

On the 4-girth-thickness of the line graph of the complete graph

Christian Rubio-Montiel *

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Abstract

The g -girth-thickness $\theta(g, G)$ of a graph G is the minimum number of planar subgraphs of girth at least g whose union is G . In this note, we give the 4-girth-thickness $\theta(4, L(K_n))$ of the line graph of the complete graph $L(K_n)$ when n is even. We also give the minimum number of subgraphs of $L(K_n)$, which are of girth at least 4 and embeddable on the projective plane, whose union is $L(K_n)$.

Keywords: girth-thickness, S -thickness, planar decomposition, line graph, token graph.

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1 Introduction

The *thickness* $\theta(G)$ of a graph G is the minimum number of elements in any partition of $E(G)$ such that the induced subgraph of each part is a planar graph. Equivalently, $\theta(G)$ is defined as the minimum number of planar subgraphs whose union is G .

The thickness has draw the attention of several researchers since its introduction in the 60s [20] because it is an NP-hard problem [16] and it has many applications, for instance, in the design of circuits [1], in the Ringel's earth-moon problem [14] and to bound the achromatic numbers of planar graphs [3], see the survey [17].

*División de Matemáticas e Ingeniería, FES Acatlán, Universidad Nacional Autónoma de México, 53150, Naucalpan, Mexico, christian.rubio@apolo.acatlan.unam.mx.

Only some exact results are known, for example, when G is a complete graph [2, 5, 6], a hypercube [15], or a complete multipartite graph [7, 13, 21, 22]. And some generalizations of the thickness also have been studied such that the outerthickness θ_o , defined similarly but with outerplanar instead of planar [12], and the S -thickness θ_S , considering the thickness on a surfaces S instead of the plane [4].

The g -girth-thickness $\theta(g, G)$ of a graph G , introduced in [18], is the minimum number of elements in any partition of $E(G)$ such that the induced subgraphs of each part is a planar graph of girth at least g . The g -girth-thickness is the usual thickness when $g = 3$ and it is the *arboricity number* when $g = \infty$. Recall that the *girth* of a graph is the size of its shortest cycle or ∞ if it is acyclic.

Exact results also are known when $g > 3$ and finite, for instance, the 4-girth-thickness of the complete graph [9, 11, 18], the 4-girth-thickness of the complete multipartite graph [11, 19] and the 6-girth-thickness of the complete graph [9]. Owing to the fact that the hypercube and the complete bipartite are triangle-free graphs, their thickness equal their 4-girth-thickness which were calculate in [15] and partially calculate in [7, 13], respectively.

We define the S - g -girth-thickness $\theta_S(g, G)$ of a graph G as the minimum number of subgraphs embeddable on a surface S of girth at least g whose union is G . Of course, if G has girth g then $\theta_S(g, G)$ is $\theta_S(G)$ as in the case of $K_{n,n}$ for $g = 4$, see [4].

In this note, we obtain the 4-girth-thickness $\theta(4, L(K_n))$ of the line graph $L(K_n)$ of the complete graph K_n when n is even. To achieve this, in Section 2 we recall some properties about token graphs $F_k(G)$. In Section 3, we determine $\theta(4, F_2(G))$ when G contains a factorization into Hamiltonian paths, in particular

$$\theta(4, L(K_n)) = \frac{n}{2} \text{ and } \theta(4, F_2(K_{n-1,n})) = \frac{n}{2}$$

for n even. Finally, in Section 4, we determine $\theta_S(4, F_2(G))$ when S is the projective plane and G contains a Hamiltonian-factorization, in consequence

$$\theta_S(4, L(K_n)) = \left\lfloor \frac{n}{2} \right\rfloor \text{ and } \theta(4, F_2(K_{2n,2n})) = n$$

for all n .

2 Token graphs

Consider the following graph $F_k(G)$ called the k -token graph introduced in [10], for given an integer $k \geq 1$ and a graph G of order n . The vertex set $V(F_k(G))$ is the family of k subsets of $V(G)$, therefore $|V(F_k(G))| = \binom{n}{k}$. Two such k -subsets X and Y are adjacent if its symmetric difference $X \Delta Y = \{x, y\}$ such that $x \in X$, $y \in Y$ and $xy \in E(G)$. The size of

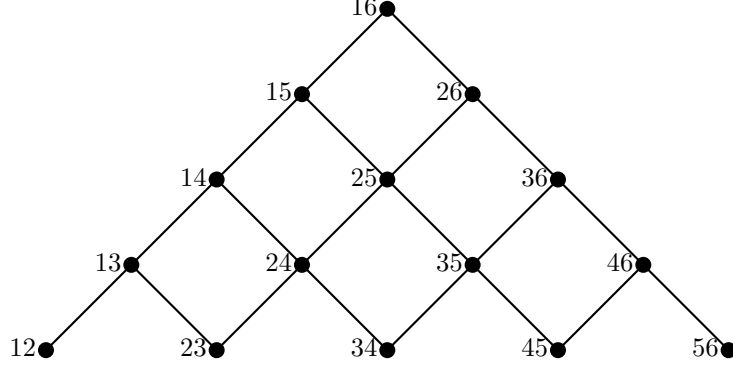


Figure 1: The 2-token graph of the path of order 6.

$F_k(G)$ is $\binom{n-2}{k-1}|E(G)|$, see [10]. An example of a 2-token graph is showed in Figure 1, which is the $F_2(P_6)$.

The 2-token graph $F_2(K_n)$ of the complete graph K_n is the line graph $L(K_n)$ of the complete graph K_n because each pair of incident edges xz and zy has symmetric difference the set $\{x, y\}$ which is the edge xy of the complete graph. In general, the Johnson graph $J(n, k) \cong F_k(K_n)$ owing to the fact that it is the graph whose vertices are the k -subsets of an n -set, where two such subsets X and Y are adjacent whenever $|X \cap Y| = k - 1$.

In [8], the authors remark that the 2-token graph $F_2(P_n)$ of the path graph P_n of n vertices is planar of girth at least 4 for every n .

Now, we prove that an edge-partition of a graph G induces an edge partition of $F_k(G)$.

Lemma 2.1. *Let G be a non empty graph and $P = \{E_1, \dots, E_l\}$ an edge-partition of G . Then the set $\{E'_1, \dots, E'_l\}$ is an edge-partition of $F_k(G)$ where $E'_i = E(F_k(G[E_i]))$ for all $i \in \{1, \dots, l\}$.*

Proof. Let XY be an edge of $F_k(G)$, that is, X and Y are k -subsets of $V(G)$ such that for some $x \in X$ and some $y \in Y$, the symmetric difference of X and Y is $\{x, y\}$, and xy is an edge of G . Let j the unique index in the set $\{1, \dots, l\}$ such that $xy \in E_j$. Thus xy is an edge of $G[E_j]$ and in consequence XY is an edge of $F_k(G[E_j]) = E'_j$. Then $XY \in E'_j$. Moreover, if $XY \in E'_i$ for some $i \in \{1, \dots, l\}$, then $xy \in E(G[E_i]) = E_i$. But $P = \{E_1, \dots, E_l\}$ is an edge-partition of G , and $xy \in E_j$, so $i = j$. Therefore each edge of $F_k(G)$ is in a unique element of $\{E'_1, \dots, E'_l\}$. In order to guarantee that every E'_i is a non empty set, we need that G has order at least $k + 1$. In that case, if $xy \in E_i$ and $U = \{g_1, \dots, g_{k-1}\} \subseteq V(G) \setminus \{x, y\}$, then $X' = U \cup \{x\}$ and $Y' = U \cup \{y\}$ are two k -subsets of $G[E_i]$ such that its symmetric difference is $\{x, y\}$, and then $E'_i \neq \emptyset$, because $X'Y' \in E'_i$. \square

3 Determining $\theta(4, L(K_n))$ for n even

A planar graph of n vertices and girth at least 4 has at most $2(n-2)$ edges for $n \geq 4$ and at most $n-1$, otherwise. In consequence, the 4-girth-thickness $\theta(4, G)$ of a graph G is at least $\left\lceil \frac{|E(G)|}{2(n-2)} \right\rceil$ for $n \geq 4$ and at least $\left\lceil \frac{|E(G)|}{n-1} \right\rceil$, otherwise.

Therefore we have the following theorem.

Theorem 3.1. *If G contains a factorization into k Hamiltonian paths, then $\theta(4, F_2(G)) = k$.*

Proof. For $G = K_2$ or $G = P_3$, it is easy to check that $\theta(4, F_2(G)) = 1$. Assume that G is a graph of order $n \geq 4$ containing a factorization into Hamiltonian paths. Then G has size $e = (n-1)k \leq \binom{n}{2}$, then $k \leq n/2$ and

$$k < n/2 + 1 + 1/(n-3).$$

Since, the 2-token graph $F_2(G)$ has order $\binom{n}{2}$ and size $(n-2)(n-1)k$, it follows that

$$\theta(4, F_2(G)) \geq \left\lceil \frac{(n-2)(n-1)k}{2\left(\binom{n}{2} - 2\right)} \right\rceil = \left\lceil k - \frac{2nk - 6k}{n^2 - n - 4} \right\rceil.$$

Because $k < \frac{n}{2} + 1 + \frac{1}{n-3} = \frac{n^2 - n - 4}{2n - 6}$ then

$$0 < \frac{k(2n-6)}{n^2 - n - 4} < 1$$

and we have

$$\theta(4, F_2(G)) \geq k.$$

By Lemma 2.1, the partition of k Hamiltonian paths $\{G_1, \dots, G_k\}$ of G induces a partition of $F_2(G)$ into k planar subgraphs of girth at least 4, $\{F_2(G_1), \dots, F_2(G_k)\}$ and the result follows. \square

We have the following corollaries.

Corollary 3.2. *If n is even then $\theta(4, F_2(K_{n-1, n})) = n/2$.*

Corollary 3.3. *If n is even then $\theta(4, L(K_n)) = n/2$.*

4 $\theta_S(4, L(K_n))$ when S is the projective plane

Although the problem of finding the minimum number of planar graphs of girth at least 4 into which the line graph of the complete graph can be decomposed remains partially

solved, the corresponding problem can be solved for the surface called the projective plane. A similar proof provide the solution.

On one hand, a maximal graph of order n and girth at least 4 embeddable in the projective plane S has size at most $2n - 2$. On the other hand, since the 2-token graph of a cycle is a graph embeddable in S with girth 4, see Figure 2 for a example, we can give the following theorem.

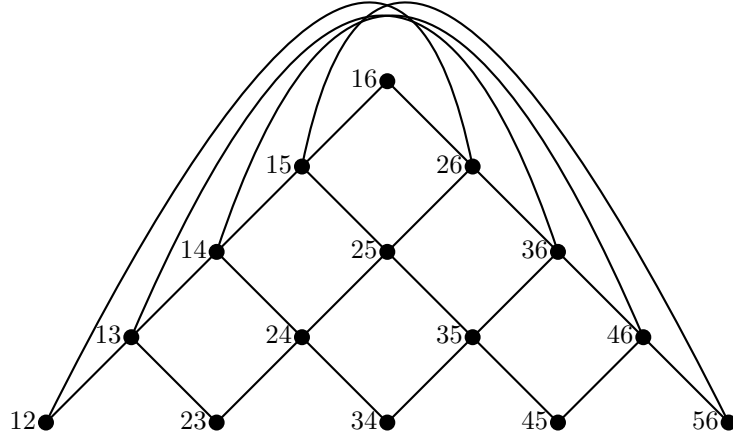


Figure 2: The 2-token graph of the cycle of order 6.

Theorem 4.1. *If G is a graph of order $n \geq 4$ and contains a factorization into k Hamiltonian cycles, then $\theta_S(4, F_2(G)) = k$ when S is the projective plane.*

Proof. Let G be a graph of order $n \geq 4$ containing a Hamiltonian-factorization, that is, a factorization into Hamiltonian cycles. Then G has size $e = nk \leq \binom{n}{2}$, then $k \leq (n - 1)/2$ and

$$k < n + 1 + 2/(n - 2).$$

Since, the 2-token graph $F_2(G)$ has order $\binom{n}{2}$ and size $(n - 2)nk$, it follows that

$$\theta_S(4, F_2(G)) \geq \left\lceil \frac{(n - 2)nk}{2\binom{n}{2} - 2} \right\rceil = \left\lceil k - \frac{nk - 2k}{n^2 - n - 2} \right\rceil.$$

Because $k < n + 1 + \frac{2}{n-2} = \frac{n^2 - n - 2}{n-2}$ then

$$0 < \frac{k(n - 2)}{n^2 - n - 2} < 1$$

and we have

$$\theta_S(4, F_2(G)) \geq k.$$

By Lemma 2.1, the partition of k Hamiltonian cycles $\{G_1, \dots, G_k\}$ of G induces a partition of $F_2(G)$ into k planar subgraphs of girth at least 4 embeddable in S , $\{F_2(G_1), \dots, F_2(G_k)\}$ and the result follows. \square

We have the following corollaries.

Corollary 4.2. *If n is even then $\theta_S(4, F_2(K_{n,n})) = n/2$.*

Corollary 4.3. *For all n , we have that $\theta_S(4, L(K_n)) = \lfloor \frac{n}{2} \rfloor$.*

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