A SYMMETRIC FORMULA FOR HYPERGEOMETRIC SERIES

CHUANAN WEI

Department of Medical Informatics
Hainan Medical University, Haikou 571199, China

Abstract. In terms of Dougall's ${}_2H_2$ series identity and the series rearrangement method, we establish an interesting symmetric formula for hypergeometric series. Then it is utilized to derive a known nonterminating form of Saalschütz's theorem. Similarly, we also show that Bailey's ${}_6\psi_6$ series identity implies the nonterminating form of Jackson's ${}_8\phi_7$ summation formula. Considering the reversibility of the proofs, it is routine to show that Dougall's ${}_2H_2$ series identity is equivalent to a known nonterminating form of Saalschütz's theorem and Bailey's ${}_6\psi_6$ series identity is equivalent to the nonterminating form of Jackson's ${}_8\phi_7$ summation formula.

1. Introduction

For an integer n and a complex number x, define the shifted factorial to be

$$(x)_n = \Gamma(x+n)/\Gamma(x),$$

where $\Gamma(x)$ is the well known gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt \quad \text{with} \quad Re(x) > 0.$$

Following Andrews, Askey and Roy [2], define the hypergeometric series by

$${}_{1+r}F_s\left[\begin{matrix} a_0, & a_1, & \cdots, & a_r \\ & b_1, & \cdots, & b_s \end{matrix} \middle| z\right] = \sum_{k=0}^{\infty} \frac{(a_0)_k(a_1)_k\cdots(a_r)_k}{k!(b_1)_k\cdots(b_s)_k} z^k.$$

Then Saalschütz's theorem (cf. [2, p. 69]) can be stated as

$${}_3F_2\left[\begin{matrix} a, b, -n \\ c, 1+a+b-c-n \end{matrix} \middle| 1\right] = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n}. \tag{1}$$

A known nonterminating form of it (cf. [2, p. 92]) reads as

$$\begin{aligned} {}_3F_2\left[\begin{matrix} a, b, c+d-a-b-1 \\ c, d \end{matrix} \middle| 1\right] &= {}_3F_2\left[\begin{matrix} 1, c-a, c-b \\ c-a-b+1, c+d-a-b \end{matrix} \middle| 1\right] \\ \times\ \frac{\Gamma(c)\Gamma(d)}{\Gamma(a)\Gamma(b)\Gamma(c+d-a-b)}\frac{1}{a+b-c} &+ \frac{\Gamma(c)\Gamma(d)\Gamma(c-a-b)\Gamma(d-a-b)}{\Gamma(c-a)\Gamma(c-b)\Gamma(d-a)\Gamma(d-b)}, \end{aligned} \tag{2}$$

provided $Re(d-a-b) > 0$. The known proof of (2) comes from a transformation formula involving three ${}_3F_2$ series given by the contour integration method. The reader is referred to [2, Section 2.4] for details.

2010 Mathematics Subject Classification: Primary 05A19 and Secondary 33C20

Key words and phrases. Hypergeometric series; Dougall's ${}_2H_2$-series identity; Basic Hypergeometric series; Bailey's ${}_6\psi_6$ series identity.

Email address: weichuanan78@163.com.

Following Slater [12], define the bilateral hypergeometric series to be

$$ {}_rH_s\left[\begin{matrix} a_1, & a_2, & \cdots, & a_r \\ b_1, & b_2, & \cdots, & b_s \end{matrix}\,\middle|\, z\right] = \sum_{k=-\infty}^{\infty}\frac{(a_1)_k(a_2)\cdots(a_r)_k}{(b_1)_k(b_2)_k\cdots(b_s)_k}z^k. $$

Thus Dougall's ${}_2H_2$ series identity (cf. [2, p. 110]) can be written as

$$ {}_2H_2\left[\begin{matrix} a, b \\ c, d \end{matrix}\,\middle|\, 1\right] = \frac{\Gamma(1-a)\Gamma(1-b)\Gamma(c)\Gamma(d)\Gamma(c+d-a-b-1)}{\Gamma(c-a)\Gamma(c-b)\Gamma(d-a)\Gamma(d-b)}, \tag{3} $$

where $Re(c+d-a-b) > 1$.

For an integer n and two complex numbers x, q with $|q| < 1$, define the q-shifted factorial by

$$ (x;q)_\infty = \prod_{i=0}^{\infty}(1-xq^i), \quad (x;q)_n = \frac{(x;q)_\infty}{(xq^n;q)_\infty}. $$

For simplification, we shall frequently adopt the following notations:

$$ (x_1,x_2,\cdots,x_r;q)_\infty = (x_1;q)_\infty(x_2;q)_\infty\cdots(x_r;q)_\infty, $$
$$ (x_1,x_2,\cdots,x_r;q)_n = (x_1;q)_n(x_2;q)_n\cdots(x_r;q)_n. $$

Following Gasper and Rahman [4], define the basic hypergeometric series and bilateral basic hypergeometric series to be

$$ {}_{1+r}\phi_s\left[\begin{matrix} a_0, & a_1, & \cdots, a_r \\ & b_1, & \cdots, b_s \end{matrix}\,\middle|\, q;z\right] = \sum_{k=0}^{\infty}\frac{(a_0,a_1,\cdots,a_r;q)_k}{(q,b_1,\cdots,b_s;q)_k}\Big\{(-1)^kq^{\binom{k}{2}}\Big\}^{s-r}z^k, $$
$$ {}_r\psi_s\left[\begin{matrix} a_1, & a_2, & \cdots, a_r \\ b_1, & b_2, & \cdots, b_s \end{matrix}\,\middle|\, q;z\right] = \sum_{k=-\infty}^{\infty}\frac{(a_1,a_2,\cdots,a_r;q)_k}{(b_1,b_2,\cdots,b_s;q)_k}\Big\{(-1)^kq^{\binom{k}{2}}\Big\}^{s-r}z^k. $$

Then the nonterminating form of Jackson's ${}_8\phi_7$ summation formula (cf. [4, p.54]) and Bailey's ${}_6\psi_6$ series identity (cf. [4, p.140]) can be expressed as

$$ \begin{aligned} &{}_8\phi_7\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, f \\ \sqrt{a}, -\sqrt{a}, qa/b, qa/c, qa/d, qa/e, qa/f \end{matrix}\,\middle|\, q;q\right] \\ &= \frac{b}{a}\frac{(qa,c,d,e,f,qb/a,qb/c,qb/d,qb/e,qb/f;q)_\infty}{(qa/b,qa/c,qa/d,qa/e,qa/f,bc/a,bd/a,be/a,bf/a,qb^2/a;q)_\infty} \\ &\times {}_8\phi_7\left[\begin{matrix} b^2/a, qb/\sqrt{a}, -qb/\sqrt{a}, b, bc/a, bd/a, be/a, bf/a \\ b/\sqrt{a}, -b/\sqrt{a}, qb/a, qb/c, qb/d, qb/e, qb/f \end{matrix}\,\middle|\, q;q\right] \\ &+ \frac{(qa,b/a,qa/cd,qa/ce,qa/cf,qa/de,qa/df,qa/ef;q)_\infty}{(qa/c,qa/d,qa/e,qa/f,bc/a,bd/a,be/a,bf/a;q)_\infty} \end{aligned} \tag{4} $$

with $qa^2 = bcdef$,

$$ \begin{aligned} &{}_6\psi_6\left[\begin{matrix} q\sqrt{a}, -q\sqrt{a}, b, c, d, e \\ \sqrt{a}, -\sqrt{a}, qa/b, qa/c, qa/d, qa/e \end{matrix}\,\middle|\, q;\frac{qa^2}{bcde}\right] \\ &= \left[\begin{matrix} q, qa, q/a, qa/bc, qa/bd, qa/be, qa/cd, qa/ce, qa/de \\ q/b, q/c, q/d, q/e, qa/b, qa/c, qa/d, qa/e, qa^2/bcde \end{matrix}\,\middle|\, q\right]_\infty, \end{aligned} \tag{5} $$

provided $|qa^2/bcde| < 1$. The original proof of (4) comes from a three term relation of ${}_8\phi_7$ series offered by the q-integration method. The reader may consult [4, Section 2.11] for details. Recently, the research of q-congruence attracts several mathematicians. Some nice results can be seen in the papers [5, 6].

In 2006, Chen and Fu [3] established some semi-finite forms of bilateral basic hypergeometric series in accordance with Cauchy's method. Subsequently, Jouhet [7] deduced (5) from (4) in the same way. Several years later, Wei, Yan and Li [13] derived similarly (3) from (2). More results related to Cauchy's method can be found in the papers [14, 15].

Inspired by the works just mentioned, it is natural to consider the inverse of Cauchy's method. According to the series rearrangement method, we shall deduce (2) from (3) in Section 2 and show that (5) implies (4) in Section 3.

2. A symmetric formula for hypergeometric series

Theorem 1. *Let a, b, c, d be complex numbers. Then*

$$\Phi(a,b;c,d)+\Phi(c,d;a,b)=\frac{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(d)\Gamma(a+b+c+d-1)}{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)},$$

where the symbol on the left hand side stands for

$$\Phi(a,b;c,d)=\sum_{k=0}^{\infty}\frac{\Gamma(a+k)\Gamma(b+k)\Gamma(a+b+c+d-1+k)}{\Gamma(1+k)\Gamma(a+b+c+k)\Gamma(a+b+d+k)}.$$

Proof. Split the bilateral series into two parts to obtain

$$\begin{aligned}
&{}_2H_2\left[\begin{matrix}1-a,1-b\\1+c,1+d\end{matrix}\,\middle|\,1\right]\\
&=\sum_{i=0}^{\infty}\frac{(1-a)_i(1-b)_i}{(1+c)_i(1+d)_i}+\sum_{i=-\infty}^{-1}\frac{(1-a)_i(1-b)_i}{(1+c)_i(1+d)_i}\\
&=\sum_{i=0}^{\infty}\frac{(1-a)_i(1-b)_i}{(1+c)_i(1+d)_i}+\frac{cd}{ab}\sum_{i=0}^{\infty}\frac{(1-c)_i(1-d)_i}{(1+a)_i(1+b)_i}\\
&=\frac{\Gamma(1+c)\Gamma(1+d)}{\Gamma(1-a)\Gamma(1-b)}\sum_{i=0}^{\infty}\frac{\Gamma(1-a+i)\Gamma(1-b+i)}{\Gamma(1+c+i)\Gamma(1+d+i)}\\
&+\frac{\Gamma(a)\Gamma(b)}{\Gamma(-c)\Gamma(-d)}\sum_{i=0}^{\infty}\frac{\Gamma(1-c+i)\Gamma(1-d+i)}{\Gamma(1+a+i)\Gamma(1+b+i)}.
\end{aligned}$$

By means of the case $d=1$ of (3):

$${}_2F_1\left[\begin{matrix}a,\ b\\c\end{matrix}\,\middle|\,1\right]=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \tag{6}$$

we can proceed as follows:

$$\begin{aligned}
&{}_2H_2\left[\begin{matrix}1-a,1-b\\1+c,1+d\end{matrix}\,\middle|\,1\right]\\
&=\frac{\Gamma(1+c)\Gamma(1+d)}{\Gamma(1-a)\Gamma(1-b)}\sum_{i=0}^{\infty}\frac{\Gamma(1-a+i)\Gamma(1-b+i)}{\Gamma(1+i)\Gamma(1+c+d+i)}{}_2F_1\left[\begin{matrix}c,d\\1+c+d+i\end{matrix}\,\middle|\,1\right]\\
&+\frac{\Gamma(a)\Gamma(b)}{\Gamma(-c)\Gamma(-d)}\sum_{i=0}^{\infty}\frac{\Gamma(1-c+i)\Gamma(1-d+i)}{\Gamma(1+i)\Gamma(1+a+b+i)}{}_2F_1\left[\begin{matrix}a,b\\1+a+b+i\end{matrix}\,\middle|\,1\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{cd}{\Gamma(1-a)\Gamma(1-b)}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{\Gamma(1-a+i)\Gamma(1-b+i)\Gamma(c+k)\Gamma(d+k)}{\Gamma(1+i)\Gamma(1+c+d+i+k)\Gamma(1+k)}\\
&+ \frac{1}{\Gamma(-c)\Gamma(-d)}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{\Gamma(1-c+i)\Gamma(1-d+i)\Gamma(a+k)\Gamma(b+k)}{\Gamma(1+i)\Gamma(1+a+b+i+k)\Gamma(1+k)}\\
&= cd\sum_{k=0}^{\infty}\frac{\Gamma(c+k)\Gamma(d+k)}{\Gamma(1+k)\Gamma(1+c+d+k)}{}_2F_1\left[\begin{matrix}1-a,1-b\\1+c+d+k\end{matrix}\,\middle|\,1\right]\\
&+ cd\sum_{k=0}^{\infty}\frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(1+k)\Gamma(1+a+b+k)}{}_2F_1\left[\begin{matrix}1-c,1-d\\1+a+b+k\end{matrix}\,\middle|\,1\right]\\
&= cd\sum_{k=0}^{\infty}\frac{\Gamma(c+k)\Gamma(d+k)\Gamma(a+b+c+d-1+k)}{\Gamma(1+k)\Gamma(a+c+d+k)\Gamma(b+c+d+k)}\\
&+ cd\sum_{k=0}^{\infty}\frac{\Gamma(a+k)\Gamma(b+k)\Gamma(a+b+c+d-1+k)}{\Gamma(1+k)\Gamma(a+b+c+k)\Gamma(a+b+d+k)}.
\end{aligned}
$$

Employing the substitutions $a \to 1-a, b \to 1-b, c \to 1+c, d \to 1+d$ in (3), we get

$$
{}_2H_2\left[\begin{matrix}1-a,\ 1-b\\1+c,\ 1+d\end{matrix}\,\middle|\,1\right] = \frac{\Gamma(a)\Gamma(b)\Gamma(1+c)\Gamma(1+d)\Gamma(a+b+c+d-1)}{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}.
$$

The combination of the last two equations produces

$$
\begin{aligned}
&cd\sum_{k=0}^{\infty}\frac{\Gamma(c+k)\Gamma(d+k)\Gamma(a+b+c+d-1+k)}{\Gamma(1+k)\Gamma(a+c+d+k)\Gamma(b+c+d+k)}\\
&+cd\sum_{k=0}^{\infty}\frac{\Gamma(a+k)\Gamma(b+k)\Gamma(a+b+c+d-1+k)}{\Gamma(1+k)\Gamma(a+b+c+k)\Gamma(a+b+d+k)}\\
&=\frac{\Gamma(a)\Gamma(b)\Gamma(1+c)\Gamma(1+d)\Gamma(a+b+c+d-1)}{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}.
\end{aligned}
$$

Dividing both sides by cd, we achieve Theorem 1. □

The symmetric formula is beautiful and it includes some known results as special cases. On the research of reciprocity formulas, the reader is referred to the papers [1, 8, 9, 10, 11].

When $b=-n$, Theorem 1 reduces to the following summation formula:

$$
{}_3F_2\left[\begin{matrix}a, a+c+d-1-n, -n\\a+c-n, a+d-n\end{matrix}\,\middle|\,1\right] = \frac{(1-c)_n(1-d)_n}{(1-a-c)_n(1-a-d)_n},
$$

which is equivalent to (1). Thus Theorem 1 can be regarded as the nonterminating form of (1).

When $c=a$ and $d=b$, Theorem 1 reduces to the following summation formula:

$$
{}_3F_2\left[\begin{matrix}a, b, 2a+2b-1\\a+2b, 2a+b\end{matrix}\,\middle|\,1\right] = \frac{1}{2}\frac{\Gamma(a)\Gamma(b)\Gamma(a+2b)\Gamma(2a+b)}{\Gamma(2a)\Gamma(2b)\Gamma(a+b)\Gamma(a+b)}.
$$

It can also be attained by letting $a \to 2a+2b-1$, $c \to a$ in Dixon's ${}_3F_2$-series identity(cf. [2, p. 72]):

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix} a,b,c \\ 1+a-b,1+a-c \end{matrix} \middle| 1\right] \\
&= \frac{\Gamma(1+\frac{a}{2})\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+\frac{a}{2}-b-c)}{\Gamma(1+a)\Gamma(1+\frac{a}{2}-b)\Gamma(1+\frac{a}{2}-c)\Gamma(1+a-b-c)},
\end{aligned}
$$

provided $Re(1+\frac{a}{2}-b-c)>0$.

Now we begin to prove (2) by using Theorem 1. In terms of (6), we have

$$
\begin{aligned}
&\Phi(c,d;a,b) \\
&= \sum_{k=0}^{\infty} \frac{\Gamma(c+k)\Gamma(d+k)\Gamma(a+b+c+d-1+k)}{\Gamma(1+k)\Gamma(a+c+d+k)\Gamma(b+c+d+k)} \\
&= \sum_{k=0}^{\infty} \frac{\Gamma(d+k)\Gamma(a+b+c+d-1+k)}{\Gamma(1+k)\Gamma(a+b+c+2d+k)} {}_2F_1\left[\begin{matrix} a+d,b+d \\ a+b+c+2d+k \end{matrix} \middle| 1\right] \\
&= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{\Gamma(d+k)\Gamma(a+b+c+d-1+k)\Gamma(a+d+j)\Gamma(b+d+j)}{\Gamma(1+k)\Gamma(1+j)\Gamma(a+b+c+2d+k+j)\Gamma(a+d)\Gamma(b+d)} \\
&= \sum_{j=0}^{\infty} \frac{\Gamma(a+d+j)\Gamma(b+d+j)\Gamma(d)\Gamma(a+b+c+d-1)}{\Gamma(1+j)\Gamma(a+b+c+2d+j)\Gamma(a+d)\Gamma(b+d)} {}_2F_1\left[\begin{matrix} d,a+b+c+d-1 \\ a+b+c+2d+j \end{matrix} \middle| 1\right] \\
&= \sum_{j=0}^{\infty} \frac{\Gamma(d)\Gamma(a+b+c+d-1)\Gamma(a+d+j)\Gamma(b+d+j)}{\Gamma(a+d)\Gamma(b+d)\Gamma(1+d+j)\Gamma(a+b+c+d+j)} \\
&= \frac{1}{d(a+b+c+d-1)} {}_3F_2\left[\begin{matrix} 1,a+d,b+d \\ 1+d,a+b+c+d \end{matrix} \middle| 1\right].
\end{aligned}
$$

Substitute the relation into Theorem 1 to obtain

$$
\begin{aligned}
&\frac{\Gamma(a)\Gamma(b)\Gamma(a+b+c+d-1)}{\Gamma(a+b+c)\Gamma(a+b+d)} {}_3F_2\left[\begin{matrix} a,b,a+b+c+d-1 \\ a+b+c,a+b+d \end{matrix} \middle| 1\right] \\
&+ \frac{1}{d(a+b+c+d-1)} {}_3F_2\left[\begin{matrix} 1,a+d,b+d \\ 1+d,a+b+c+d \end{matrix} \middle| 1\right] \\
&= \frac{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(d)\Gamma(a+b+c+d-1)}{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}.
\end{aligned}
$$

Performing the replacements $c \to d-a-b$, $d \to c-a-b$ in the last equation, we get (2) to complete the proof.

In a word, we have derived the nonterminating form of Saalschütz's theorem (2) from Dougall's ${}_2H_2$ series identity (3) via the series rearrangement method. By reversing the process, it is not difficult to realize that we can also deduce (3) from (2) through the series rearrangement method. In this sense, (2) and (3) are equivalent with each other.

3. Bailey's ${}_6\psi_6$ series identity implies the nonterminating form of Jackson's ${}_8\phi_7$ summation formula

The case $e=a$ of (5) is

$$ {}_6\phi_5\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, d \\ \sqrt{a}, -\sqrt{a}, qa/b, qa/c, qa/d \end{matrix}\,\middle|\, q; \frac{qa}{bcd}\right] = \frac{(qa, qa/bc, qa/bd, qa/cd; q)_\infty}{(qa/b, qa/c, qa/d, qa/bcd; q)_\infty}, \tag{7}$$

where $|qa/bcd|<1$. It is well known that Jackson's ${}_8\phi_7$ summation formula (cf. [4, p. 43])

$$\begin{aligned} &{}_8\phi_7\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, d, q^{1+n}a^2/bcd, q^{-n} \\ \sqrt{a}, -\sqrt{a}, qa/b, qa/c, qa/d, q^{-n}bcd/a, q^{1+n}a \end{matrix}\,\middle|\, q; q\right] \\ &= \frac{(qa, qa/bc, qa/bd, qa/cd; q)_n}{(qa/b, qa/c, qa/d, qa/bcd; q)_n} \end{aligned} \tag{8}$$

can be derived from (7) in accordance with the series rearrangement method. Now we start to prove (4) according to (5), (7) and (8). Split the bilateral series on the left hand side of (5) into two parts to achieve

$$\begin{aligned} &\frac{(q, qa/cd, qa/ce, qa/cf, qa/de, qa/df, qa/ef, qa/cdef, qcdef/a; q)_\infty}{(qc, qd, qe, qf, qa/cde, qa/cdf, qa/cef, qa/def, qa^2/cdef; q)_\infty} \\ &= {}_6\psi_6\left[\begin{matrix} q\sqrt{\frac{cdef}{a}}, -q\sqrt{\frac{cdef}{a}}, cde/a, cdf/a, cef/a, def/a \\ \sqrt{\frac{cdef}{a}}, -\sqrt{\frac{cdef}{a}}, qf, qe, qd, qc \end{matrix}\,\middle|\, q; \frac{qa^2}{cdef}\right] \\ &= \sum_{k=0}^{\infty} \frac{1-q^{2k}cdef/a}{1-cdef/a} \frac{(cde/a, cdf/a, cef/a, def/a; q)_k}{(qf, qe, qd, qc; q)_k} \left(\frac{qa^2}{cdef}\right)^k \\ &+ \sum_{k=-\infty}^{-1} \frac{1-q^{2k}cdef/a}{1-cdef/a} \frac{(cde/a, cdf/a, cef/a, def/a; q)_k}{(qf, qe, qd, qc; q)_k} \left(\frac{qa^2}{cdef}\right)^k \\ &= \sum_{k=0}^{\infty} \frac{1-q^{2k}cdef/a}{1-cdef/a} \frac{(cde/a, cdf/a, cef/a, def/a; q)_k}{(qf, qe, qd, qc; q)_k} \left(\frac{qa^2}{cdef}\right)^k \\ &+ \frac{qa^2}{cdef} \frac{(1-q^2a/cdef)(1-1/c)(1-1/d)(1-1/e)(1-1/f)}{(1-a/cdef)(1-qa/def)(1-qa/cef)(1-qa/cdf)(1-qa/cde)} \\ &\times \sum_{k=0}^{\infty} \frac{1-q^{2+2k}a/cdef}{1-q^2a/cdef} \frac{(q/c, q/d, q/e, q/f; q)_k}{(q^2a/def, q^2a/cef, q^2a/cdf, q^2a/cde; q)_k} \left(\frac{qa^2}{cdef}\right)^k. \end{aligned} \tag{9}$$

Denote the two sums on the right hand side by $\Omega(a,c,d,e,f)$ and $\Theta(a,c,d,e,f)$, respectively. Above all, we can calculate $\Omega(a,c,d,e,f)$ as follows:

$$\begin{aligned} &\Omega(a,c,d,e,f) \\ &= \sum_{k=0}^{\infty} \frac{1-q^{2k}cdef/a}{1-cdef/a} \frac{(cdef/a, cef/a, def/a, cd/a; q)_k}{(q, qd, qc, qef; q)_k} \left(\frac{qa^2}{cdef}\right)^k \\ &\times {}_8\phi_7\left[\begin{matrix} ef, q\sqrt{ef}, -q\sqrt{ef}, e, f, qa/cd, q^k cdef/a, q^{-k} \\ \sqrt{ef}, -\sqrt{ef}, qf, qe, cdef/a, q^{1-k}a/cd, q^{1+k}ef \end{matrix}\,\middle|\, q; q\right] \end{aligned}$$

$$= \sum_{i=0}^{\infty}\sum_{k=i}^{\infty}\frac{1-q^{2i}ef}{1-ef}\frac{(ef,e,f,qa/cd,q^kcdef/a,q^{-k};q)_i}{(q,qf,qe,cdef/a,q^{1-k}a/cd,q^{1+k}ef;q)_i}q^i$$

$$\times\frac{1-q^{2k}cdef/a}{1-cdef/a}\frac{(cdef/a,cef/a,def/a,cd/a;q)_k}{(q,qd,qc,qef;q)_k}\left(\frac{qa^2}{cdef}\right)^k$$

$$= \sum_{i=0}^{\infty}\frac{1-q^{2i}ef}{1-ef}\frac{(ef,e,f,qa/cd,cef/a,def/a;q)_i(qcdef/a;q)_{2i}}{(q,qf,qe,cdef/a,qd,qc;q)_i(qef;q)_{2i}}\left(\frac{qa}{ef}\right)^i$$

$$\times\ {}_6\phi_5\left[\begin{matrix}q^{2i}cdef/a,q^{1+i}\sqrt{cdef/a},-q^{1+i}\sqrt{cdef/a},q^icef/a,q^idef/a,cd/a\\ q^i\sqrt{cdef/a},-q^i\sqrt{cdef/a},q^{1+i}d,q^{1+i}c,q^{1+2i}ef\end{matrix}\ \middle|\ q;\frac{qa^2}{cdef}\right]$$

$$= \frac{(qa/c,qa/d,qa/ef,qcdef/a;q)_\infty}{(qc,qd,qef,qa^2/cdef;q)_\infty}\sum_{i=0}^{\infty}\frac{1-q^{2i}ef}{1-ef}\frac{(ef,e,f,ef/a;q)_i}{(q,qf,qe,qa;q)_i}\left(\frac{qa}{ef}\right)^i$$

$$\times\ {}_8\phi_7\left[\begin{matrix}a,q\sqrt{a},-q\sqrt{a},qa^2/cdef,c,d,q^ief,q^{-i}\\ \sqrt{a},-\sqrt{a},cdef/a,qa/c,qa/d,q^{1-i}a/ef,q^{1+i}a\end{matrix}\ \middle|\ q;q\right]$$

$$= \frac{(qa/c,qa/d,qa/ef,qcdef/a;q)_\infty}{(qc,qd,qef,qa^2/cdef;q)_\infty}\sum_{j=0}^{\infty}\frac{1-q^{2j}a}{1-a}\frac{(a,qa^2/cdef,c,d,e,f;q)_j}{(q,cdef/a,qa/c,qa/d,qf,qe;q)_j}$$

$$\times\frac{(qef;q)_{2j}}{(qa;q)_{2j}}q^j\ {}_6\phi_5\left[\begin{matrix}q^{2j}ef,q^{1+j}\sqrt{ef},-q^{1+j}\sqrt{ef},q^je,q^jf,ef/a\\ q^j\sqrt{ef},-q^j\sqrt{ef},q^{1+j}f,q^{1+j}e,q^{1+2j}a\end{matrix}\ \middle|\ q;\frac{qa}{ef}\right]$$

$$= \frac{(q,qa/c,qa/d,qa/e,qa/f,qcdef/a;q)_\infty}{(qa,qc,qd,qe,qf,qa^2/cdef;q)_\infty}$$

$$\times\ {}_8\phi_7\left[\begin{matrix}a,q\sqrt{a},-q\sqrt{a},qa^2/cdef,c,d,e,f\\ \sqrt{a},-\sqrt{a},cdef/a,qa/c,qa/d,qa/e,qa/f\end{matrix}\ \middle|\ q;q\right].$$

Similarly, $\Theta(a,c,d,e,f)$ can be manipulated as

$$\Theta(a,c,d,e,f)$$

$$= \frac{qa^2}{cdef}\frac{(1-q^2a/cdef)(1-1/c)(1-1/d)(1-1/e)(1-1/f)}{(1-a/cdef)(1-qa/def)(1-qa/cef)(1-qa/cdf)(1-qa/cde)}$$

$$\times\frac{(q,q^3a/cdef,q^2a^2/c^2def,q^2a^2/cd^2ef,q^2a^2/cde^2f,q^2a^2/cdef^2;q)_\infty}{(q^2a/cde,q^2a/cdf,q^2a/cef,q^2a/def,qa^2/cdef,q^3a^3/c^2d^2e^2f^2;q)_\infty}$$

$$\times\ {}_8\phi_7\left[\begin{matrix}\frac{q^2a^3}{c^2d^2e^2f^2},q\sqrt{\frac{q^2a^3}{c^2d^2e^2f^2}},-q\sqrt{\frac{q^2a^3}{c^2d^2e^2f^2}},\frac{qa^2}{cdef},\frac{qa}{cde},\frac{qa}{cdf},\frac{qa}{cef},\frac{qa}{def}\\ \sqrt{\frac{q^2a^3}{c^2d^2e^2f^2}},-\sqrt{\frac{q^2a^3}{c^2d^2e^2f^2}},\frac{q^2a}{cdef},\frac{q^2a^2}{cdef^2},\frac{q^2a^2}{cde^2f},\frac{q^2a^2}{cd^2ef},\frac{q^2a^2}{c^2def}\end{matrix}\ \middle|\ q;q\right].$$

Substituting the last two equations into (9), we attain (4) after some regular simplifications.

In conclusion, we have deduced the nonterminating form of Jackson's ${}_8\phi_7$ summation formula (4) from Bailey's ${}_6\psi_6$ series identity (5) via the series rearrangement method. By reversing the process, it is conventional to understand that we can also derive (5) from (4) through the series rearrangement method. In this sense, (4) and (5) are equivalent with each other.

Acknowledgments

The work is supported by the National Natural Science Foundation of China (No. 11661032).

References

[1] G.E. Andrews, Ramanujan's "lost" notebook. I. Partial θ-function, Adv. Math. 41 (1981), 137–172.

[2] G.E. Andrews, R. Askey, R. Roy, Special Functions, Cambridge University Press, Cambridge, 2000.

[3] W.Y.C. Chen, A.M. Fu, Semi-finite forms of bilateral basic hypergeometric series, Prop. Amer. Math. Soc. 134 (2006), 1719–1725.

[4] G. Gasper, M. Rahman, Basic Hypergeometric Series (2nd edition), Cambridge University Press, Cambridge, 2004.

[5] V.J.W. Guo, A q-analogue of the (I.2) supercongruence of Van Hamme, Int. J. Number Theory 15 (2019), 29–36.

[6] V.J.W. Guo, W. Zudilin, A q-microscope for supercongruences, Adv. Math. 346 (2019), 329–358.

[7] F. Jouhet, Some more Semi-finite forms of bilateral basic hypergeometric series, Ann. Combin. 11 (2007), 47–57.

[8] S.Y. Kang, Generalizations of Ramanujan's reciprocity theorem and their applications, J. London Math. Soc. (2) 75 (2007), 18–34.

[9] Z.-G. Liu, Some operator identities and q-series transformation formulas, Discrete Math. 265 (2003), 119–139.

[10] Z.-G. Liu, Extensions of Ramanujan's reciprocity theorem and the Andrews-Askey integral, J. Math. Anal. Appl. 443 (2016), 1110–1229.

[11] X.R. Ma, Six-variable generalization of Ramanujan's reciprocity theorem, J. Math. Anal. Appl. 353 (2009), 320–328.

[12] L.J. Slater, Generalized Hypergeometric Functions, Cambridge University Press, Cambridge, 1966.

[13] C.A. Wei, Q.L. Yan, J.B. Li, A short proof for Dougall's ${}_2H_2$-series identity, Discrete Math. 312 (2012), 2997–2999.

[14] C.H. Zhang, Z.Z. Zhang, Two new transformation formulas of basichypergeometric series, J. Math. Anal. Appl. 336 (2007), 777–787.

[15] Z.Z. Zhang, Q.X. Hu, On the very-well-poised bilateral basic hypergeometric ${}_7\psi_7$ series, J. Math. Anal. Appl. 367 (2010), 657–668.