_{+, n}^* H_{+, \rho} = H_{+, \rho} S_{-, m}.$$

With some ambiguity in terminology we call the operator  $L_\rho$  given by (2.1) the *Laurent operator* defined by  $\rho$ . Since  $\rho^*(\zeta) = \sum_{\nu=-\infty}^{\infty} \zeta^\nu r_{-\nu}^*$ , formulas (2.2), (2.3) and (2.4), (2.5) yield the following identities:

$$(2.10) \quad T_{+,\rho}^* = T_{+,\rho^*}, \quad T_{-,\rho}^* = T_{-,\rho^*}, \quad H_{+,\rho}^* = H_{-,\rho^*}.$$

In the sequel,  $\ell$  stands for the scalar function  $\ell$  given by  $\ell(\zeta) = \zeta$  for each  $\zeta \in \mathbb{T}$ . Note that  $\ell^*(\zeta) = \ell(\zeta)^* = \ell(\zeta)^{-1}$ ,  $\zeta \in \mathbb{T}$ . It follows that

$$(2.11) \quad (\ell\rho)(\zeta) = \sum_{\nu=-\infty}^{\infty} \zeta^\nu r_{\nu-1} \quad \text{and} \quad (\ell^*\rho)(\zeta) = \sum_{\nu=-\infty}^{\infty} \zeta^\nu r_{\nu+1} \quad (\zeta \in \mathbb{T}).$$

But then (2.8) can be rewritten as

$$(2.12) \quad S_{+,\rho}^* H_{-,\rho} = H_{-,\ell\rho} \quad \text{and} \quad S_{+,\rho}^* H_{+,\rho} = H_{+,\ell^*\rho}.$$

Note that the identities in (2.12) allow us to rewrite the  $2 \times 2$  operator matrix defining  $L_\rho$  in (2.1) in the following way:

$$(2.13) \quad L_\rho = \begin{bmatrix} T_{-,\rho} & H_{-,\ell\rho} \\ H_{+,\ell^*\rho} & T_{+,\rho} \end{bmatrix}.$$

Since  $\mathcal{W} = \mathcal{W}^{1 \times 1}$  is an algebra, we know that  $\rho \in \mathcal{W}^{n \times m}$  and  $\phi \in \mathcal{W}^{m \times k}$  implies that  $\rho\phi \in \mathcal{W}^{n \times k}$ , and hence by the theory of Laurent operators we have

$$(2.14) \quad L_{\rho\phi} = L_\rho L_\phi.$$

Using the representation (2.13) for  $\rho$ , for  $\phi$  in place of  $\rho$ , and for  $\rho\phi$  in place of  $\rho$  we see that the product formula (2.14) is equivalent to the following four identities:

$$(2.15) \quad T_{+,\rho\phi} = T_{+,\rho} T_{+,\phi} + H_{+,\ell^*\rho} H_{-,\ell\phi},$$

$$(2.16) \quad H_{+,\ell^*\rho\phi} = H_{+,\ell^*\rho} T_{-,\phi} + T_{+,\rho} H_{+,\ell^*\phi},$$

$$(2.17) \quad H_{-,\ell\rho\phi} = T_{-,\rho} H_{-,\ell\phi} + H_{-,\ell\rho} T_{+,\phi},$$

$$(2.18) \quad T_{-,\rho\phi} = T_{-,\rho} T_{-,\phi} + H_{-,\ell\rho} H_{+,\ell^*\phi}.$$

Finally, if  $r_0$  is an  $n \times n$  matrix, then  $\Delta_{r_0}$  denotes the diagonal operator acting on  $\ell_-^2(\mathbb{C}^n)$  or  $\ell_+^2(\mathbb{C}^n)$ . For  $\rho \in \mathcal{W}^{n \times m}$  one has

$$(2.19) \quad T_{\pm,\rho r_0} = T_{\pm,\rho} \Delta_{r_0} \quad \text{and} \quad H_{\pm,\rho r_0} = H_{\pm,\rho} \Delta_{r_0}.$$

In the remaining part of this section we deal with the functions  $\alpha, \beta, \gamma, \delta$  given by (1.1). The fact that  $\alpha \in \mathcal{W}_+^{p \times p}$ ,  $\beta \in \mathcal{W}_+^{p \times q}$ ,  $\gamma \in \mathcal{W}_-^{q \times p}$ ,  $\delta \in \mathcal{W}_-^{q \times q}$  implies that

$$(2.20) \quad H_{-,\ell\alpha} = 0, \quad H_{-,\ell\beta} = 0, \quad H_{+,\ell^*\gamma} = 0, \quad H_{+,\ell^*\delta} = 0,$$

$$(2.21) \quad H_{+,\ell^*\alpha^*} = 0, \quad H_{+,\ell^*\beta^*} = 0, \quad H_{-,\ell\gamma^*} = 0, \quad H_{-,\ell\delta^*} = 0.$$

Note that the identities in (2.21) follow from those in (2.20) by taking adjoints.

The next proposition presents some implications of the inclusions in (1.2) and (1.3) in operator language.

**Proposition 2.1.** *Let  $\alpha, \beta, \gamma, \delta$  be the functions given by (1.1), and let  $g$  be an arbitrary function in the Wiener space  $\mathcal{W}_+^{p \times q}$ . Then the inclusions in (1.2) and (1.3) imply the following identities*

$$(2.22) \quad H_{+,\rho} T_{-,\ell^*\gamma} = -H_{+,\ell^*\alpha} \quad \text{and} \quad T_{+,\alpha^*} H_{+,\rho} = -H_{+,\rho^*};$$

$$(2.23) \quad T_{+,\ell^*\beta^*} H_{+,\rho} = -H_{+,\ell^*\delta^*} \quad \text{and} \quad H_{+,\rho} T_{-,\delta} = -H_{+,\rho}.$$

More precisely, the first inclusions in (1.2) and (1.3) imply the first identities in (2.22) and (2.23), respectively, and similarly with second in place of first.

**Proof.** We shall only prove that the first inclusion in (1.2) implies the first identity in (2.22). The other implications are proved in a similar way.

First we apply (2.16) with  $\rho = \ell g$  and  $\phi = \ell^* \gamma$ . Using  $\ell$  is scalar we obtain  $\rho \phi = \ell g \ell^* \gamma = \ell \ell^* g \gamma = g \gamma$ . This yields

$$H_{+,g} T_{-, \ell^* \gamma} = H_{+, \ell^* g \gamma} - T_{+, \ell g} H_{+, \ell^* \ell^* \gamma}.$$

Since  $\ell^* \ell^* \gamma \in \mathcal{W}_{-,0}^{q \times p}$ , we see that  $H_{+, \ell^* \ell^* \gamma} = 0$ , and thus

$$(2.24) \quad H_{+,g} T_{-, \ell^* \gamma} = H_{+, \ell^* g \gamma}.$$

The first inclusion in (1.2) tells us that  $\ell^* g \gamma + \ell^* \alpha = \ell^* (g \gamma + \alpha) \in \mathcal{W}_{-,0}^{p \times p}$ , and hence  $H_{+, \ell^* g \gamma} = -H_{+, \ell^* \alpha}$ . Using the latter identity in (2.24) we obtain the first identity in (2.22).  $\square$

### 3. THE INVERSION THEOREM

Let  $\alpha, \beta, \gamma, \delta$  be the functions given by (1.1), and let  $a, b, c, d$  be the associate linear maps given by formulas (1.6)–(1.9). Assume that the two matrices  $a_0$  and  $d_0$  are invertible. Using Toeplitz operators and Hankel operators of the type defined in the previous section we introduce the following operators:

$$(3.1) \quad M_{11} = T_{+, \alpha} \Delta_{a_0^{-1}} T_{+, \alpha}^* - S_{+, p} T_{+, \beta} \Delta_{d_0^{-1}} T_{+, \beta}^* S_{+, p}^* : \ell_+^2(\mathbb{C}^p) \rightarrow \ell_+^2(\mathbb{C}^p),$$

$$(3.2) \quad M_{21} = H_{-, \gamma} \Delta_{a_0^{-1}} T_{+, \alpha}^* - S_{-, q}^* H_{-, \delta} \Delta_{d_0^{-1}} T_{+, \beta}^* S_{+, p}^* : \ell_+^2(\mathbb{C}^p) \rightarrow \ell_-^2(\mathbb{C}^q),$$

$$(3.3) \quad M_{12} = H_{+, \beta} \Delta_{d_0^{-1}} T_{-, \delta}^* - S_{+, p}^* H_{+, \alpha} \Delta_{a_0^{-1}} T_{-, \gamma}^* S_{-, q}^* : \ell_-^2(\mathbb{C}^q) \rightarrow \ell_+^2(\mathbb{C}^p),$$

$$(3.4) \quad M_{22} = T_{-, \delta} \Delta_{d_0^{-1}} T_{-, \delta}^* - S_{-, q} T_{-, \gamma} \Delta_{a_0^{-1}} T_{-, \gamma}^* S_{-, q}^* : \ell_-^2(\mathbb{C}^q) \rightarrow \ell_-^2(\mathbb{C}^q).$$

Notice that the operators  $M_{ij}$ ,  $1 \leq i, j \leq 2$ , are uniquely determined by the data. If  $a_0$  and  $d_0$  are selfadjoint, then formulas (3.1) and (3.4) show that  $M_{11}^* = M_{11}$  and  $M_{22}^* = M_{22}$ . Later (see Lemma 5.3) we shall prove that under certain additional conditions  $M_{12}^* = M_{21}$ .

We are now ready to state the inversion theorem.

**Theorem 3.1.** *Let  $\alpha, \beta, \gamma, \delta$  be the functions given by (1.1), and let  $a, b, c, d$  be the associate linear maps given by formulas (1.6)–(1.9) with both matrices  $a_0$  and  $d_0$  invertible. Assume  $g \in \mathcal{W}_+^{p \times q}$  is a solution to the twofold EG inverse problem associated with the data set  $\{\alpha, \beta, \gamma, \delta\}$ . Then  $g$  is the only solution and the operator  $\Omega$  given by the  $2 \times 2$  operator matrix*

$$(3.5) \quad \Omega = \begin{bmatrix} I & H_{+,g} \\ H_{-,g^*} & I \end{bmatrix} : \begin{bmatrix} \ell_+^2(\mathbb{C}^p) \\ \ell_-^2(\mathbb{C}^q) \end{bmatrix} \rightarrow \begin{bmatrix} \ell_+^2(\mathbb{C}^p) \\ \ell_-^2(\mathbb{C}^q) \end{bmatrix}$$

is invertible and its inverse is given by

$$(3.6) \quad \Omega^{-1} = M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix},$$

where  $M_{11}, M_{21}, M_{12}, M_{22}$  are the operators given by (3.1) – (3.4).

Conversely, if  $g \in \mathcal{W}_+^{p \times q}$ , and the operator  $\Omega$  given by (3.5) is invertible, then there exists a unique data set  $\{\alpha, \beta, \gamma, \delta\}$  such that  $g$  is a solution to the twofold EG inverse problem associated with this data set.

**Remark 3.2.** As is mentioned in the introduction, the above theorem is known. In Section 6 we shall give a direct proof based on the analysis of the operators  $M_{ij}$ ,  $1 \leq i, j \leq 2$ , given in Section 5.

For later purposes we mention that the operators  $M_{ij}$ ,  $1 \leq i, j \leq 2$ , are also given by

$$(3.7) \quad M_{11} = T_{+, \alpha} \Delta_{a_0^{-1}} T_{+, \alpha}^* - T_{+, \ell\beta} \Delta_{d_0^{-1}} T_{+, \ell\beta}^* : \ell_+^2(\mathbb{C}^p) \rightarrow \ell_+^2(\mathbb{C}^p),$$

$$(3.8) \quad M_{21} = H_{-, \gamma} \Delta_{a_0^{-1}} T_{+, \alpha}^* - H_{-, \ell\delta} \Delta_{d_0^{-1}} T_{+, \ell\beta}^* : \ell_+^2(\mathbb{C}^p) \rightarrow \ell_-^2(\mathbb{C}^q),$$

$$(3.9) \quad M_{12} = H_{+, \beta} \Delta_{d_0^{-1}} T_{-, \delta}^* - H_{+, \ell^*\alpha} \Delta_{a_0^{-1}} T_{-, \ell^*\gamma}^* : \ell_-^2(\mathbb{C}^q) \rightarrow \ell_+^2(\mathbb{C}^p),$$

$$(3.10) \quad M_{22} = T_{-, \delta} \Delta_{d_0^{-1}} T_{-, \delta}^* - T_{-, \ell^*\alpha} \Delta_{a_0^{-1}} T_{-, \ell^*\gamma}^* : \ell_-^2(\mathbb{C}^q) \rightarrow \ell_-^2(\mathbb{C}^q).$$

Here  $\ell$  is the scalar function defined in the paragraph directly after (2.10).

To derive the above formulas note that

$$S_{+, p} T_{+, \beta} = S_{+, p} \begin{bmatrix} b_0 & 0 & 0 & \cdots \\ b_1 & b_0 & 0 & \cdots \\ b_2 & b_1 & b_0 & \\ \vdots & \vdots & & \ddots \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ b_0 & 0 & 0 & \cdots \\ b_1 & b_0 & 0 & \\ \vdots & \vdots & & \ddots \end{bmatrix} = T_{+, \ell\beta},$$

$$S_{-, q} T_{-, \gamma} = S_{-, q} \begin{bmatrix} \cdots & \vdots & \vdots & \vdots \\ \cdots & c_0 & c_{-1} & c_{-2} \\ \cdots & 0 & c_0 & c_{-1} \\ \cdots & 0 & 0 & c_0 \end{bmatrix} = \begin{bmatrix} \cdots & \vdots & \vdots & \vdots \\ \cdots & 0 & c_0 & c_{-1} \\ \cdots & 0 & 0 & c_0 \\ \cdots & 0 & 0 & 0 \end{bmatrix} = T_{-, \ell^*\gamma}.$$

Furthermore, from the identities in (2.12) we know that

$$S_{-, q}^* H_{-, \delta} = H_{-, \ell\delta} \quad \text{and} \quad S_{+, p}^* H_{+, \alpha} = H_{+, \ell^*\alpha}.$$

Using the above identities in (3.1)–(3.4) we obtain (3.7)–(3.10).

**Remark 3.3.** In Theorem 3.1 the condition that both  $a_0$  and  $d_0$  are invertible can be replaced by the weaker condition that at least one of the two is invertible. On the other hand, in that case the assumption  $g \in \mathcal{W}_+^{p \times q}$  is a solution to the twofold EG inverse problem implies that both are invertible; see, e.g., [4, Section 3 and 4], [1, Proposition 5.2] or [10, Theorem 1.1], and see Section A for further details.

#### 4. THE SOLUTION OF THE INVERSE PROBLEM

In this section we present our solution to the twofold EG inverse problem, as well as a characterization of the case where a solution exists with a strictly contractive Hankel operator.

**Theorem 4.1.** *Let  $\alpha, \beta, \gamma, \delta$  be the functions given by (1.1), and let  $a, b, c, d$  be the associate linear maps given by formulas (1.6)–(1.9) with both matrices  $a_0$  and  $d_0$  invertible. Then the twofold EG inverse problem associated with the data set  $\{\alpha, \beta, \gamma, \delta\}$  has a solution if and only if the following conditions are satisfied:*

- (D1) *the identities in (1.4) hold true;*
- (D2) *the operators  $M_{11}$  and  $M_{22}$  defined by (3.1) and (3.4) are one-to-one.*

*Furthermore, in that case  $M_{11}$  and  $M_{22}$  are invertible, the solution is unique and the unique solution  $g$  and its adjoint are given by*

$$(4.1) \quad g = -\mathcal{F}(M_{11}^{-1}b) \quad \text{and} \quad g^* = -\mathcal{F}(M_{22}^{-1}c).$$



**Remark 4.2.** We shall see that  $M_{11}^{-1}b$  is a linear map from  $\mathbb{C}^q$  to  $\ell_+^1(\mathbb{C}^p)$  and  $M_{22}^{-1}c$  is a linear map from  $\mathbb{C}^p$  to  $\ell_-^1(\mathbb{C}^p)$ . Thus the inverse Fourier transforms in (4.1) are well-defined.

**Remark 4.3.** The necessity of condition (D1) we know from [11, Theorem 5]. Furthermore, we can use Theorem 3.1 to prove the necessity of condition (D2), the uniqueness of the solution and a formula for the solution. Indeed, as we shall see in Section 6, formula (3.6) implies that  $H_{+,g} = -M_{11}^{-1}M_{12}$ . New in the above theorem are the sufficiency of conditions (D1) and (D2), and the formulas for the solution given in (4.1).

The next theorem is a generalisation of Theorem 5.1 in [4] which deals with the inverse problem for the onefold case; see also [12, Theorem 3.1].

**Theorem 4.4.** *Let  $\alpha, \beta, \gamma, \delta$  be the functions given by (1.1), and let  $a, b, c, d$  be the associate linear maps given by formulas (1.6)–(1.9). Then the twofold EG inverse problem associated with the data set  $\{\alpha, \beta, \gamma, \delta\}$  has a solution  $g$  with the additional property that  $H_{+,g}$  is a strict contraction if and only if the following three conditions are satisfied:*

- (i)  $a_0$  and  $d_0$  are positive definite,
- (ii)  $\alpha^*\alpha - \gamma^*\gamma = a_0$ ,  $\delta^*\delta - \beta^*\beta = d_0$ ,  $\alpha^*\beta = \gamma^*\delta$ ,
- (iii)  $\det \alpha$  and  $\det \delta$  have no zeros in  $|\lambda| \leq 1$  and  $|\lambda| \geq 1$ , respectively.

## 5. TOWARDS THE PROOFS OF THE THEOREMS

Throughout this section  $\alpha, \beta, \gamma, \delta$  are the functions given by (1.1), and  $a, b, c, d$  are the associate linear maps given by formulas (1.6)–(1.9). Moreover it will be assumed that the matrices  $a_0$  and  $d_0$  are invertible. In the four lemmas presented in this section we also assume that the identities in (1.4) are satisfied, that is,

$$(5.1) \quad \alpha^*\alpha - \gamma^*\gamma = a_0, \quad \delta^*\delta - \beta^*\beta = d_0, \quad \alpha^*\beta = \gamma^*\delta.$$

The first two identities in (5.1) imply that the matrices  $a_0$  and  $d_0$  are selfadjoint. Hence formulas (3.1) and (3.4) give that  $M_{11}^* = M_{11}$  and  $M_{22}^* = M_{22}$  (cf., the comment after (3.4)).

Together the identities in (5.1) are equivalent to

$$(5.2) \quad \begin{bmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{bmatrix} \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} a_0 & 0 \\ 0 & -d_0 \end{bmatrix}.$$

Since all entries are matrices of functions, it follows that

$$(5.3) \quad \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} a_0^{-1} & 0 \\ 0 & -d_0^{-1} \end{bmatrix} \begin{bmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{bmatrix} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix},$$

which in turn is equivalent to the following three identities

$$(5.4) \quad \alpha a_0^{-1} \alpha^* - \beta d_0^{-1} \beta^* = I_p, \quad \delta d_0^{-1} \delta^* - \gamma a_0^{-1} \gamma^* = I_q,$$

$$(5.5) \quad \alpha a_0^{-1} \gamma^* = \beta d_0^{-1} \delta^*.$$

By a similar argument, it follows that the identities in (5.4) and (5.5) imply those in (5.1), hence we conclude that the three identities in (5.4) and (5.5) are equivalent to the three identities in (5.1) (provided, as is assumed throughout this section,  $a_0$  and  $d_0$  are invertible).

The following lemma is an addition to the final part of Section 2.

**Lemma 5.1.** *Assume that condition (5.1) is satisfied. Then*

$$(5.6) \quad \begin{bmatrix} T_{+, \alpha^*} \\ T_{+, \ell^* \beta^*} \end{bmatrix} [H_{+, \ell^* \alpha} \quad H_{+, \beta}] = \begin{bmatrix} H_{+, \gamma^*} \\ H_{+, \ell^* \delta^*} \end{bmatrix} [T_{-, \ell^* \gamma} \quad T_{-, \delta}],$$

$$(5.7) \quad \begin{bmatrix} T_{+, \alpha^*} \\ T_{+, \ell^* \beta^*} \end{bmatrix} S_{+, p}^* [H_{+, \ell^* \alpha} \quad H_{+, \beta}] = \begin{bmatrix} H_{+, \gamma}^* \\ H_{+, \ell^* \delta^*} \end{bmatrix} S_{-, q} [T_{-, \delta} \quad T_{-, \ell^* \gamma}].$$

**Proof.** In order to prove (5.6) we have to check the following four identities:

$$(5.8) \quad T_{+, \alpha^*} H_{+, \ell^* \alpha} = H_{+, \gamma^*} T_{-, \ell^* \gamma} \quad \text{and} \quad T_{+, \alpha^*} H_{+, \beta} = H_{+, \gamma^*} T_{-, \delta},$$

$$(5.9) \quad T_{+, \ell^* \beta^*} H_{+, \ell^* \alpha} = H_{+, \ell^* \delta^*} T_{-, \ell^* \gamma} \quad \text{and} \quad T_{+, \ell^* \beta^*} H_{+, \beta} = H_{+, \ell^* \delta^*} T_{-, \delta}.$$

STEP 1. We prove the second identity in (5.8). First we apply (2.16) with  $\rho = \alpha^*$  and  $\phi = \ell\beta$ , and we use the first identity in (2.21). We obtain

$$H_{+, \alpha^* \beta} = H_{+, \ell^* \alpha^* \ell \beta} = H_{+, \ell^* \alpha^*} T_{-, \ell \beta} + T_{+, \alpha^*} H_{+, \beta} = T_{+, \alpha^*} H_{+, \beta}.$$

On the other hand, by applying (2.16) with  $\rho = \ell\gamma^*$  and  $\phi = \delta$ , and using the last equality in (2.20), we get

$$H_{+, \gamma^* \delta} = H_{+, \ell^* \ell \gamma^* \delta} = H_{+, \gamma^*} T_{-, \delta} + T_{+, \ell \gamma^*} H_{+, \ell^* \delta} = H_{+, \gamma^*} T_{-, \delta}.$$

Now notice that the third identity in (5.1) implies that  $H_{+, \alpha^* \beta} = H_{+, \gamma^* \delta}$ . It follows that  $T_{+, \alpha^*} H_{+, \beta} = H_{+, \gamma^*} T_{-, \delta}$  as desired.

STEP 2. We prove the first identity in (5.8). We first apply (2.16) with  $\rho = \alpha^*$  and  $\phi = \alpha$ , and we use again the first identity in (2.21). This yields

$$H_{+, \ell^* \alpha^* \alpha} = H_{+, \ell^* \alpha^*} T_{-, \alpha} + T_{+, \alpha^*} H_{+, \ell^* \alpha} = T_{+, \alpha^*} H_{+, \ell^* \alpha}.$$

On the other hand, using the first identity in (5.1), we see that  $\ell^* \alpha^* \alpha = \ell^* \gamma^* \gamma + \ell^* a_0$ . Since the Hankel operator  $H_{+, \ell^* a_0}$  is zero, we obtain

$$H_{+, \ell^* \alpha^* \alpha} = H_{+, \ell^* \gamma^* \gamma}.$$

Next we apply (2.16) with  $\rho = \ell\gamma^*$  and  $\phi = \ell^* \gamma$ , which yields

$$H_{+, \ell^* \gamma^* \gamma} = H_{+, \ell^* \ell \gamma^* \ell^* \gamma} = H_{+, \gamma^*} T_{-, \ell^* \gamma} + T_{+, \ell \gamma^*} H_{+, \ell^* \ell^* \gamma}.$$

Note that the Hankel operator  $H_{+, \ell^* \ell^* \gamma}$  is zero. We conclude that

$$T_{+, \alpha^*} H_{+, \ell^* \alpha} = H_{+, \ell^* \alpha^* \alpha} = H_{+, \ell^* \gamma^* \gamma} = H_{+, \gamma^*} T_{-, \ell^* \gamma}.$$

We proved that  $T_{+, \alpha^*} H_{+, \ell^* \alpha} = H_{+, \gamma^*} T_{-, \ell^* \gamma}$ .

STEP 3. We prove the first identity in (5.9). First we apply (2.16) with  $\rho = \ell^* \beta^*$  and  $\phi = \alpha$ , and we use that  $H_{+, \ell^* \ell^* \beta^*} = 0$ . This yields

$$H_{+, \ell^* \ell^* \beta^* \alpha} = H_{+, \ell^* \ell^* \beta^*} T_{-, \alpha} + T_{+, \ell^* \beta^*} H_{+, \ell^* \alpha} = T_{+, \ell^* \beta^*} H_{+, \ell^* \alpha}.$$

Next we apply (2.16) with  $\rho = \delta^*$  and  $\phi = \ell^* \gamma$ , and we use that  $H_{+, \ell^* \ell^* \gamma} = 0$ . This yields

$$H_{+, \ell^* \delta^* \ell^* \gamma} = H_{+, \ell^* \delta^*} T_{-, \ell^* \gamma} + T_{+, \delta^*} H_{+, \ell^* \ell^* \gamma} = H_{+, \ell^* \delta^*} T_{-, \ell^* \gamma}.$$

Notice that the third identity in (5.1) implies that  $H_{+, \ell^* \ell^* \beta^* \alpha} = H_{+, \ell^* \delta^* \ell^* \gamma}$ . Thus we proved the first identity in (5.9).

STEP 4. We prove the second identity in (5.9). To do this we apply (2.16) with  $\rho = \ell^* \beta^*$  and  $\phi = \ell\beta$ , and we use that  $H_{+, \ell^* \ell^* \beta^*} = 0$ . This yields

$$H_{+, \ell^* \beta^* \beta} = H_{+, \ell^* \ell^* \beta^* \ell \beta} = H_{+, \ell^* \ell^* \beta^*} T_{-, \ell \beta} + T_{+, \ell^* \beta^*} H_{+, \beta} = T_{+, \ell^* \beta^*} H_{+, \beta}.$$

Next we apply (2.16) with  $\rho = \delta^*$  and  $\phi = \delta$ . This yields

$$H_{+, \ell^* \delta^* \delta} = H_{+, \ell^* \delta^*} T_{-, \delta} + T_{+, \delta^*} H_{+, \ell^* \delta} = H_{+, \ell^* \delta^*} T_{-, \delta},$$

where the last equality follows from  $H_{+, \ell^* \delta} = 0$ . To complete the proof of this step we use the second identity in (5.9) to show that

$$H_{+, \ell^* \delta^* \delta} - H_{+, \ell^* \beta^* \beta} = H_{+, \ell^* d_0} = 0.$$

STEP 5. It remains to prove (5.7). Since  $H_{+, \ell^* \alpha}$  and  $H_{+, \beta}$  are Hankel operators, and the operators  $T_{-, \delta}$  and  $T_{-, \ell^* \gamma}$  are Toeplitz operators, the following intertwining relations hold:

$$\begin{aligned} S_{+, p}^* \begin{bmatrix} H_{+, \ell^* \alpha} & H_{+, \beta} \end{bmatrix} &= \begin{bmatrix} H_{+, \ell^* \alpha} & H_{+, \beta} \end{bmatrix} \begin{bmatrix} S_{-q} & 0 \\ 0 & S_{-p} \end{bmatrix} \\ S_{-, q} \begin{bmatrix} T_{-, \delta} & T_{-, \ell^* \gamma} \end{bmatrix} &= \begin{bmatrix} T_{-, \delta} & T_{-, \ell^* \gamma} \end{bmatrix} \begin{bmatrix} S_{-q} & 0 \\ 0 & S_{-p} \end{bmatrix}. \end{aligned}$$

But then using (5.6) we obtain:

$$\begin{aligned} \begin{bmatrix} T_{+, \alpha^*} \\ T_{+, \ell^* \beta^*} \end{bmatrix} S_{+, p}^* \begin{bmatrix} H_{+, \ell^* \alpha} & H_{+, \beta} \end{bmatrix} &= \\ &= \begin{bmatrix} T_{+, \alpha^*} \\ T_{+, \ell^* \beta^*} \end{bmatrix} \begin{bmatrix} H_{+, \ell^* \alpha} & H_{+, \beta} \end{bmatrix} \begin{bmatrix} S_{-q} & 0 \\ 0 & S_{-p} \end{bmatrix} \\ &= \begin{bmatrix} H_{+, \gamma^*} \\ H_{+, \ell^* \delta^*} \end{bmatrix} \begin{bmatrix} T_{-, \ell^* \gamma} & T_{-, \delta} \end{bmatrix} \begin{bmatrix} S_{-q} & 0 \\ 0 & S_{-p} \end{bmatrix} \\ &= \begin{bmatrix} H_{+, \gamma^*} \\ H_{+, \ell^* \delta^*} \end{bmatrix} S_{-, q} \begin{bmatrix} T_{-, \ell^* \gamma} & T_{-, \delta} \end{bmatrix}, \end{aligned}$$

and (5.7) is proved.  $\square$

By taking adjoints and using the identities in (2.10) it is straightforward to see that (5.6) yields the following identity:

$$(5.10) \quad \begin{bmatrix} T_{-, \delta^*} \\ T_{-, \ell^* \gamma^*} \end{bmatrix} \begin{bmatrix} H_{-, \gamma} & H_{-, \ell \delta} \end{bmatrix} = \begin{bmatrix} H_{-, \beta^*} \\ H_{-, \ell \alpha^*} \end{bmatrix} \begin{bmatrix} T_{+, \alpha} & T_{+, \ell \beta} \end{bmatrix}.$$

Furthermore, note that the identity (5.7) remains true if  $S_{+, p}^*$  and  $S_{-, q}$  are replaced by  $(S_{+, p}^*)^n$  and  $(S_{-, q})^n$ , respectively, where  $n$  is any nonnegative integer.

The following three lemmas contain basic identities for the operators  $M_{ij}$ , defined by (3.1)–(3.4). These identities will play an essential role in the proofs of Theorems 3.1 and 4.1. In fact, they will allow us to reduce the proofs of those two theorems to a matter of direct checking.

**Lemma 5.2.** *Assume that condition (5.1) is satisfied. Then the following identities hold true:*

$$(5.11) \quad M_{11} \varepsilon_{+, p} = a, \quad M_{12} \varepsilon_{-, q} = b, \quad M_{21} \varepsilon_{+, p} = c, \quad M_{22} \varepsilon_{-, q} = d.$$

**Proof.** From the block matrix representation of  $T_{+, \ell \beta}$  one sees that the first column of  $T_{+, \ell \beta}^*$  consists of zero entries only, and thus  $T_{+, \ell \beta}^* \varepsilon_{+, p} = 0$ . Using this fact together with  $a_0^* = a_0$  in (3.7) yields

$$\begin{aligned} M_{11} \varepsilon_{+, p} &= T_{+, \alpha} \Delta_{a_0^{-1}} T_{+, \alpha}^* \varepsilon_{+, p} = T_{+, \alpha} \Delta_{a_0^{-1}} \varepsilon_{+, p} a_0^* = T_{+, \alpha} \varepsilon_{+, p} a_0^{-1} a_0 \\ &= T_{+, \alpha} \varepsilon_{+, p} = a. \end{aligned}$$

This proves the first identity in (5.11). Again using  $T_{+,\ell\beta}^*e_{+,p} = 0$  and (3.8) one obtains

$$\begin{aligned} M_{21}\varepsilon_{+,p} &= H_{-,\gamma}\Delta_{a_0^{-1}}T_{+,\alpha}^*\varepsilon_{+,p} = H_{-,\gamma}\Delta_{a_0^{-1}}\varepsilon_{+,p}a_0^* = H_{-,\gamma}\varepsilon_{+,p}a_0^{-1}a_0 \\ &= H_{-,\gamma}\varepsilon_{+,p} = c, \end{aligned}$$

which proves the third identity in (5.11). The other two identities are proved in a similar way.  $\square$

**Lemma 5.3.** *Assume that condition (5.1) is satisfied. Then*

$$(5.12) \quad M_{11} = I_{\ell_+^2(\mathbb{C}^p)} - H_{+,\ell^*\alpha}\Delta_{a_0^{-1}}H_{+,\ell^*\alpha}^* + H_{+,\beta}\Delta_{d_0^{-1}}H_{+,\beta}^*,$$

$$(5.13) \quad M_{21} = T_{-,\delta}\Delta_{d_0^{-1}}H_{+,\beta}^* - T_{-,\ell^*\gamma}\Delta_{a_0^{-1}}H_{+,\ell^*\alpha}^*,$$

$$(5.14) \quad M_{12} = T_{+,\alpha}\Delta_{a_0^{-1}}H_{-,\gamma}^* - T_{+,\ell\beta}\Delta_{d_0^{-1}}H_{-,\ell\delta}^*,$$

$$(5.15) \quad M_{22} = I_{\ell_-^2(\mathbb{C}^q)} - H_{-,\ell\delta}\Delta_{d_0^{-1}}H_{-,\ell\delta}^* + H_{-,\gamma}\Delta_{a_0^{-1}}H_{-,\gamma}^*.$$

*In particular  $M_{12}^* = M_{21}$ . Hence  $M$  in (3.6) is selfadjoint.*

**Proof.** First notice that condition (5.1) yields the first identity in (5.4), and therefore

$$(5.16) \quad T_{+,\alpha a_0^{-1}\alpha^*} - T_{+,\beta d_0^{-1}(\beta)^*} - I_{\ell_+^2(\mathbb{C}^p)} = T_{+,\alpha a_0^{-1}\alpha^* - \beta d_0^{-1}\beta^* - \varepsilon_{+,p}} = 0.$$

Since  $\ell\ell^*$  is identically equal to 1, we have  $\ell\beta d_0^{-1}(\ell\beta)^* = \beta\ell d_0^{-1}\ell^*\beta^* = \beta d_0^{-1}\beta^*$ . By using this in (5.16) we obtain

$$T_{+,\alpha a_0^{-1}\alpha^*} - T_{+,\ell\beta d_0^{-1}(\ell\beta)^*} = I_{\ell_+^2(\mathbb{C}^p)}.$$

By applying the product rule (2.15) and the identities in (2.19) one sees that

$$\begin{aligned} T_{+,\alpha}\Delta_{a_0^{-1}}T_{+,\alpha}^* - T_{+,\ell\beta}\Delta_{d_0^{-1}}T_{+,\ell\beta}^* &= \\ &= I_{\ell_+^2(\mathbb{C}^p)} - H_{+,\ell^*\alpha}\Delta_{a_0^{-1}}H_{+,\ell^*\alpha}^* + H_{+,\beta}\Delta_{d_0^{-1}}H_{+,\beta}^*. \end{aligned}$$

We conclude that the operator  $M_{11}$  defined by (3.7) is also given by (5.12).

In a similar way one shows that the second identity in (5.4) yields the identity (5.15).

Next we apply (2.16) with  $\rho = \alpha a_0^{-1}$  and  $\phi = \ell\gamma^*$ . This yields

$$\begin{aligned} H_{+,\alpha a_0^{-1}\gamma^*} &= H_{+,\ell^*\alpha a_0^{-1}\ell\gamma^*} \\ &= T_{+,\alpha}\Delta_{a_0^{-1}}H_{+,\gamma^*} + H_{+,\ell^*\alpha}\Delta_{a_0^{-1}}T_{-,\ell\gamma^*} \\ (5.17) \quad &= T_{+,\alpha}\Delta_{a_0^{-1}}H_{-,\gamma}^* + H_{+,\ell^*\alpha}\Delta_{a_0^{-1}}T_{-,\ell^*\gamma}^*. \end{aligned}$$

Similarly, by applying (2.16) with  $\rho = \ell\beta d_0^{-1}$  and  $\phi = \delta^*$ , we obtain

$$\begin{aligned} H_{+,\beta d_0^{-1}\delta^*} &= H_{+,\ell^*\ell\beta d_0^{-1}\delta^*} \\ &= T_{+,\ell\beta}\Delta_{d_0^{-1}}H_{+,\ell^*\delta^*} + H_{+,\beta}\Delta_{d_0^{-1}}T_{-,\delta^*} \\ (5.18) \quad &= T_{+,\ell\beta}\Delta_{d_0^{-1}}H_{-,\ell\delta}^* + H_{+,\beta}\Delta_{d_0^{-1}}T_{-,\delta}^*. \end{aligned}$$

Now note that (5.5) implies that  $H_{+,\alpha a_0^{-1}\gamma^*} = H_{+,\beta d_0^{-1}\delta^*}$ . But then equalities (5.17) and (5.18) show that the right hand sides of (3.9) and (5.14) are equal.

From (3.8) and (5.14) we obtain  $M_{21}^* = M_{12}$ . Using this fact along with formula (3.9) for  $M_{12}$  we see that  $M_{21} = M_{12}^*$  is given by (5.13). We already know (see the first paragraph of the present section) that  $M_{11}$  and  $M_{22}$  are selfadjoint. Hence  $M$  in (3.6) is selfadjoint.  $\square$

**Lemma 5.4.** *Assume that condition (5.1) is satisfied. Let the operators  $M_{ij}$ ,  $i, j = 1, 2$ , be given by (3.1)–(3.4). Then*

$$(5.19) \quad \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} I_{\ell_+^2(\mathbb{C}^p)} & 0 \\ 0 & -I_{\ell_-^2(\mathbb{C}^q)} \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = \begin{bmatrix} M_{11} & 0 \\ 0 & -M_{22} \end{bmatrix}.$$

In particular,

$$(5.20) \quad M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} : \begin{bmatrix} \ell_+^2(\mathbb{C}^p) \\ \ell_-^2(\mathbb{C}^q) \end{bmatrix} \rightarrow \begin{bmatrix} \ell_+^2(\mathbb{C}^p) \\ \ell_-^2(\mathbb{C}^q) \end{bmatrix}$$

is invertible if and only if  $M_{11}$  and  $M_{22}$  are invertible, and in that case

$$(5.21) \quad M^{-1} = \begin{bmatrix} I_{\ell_+^2(\mathbb{C}^p)} & -M_{12}M_{22}^{-1} \\ -M_{21}M_{11}^{-1} & I_{\ell_-^2(\mathbb{C}^q)} \end{bmatrix} = \begin{bmatrix} I_{\ell_+^2(\mathbb{C}^p)} & -M_{11}^{-1}M_{12} \\ -M_{22}^{-1}M_{21} & I_{\ell_-^2(\mathbb{C}^q)} \end{bmatrix}.$$

Finally,

$$(5.22) \quad M_{11}S_{+,p}^*M_{12} = M_{12}S_{-,q}M_{22}.$$

**Proof.** To check (5.19) we will prove the four identities

$$(5.23) \quad M_{11}M_{12} = M_{12}M_{22}, \quad M_{22}M_{21} = M_{21}M_{11},$$

$$(5.24) \quad M_{11}M_{11} - M_{12}M_{21} = M_{11}, \quad M_{22}M_{22} - M_{21}M_{12} = M_{22}.$$

From (3.7) and (3.9) it follows that

$$\begin{aligned} M_{11}M_{12} &= [T_{+, \alpha} \quad T_{+, \ell\beta}] \begin{bmatrix} \Delta_{a_0^{-1}} & 0 \\ 0 & -\Delta_{d_0^{-1}} \end{bmatrix} \begin{bmatrix} T_{+, \alpha}^* \\ T_{+, \ell\beta}^* \end{bmatrix} \times \\ &\quad \times [H_{+, \ell^*\alpha} \quad H_{+, \beta}] \begin{bmatrix} -\Delta_{a_0^{-1}} & 0 \\ 0 & \Delta_{d_0^{-1}} \end{bmatrix} \begin{bmatrix} T_{-, \ell^*\gamma}^* \\ T_{-, \delta}^* \end{bmatrix}. \end{aligned}$$

Furthermore, from (5.14) and (3.10) it follows that

$$\begin{aligned} M_{12}M_{22} &= [T_{+, \alpha} \quad T_{+, \ell\beta}] \begin{bmatrix} \Delta_{a_0^{-1}} & 0 \\ 0 & -\Delta_{d_0^{-1}} \end{bmatrix} \begin{bmatrix} H_{-, \gamma}^* \\ H_{-, \ell\delta}^* \end{bmatrix} \times \\ &\quad \times [T_{-, \ell^*\gamma} \quad T_{-, \delta}] \begin{bmatrix} -\Delta_{a_0^{-1}} & 0 \\ 0 & \Delta_{d_0^{-1}} \end{bmatrix} \begin{bmatrix} T_{-, \ell^*\gamma}^* \\ T_{-, \delta}^* \end{bmatrix}. \end{aligned}$$

But then (5.6) shows that  $M_{11}M_{12} = M_{12}M_{22}$ . In a similar way, using (5.10) one proves that  $M_{22}M_{21} = M_{21}M_{11}$ .

Next, using (3.7) and (5.12), observe that

$$\begin{aligned} M_{11}(M_{11} - I_{\ell_+^2(\mathbb{C}^p)}) &= -[T_{+, \alpha} \quad T_{+, \ell\beta}] \begin{bmatrix} \Delta_{a_0^{-1}} & 0 \\ 0 & -\Delta_{d_0^{-1}} \end{bmatrix} \begin{bmatrix} T_{+, \alpha}^* \\ T_{+, \ell\beta}^* \end{bmatrix} \times \\ &\quad \times [H_{+, \ell^*\alpha} \quad H_{+, \beta}] \begin{bmatrix} -\Delta_{a_0^{-1}} & 0 \\ 0 & \Delta_{d_0^{-1}} \end{bmatrix} \begin{bmatrix} H_{+, \ell^*\alpha}^* \\ H_{+, \beta}^* \end{bmatrix}. \end{aligned}$$

Furthermore, using (5.14) and (5.13), we see that

$$\begin{aligned} M_{12}M_{21} &= [T_{+, \alpha} \quad T_{+, \ell\beta}] \begin{bmatrix} \Delta_{a_0^{-1}} & 0 \\ 0 & -\Delta_{d_0^{-1}} \end{bmatrix} \begin{bmatrix} H_{-, \gamma}^* \\ H_{-, \ell\delta}^* \end{bmatrix} \times \\ &\quad \times [T_{-, \ell^*\gamma} \quad T_{-, \delta}] \begin{bmatrix} -\Delta_{a_0^{-1}} & 0 \\ 0 & \Delta_{d_0^{-1}} \end{bmatrix} \begin{bmatrix} H_{+, \ell^*\alpha}^* \\ H_{+, \beta}^* \end{bmatrix}. \end{aligned}$$

But then (5.6) implies that  $M_{11}M_{11} - M_{12}M_{21} = M_{11}$ . Similarly, using (5.10) one proves that  $M_{22}M_{22} - M_{21}M_{12} = M_{22}$ .

Given (5.20) the equalities in (5.21) are immediate from (5.19). Finally, by multiplying (5.7) from the left and the right by

$$\begin{bmatrix} T_{+, \alpha}\Delta_{a_0^{-1}} & T_{+, \ell\beta}\Delta_{d_0^{-1}} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} T_{-, \delta}^* \\ T_{-, \ell^*\gamma}^* \end{bmatrix},$$

respectively, one obtains the equality (5.22).  $\square$

## 6. DIRECT PROOF OF THEOREM 3.1

Let  $\alpha, \beta, \gamma, \delta$  be the functions given by (1.1), and let  $a, b, c, d$  be the associate linear maps given by formulas (1.6)–(1.9) with both matrices  $a_0$  and  $d_0$  invertible. Throughout this section  $g$  is a function in  $\mathcal{W}_+^{p \times q}$ .

**Proof of Theorem 3.1.** We split the proof into two parts.

PART 1. First we assume that  $g$  is a solution of the twofold EG inverse problem associated with  $\{\alpha, \beta, \gamma, \delta\}$ , that is, we assume that the inclusions in (1.2) and (1.3) are satisfied. Our aim is to prove the identity

$$(6.1) \quad \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} I_{\ell_+^2(\mathbb{C}^p)} & H_{+, g} \\ H_{-, g^*} & I_{\ell_-^2(\mathbb{C}^q)} \end{bmatrix} = \begin{bmatrix} I_{\ell_+^2(\mathbb{C}^p)} & 0 \\ 0 & I_{\ell_-^2(\mathbb{C}^q)} \end{bmatrix}.$$

According to Proposition 2.1 our assumptions imply that the operator identities in (2.22) and (2.23) hold true. Furthermore, since  $g$  is a solution of the twofold EG inverse problem associated with  $\{\alpha, \beta, \gamma, \delta\}$ , the identities in (1.4) (and (5.1)) are satisfied, and hence we may use Lemma 5.3. We proceed in five steps.

STEP 1.1. According to (5.12) and (3.9) we have

$$\begin{aligned} M_{11} - I_{\ell_+^2(\mathbb{C}^p)} + M_{12}H_{-, g^*} &= \\ &= H_{+, \beta}\Delta_{d_0^{-1}}(T_{-, \delta}^*H_{-, g^*} + H_{+, \beta}^*) - H_{+, \ell^*\alpha}\Delta_{a_0^{-1}}(T_{-, \ell^*\gamma}^*H_{-, g^*} + H_{+, \ell^*\alpha}^*). \end{aligned}$$

Taking the adjoints of the second identity in (2.23) and of the first identity in (2.22) (using identities from (2.10) when necessary), we see that

$$T_{-, \delta}^*H_{-, g^*} + H_{+, \beta}^* = 0 \quad \text{and} \quad T_{-, \ell^*\gamma}^*H_{-, g^*} + H_{+, \ell^*\alpha}^* = 0.$$

It follows that  $M_{11} + M_{12}H_{-, g^*} = I_{\ell_+^2(\mathbb{C}^p)}$ .

STEP 1.2. According to (3.7) and (5.14) one has that

$$\begin{aligned} M_{11}H_{+, g} + M_{12} &= \\ &= T_{+, \alpha}\Delta_{a_0^{-1}}(T_{+, \alpha}^*H_{+, g} + H_{-, \gamma}^*) - T_{+, \ell\beta}\Delta_{d_0^{-1}}(T_{+, \ell\beta}^*H_{+, g} + H_{-, \ell\delta}^*). \end{aligned}$$

Using the second identity (2.22) and the first in (2.23) (together with the identities in (2.10)) we see that

$$(6.2) \quad T_{+,\alpha}^* H_{+,g} + H_{-,\gamma}^* = 0 \quad \text{and} \quad T_{+,\ell\beta}^* H_{+,g} + H_{-,\ell\delta}^* = 0.$$

It follows that  $M_{11}H_{+,g} + M_{12} = 0$ .

STEP 1.3. According to (5.15) and (3.8) we have

$$\begin{aligned} M_{21}H_{+,g} + M_{22} - I_{\ell^2(\mathbb{C}^q)} &= \\ &= H_{-,\gamma} \Delta_{a_0^{-1}} (H_{-,\gamma}^* + T_{+,\alpha}^* H_{+,g}) - H_{-,\ell\delta} \Delta_{d_0^{-1}} (H_{-,\ell\delta}^* + T_{+,\ell\beta}^* H_{+,g}). \end{aligned}$$

But then, using the two identities in (6.2), we have

$$H_{-,\gamma}^* + T_{+,\alpha}^* H_{+,g} = 0 \quad \text{and} \quad H_{-,\ell\delta}^* + T_{+,\ell\beta}^* H_{+,g} = 0.$$

It follows that  $M_{21}H_{+,g} + M_{22} = I_{\ell^2(\mathbb{C}^q)}$ .

STEP 1.4. According to (5.13) and (3.10) we have

$$\begin{aligned} M_{21} + M_{22}H_{-,g^*} &= \\ &= T_{-,\delta} \Delta_{d_0^{-1}} (T_{-,\delta}^* H_{-,g^*} + H_{+,\beta}^*) - T_{-,\ell^*\gamma} \Delta_{a_0^{-1}} (T_{-,\ell^*\gamma}^* H_{-,g^*} + H_{+,\ell^*\alpha}^*). \end{aligned}$$

Taking adjoints of the second identity in (2.23) and of the first identity in (2.22) (using identities from (2.10) when necessary) we see that

$$T_{-,\delta}^* H_{-,g^*} + H_{+,\beta}^* = 0 \quad \text{and} \quad T_{-,\ell^*\gamma}^* H_{-,g^*} + H_{+,\ell^*\alpha}^* = 0.$$

It follows that  $M_{21} + M_{22}H_{-,g^*} = 0$ .

STEP 1.5. Putting together the results in the preceding four steps we have proved (6.1). Since both factors in the left hand side of (6.1) are selfadjoint, we have proved (3.6). Furthermore, since the operator matrix  $M$  in (3.6) only depends on the given data set  $\{\alpha, \beta, \gamma, \delta\}$ , so does  $\Omega = M^{-1}$  in (3.5). Hence  $H_{+,g}$  is uniquely determined by  $\{\alpha, \beta, \gamma, \delta\}$ , and, consequently, it follows that the solution  $g$  is unique.

PART 2. Conversely, let  $g \in \mathcal{W}_+^{p \times q}$ , and assume that the operator  $\Omega$  given by (3.5) is invertible. Then the equations in the right hand sides of (1.12) and (1.13) have a unique solution. Given the solution  $\{a, b, c, d\}$  define  $\{\alpha, \beta, \gamma, \delta\}$  by (1.10). The equivalences in (1.12) and (1.13) imply that  $\{\alpha, \beta, \gamma, \delta\}$  satisfies (1.2) and (1.3). Thus indeed  $g \in \mathcal{W}_+^{p \times q}$  is a solution of the twofold EG inverse problem associated with  $\{\alpha, \beta, \gamma, \delta\}$ .  $\square$

## 7. PROOF OF THEOREM 4.1

Throughout this section  $\alpha, \beta, \gamma, \delta$  are the functions given by (1.1), and  $a, b, c, d$  are the associate linear maps given by formulas (1.6)–(1.9). We assume that both matrices  $a_0$  and  $d_0$  are invertible.

**Proof of Theorem 4.1.** We split the proof into two parts. The first part concerns the necessity of conditions (D1) and (D2) in Theorem 4.1.

PART 1. Suppose that the twofold EG inverse problem has a solution. Then, as we mentioned in the introduction (see (1.4)), condition (D1) follows from [11, Theorem 1.2]. Theorem 3.1 above states that the operator  $M$  given by

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

is invertible. It follows from Lemma 5.4 that the operators  $M_{11}$  and  $M_{22}$  are invertible, hence one-to-one. In particular, condition (D2) is satisfied.

PART 2. In this part we assume that conditions (D1) and (D2) are satisfied. We show that  $M_{11}$  and  $M_{22}$  are invertible, and we prove the reverse implications. This will be done in five steps.

STEP 2.1. We show that  $M_{11}$  and  $M_{22}$  are invertible as operators on  $\ell^2$ -spaces as well as operators on  $\ell^1$ -spaces. Since  $\alpha, \beta, \gamma, \delta$  are Wiener class functions, the corresponding Hankel operators are compact, and hence (5.12) and (5.15) imply that both  $M_{11}$  and  $M_{22}$  are of the form  $I - K$  with  $I$  an identity operator and  $K$  a compact operator. Thus both  $M_{11}$  and  $M_{22}$  are Fredholm operators of index zero, and hence condition (D2) tells us that these operators are invertible.

Recall that  $\ell_+^1(\mathbb{C}^p)$  is contained in  $\ell_+^2(\mathbb{C}^p)$  and  $\ell_-^1(\mathbb{C}^q)$  in  $\ell_-^2(\mathbb{C}^q)$ . The fact that  $\alpha, \beta, \gamma, \delta$  are Wiener class functions implies that  $M_{11}$  maps  $\ell_+^1(\mathbb{C}^p)$  into itself and  $M_{22}$  maps  $\ell_-^1(\mathbb{C}^q)$  into itself. Thus, by condition (D2), the induced operators

$$(7.1) \quad \tilde{M}_{11} : \ell_+^1(\mathbb{C}^p) \rightarrow \ell_+^1(\mathbb{C}^p) \quad \text{and} \quad \tilde{M}_{22} : \ell_-^1(\mathbb{C}^q) \rightarrow \ell_-^1(\mathbb{C}^q)$$

are also one-to-one. Moreover, again using that  $\alpha, \beta, \gamma, \delta$  are Wiener class functions, the induced operators  $\tilde{M}_{11}$  and  $\tilde{M}_{22}$  are of the form identity minus a compact operator. Hence these operators are also invertible. In particular,  $M_{11}^{-1}$  maps  $\ell_+^1(\mathbb{C}^p)$  into itself and  $M_{22}^{-1}$  maps  $\ell_-^1(\mathbb{C}^q)$  into itself.

Finally, with  $M_{11}$  and  $M_{22}$  invertible we obtain from Lemma 5.4 that  $M$  is invertible and

$$(7.2) \quad M^{-1} = \begin{bmatrix} I_{\ell_+^2(\mathbb{C}^p)} & -M_{12}M_{22}^{-1} \\ -M_{21}M_{11}^{-1} & I_{\ell_-^2(\mathbb{C}^q)} \end{bmatrix} = \begin{bmatrix} I_{\ell_+^2(\mathbb{C}^p)} & -M_{11}^{-1}M_{12} \\ -M_{22}^{-1}M_{21} & I_{\ell_-^2(\mathbb{C}^q)} \end{bmatrix}.$$

STEP 2.2. The next step is to show that  $M_{11}^{-1}M_{12}$  and  $M_{22}^{-1}M_{21}$  are Hankel operators. From (5.22) we know that  $M_{12}S_{-,q}M_{22} = M_{11}S_{+,p}^*M_{12}$ . Therefore

$$M_{11}^{-1}M_{12}S_{-,q} = S_{+,p}^*M_{12}M_{22}^{-1} = S_{+,p}^*M_{11}^{-1}M_{12},$$

with the last identity following from the equality of the right upper corners in (7.2). This intertwining relation proves that  $M_{11}^{-1}M_{12}$  is a Hankel operator. Since  $M_{11}^* = M_{11}$ ,  $M_{22}^* = M_{22}$ , and  $M_{12}^* = M_{21}$ , we have that

$$(M_{11}^{-1}M_{12})^* = M_{21}M_{11}^{-1} = M_{22}^{-1}M_{21}.$$

It follows that  $M_{22}^{-1}M_{21}$  is also a Hankel operator.

STEP 2.3. Let  $g$  be defined by the first identity in (4.1). We shall show that  $g \in \mathcal{W}_+^{p \times q}$  and that  $H_{+,g} = -M_{11}^{-1}M_{12}$ . To do this, put  $h = -M_{11}^{-1}M_{12}\varepsilon_{-,q}$ . From the second identity in (5.11) we know that  $M_{12}\varepsilon_{-,q} = b$ . Recall that  $b$  is a linear map from  $\mathbb{C}^q$  to  $\ell_+^1(\mathbb{C}^p)$ . As we have seen in Step 2.1 the operator  $M_{11}^{-1}$  maps  $\ell_+^1(\mathbb{C}^p)$  into  $\ell_+^1(\mathbb{C}^p)$ . Hence

$$h = -M_{11}^{-1}b : \mathbb{C}^q \rightarrow \ell_+^1(\mathbb{C}^p) \quad \text{and} \quad g = \mathcal{F}h \in \mathcal{W}_+^{p \times q}.$$

Since  $M_{11}^{-1}M_{12}$  is a Hankel operator, the identities  $h = -M_{11}^{-1}M_{12}\varepsilon_{-,q}$  and  $g = \mathcal{F}h$  imply that  $-M_{11}^{-1}M_{12} = H_{+,g}$ .

STEP 2.4. In this part we prove the second identity in (4.1). Put  $\tilde{h} = -M_{22}^{-1}M_{21}\varepsilon_{+,p}$ . From the third identity in (5.11) we know that  $M_{21}\varepsilon_{+,p} = c$ . Repeating the argument of the previous step, with  $c$  in place of  $b$  and  $M_{22}$  in place of  $M_{11}$  we see



that

$$\tilde{h} = -M_{22}^{-1}c : \mathbb{C}^p \rightarrow \ell^1(\mathbb{C}^q) \quad \text{and} \quad \tilde{g} := \mathcal{F}\tilde{h} \in \mathcal{W}_-^{q \times p}.$$

But then, since  $M_{22}^{-1}M_{21}$  is a Hankel operator, we conclude that  $M_{22}^{-1}M_{21} = -H_{-, \tilde{g}}$ , where  $\tilde{g} := \mathcal{F}\tilde{h} = -\mathcal{F}(M_{22}^{-1}c)$ . Summarizing, using the results of the two previous steps, we have

$$H_{-, \tilde{g}} = -M_{22}^{-1}M_{21} = -(M_{11}^{-1}M_{12})^* = H_{+, g}^* = H_{-, g^*}.$$

Thus  $g^* = \tilde{g} = \mathcal{F}(-M_{22}^{-1}c)$ , and the second identity in (4.1) is proved.

STEP 2.5. Finally, we show that  $g$  is a solution to the twofold EG inverse problem associated with the data  $\{\alpha, \beta, \gamma, \delta\}$ . By Lemma 5.2 we have

$$M \begin{bmatrix} \varepsilon_{+, p} \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} \quad \text{and} \quad M \begin{bmatrix} 0 \\ \varepsilon_{-, q} \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}.$$

Since  $M_{11}$  and  $M_{22}$  are invertible, the results of the previous steps and the second part of formula (5.21) show that

$$\Omega := M^{-1} = \begin{bmatrix} I_{\ell_+^2(\mathbb{C}^p)} & -M_{11}^{-1}M_{12} \\ -M_{22}^{-1}M_{21} & I_{\ell_-^2(\mathbb{C}^q)} \end{bmatrix} = \begin{bmatrix} I_{\ell_+^2(\mathbb{C}^p)} & H_{+, g} \\ H_{-, g^*} & I_{\ell_-^2(\mathbb{C}^q)} \end{bmatrix}.$$

We obtain that

$$\Omega \begin{bmatrix} a \\ c \end{bmatrix} = \Omega M \begin{bmatrix} \varepsilon_{+, p} \\ 0 \end{bmatrix} = \begin{bmatrix} \varepsilon_{+, p} \\ 0 \end{bmatrix},$$

and similarly

$$\Omega \begin{bmatrix} b \\ d \end{bmatrix} = \Omega M \begin{bmatrix} 0 \\ \varepsilon_{-, q} \end{bmatrix} = \begin{bmatrix} 0 \\ \varepsilon_{-, q} \end{bmatrix}.$$

Note that  $H_{+, g} = G$ , as in (1.11), and  $H_{-, g^*} = G^*$ . Hence, via the implications (1.12) and (1.13) we obtain that  $g$  satisfies (1.2) and (1.3). This shows  $g$  is a solution of the twofold EG inverse problem associated with the data  $\{\alpha, \beta, \gamma, \delta\}$ .  $\square$

## 8. PROOF OF THEOREM 4.4

In order to prove Theorem 4.4 we start with a proposition that on the one hand covers part of Theorem 4.4, but on the other hand is more detailed. To state this proposition we need some additional notation. Recall that  $\mathcal{W}^{n \times m}$  decomposes as

$$\mathcal{W}^{n \times m} = \mathcal{W}_+^{n \times m} \dot{+} \mathcal{W}_{-, 0}^{n \times m}, \quad \text{and} \quad \mathcal{W}^{n \times m} = \mathcal{W}_{+, 0}^{n \times m} \dot{+} \mathcal{W}_-^{n \times m}.$$

Now let  $\rho \in \mathcal{W}^{n \times m}$ . Then, using the above decompositions, we can write  $\rho$  in a unique way as

$$\rho = \rho_+ + \rho_{-, 0}, \quad \text{and} \quad \rho = \rho_{+, 0} + \rho_-,$$

where  $\rho_+ \in \mathcal{W}_+^{n \times m}$ ,  $\rho_{-, 0} \in \mathcal{W}_{-, 0}^{n \times m}$ ,  $\rho_{+, 0} \in \mathcal{W}_{+, 0}^{n \times m}$  and  $\rho_- \in \mathcal{W}_-^{n \times m}$ . These direct sum decompositions will play a role in the next proposition.

**Proposition 8.1.** *Let  $\{a, b, c, d\}$  be the data given by formulas (1.6)–(1.9), and let  $\alpha, \beta, \gamma$ , and  $\delta$  be the functions defined by (1.10).*

- (i) *If  $a_0$  is positive definite,  $\alpha^* \alpha - \gamma^* \gamma = a_0$  and  $\det \alpha(\lambda)$  has no zero in  $\lambda \leq 1$ , then  $g_1 = -(\alpha^{-*} \gamma^*)_+$  is the unique element in  $\mathcal{W}_+^{p \times q}$  that satisfies the inclusions in (1.2).*
- (ii) *If  $d_0$  is positive definite,  $\delta^* \delta - \beta^* \beta = d_0$  and  $\det \delta(\lambda)$  has no zero in  $\lambda \geq 1$ , then  $g_2 = -(\beta \delta^{-1})_+$  is the unique element in  $\mathcal{W}_+^{p \times q}$  that satisfies the inclusions in (1.3).*

Moreover, if in addition the third condition in (1.4) is also satisfied, that is  $\alpha^*\beta = \gamma^*\delta$ , then  $g_1$  in item (i) and  $g_2$  in item (ii) are equal.

**Proof.** It follows from Theorem 3.1 in [12] that the Fourier coefficients of the unique function  $g_1$  that satisfies the inclusions in (1.2) are given by  $-(T_{\alpha^*})^{-1}c^*$ . Therefore  $g_1 = -(\alpha^{-*}\gamma^*)_+$ . This proves the first item.

The second item we derive from the first as follows. Put

$$\tilde{\alpha}(\lambda) = \delta(\lambda^{-1}), \quad \tilde{\gamma}(\lambda) = \beta(\lambda^{-1}), \quad \tilde{p} = q, \quad \tilde{q} = p \quad \text{and} \quad \tilde{a}_0 = d_0.$$

Then item (i) gives that  $\tilde{g}_2 = -(\tilde{\alpha}^{-*}\tilde{\gamma}^*)_+$  is the unique element in  $\mathcal{W}_+^{q \times p}$  such that

$$\tilde{\alpha} + \tilde{g}_2\tilde{\gamma} - e_q \in \mathcal{W}_{-,0}^{q \times q}, \quad \tilde{g}_2^*\tilde{\alpha} + \tilde{\gamma} \in \mathcal{W}_{+,0}^{q \times p}.$$

Put  $g_2(\lambda) = \tilde{g}_2(\lambda^{-1})^*$ . Then  $g_2 \in \mathcal{W}_+^{p \times q}$  and

$$\delta + g_2^*\beta - e_q \in \mathcal{W}_{+,0}^{q \times q}, \quad g_2\delta + \beta \in \mathcal{W}_{-,0}^{p \times q}.$$

To finish the proof of item (ii) notice that

$$\begin{aligned} g_2(\lambda) &= \tilde{g}_2(\lambda^{-1})^* = - \left[ \left( \tilde{\alpha}(\frac{1}{\lambda})^{-*} \tilde{\gamma}(\frac{1}{\lambda})^* \right)_- \right]^* = - \left[ (\delta(\lambda)^{-*} \beta(\lambda)^*)_- \right]^* \\ &= -(\beta\delta^{-1})_+(\lambda). \end{aligned}$$

Finally, notice that the conditions in (i) and (ii) imply that  $\alpha^{-*}$  and  $\delta^{-1}$  are well defined and hence in that case  $\alpha^*\beta = \gamma^*\delta$  implies  $\alpha^{-*}\gamma^* = \beta\delta^{-1}$ . Therefore  $\alpha^*\beta = \gamma^*\delta$  implies  $g_1 = g_2$ .  $\square$

In the following proof we refer to results in Section A that are basically taken from Sections 3 and 4 in [4].

**Proof of Theorem 4.4.** Assume that  $g$  is a solution of the EG inverse problem with  $G := H_{+,g}$  strictly contractive. According to Lemma A.2 the matrices  $a_0$  and  $d_0$  are positive definite. We know from Theorem 1.2 in [11] that condition (ii) is satisfied. It follows from Proposition A.3 that  $\det \alpha$  has no zeros in the unit disk and  $\det \delta(\lambda)$  has no zeros with  $|\lambda| \geq 1$ .

Conversely, assume that the conditions (i), (ii), and (iii) are satisfied. If the EG inverse problem has a solution, then Proposition A.3 tells us that the operator  $\Omega_1$  defined in (A.4) is positive definite, and hence by Lemma A.2 we conclude that  $G$  is strictly contractive. So it remains to show that there exists a solution. According to Proposition 8.1 the function  $g_1 = -(\alpha^{-*}\gamma^*)_+$  is the unique function in  $\mathcal{W}_+^{p \times q}$  that satisfies the inclusions in (1.2). Also  $g_2 = -(\beta\delta^{-1})_+$  is the unique function in  $\mathcal{W}_+^{p \times q}$  that satisfies the inclusions in (1.3). The third statement in Proposition 8.1 gives that  $g_1 = g_2$ . Hence  $g = g_1 = g_2$  is the unique solution of the twofold EG inverse problem.  $\square$

## 9. THE POLYNOMIAL CASE.

In this section we treat the case where the functions  $\alpha$  and  $\beta$  are polynomials in  $\lambda$ , and  $\gamma$  and  $\delta$  are polynomials in  $\lambda^{-1}$ . We will prove the following theorem.

**Theorem 9.1.** *Let  $\alpha$  and  $\beta$  be matrix polynomials in  $\lambda$ , let  $\gamma$  and  $\delta$  be matrix polynomials in  $\lambda^{-1}$ , and let  $m$  be an upper bound of the degrees of  $\alpha$ ,  $\beta$ ,  $\gamma$  and  $\delta$ . Assume  $a_0$  and  $d_0$  are invertible. Then there exists a solution  $g$  to the twofold EG-inverse problem associated with  $\alpha$ ,  $\beta$ ,  $\gamma$ ,  $\delta$  if and only if the identities in (1.4)*

are satisfied. Moreover, this solution  $g$  is unique, it is a matrix polynomial with  $\deg g \leq m$  and its coefficients  $g_0, \dots, g_m$  are given by

$$-\begin{bmatrix} b_m & \cdots & b_0 \\ & \ddots & \vdots \\ & & b_m \end{bmatrix} \begin{bmatrix} e_m \\ \vdots \\ e_0 \end{bmatrix} = \begin{bmatrix} g_0 \\ \vdots \\ g_m \end{bmatrix} = -\begin{bmatrix} a_0^* & \cdots & a_m^* \\ & \ddots & \vdots \\ & & a_0^* \end{bmatrix}^{-1} \begin{bmatrix} c_0^* \\ \vdots \\ c_{-m}^* \end{bmatrix}.$$

Here

$$(9.1) \quad \begin{bmatrix} e_m \\ \vdots \\ e_0 \end{bmatrix} := \begin{bmatrix} d_0 & \cdots & d_{-m} \\ & \ddots & \vdots \\ & & d_0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_m \end{bmatrix}.$$

This theorem is closely related to Theorem 2.4 in [5]. To see this note that under the conditions mentioned in Theorem 9.1 the equations

$$\begin{bmatrix} I & G \\ G^* & I \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} e_{+,p} \\ 0 \end{bmatrix}, \quad \begin{bmatrix} I & G \\ G^* & I \end{bmatrix} \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ e_{-,q} \end{bmatrix}$$

are equivalent to

$$\begin{bmatrix} I & G_m \\ G_m^* & I \end{bmatrix} \begin{bmatrix} a(m) \\ c(m) \end{bmatrix} = \begin{bmatrix} e_{+,p}(m) \\ 0 \end{bmatrix}, \quad \begin{bmatrix} I & G_m \\ G_m^* & I \end{bmatrix} \begin{bmatrix} b(m) \\ d(m) \end{bmatrix} = \begin{bmatrix} 0 \\ e_{-,q}(m) \end{bmatrix},$$

where  $m$  is the upper bound for the degrees given above. In these equations  $G_m$  denotes the right upper  $(m+1) \times (m+1)$  submatrix of  $G$ ,  $a(m) = [a_0 \ \cdots \ a_m]^\top$ ,  $b(m) = [b_0 \ \cdots \ b_m]^\top$ ,  $c(m) = [c_{-m} \ \cdots \ c_0]^\top$  and  $d(m) = [d_{-m} \ \cdots \ d_0]^\top$ . The symbol  $e_{+,p}(m)$  denotes the first column of the  $(m+1) \times (m+1)$  identity  $p \times p$  block matrix and  $e_{-,q}(m)$  denotes the last column of the  $(m+1) \times (m+1)$  identity  $q \times q$  block matrix. The latter set of equations is solved in [5, Theorem 2.4] for the case when  $p = q$ .

Here we present a proof of Theorem 9.1 using Theorem 3.3 in [12]. Note that the uniqueness of the solution is covered by Theorem 3.1 above.

We derive Theorem 9.1 as a corollary of the following proposition.

**Proposition 9.2.** *Let  $\alpha$  and  $\beta$  be matrix polynomials in  $\lambda$ , let  $\gamma$  and  $\delta$  be matrix polynomials in  $\lambda^{-1}$ , and let  $m$  be an upper bound of the degrees of  $\alpha$ ,  $\beta$ ,  $\gamma$  and  $\delta$ . Assume  $a_0$  and  $d_0$  are invertible.*

- (i) *If  $\alpha^* \alpha - \gamma^* \gamma = a_0$ , then the onefold EG inverse problem associated with  $\alpha$  and  $\gamma$  has a unique polynomial solution  $g$ , which has degree at most  $m$ . Moreover the Fourier coefficients  $g_0, \dots, g_m$  of  $g$  are given by*

$$\begin{bmatrix} g_0 \\ \vdots \\ g_m \end{bmatrix} = -\begin{bmatrix} a_0^* & \cdots & a_m^* \\ & \ddots & \vdots \\ & & a_0^* \end{bmatrix}^{-1} \begin{bmatrix} c_0^* \\ \vdots \\ c_{-m}^* \end{bmatrix}.$$

- (ii) *If  $\delta^* \delta - \beta^* \beta = d_0$ , then there exists a unique polynomial  $\varphi(\lambda) = \sum_{j=0}^m \varphi_j \lambda^{-j}$  in  $\mathcal{W}_{-}^{q \times p}$  such that*

$$\delta + \varphi \beta - e_q \in \mathcal{W}_{+,0}^{q \times q} \quad \text{and} \quad \varphi^* \delta + \beta \in \mathcal{W}_{-,0}^{p \times q}.$$

Moreover the Fourier coefficients  $\varphi_0, \dots, \varphi_{-m}$  are given by

$$(9.2) \quad \begin{aligned} & [\varphi_0 \quad \cdots \quad \varphi_m] = \\ & - [0 \quad \cdots \quad 0 \quad I_q] \begin{bmatrix} d_0^* & & \\ \vdots & \ddots & \\ d_{-m}^* & \cdots & d_0^* \end{bmatrix}^{-1} \begin{bmatrix} b_m^* & & \\ \vdots & \ddots & \\ b_0^* & \cdots & b_m^* \end{bmatrix}. \end{aligned}$$

Moreover, if all three conditions in (1.4) are satisfied, then for  $g$  and  $\varphi$  as in items (i) and (ii) one has  $\varphi^* = g$ .

The first statement of this proposition can be found in [5, Theorem 2.1] for the case when  $p = q$ .

Before we start the proof we introduce some notation. We denote the compressions to the first  $m+1$  components of  $T_{+, \alpha}$  by  $T_{+, \alpha, m}$ , of  $H_{+, \beta}$  by  $H_{+, \beta, m}$ , of  $H_{-, \gamma}$  by  $H_{-, \gamma, m}$ , and of  $T_{-, \delta}$  by  $T_{-, \delta, m}$ . Thus  $y = T_{+, \alpha, m}x$  and  $y = H_{+, \beta, m}x$  if and only if

$$\begin{bmatrix} y_0 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} a_0 & & \\ \vdots & \ddots & \\ a_m & \cdots & a_0 \end{bmatrix} \begin{bmatrix} x_0 \\ \vdots \\ x_m \end{bmatrix}, \quad \begin{bmatrix} y_0 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} b_m & \cdots & b_0 \\ & \ddots & \vdots \\ & & b_m \end{bmatrix} \begin{bmatrix} x_{-m} \\ \vdots \\ x_0 \end{bmatrix},$$

respectively. Similarly,  $y = H_{-, \gamma, m}x$  and  $y = T_{-, \delta, m}x$  if and only if

$$\begin{bmatrix} y_{-m} \\ \vdots \\ y_0 \end{bmatrix} = \begin{bmatrix} c_{-m} & & \\ \vdots & \ddots & \\ c_0 & \cdots & c_{-m} \end{bmatrix} \begin{bmatrix} x_0 \\ \vdots \\ x_m \end{bmatrix}, \quad \begin{bmatrix} y_{-m} \\ \vdots \\ y_0 \end{bmatrix} = \begin{bmatrix} d_0 & \cdots & d_{-m} \\ & \ddots & \vdots \\ & & d_0 \end{bmatrix} \begin{bmatrix} x_{-m} \\ \vdots \\ x_0 \end{bmatrix},$$

respectively.

**Proof of Proposition 9.2.** Item (i) is a direct consequence of Theorem 3.3 in [12]. For later purpose we remark that

$$\begin{bmatrix} g_0 \\ \vdots \\ g_m \end{bmatrix} = -T_{+, \alpha, m}^{-*} H_{-, \gamma, m}^* \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_q \end{bmatrix}.$$

The next step is to prove item (ii). Define polynomials  $\tilde{\alpha}(\lambda) = \delta(\lambda^{-1})$  and  $\tilde{\gamma}(\lambda) = \beta(\lambda^{-1})$  in  $\lambda$ . Note that  $\tilde{\alpha}(0) = d_0$  is invertible, and that we have that

$$\tilde{\alpha}^* \tilde{\alpha} - \tilde{\gamma}^* \tilde{\gamma} = d_0.$$

Again applying Theorem 3.3 from [12] we obtain that there exists a unique matrix polynomial  $\tilde{\varphi}(\lambda) = \sum_{j=0}^m \varphi_j \lambda^j$  so that

$$\tilde{\alpha} + \tilde{\varphi} \tilde{\gamma} - e_q \in \mathcal{W}_{-, 0}^{q \times q} \quad \text{and} \quad \tilde{\varphi}^* \tilde{\alpha} + \tilde{c} \in \mathcal{W}_{+, 0}^{p \times q}.$$

Moreover, this matrix polynomial  $\tilde{\varphi}$  satisfies  $\deg \tilde{\varphi} \leq m$  and its coefficients are given by

$$\begin{bmatrix} \varphi_0 \\ \vdots \\ \varphi_m \end{bmatrix} = - \begin{bmatrix} d_0^* & \cdots & d_{-m}^* \\ & \ddots & \vdots \\ & & d_0^* \end{bmatrix}^{-1} \begin{bmatrix} b_0^* \\ \vdots \\ b_m^* \end{bmatrix},$$

or equivalently

$$\begin{aligned}
[\varphi_0 \quad \cdots \quad \varphi_m] &= -[0 \quad \cdots \quad 0 \quad I_q] \begin{bmatrix} d_0^* & & \\ \vdots & \ddots & \\ d_{-m}^* & \cdots & d_0^* \end{bmatrix}^{-1} \begin{bmatrix} b_m^* & & \\ \vdots & \ddots & \\ b_0^* & \cdots & b_m^* \end{bmatrix} \\
(9.3) \qquad &= -e_{-,q}(m)^* T_{-, \delta, m}^{-*} H_{+, \beta, m}^*.
\end{aligned}$$

Here we use that the inverse of an invertible block triangular Toeplitz matrix is again a block triangular Toeplitz matrix. So there exist  $q \times q$  matrices  $e_0, \dots, e_m$  such that

$$\begin{bmatrix} d_0^* & \cdots & d_{-m}^* \\ & \ddots & \vdots \\ & & d_0^* \end{bmatrix}^{-1} = \begin{bmatrix} e_0^* & \cdots & e_m^* \\ & \ddots & \vdots \\ & & e_0^* \end{bmatrix},$$

and hence

$$\begin{bmatrix} d_0^* & & \\ \vdots & \ddots & \\ d_{-m}^* & \cdots & d_0^* \end{bmatrix}^{-1} = \begin{bmatrix} e_0^* & & \\ \vdots & \ddots & \\ e_m^* & \cdots & e_0^* \end{bmatrix}.$$

Put  $\varphi(\lambda) = \tilde{\varphi}(\lambda^{-1})$ . Then

$$\delta + \varphi\beta - e_q \in \mathcal{W}_{+,0}^{q \times q} \quad \text{and} \quad \varphi^* \delta + \beta \in \mathcal{W}_{-,0}^{p \times q}.$$

To finish the proof we need to show that  $\varphi(\lambda)^* = g(\lambda)$ . In other words we need to prove that

$$\psi^* = \begin{bmatrix} \varphi_0^* \\ \vdots \\ \varphi_m^* \end{bmatrix} = \begin{bmatrix} g_0 \\ \vdots \\ g_m \end{bmatrix} = h.$$

We claim that this is the case as a result of the third identity in (1.4). As observed in Lemma 5.1 we have

$$T_{+, \alpha}^* H_{+, \beta} = T_{+, \alpha}^* H_{+, \beta} = H_{+, \gamma}^* T_{-, \delta} = H_{-, \gamma}^* T_{-, \delta}.$$

Since

$$H_{+, \beta} = \begin{bmatrix} 0 & H_{+, \beta, m} \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad H_{-, \gamma} = \begin{bmatrix} 0 & 0 \\ H_{-, \gamma, m} & 0 \end{bmatrix},$$

we have  $T_{+, \alpha, m}^* H_{+, \beta, m} = H_{-, \gamma, m}^* T_{-, \delta, m}$ . It follows that

$$\psi^* = -H_{+, \beta, m} T_{-, \delta, m}^{-*} e_{-,q}(m) = -T_{+, \alpha, m}^* H_{-, \gamma, m}^* e_{-,q}(m) = h.$$

□

#### APPENDIX A. THE ROLE OF THE MATRICES $a_0$ AND $d_0$

In the main theorems of this paper it is assumed that the matrices  $a_0$  and  $d_0$  are invertible. On the one hand these conditions can be weakened. For example, in many cases it suffices to assume that only one of the two is invertible. On the other hand additional conditions on  $a_0$  and  $d_0$  yield additional properties of the solution of the EG inverse problem. These facts can be found in a somewhat less general form in Sections 3 and 4 in [4], where the connections between polynomials and finite Toeplitz matrices is a starting point (cf., Chapter 1 in [3]). For the sake of completeness these results are reviewed in this section. We apply them in, e.g., the proof of Proposition 8.1.

Let  $g$  be any function in  $\mathcal{W}_+^{p \times q}$ , and let  $g(\lambda) = \sum_{\nu=0}^{\infty} \lambda^\nu g_\nu$ ,  $\lambda \in \mathbb{T}$ . With  $g$  we associate the Hankel operator  $G$  defined by (1.11). Put

$$(A.1) \quad G_1 = \begin{bmatrix} \cdots & g_3 & g_2 & g_1 \\ \cdots & g_4 & g_3 & g_2 \\ \cdots & g_5 & g_4 & g_3 \\ & \vdots & \vdots & \vdots \end{bmatrix} : \ell_-^2(\mathbb{C}^q) \rightarrow \ell_+^2(\mathbb{C}^p).$$

Note that

$$G_1 = S_{+,p}^* G = G S_{-,q},$$

where  $S_{+,p}$  and  $S_{-,q}$  are defined by (2.6) and (2.7), respectively. We have

$$(A.2) \quad G = \begin{bmatrix} R_1 \\ G_1 \end{bmatrix}, \quad \text{with } R_1 = [\cdots \quad g_2 \quad g_1 \quad g_0] : \ell_-^2(\mathbb{C}^q) \rightarrow \mathbb{C}^p,$$

$$(A.3) \quad G = [G_1 \quad K_1], \quad \text{with } K_1 = \begin{bmatrix} g_0 \\ g_1 \\ g_2 \\ \vdots \end{bmatrix} : \mathbb{C}^q \rightarrow \ell_+^2(\mathbb{C}^p).$$

Next define

$$(A.4) \quad \Omega := \begin{bmatrix} I & G \\ G^* & I \end{bmatrix} \quad \text{and} \quad \Omega_1 := \begin{bmatrix} I & G_1 \\ G_1^* & I \end{bmatrix}.$$

**Proposition A.1.** *Assume that  $g$  is a solution to the twofold EG inverse problem associated with the data set  $\{\alpha, \beta, \gamma, \delta\}$ .*

- (i) *Then  $a_0$  and  $d_0$  are selfadjoint. If  $a_0$  or  $d_0$  is invertible, then  $\Omega$  is invertible.*
- (ii) *Conversely, if  $\Omega$  is invertible, then  $\Omega_1$  is invertible if and only if  $a_0$  or  $d_0$  is invertible, and in that case both  $a_0$  and  $d_0$  are invertible.*

**Proof.** First recall that Theorem 1.2 in [11] gives that (1.4) is satisfied and hence  $a_0$  and  $d_0$  are selfadjoint. Theorem 1.1 in [10] gives that if  $a_0$  or  $d_0$  is invertible, then  $\Omega$  is invertible. This proves item (i).

In order to prove item (ii) we need some preliminaries. Using the partitionings in (A.2) and (A.3) we see that the operator  $\Omega$  admits the following  $3 \times 3$  block partitionings :

$$(A.5) \quad \Omega = \left[ \begin{array}{cc|c} I_p & 0 & R_1 \\ 0 & I & G_1 \\ \hline \text{---} & \text{---} & \text{---} \\ R_1^* & G_1^* & I \end{array} \right] \quad \text{and} \quad \Omega = \left[ \begin{array}{c|cc} I & G_1 & K_1 \\ \hline G_1^* & I & 0 \\ K_1^* & 0 & I_q \end{array} \right]$$

Using the definition of  $\Omega_1$  in the right hand side of (A.4) we obtain two alternative  $2 \times 2$  block operator matrix representations of  $\Omega$ :

$$(A.6) \quad \Omega = \begin{bmatrix} I_p & R \\ R^* & \Omega_1 \end{bmatrix} \quad \text{and} \quad \Omega = \begin{bmatrix} \Omega_1 & K \\ K^* & I_q \end{bmatrix}$$

Here

$$(A.7) \quad R = [0 \quad R_1] \quad \text{and} \quad K = \begin{bmatrix} K_1 \\ 0 \end{bmatrix}.$$

Now use that  $g \in \mathcal{W}_+^{p \times q}$  is a solution to the twofold EG inverse problem associated with the data set  $\{\alpha, \beta, \gamma, \delta\}$ . Thus

$$(A.8) \quad \begin{bmatrix} I & G \\ G^* & I \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} \varepsilon_{+,p} \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} I & G \\ G^* & I \end{bmatrix} \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ \varepsilon_{-,q} \end{bmatrix}$$

Using the partitions in (A.5), the identities in (A.8) can be rewritten as

$$(A.9) \quad \begin{bmatrix} I_p & R \\ R^* & \Omega_1 \end{bmatrix} \begin{bmatrix} a_0 \\ X \end{bmatrix} = \begin{bmatrix} I_p \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \Omega_1 & K \\ K^* & I_q \end{bmatrix} \begin{bmatrix} Y \\ d_0 \end{bmatrix} = \begin{bmatrix} 0 \\ I_q \end{bmatrix}$$

Assume now that  $\Omega$  is invertible. Then the two identities in (A.9) tell us that

$$(A.10) \quad \begin{bmatrix} I_p & 0 \end{bmatrix} \Omega^{-1} \begin{bmatrix} I_p \\ 0 \end{bmatrix} = a_0 \quad \text{and} \quad \begin{bmatrix} 0 & I_q \end{bmatrix} \Omega^{-1} \begin{bmatrix} 0 \\ I_q \end{bmatrix} = d_0.$$

Moreover, using the two identities in (A.10), a standard Schur complement argument shows that

$$\begin{aligned} a_0 \text{ is invertible} &\iff \Omega_1 \text{ is invertible,} \\ d_0 \text{ is invertible} &\iff \Omega_1 \text{ is invertible.} \end{aligned}$$

Hence item (ii) is proved.  $\square$

**Lemma A.2.** *Assume that  $g \in \mathcal{W}_+^{p \times q}$  is a solution to the twofold EG inverse problem associated with the data set  $\{\alpha, \beta, \gamma, \delta\}$ . If  $G$  in (1.11) is strictly contractive, then both  $a_0$  and  $d_0$  are positive definite and  $\Omega_1$  is strictly positive. Conversely, if  $\Omega_1$  is strictly positive and  $a_0$  or  $d_0$  is positive definite, then  $G$  is strictly contractive.*

**Proof.** The two identities in (A.9) also yield the following identities:

$$(A.11) \quad \begin{bmatrix} a_0 & X^* \\ 0 & I \end{bmatrix} \Omega \begin{bmatrix} a_0 & 0 \\ X & I \end{bmatrix} = \begin{bmatrix} a_0 & 0 \\ 0 & \Omega_1 \end{bmatrix},$$

$$(A.12) \quad \begin{bmatrix} I & 0 \\ Y^* & d_0 \end{bmatrix} \Omega \begin{bmatrix} I & Y \\ 0 & d_0 \end{bmatrix} = \begin{bmatrix} \Omega_1 & 0 \\ 0 & d_0 \end{bmatrix}.$$

Indeed,

$$\begin{bmatrix} a_0 & X^* \\ 0 & I \end{bmatrix} \Omega \begin{bmatrix} a_0 & 0 \\ X & I \end{bmatrix} = \begin{bmatrix} a_0 & X^* \\ 0 & I \end{bmatrix} \begin{bmatrix} I_p & R \\ 0 & \Omega_1 \end{bmatrix} = \begin{bmatrix} a_0 & a_0 R + X^* \Omega_1 \\ 0 & \Omega_1 \end{bmatrix}.$$

From the first identity in (A.9) we know that  $R^* a_0 + \Omega_1 X = 0$ . Since both  $a_0$  and  $\Omega_1$  are selfadjoint, it follows that  $a_0 R + X^* \Omega_1 = 0$ , and (A.11) is proved. The identity (A.12) is proved in a similar way.

Note that  $G$  is strictly contractive if and only if  $\Omega$  is strictly positive. The identity (A.11) gives that  $\Omega$  is strictly positive if and only if  $\Omega_1$  is strictly positive and  $a_0$  is positive definite. Similarly, the identity (A.12) gives that  $\Omega$  is strictly positive if and only if  $\Omega_1$  is strictly positive and  $d_0$  is positive definite.  $\square$

For  $p = q$  the following proposition is an immediate consequence of Theorems 4.1 and 4.2 in [4]. If  $p \neq q$ , with a minor modification of the data  $\alpha, \beta, \gamma, \delta$  the arguments used to prove Theorems 4.1 and 4.2 in [4] also yield the result below.

**Proposition A.3.** *Let  $g \in \mathcal{W}_+^{p \times q}$  be a solution to the twofold EG inverse problem associated with the data set  $\{\alpha, \beta, \gamma, \delta\}$  with  $\alpha, \beta, \gamma$ , and  $\delta$  the functions defined by (1.10).*

- (i) Assume  $a_0$  is positive definite. Then  $\det \alpha$  has no zero on  $\mathbb{T}$ , and  $\Omega_1$  is strictly positive if and only if  $\det \alpha$  has no zero inside the unit circle.
- (ii) Assume  $d_0$  is positive definite. Then  $\det \delta$  has no zero on  $\mathbb{T}$ , and  $\Omega_1$  is strictly positive if and only if  $\det \delta$  has no zero outside the unit circle.

**Proof.** We only have to consider the case when  $p \neq q$ . Assume  $p > q$ . Let  $\alpha, \beta, \gamma, \delta$  be the functions defined by (1.10). Put

$$\tilde{\alpha} = \alpha, \quad \tilde{\beta} = \begin{bmatrix} \beta & 0 \end{bmatrix}, \quad \tilde{\gamma} = \begin{bmatrix} \gamma \\ 0 \end{bmatrix}, \quad \tilde{\delta} = \begin{bmatrix} \delta & 0 \\ 0 & I_{p-q} \end{bmatrix}, \quad \tilde{g} = \begin{bmatrix} g & 0 \end{bmatrix}.$$

Here 0 stands for the a zero matrix which each time is chosen such that the extended matrix is of size  $p \times p$ . Thus  $\tilde{\alpha} \in \mathcal{W}_+^{p \times p}$ ,  $\tilde{\beta} \in \mathcal{W}_+^{p \times p}$ ,  $\tilde{\gamma} \in \mathcal{W}_-^{p \times p}$ ,  $\tilde{\delta} \in \mathcal{W}_-^{p \times p}$ , and  $\tilde{g} \in \mathcal{W}_+^{p \times p}$ . Let  $\{\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}\}$  be the dataset corresponding to  $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}$  as in (1.10). Then it follows by direct verification that  $\tilde{g}$  is the solution of the twofold EG inverse problem associated to the dataset  $\{\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}\}$ . Let  $\tilde{\Omega}_1$  be defined as  $\Omega_1$  with  $g$  replaced by  $\tilde{g}$ . Now assume that  $\tilde{a}_0 = a_0$  is positive definite. Then it follows from Theorem 4.1 in [4] that  $\det \tilde{\alpha} = \det \alpha$  has no zero on  $\mathbb{T}$ , and  $\tilde{\Omega}_1$  is strictly positive if and only if  $\det \tilde{\alpha}$  has no zero inside  $\mathbb{T}$ . Notice that there exists an invertible transformation  $E$  such that

$$\tilde{\Omega}_1 = E \begin{bmatrix} \Omega_1 & 0 \\ 0 & I \end{bmatrix} E^{-1}.$$

In particular we get that  $\tilde{\Omega}_1$  is positive definite if and only if  $\Omega_1$  is.

The case when  $p < q$  and the item (ii) are proved in a similar way. □

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