MASS CONCENTRATION AND CHARACTERIZATION OF FINITE TIME BLOW-UP SOLUTIONS FOR THE NONLINEAR SCHRODINGER EQUATION ¨ WITH INVERSE-SQUARE POTENTIAL

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ABSTRACT. We consider the L^2 -critical NLS with inverse-square potential

 $i\partial_t u + \Delta u + c|x|^{-2}u = -|u|^{\frac{4}{d}}u, \quad u(0) = u_0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^d,$

where $d \geq 3$ and $c \neq 0$ satisfies $c < \lambda(d) := \left(\frac{d-2}{2}\right)^2$. We extend the mass concentration of finite time blow-up solutions established by the first author in [\[2\]](#page-15-0) to $c < \lambda(d)$. Using the profile decomposition, we give a short and simple proof of a limiting profile theorem that yields the same characterization of finite time blow-up solutions with minimal mass obtained by Csobo-Genoud in [\[7\]](#page-15-1). We also extend the characterization obtained by Csobo-Genoud to $c < \lambda(d)$.

1. Introduction

Consider the Cauchy problem for the focusing L^2 -critical nonlinear Schrödinger equation with inversesquare potential

$$
\begin{cases}\ni\partial_t u + \Delta u + c|x|^{-2}u &= -|u|^{\frac{4}{d}}u, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^d, \\
u(0) &= u_0,\n\end{cases}
$$
\n(1.1)

where $d \geq 3$, $u : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{C}$, $u_0 : \mathbb{R}^d \to \mathbb{C}$ and $c \neq 0$ satisfies $c < \lambda(d) := \left(\frac{d-2}{2}\right)^2$. The Schrödinger equation with inverse-square potential appears in a variety of physical settings, such as in quantum field equations or black hole solutions of the Einstein's equations (see e.g. $[5]$ or $[10]$). The study of nonlinear Schrödinger equation with inverse-square potential and power-type nonlinearity has attracted a lot of interests in the last several years (see e.g. [\[10,](#page-15-3) [4,](#page-15-4) [19,](#page-15-5) [23,](#page-15-6) [20,](#page-15-7) [12,](#page-15-8) [13,](#page-15-9) [11,](#page-15-10) [14,](#page-15-11) [7,](#page-15-1) [8,](#page-15-12) [2\]](#page-15-0) and references therein).

Denote P_c the self-adjoint extension of $-\Delta - c|x|^{-2}$. It is known (see e.g. [\[10\]](#page-15-3)) that in the range $\lambda(d) - 1 < c < \lambda(d)$, the extension is not unique. In this case, we do make a choice among possible extensions, such as Friedrichs extension. The restriction $c < \lambda(d)$ comes from the sharp Hardy inequality

$$
\lambda(d) \int |x|^{-2} |f(x)|^2 dx \le \int |\nabla f(x)|^2 dx, \quad \forall f \in H^1,
$$
\n(1.2)

which ensures that P_c is a positive operator. We define the homogeneous Sobolev space \dot{H}_c^1 as a completion of $C_0^{\infty}(\mathbb{R}^d \setminus \{0\})$ under the norm

$$
||f||_{\dot{H}_c^1} := ||\sqrt{P_c}f||_{L^2} = \left(\int |\nabla f(x)|^2 - c|x|^{-2}|f(x)|^2 dx\right)^{1/2}.
$$
 (1.3)

The sharp Hardy inequality implies that for $c < \lambda(d)$, $||f||_{\dot{H}_c^1} \sim ||f||_{\dot{H}^1}$, and the homogeneous Sobolev space \dot{H}_c^1 is equivalent to the usual homogenous Sobolev space \dot{H}^1 .

The local well-posedness for (1*.*[1\)](#page-0-0) was established by Okazawa-Suzuki-Yokota [\[19\]](#page-15-5).

Theorem 1.1 (Local well-posedness [\[19\]](#page-15-5)). Let $d \geq 3$ and $c \neq 0$ be such that $c < \lambda(d)$. Then for any $u_0 \in H^1$, there exists $T \in (0, +\infty]$ and a maximal solution $u \in C([0, T), H^1)$ $u \in C([0, T), H^1)$ of (1.1) *. The maximal time of existence satisfies either* $T = +\infty$ *or* $T < +\infty$ *and* $\lim_{t \uparrow T} ||\nabla u(t)||_{L^2} = \infty$ *. Moreover, the local solution enjoys the conservation of mass and energy*

$$
M(u(t)) = \int |u(t,x)|^2 dx = M(u_0),
$$

\n
$$
E(u(t)) = \frac{1}{2} \int |\nabla u(t,x)|^2 dx - \frac{c}{2} \int |x|^{-2} |u(t,x)|^2 dx - \frac{d}{2d+4} \int |u(t,x)|^{\frac{4}{d}+2} dx,
$$

for any $t \in [0, T)$ *.*

We refer the reader to [\[19,](#page-15-5) Theorem 5.1] for the proof of the above local well-posedness result. Note that the existence of local solutions is based on a refined energy method, and the uniqueness follows from Strichartz estimates which are shown by Burq-Planchon-Stalker-Zadeh in [\[4\]](#page-15-4).

The main purpose of this paper is to study dynamical properties of blow-up solutions to (1.1) , including mass concentration, limiting profile and the characterization of finite time blow-up solutions with minimal mass. Such phenomena were extensively studied in the last decades especially for the mass-critical nonlinear Schrödinger equation (NLS) (i.e. $c = 0$ in [\(1](#page-0-0).1)). For the mass-critical NLS, the mass concentration was first established by Tsutsumi [\[17\]](#page-15-13) and Merle-Tsutsumi [\[18\]](#page-15-14). The limiting profile of finite time blow-up solutions was obtained by Weinstein in [\[22\]](#page-15-15). The characterization of finite time blow-up solutions with minimal mass was obtained by Merle in [\[16\]](#page-15-16). Based on a refined compactness lemma, Hmidi-Keraani in [\[9\]](#page-15-17) gave much simpler proofs of all the aforementioned results. It is their approach that we are going to pursue in the sequel.

Following the idea of Hmidi-Keraani in [\[9\]](#page-15-17), to study dynamical properties of finite time blow-up solutions for (1.1) (1.1) , we first need the profile decomposition of bounded sequences in $H¹$ related to (1.1) . This profile decomposition was proved recently by the first author in [\[2\]](#page-15-0). Thanks to this profile decomposition, a refined version of compactness lemma related to (1*.*[1\)](#page-0-0) was shown. With the help of this refined compactness lemma, we are able to study dynamical properties of finite time blow-up solutions for (1*.*[1\)](#page-0-0).

The mass concentration for non-radial blow-up solutions was established by the first author in [\[2\]](#page-15-0) for the case $0 < c < \lambda(d)$. Here we extend this result to $c < \lambda(d)$. We also give an improvement of the mass concentration for radial blow-up solutions in the case *c <* 0. This improvement is due to the sharp radial Gagliardo-Nirenberg inequality related to (1*.*[1\)](#page-0-0) for *c <* 0. More precisely, we prove the following result.

Theorem 1.2 (Mass concentration). Let $d \geq 3$, $c \neq 0$ and $c < \lambda(d)$. Let $u_0 \in H^1$ be such that the *corresponding solution u to* (1.[1\)](#page-0-0) *blows up at finite time* $0 < T < +\infty$ *. Let* $a(t) > 0$ *be such that*

$$
a(t)\|\nabla u(t)\|_{L^2} \to \infty,\tag{1.4}
$$

 $as \ t \uparrow T$. Then there exists $x(t) \in \mathbb{R}^d$ such that

$$
\liminf_{t \uparrow T} \int_{|x - x(t)| \le a(t)} |u(t, x)|^2 dx \ge ||Q_{\overline{c}}||_{L^2}^2,
$$
\n(1.5)

where $\overline{c} = \max\{c, 0\}$ *. Moreover, in the case* $c < 0$ *, if we assume in addition that* u_0 *is radial, then* (1.[5\)](#page-1-0) *can be improved to*

$$
\liminf_{t \uparrow T} \int_{|x| \le a(t)} |u(t,x)|^2 dx \ge \|Q_{c, \text{rad}}\|_{L^2}^2.
$$
\n(1.6)

.

*Here Q^c and Qc,*rad *are given in Theorem* 2*.*[1](#page-2-0)*.*

Remark 1.3. • By using a standard argument of Merle-Raphaël [\[15\]](#page-15-18), we have the following blowup rate: if *u* is a solution to (1.[1\)](#page-0-0) blows up at finite time $0 < T < +\infty$, then there exists $C > 0$ such that

$$
\|\nabla u(t)\|_{L^2} > \frac{C}{\sqrt{T-t}}
$$

• Rewriting

we see

$$
\frac{1}{a(t)\|\nabla u(t)\|_{L^2}} = \frac{\sqrt{T-t}}{a(t)} \frac{1}{\sqrt{T-t}\|\nabla u(t)\|_{L^2}} < C\frac{\sqrt{T-t}}{a(t)},
$$

that any function $a(t) > 0$ satisfying $\frac{\sqrt{T-t}}{a(t)} \to 0$ as $t \to T$ fulfills (1.4).

The characterization of finite time blow-up solutions for (1*.*[1\)](#page-0-0) with minimal mass was recently estab-lished by Csobo-Genoud in [\[7\]](#page-15-1) in the case $0 < c < \lambda(d)$. They showed that up to symmetries of the equation, the only finite time blow-up solutions for (1*.*[1\)](#page-0-0) with minimal mass are the pseudo-conformal transformation of ground state standing waves. Note that since the uniqueness of ground states for (1*.*[1\)](#page-0-0) is not yet known, one needs to define properly a notion of ground states for (1*.*[1\)](#page-0-0). The proof of their result is based on the concentration-compactness lemma (see e.g. [\[6,](#page-15-19) Proposition 1.7.6]). The key point is the limiting profile result (see [\[7,](#page-15-1) Proposition 4, p.120]). In this paper, we aim to give a simple proof for the above result of Csobo-Genoud in the case $0 < c < \lambda(d)$. Our approach is based on the profile decomposition of [\[2\]](#page-15-0). This allows us to give a simple version of the limiting profile compared to the one of [\[7\]](#page-15-1). We also extend Csobo-Genoud's result to negative values of *c*. Since the sharp non-radial Gagliardo-Nirenberg inequality for $c < 0$ is never attained for $c < 0$. We need to restrict our attention only to finite time radial blow-up solutions. More precisely, we prove the following result.

Theorem 1.4 (Characterization of finite time blow-up solutions with minimal mass). • Let $d \geq 3$ α *and* $0 < c < \lambda(d)$. Let $u_0 \in H^1$ be such that $||u_0||_{L^2} = M_{gs}$. Suppose that the corresponding so*lution u to* [\(1](#page-0-0).1) *blows up at finite time* $0 < T < +\infty$ *. Then there exist* $Q \in \mathcal{G}, \theta \in \mathbb{R}$ *and* $\lambda > 0$

such that

$$
u_0(x) = e^{i\theta} e^{i\frac{\lambda^2}{T}} e^{-i\frac{|x|^2}{4T}} \left(\frac{\lambda}{T}\right)^{\frac{d}{2}} Q\left(\frac{\lambda x}{T}\right).
$$
 (1.7)

In particular, $u(t, x) = S_{Q, T, \theta, \lambda}(t, x)$ *, where*

$$
S_{Q,T,\theta,\lambda}(t,x) := e^{i\theta} e^{i\frac{\lambda^2}{T-t}} e^{-i\frac{|x|^2}{4(T-t)}} \left(\frac{\lambda}{T-t}\right)^{\frac{d}{2}} Q\left(\frac{\lambda x}{T-t}\right).
$$

• Let $d \geq 3$ and $c < 0$. Let $u_0 \in H_{rad}^1$ be such that $||u_0||_{L^2} = M_{gs,rad}$. Suppose that the corresponding *solution u to* [\(1](#page-0-0).1) *blows up at finite time* $0 < T < +\infty$ *. Then there exist* $Q_{\text{rad}} \in \mathcal{G}_{\text{rad}}$, $\vartheta \in \mathbb{R}$ and *ρ >* 0 *such that*

$$
u_0(x) = e^{i\vartheta} e^{i\frac{\rho^2}{T}} e^{-i\frac{|x|^2}{4T}} \left(\frac{\rho}{T}\right)^{\frac{d}{2}} Q_{\text{rad}} \left(\frac{\rho x}{T}\right).
$$
 (1.8)

In particular, $u(t, x) = S_{Q_{rad}, T, \vartheta, \rho}(t, x)$ *, where*

$$
S_{Q_{\text{rad}},T,\vartheta,\rho}(t,x) := e^{i\vartheta} e^{i\frac{\rho^2}{T-t}} e^{-i\frac{|x|^2}{4(T-t)}} \left(\frac{\rho}{T-t}\right)^{\frac{d}{2}} Q_{\text{rad}}\left(\frac{\rho x}{T-t}\right).
$$

We refer the reader to Section [4](#page-8-0) for the notations M_{gs} , \mathcal{G} , $M_{gs,rad}$ and \mathcal{G}_{rad} .

The paper is organized as follows. In Section 2, we recall sharp Gagliardo-Nirenberg inequalities and the compactness lemma related to [\(1](#page-0-0)*.*1). In Section 3, we give the proof of the mass concentration given in Theorem 1*.*[2.](#page-1-2) In Section 4, we prove a simple version of the limiting profile result compared to the one in [\[7\]](#page-15-1). Using this limiting profile, we give the proof of the characterization of finite time blow-up solutions with minimal mass given in Theorem 1*.*4.

2. Preliminaries

2.1. **Sharp Gagliardo-Nirenberg inequalities.** In this subsection, we recall sharp Gagliardo-Nirenberg inequalities related to [\(1](#page-0-0)*.*1). Let us start with the sharp non-radial Gagliardo-Nirenberg inequality

$$
||u||_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2} \leq C_{\text{GN}}(c)||u||_{L^2}^{\frac{4}{d}}||u||_{\dot{H}_c^1}^2,
$$
\n(2.1)

where the sharp constant $C_{GN}(c)$ is defined by

$$
C_{GN}(c) := \sup \{ J_c(u) : u \in H^1 \setminus \{0\} \}.
$$

Here $J_c(u)$ is the Weinstein functional

$$
J_c(u) := \|u\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2} \div \left[\|u\|_{L^2}^{\frac{4}{d}} \|u\|_{\dot{H}_c}^2 \right]. \tag{2.2}
$$

We also recall the sharp radial Gagliardo-Nirenberg inequality

$$
||u||_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2} \leq C_{\text{GN}}(c, \text{rad})||u||_{L^2}^{\frac{4}{d}}||u||_{\dot{H}_c^1}^2,
$$
\n(2.3)

where the sharp constant $C_{GN}(c, rad)$ is defined by

$$
C_{\text{GN}}(c,\text{rad}) := \sup \left\{ J_c(u) : u \in H^1_{\text{rad}} \backslash \{0\} \right\},\,
$$

where H_{rad}^1 is the space of radial H^1 functions. When $c = 0$, Weinstein in [\[21\]](#page-15-20) proved that the sharp constant $C_{GN}(0)$ is attained by the fuction Q_0 which is the unique (up to symmetries) positive radial solution of

$$
\Delta Q_0 - Q_0 + |Q_0|^{\frac{4}{d}} Q_0 = 0. \tag{2.4}
$$

We have the following result (see $[11]$ and also $[8]$).

Theorem 2.1 (Sharp Gagliardo-Nirenberg inequalities). Let $d > 3$ and $c \neq 0$ be such that $c < \lambda(d)$. *Then* $C_{GN}(c) \in (0, \infty)$ *and*

• *if* $0 < c < \lambda(d)$, then the equality in (2.[1\)](#page-2-1) *is attained by a function* $Q_c \in H^1$ *which is a positive radial solution to the elliptic equation*

$$
\Delta Q_c + c|x|^{-2}Q_c - Q_c + |Q_c|^{\frac{4}{d}}Q_c = 0.
$$
\n(2.5)

• *if* $c < 0$, then $C_{GN}(c) = C_{GN}(0)$ and the equality in (2.1) (2.1) is never attained. However, the equality *in* (2.[3\)](#page-2-2) *is attained by a function* $Q_{c,rad} \in H_{rad}^1$ *which is a positive solution to the elliptic equation*

$$
\Delta Q_{c, \text{rad}} + c|x|^{-2}Q_{c, \text{rad}} - Q_{c, \text{rad}} + |Q_{c, \text{rad}}|^{\frac{4}{d}}Q_{c, \text{rad}} = 0. \tag{2.6}
$$

We refer the reader to [\[11,](#page-15-10) Theorem 3.1] (see also [\[8,](#page-15-12) Theorem 4.1]) for the proof of the above result.

Remark [2](#page-2-0).2. • In the case $0 < c < \lambda(d)$, Theorem 2.1 shows that there exist positive radial solutions to the elliptic equation (2.[5\)](#page-2-3). However, unlike the case $c = 0$, the uniqueness up to symmetries of these solutions is not known yet. We also have the following Pohozaev's identities [1](#page-3-0) :

$$
\|Q_c\|_{L^2}^2=\frac{2}{d}\|Q_c\|_{\dot{H}_c^1}^2=\frac{2}{d+2}\|Q_c\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2}.
$$

In particular,

$$
C_{\text{GN}}(c) = \frac{d+2}{d} \frac{1}{\|Q_c\|_{L^2}^{\frac{4}{d}}}.
$$

• Since the above identities still hold true for $c = 0$, we get from Theorem [2](#page-2-0).1 that for any $c < \lambda(d)$,

$$
C_{\text{GN}}(c) = \frac{d+2}{d} \frac{1}{\|Q_{\overline{c}}\|_{L^2}^{\frac{4}{d}}},\tag{2.7}
$$

where $\overline{c} = \max\{c, 0\}.$

• In the case $c < 0$, we also have

$$
||Q_{c,\text{rad}}||_{L^2}^2 = \frac{2}{d}||Q_{c,\text{rad}}||_{\dot{H}_c^1}^2 = \frac{2}{d+2}||Q_{c,\text{rad}}||_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2}.
$$

In particular,

$$
C_{\rm GN}(c, \text{rad}) = \frac{d+2}{d} \frac{1}{\|Q_{c, \text{rad}}\|_{L^2}^{\frac{4}{d}}}.
$$
\n(2.8)

Note that since $C_{GN}(c, rad) < C_{GN}(c)$, we see that for any $c < 0$,

$$
||Q_0||_{L^2} < ||Q_{c,\text{rad}}||_{L^2}.
$$

2.2. **Profile decomposition.** In this subsection, we recall the profile decomposition related to the nonlinear Schrödinger equation with inverse-square potential. This profile decomposition was established recently by the first author in [\[2\]](#page-15-0) for $0 < c < \lambda(d)$. There is no difficulty to extend this result for negative values of *c*.

Proposition 2.3 (Profile decomposition). Let $d \geq 3$ and $c < \lambda(d)$. Let $(v_n)_{n \geq 1}$ be a bounded sequence *in* H^1 . Then there exist a subsequence still denoted by $(v_n)_{n\geq 1}$, a family $(x_n^j)_{n\geq 1}$ of sequences in \mathbb{R}^d and *a* sequence $(V^j)_{j \geq 1}$ of H^1 -functions such that

i) *for every* $i \neq k$ *,*

$$
|x_n^j - x_n^k| \to \infty,\tag{2.9}
$$

 $as n \rightarrow \infty$ *;* ii) *for every* $l \geq 1$ *and every* $x \in \mathbb{R}^d$, *we have*

$$
v_n(x) = \sum_{j=1}^{l} V^j(x - x_n^j) + v_n^l(x),
$$

with

$$
\limsup_{n \to \infty} ||v_n^l||_{L^q} \to 0,
$$
\n(2.10)

 $as l \rightarrow \infty$ *for every* $2 < q < \frac{2d}{d-2}$. *Moreover, for every* $l \geq 1$ *,*

$$
||v_n||_{L^2}^2 = \sum_{j=1}^l ||V^j||_{L^2}^2 + ||v_n||_{L^2}^2 + o_n(1),
$$
\n(2.11)

$$
||v_n||_{\dot{H}_c^1}^2 = \sum_{j=1}^l ||V^j(\cdot - x_n^j)||_{\dot{H}_c^1} + ||v_n^l||_{\dot{H}_c^1} + o_n(1),
$$
\n(2.12)

 $as n \rightarrow \infty$.

¹These identities can be proved rigorously by using the technique of [\[3,](#page-15-21) Proposition 1]: first, considering Pohozaev's identities in $\Omega_{r,R} := \{x : r < |x| < R\}$, and then showing the boundary term (on $\partial \Omega_{r,R}$) to converge to 0 as $r \to 0$ and $R \rightarrow +\infty$.

Proof. For reader's convenience, we recall some details. Since H^1 is a Hilbert space, we denote $\Omega(v_n)$ the set of functions obtained as weak limits of sequences of the translated $v_n(\cdot+x_n)$ with $(x_n)_{n\geq 1}$ a sequence in R *d* . Denote

$$
\eta(v_n) := \sup \{ ||v||_{L^2} + ||\nabla v||_{L^2} : v \in \Omega(v_n) \}.
$$

Clearly,

$$
\eta(v_n) \le \limsup_{n \to \infty} ||v_n||_{L^2} + ||\nabla v_n||_{L^2}.
$$

We shall prove that there exist a sequence $(V^j)_{j\geq 1}$ of $\Omega(v_n)$ and a family $(x_n^j)_{j\geq 1}$ of sequences in \mathbb{R}^d such that for every $k \neq j$,

$$
|x_n^k - x_n^j| \to \infty, \quad \text{as } n \to \infty,
$$

and up to a subsequence, the sequence $(v_n)_{n\geq 1}$ can be written, for every $l \geq 1$ and every $x \in \mathbb{R}^d$, as

$$
v_n(x) = \sum_{j=1}^{l} V^j(x - x_n^j) + v_n^l(x),
$$

with $\eta(v_n^l) \to 0$ as $l \to \infty$. Moreover, the identities (2.[11\)](#page-3-1) and (2.[12\)](#page-3-2) hold as $n \to \infty$.

Indeed, if $\eta(v_n) = 0$, then we can take $V^j = 0$ for all $j \ge 1$. Otherwise we choose $V^1 \in \Omega(v_n)$ such that

$$
||V^1||_{L^2} + ||\nabla V^1||_{L^2} \ge \frac{1}{2}\eta(v_n) > 0.
$$

By the definition of $\Omega(v_n)$, there exists a sequence $(x_n^1)_{n\geq 1} \subset \mathbb{R}^d$ such that up to a subsequence,

$$
v_n(\cdot + x_n^1) \rightharpoonup V^1
$$
 weakly in H^1 .

Set $v_n^1(x) := v_n(x) - V^1(x - x_n^1)$. We see that $v_n^1(\cdot + x_n^1) \to 0$ weakly in H^1 and thus $||v_n||_{L^2}^2 = ||V^1||_{L^2}^2 + ||v_n^1||_{L^2}^2 + o_n(1),$

$$
\|\nabla v_n\|_{L^2}^2 = \|\nabla V^1\|_{L^2}^2 + \|\nabla v_n^1\|_{L^2}^2 + o_n(1),
$$

as $n \to \infty$. We next show that

$$
\int |x|^{-2} |v_n(x)|^2 dx = \int |x|^{-2} |V^1(x - x_n^1)|^2 dx + \int |x|^{-2} |v_n^1(x)|^2 dx + o_n(1),
$$

as $n \to \infty$. Using the fact

$$
|v_n(x)|^2 = |V^1(x - x_n^1)|^2 + |v_n^1(x)|^2 + 2\text{Re}(V^1(x - x_n^1)\overline{v}_n^1(x)),
$$

it suffices to show that

$$
\int |x|^{-2} V^{1}(x - x_{n}^{1}) \overline{v}_{n}^{1}(x) dx \to 0, \qquad (2.13)
$$

as $n \to \infty$. Without loss of generality, we may assume that V^1 is continuous and compactly supported. Moreover, up to a subsequence, we assume that $|x_n^1| \to \{0, \infty\}$ as $n \to \infty$.

• Case 1: $|x_n^1| \to \infty$. Since $|x_n^1| \to \infty$ as $n \to \infty$, we see that $|x + x_n^1| \ge 1$ for all $x \in \text{supp}(V^1)$ and all $n \geq n_0$ with n_0 large enough. Therefore, for $n \geq n_0$,

$$
\left| \int |x|^{-2} V^1(x - x_n^1) \overline{v}_n^1(x) dx \right| = \int_{\text{supp}(V^1)} |x + x_n^1|^{-2} |V^1(x)| |v_n^1(x + x_n^1)| dx
$$

$$
\leq \int |V^1(x)| |v_n^1(x + x_n^1)| dx.
$$

Since $v_n^1(\cdot + x_n^1) \to 0$ in H^1 as $n \to \infty$, the last term tends to zero as $n \to \infty$. • Case 2: $|x_n^1| \to 0$. Let $\epsilon > 0$. For $\eta > 0$ small to be chosen later, we split

$$
\int_{\text{supp}(V^1)} |x + x_n^1|^{-2} |V^1(x)| |v_n^1(x + x_n^1)| dx = \int_{B(0,\eta)} |x + x_n^1|^{-2} |V^1(x)| |v_n^1(x + x_n^1)| dx \tag{2.14}
$$
\n
$$
+ \int_{\text{supp}(V^1) \backslash B(0,\eta)} |x + x_n^1|^{-2} |V^1(x)| |v_n^1(x + x_n^1)| dx.
$$

 $\text{Since } |x_n^1| \to 0$, we see that for all $n \geq n_1$ with n_1 large enough, $|x+x_n^1| \geq \eta/2$ for all $x \in \text{supp}(V^1) \setminus B(0, \eta)$. Thus

$$
\int_{\text{supp}(V^1)\backslash B(0,\eta)} |x+x_n^1|^{-2}|V^1(x)||v_n^1(x+x_n^1)|dx \lesssim \eta^{-2} \int |V^1(x)||v_n^1(x+x_n^1)|dx.
$$

We next learn from the fact $v_n^1(\cdot + x_n^1) \to 0$ in H^1 as $n \to \infty$ that for $n \ge n_1$ (increasing n_1 if necessary),

$$
\int_{\text{supp}(V^1)\backslash B(0,\eta)} |x+x_n^1|^{-2} |V^1(x)| |v_n^1(x+x_n^1)| dx < \frac{\epsilon}{2}.
$$
\n(2.15)

We next use the Cauchy-Schwarz inequality, Hardy's inequality (1.2) and the fact $(v_n^1)_{n\geq 1}$ is bounded in H^1 to get

$$
\int_{B(0,\eta)} |x + x_n^1|^{-2} |V^1(x)| |v_n^1(x + x_n^1)| dx \le \left(\int_{B(0,\eta)} |x + x_n^1|^{-2} |V^1(x)|^2 dx \right)^{1/2} \times \left(\int |x + x_n^1|^{-2} |v_n^1(x + x_n^1)|^2 dx \right)^{1/2} \le \left(\int_{B(0,\eta)} |x + x_n^1|^{-2} |V^1(x)|^2 dx \right)^{1/2} ||\nabla v_n^1||_{L^2} \le \left(\int_{B(0,\eta)} |x + x_n^1|^{-2} |V^1(x)|^2 dx \right)^{1/2} . \tag{2.16}
$$

Since $|V^1(x)|^2$ is continuous on the compact set $\overline{B}(0,3\eta)$, hence it is uniformly continuous on $\overline{B}(0,3\eta)$. Thus, there exists $\delta > 0$ such that for all $x, y \in \overline{B}(0, 3\eta)$ satisfying $|x - y| < \delta$, we have

$$
||V^1(x)|^2 - |V^1(y)|^2| < \frac{\epsilon^2}{8K(\eta)},
$$

where

$$
K(\eta) := \int_{B(0,2\eta)} |x|^{-2} dx = \frac{(2\eta)^{d-2}}{d-2} |\mathbb{S}^{d-1}|.
$$

Note that we can take $\delta \in (0, \eta)$. Since $|x_n^1| \to 0$, we have for $n \geq n_2$ with n_2 large enough that

$$
|x_n^1| < \delta < \eta.
$$

This implies that for all $x \in B(0, 2\eta)$ and all $n \geq n_2$,

$$
||V^1(x - x_n^1)|^2 - |V^1(x)|^2| < \frac{\epsilon^2}{8K(\eta)}.\tag{2.17}
$$

Since $B(x_n^1, \eta) \subset B(0, 2\eta)$ for all $n \ge n_2$, we use (2.17) (2.17) to get

$$
\int_{B(0,\eta)} |x + x_n^1|^{-2} |V^1(x)|^2 dx = \int_{B(x_n^1,\eta)} |x|^{-2} |V^1(x - x_n^1)|^2 dx
$$

\n
$$
\leq \int_{B(0,2\eta)} |x|^{-2} |V^1(x)|^2 dx + \int_{B(0,2\eta)} |x|^{-2} \frac{\epsilon^2}{8K(\eta)} dx
$$

\n
$$
= \int_{B(0,2\eta)} |x|^{-2} |V^1(x)|^2 dx + \frac{\epsilon^2}{8}.
$$

Using Hardy's inequality (1.2) (1.2) with $V^1 \in H^1$, the dominated convergence allows to choose $\eta > 0$ small enough so that

$$
\int_{B(0,2\eta)} |x|^{-2} |V^1(x)|^2 dx < \frac{\epsilon^2}{8}.
$$

We thus obtain

$$
\int_{B(0,\eta)} |x + x_n^1|^{-2} |V^1(x)|^2 dx < \frac{\epsilon^2}{4},
$$

which together with (2.16) (2.16) yield for $n \geq n_2$,

$$
\int_{B(0,\eta)} |x + x_n^1|^{-2} |V^1(x)| |v_n^1(x + x_n^1)| dx < \frac{\epsilon}{2}.
$$
\n(2.18)

Combining (2.[14\)](#page-4-0), (2.[15\)](#page-5-2) and (2.[18\)](#page-5-3), we have for $n \ge \max\{n_1, n_2\}$,

$$
\int_{\text{supp}(V^1)} |x + x_n^1|^{-2} |V^1(x)| |v_n^1(x + x_n^1)| dx < \epsilon.
$$

Therefore, (2*.*[13\)](#page-4-1) is proved in both cases.

We now replace $(v_n)_{n\geq 1}$ by $(v_n^1)_{n\geq 1}$ and repeat the same process. If $\eta(v_n^1)=0$, then we choose $V^j=0$ for all $j \ge 2$. Otherwise there exist $\overline{V}^2 \in \Omega(v_n^1)$ and a sequence $(x_n^2)_{n \ge 1} \subset \mathbb{R}^d$ such that

$$
||V^2||_{L^2} + ||\nabla V^2||_{L^2} \ge \frac{1}{2}\eta(v_n^1) > 0,
$$

and

$$
v_n^1(\cdot + x_n^2) \rightharpoonup V^2
$$
 weakly in H^1 .
Set $v_n^2(x) := v_n^1(x) - V^2(x - x_n^2)$. We thus have $v_n^2(\cdot + x_n^2) \rightharpoonup 0$ weakly in H^1 and

$$
||v_n^1||_{L^2}^2 = ||V^2||_{L^2}^2 + ||v_n^2||_{L^2}^2 + o_n(1),
$$

$$
||v_n^1||_{\dot{H}_c^1}^2 = ||V^2||_{\dot{H}_c^1}^2 + ||v_n^2||_{\dot{H}_c^1}^2 + o_n(1),
$$

as $n \to \infty$. We claim that

$$
|x_n^1 - x_n^2| \to \infty, \quad \text{as } n \to \infty.
$$

In fact, if it is not true, then up to a subsequence, $x_n^1 - x_n^2 \to x_0$ as $n \to \infty$ for some $x_0 \in \mathbb{R}^d$. Since $v_n^1(x + x_n^2) = v_n^1(x + (x_n^2 - x_n^1) + x_n^1),$

and $v_n^1(\cdot + x_n^1)$ converges weakly to 0, we see that $V^2 = 0$. This implies that $\eta(v_n^1) = 0$ and it is a contradiction. An argument of iteration and orthogonal extraction allows us to construct the family $(x_n^j)_{j\geq 1}$ of sequences in \mathbb{R}^d and the sequence $(V^j)_{j\geq 1}$ of H^1 functions satisfying the claim above. Furthermore, the convergence of the series $\sum_{j\geq 1}^{\infty} ||V^j||^2_{L^2} + ||\nabla V^j||^2_{L^2}$ implies that

$$
||V^j||_{L^2}^2 + ||\nabla V^j||_{L^2}^2 \to 0, \text{ as } j \to \infty.
$$

By construction, we have

$$
\eta(v_n^j) \le 2 \left(\|V^{j+1}\|_{L^2} + \|\nabla V^{j+1}\|_{L^2} \right),\,
$$

which proves that $\eta(v_n^j) \to 0$ as $j \to \infty$. The proof of [2](#page-3-3).3 follows by the same lines as in [\[9,](#page-15-17) Proposition 2.3]. We thus omit the details.

2.3. **Compactness lemma.** In this subsection, we recall a compactness lemma related to the nonlinear Schrödinger equation with inverse-square potential.

Lemma 2.4 (Compactness lemma). Let $d \geq 3$, $c \neq 0$ and $c < \lambda(d)$. Let $(v_n)_{n \geq 1}$ be a bounded sequence *in H*¹ *such that*

$$
\limsup_{n \to \infty} ||v_n||_{\dot{H}_c^1} \le M, \quad \limsup_{n \to \infty} ||v_n||_{L^{\frac{4}{d}+2}} \ge m.
$$
\n(2.19)

Then there exists $(x_n)_{n\geq 1}$ *in* \mathbb{R}^d *such that up to a subsequence,* $v_n(\cdot + x_n) \to V$ *weakly in* H^1 *for some* $V \in H^1$ satisfying

$$
||V||_{L^{2}}^{\frac{4}{d}} \geq \frac{d}{d+2} \frac{m^{\frac{4}{d}+2}}{M^{2}} ||Q_{\overline{c}}||_{L^{2}}^{\frac{4}{d}},
$$

where \overline{c} = max{*c,* 0} *and* Q_c *is given in Theorem* [2](#page-2-0)*.1. Moreover, in the case* $c < 0$ *, if we assume in addition* $(v_n)_{n\geq 1}$ *are radially symmetric, then up to a subsequence* $v_n \rightharpoonup V$ *weakly in* H^1 *for some* $V \in H_{\text{rad}}^1$ *satisfying*

$$
||V||_{L^{2}}^{\frac{4}{d}} \geq \frac{d}{d+2} \frac{m^{\frac{4}{d}+2}}{M^{2}} ||Q_{c,\text{rad}}||_{L^{2}}^{\frac{4}{d}},
$$

*where Qc,*rad *is also given in Theorem* [2](#page-2-0)*.*1*.*

Proof. In the case $c < \lambda(d)$ and v_n non-radial, the proof is given in [\[2,](#page-15-0) Lemma 5] using the profile decomposition, the sharp Gagliardo-Nirenberg inequality (2*.*[1\)](#page-2-1) and [\(2](#page-3-4)*.*7).

Let us now consider the case $c < 0$ and $(v_n)_{n \geq 1}$ a bouded sequence in H_{rad}^1 satisfying (2.[19\)](#page-6-0). Thanks to the fact

$$
H_{\text{rad}}^1 \hookrightarrow L^{\frac{4}{d}+2} \text{ compactly},
$$

we see that there exists $V \in H_{rad}^1$ such that up to a subsequence, $v_n \to V$ weakly in H^1 as well as strongly in $L^{\frac{4}{d}+2}$. In particular, we have from the second condition in (2.19) (2.19) that $m \leq ||V||_{L^{\frac{4}{d}+2}}$. By the sharp radial Gagliardo-Nirenberg inequality and (2*.*[8\)](#page-3-5), we have

$$
m^{\frac{4}{d}+2} \le \|V\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2} \le \frac{d+2}{d} \frac{1}{\|Q_{c,\text{rad}}\|_{L^2}^{\frac{4}{d}}}\|V\|_{L^2}^{\frac{4}{d}}\|V\|_{\dot{H}^1_c}^2.
$$

8 ABDELWAHAB BENSOUILAH AND VAN DUONG DINH

By the lower semi continuity of Hardy's functional, the first condition in (2*.*[19\)](#page-6-0) implies

$$
||V||_{\dot{H}_c^1} \le \limsup_{n \to \infty} ||v_n||_{\dot{H}_c^1} \le M.
$$

We thus obtain

$$
||V||_{L^{2}}^{\frac{4}{d}} \ge \frac{d}{d+2} \frac{m^{\frac{4}{d}+2}}{M^{2}} ||Q_{c,\text{rad}}||_{L^{2}}^{\frac{4}{d}}.
$$

The proof is complete.

3. Mass concentration

In this short section, we give the proof of the mass concentration given in Theorem 1*.*[2.](#page-1-2) *Proof of Theorem* [1](#page-1-2).2. The proof is similar to the one of [\[2,](#page-15-0) Theorem 1]. For the sake of completeness, we recall some details. Let $(t_n)_{n\geq 1}$ be a time sequence such that $t_n \uparrow T$ as $n \to \infty$. Set

$$
\lambda_n := \frac{\|Q_{\overline{c}}\|_{\dot{H}_c^1}}{\|u(t_n)\|_{\dot{H}_c^1}}, \quad v_n(x) := \lambda_n^{\frac{d}{2}} u(t_n, \lambda_n x).
$$

By the local well-posedness theory given in Theorem [1](#page-0-2).1 and the equivalence between \dot{H}_c^1 and \dot{H}^1 , we see that $\lambda_n \to 0$ as $n \to \infty$. Moreover, a direct computation combined with the conservation of mass and energy show

$$
||v_n||_{L^2} = ||u(t_n)||_{L^2} = ||u_0||_{L^2}, \quad ||v_n||_{\dot{H}_c^1} = \lambda_n ||u(t_n)||_{\dot{H}_c^1} = ||Q_{\overline{c}}||_{\dot{H}_c^1},
$$

and

$$
E(v_n) = \lambda_n^2 E(u(t_n)) = \lambda_n^2 E(u_0) \to 0,
$$

as $n \to \infty$. In particular,

$$
||v_n||_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2} \to \frac{d+2}{d} ||Q_{\overline{c}}||_{\dot{H}^1_c}^2,
$$

as $n \to \infty$. This implies in particular that $(v_n)_{n\geq 1}$ satisfies conditions of Lemma [2](#page-6-1).4 with

$$
m^{\frac{4}{d}+2} = \frac{d+2}{d} ||Q_{\overline{c}}||_{\dot{H}_c^1}^2, \quad M^2 = ||Q_{\overline{c}}||_{\dot{H}_c^1}^2.
$$

Therefore, there exist a sequence $(x_n)_{n\geq 1}$ in \mathbb{R}^d and $V \in H^1$ such that up to a subsequence

$$
v_n(\cdot + x_n) = \lambda_n^{\frac{d}{2}} u(t_n, \lambda_n \cdot + x_n) \rightharpoonup V \text{ weakly in } H^1,
$$

as $n \to \infty$ with $||V||_{L^2} \ge ||Q_{\overline{c}}||_{L^2}$. This implies for every $R > 0$,

$$
\liminf_{n \to \infty} \int_{|x| \le R} \lambda_n^d |u(t_n, \lambda_n x + x_n)|^2 dx \ge \int_{|x| \le R} |V(x)|^2 dx,
$$

hence

$$
\liminf_{n\to\infty}\int_{|x-x_n|\leq R\lambda_n}|u(t_n,x)|^2dx\geq \int_{|x|\leq R}|V(x)|^2dx.
$$

Since

$$
a(t_n) \|\nabla u(t_n)\|_{L^2} = \frac{a(t_n)}{\lambda_n} \frac{\|\nabla u(t_n)\|_{L^2}}{\|u(t_n)\|_{\dot{H}^1_c}} \|Q_{\overline{c}}\|_{\dot{H}^1_c},
$$

the equivalence $\|\nabla u(t_n)\|_{L^2} \sim \|u(t_n)\|_{\dot{H}_c^1}$ and the condition (1.[4\)](#page-1-1) yield $\frac{a(t_n)}{\lambda_n} \to \infty$ as $n \to \infty$. We thus get for every $R > 0$,

$$
\liminf_{n\to\infty}\sup_{y\in\mathbb{R}^d}\int_{|x-y|\le a(t_n)}|u(t_n,x)|^2dx\ge\int_{|x|\le R}|V(x)|^2dx,
$$

which means that

$$
\liminf_{n\to\infty}\sup_{y\in\mathbb{R}^d}\int_{|x-y|\le a(t_n)}|u(t_n,x)|^2dx\ge\int|V(x)|^2dx\ge\int|Q_{\overline{c}}(x)|^2dx.
$$

Since the sequence $(t_n)_{n\geq 1}$ is arbitrary, we infer that

$$
\liminf_{t \uparrow T} \sup_{y \in \mathbb{R}^d} \int_{|x-y| \le a(t)} |u(t,x)|^2 dx \ge \int |Q_{\overline{c}}(x)|^2 dx.
$$

Moreover, since for every $t \in (0,T)$, the function $u \mapsto \int_{|x-y| \leq a(t)} |u(t,x)|^2 dx$ is continuous and goes to zero at inifinity, there exists $x(t) \in \mathbb{R}^d$ such that

$$
\sup_{y \in \mathbb{R}^d} \int_{|x-y| \le a(t)} |u(t,x)|^2 dx = \int_{|x-x(t)| \le a(t)} |u(t,x)|^2 dx.
$$

This completes the first part of Theorem 1*.*[2.](#page-1-2)

We now consider the case $c < 0$ and assume $u_0 \n\t\in H_{rad}^1$. It is well-known that the corresponding solution $u(t)$ to [\(1](#page-0-0).1) with initial data u_0 is also in H_{rad}^1 for any t in the existence time. Let $(t_n)_{n\geq 1}$ be such that $t_n \uparrow T$ as $n \to \infty$. Denote

$$
\rho_n := \frac{\|Q_{c,\text{rad}}\|_{\dot{H}_c^1}}{\|u(t_n)\|_{\dot{H}_c^1}}, \quad v_n(x) := \rho_n^{\frac{d}{2}} u(t_n, \rho_n x).
$$

As above, the blow-up alternative implies $\rho_n \to 0$ as $n \to \infty$. We also have

$$
||v_n||_{L^2} = ||u_0||_{L^2}, \quad ||v_n||_{\dot{H}_c^1} = \rho_n ||u(t_n)||_{\dot{H}_c^1} = ||Q_{c,\text{rad}}||_{\dot{H}_c^1},
$$

and

$$
E(v_n) = \rho_n^2 E(u(t_n)) = \rho_n^2 E(u_0) \to 0,
$$

as $n \to \infty$. This implies in particular that

$$
||v_n||_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2} \to \frac{d+2}{d} ||Q_{c,\text{rad}}||_{\dot{H}_c^1}^2,
$$

as $n \to \infty$. We thus obtain a bounded sequence $(v_n)_{n\geq 1}$ in H_{rad}^1 satisfying conditions of Lemma [2](#page-6-1).4 with

$$
m^{\frac{4}{d}+2} = \frac{d+2}{d} ||Q_{c,\text{rad}}||_{\dot{H}_c^1}^2, \quad M^2 = ||Q_{c,\text{rad}}||_{\dot{H}_c^1}^2.
$$

Thus, there exists $V \in H^1_{rad}$ such that

 $v_n \rightharpoonup V$ weakly in H^1 ,

as $n \to \infty$ with $||V||_{L^2} \ge ||Q_{c,\text{rad}}||_{L^2}$. The rest of the proof follows by the same argument as in the first case. The proof is complete case. The proof is complete.

4. Characterization of finite time blow-up solutions with minimal mass

In this section, we give the proof of the characterization of finite time blow-up solutions with minimal mass given in Theorem 1*.*4. Let us start with the following variational structure of ground states.

4.1. **Variational structure of ground states.** In this subsection, we show the variational structure of ground states which is neccessary in the study of limiting profile of finite time blow-up solutions with minimal mass. To sucessfully study the variational structure of ground states, we need to define a proper notion of ground states. To do this, we follow the idea of Csobo-Genoud in [\[7\]](#page-15-1).

- **Definition 4.1** (Ground states). \bullet In the case $0 < c < \lambda(d)$, we call **ground states** the maximizers of J_c (see [\(2](#page-2-3).2)) which are positive radial solutions to the elliptic equation (2.5). The set of ground states is denoted by \mathcal{G} .
	- In the case *c <* 0, we call **radial ground states** the maximizers of *J^c* which are positive radial solutions to the elliptic equation (2.6) (2.6) . The set of radial ground states is denoted by \mathcal{G}_{rad} .

Remark 4.2. • The reason for introducing the above notion of ground states is that the uniqueness (up to symmetries) of positive radial solutions to [\(2](#page-2-3)*.*5) and (2*.*[6\)](#page-2-5) are not yet known.

- By definition, the function Q_c (resp. $Q_{c,\text{rad}}$) given in Theorem [2](#page-2-0).1 belongs to $\mathcal G$ (resp. $\mathcal G_{\text{rad}}$).
- It follows from the proof of Theorem [2](#page-2-0)*.*1 and [\(2](#page-3-4)*.*7) that all ground states have the same mass. Hence, there exists $M_{gs} > 0$ such that $||Q||_{L^2} = M_{gs}$ for all $Q \in \mathcal{G}$. The constant M_{gs} is called **minimal mass**.
- Similarly, it follows from the proof of Theorem [2](#page-2-0)*.*1 and [\(2](#page-3-5)*.*8) that all radial ground states have the same mass. Hence there exists $M_{gs,rad} > 0$ such that $||Q_{rad}||_{L^2} = M_{gs,rad}$ for all $Q_{rad} \in \mathcal{G}_{rad}$. The constant *M*gs,rad is called **radial minimal mass**.

Using Definition 4.1, we have the following sharp Gagliardo-Nirenberg inequality: for $0 < c < \lambda(d)$,

$$
||u||_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2} \leq C_{\text{GN}}(c)||u||_{L^2}^{\frac{4}{d}}||u||_{\dot{H}_c^1}^2,
$$
\n(4.1)

for any $u \in H^1 \backslash \{0\}$, where

$$
C_{\text{GN}}(c) = \frac{d+2}{d} \frac{1}{M_{\text{gs}}^{\frac{4}{d}}},
$$

and also the following sharp radial Gagliardo-Nirenberg inequality: for *c <* 0,

$$
||u||_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2} \le C_{GN}(c, \text{rad})||u||_{L^{2}}^{\frac{4}{d}}||u||_{\dot{H}^{1}_{c}}^{2},\tag{4.2}
$$

for any $u \in H_{\text{rad}}^1 \setminus \{0\}$, where

$$
C_{\text{GN}}(c,\text{rad}) = \frac{d+2}{d} \frac{1}{M_{\text{gs,rad}}^{\frac{4}{d}}}.
$$

We have the following variational structure of ground states.

Lemma 4.3 (Variational structure of ground states). • *Let* $d > 3$ *and* $0 < c < \lambda(d)$ *. If* $v \in H^1$ *satisfies*

$$
||v||_{L^2} = M_{\rm gs}, \quad E(v) = 0,
$$

then there exists $Q \in \mathcal{G}$ *such that v is of the form*

$$
u(x) = e^{i\theta} \lambda^{\frac{d}{2}} Q(\lambda x),
$$

for some $\theta \in \mathbb{R}$ *and* $\lambda > 0$ *.*

• Let $d \geq 3$ and $c < 0$. If $v \in H_{rad}^1$ satisfies

$$
||v||_{L^2} = M_{gs,rad}, \quad E(v) = 0,
$$

then there exists $Q_{rad} \in \mathcal{G}_{rad}$ *such that v is of the form*

$$
v(x) = e^{i\vartheta} \rho^{\frac{d}{2}} Q_{\text{rad}}(\rho x),
$$

for some $\vartheta \in \mathbb{R}$ *and* $\rho > 0$ *.*

Proof. In the case $0 < c < \lambda(d)$, the proof of the above result is given in [\[7,](#page-15-1) Proposition 3, p.119]. The one for $c < 0$ is similar. We thus omit the details.

4.2. **Limiting profile of finite time minimal mass blow-up solutions.** Using the variational structure of ground states given in Lemma 4*.*3, we obtain the following limiting profile of finite time blow-up solutions with minimal mass. This limiting profile plays a same role as the one proved by Csobo-Genoud in [\[7,](#page-15-1) Proposition 4, p.120]. With the help of this limiting profile, we show the classification of finite time blow-up solutions with minimal mass for [\(1](#page-0-0)*.*1).

Theorem 4.4 (Limiting profile with minimal mass). • *Let* $d \geq 3$ *and* $0 < c < \lambda(d)$ *. Let* $u_0 \in H^1$ *be such that* $||u_0||_{L^2} = M_{gs}$ *. Suppose that the corresponding solution u to* [\(1](#page-0-0).1) *blows up at finite time* $0 < T < +\infty$ *. Then for any time sequence* $(t_n)_{n\geq 1}$ *satisfying* $t_n \uparrow T$ *, there exist a subsequence still denoted by* $(t_n)_{n\geq 1}$ *, a function* $Q \in \mathcal{G}$ *, sequences of* $\theta_n \in \mathbb{R}$ *,* $\lambda_n > 0$ *,* $\lambda_n \to 0$ and $x_n \in \mathbb{R}^d$ *such that*

$$
e^{it\theta_n} \lambda_n^{\frac{d}{2}} u(t_n, \lambda_n \cdot + x_n) \to Q \ \text{strongly in } H^1,
$$
\n
$$
(4.3)
$$

 $as n \to \infty$.

• Let $d \geq 3$ and $c < 0$. Let $u_0 \in H_{rad}^1$ satisfy $||u_0||_{L^2} = M_{gs, rad}$. Suppose that the corresponding *solution u to* (1.[1\)](#page-0-0) *blows up at finite time* $0 < T < +\infty$ *. Then for any time sequence* $(t_n)_{n\geq 1}$ *satisfying* $t_n \uparrow T$ *, there exist a subsequence still denoted by* $(t_n)_{n\geq 1}$ *, a function* $Q_{rad} \in \mathcal{G}_{rad}$ *, sequences of* $\vartheta_n \in \mathbb{R}$ *and* $\rho_n > 0$, $\rho_n \to 0$ *such that*

$$
e^{it\vartheta_n} \rho_n^{\frac{d}{2}} u(t_n, \rho_n \cdot) \to Q_{\text{rad}} \ \text{strongly in } H^1,\tag{4.4}
$$

 $as n \to \infty$.

Proof. Let us firstly consider the case $0 < c < \lambda(d)$. Let $(t_n)_{n>1}$ be a sequence such that $t_n \uparrow T$. Set

$$
\lambda_n := \frac{\|Q_c\|_{\dot{H}_c^1}}{\|u(t_n)\|_{\dot{H}_c^1}}, \quad v_n(x) := \lambda_n^{\frac{d}{2}} u(t_n, \lambda_n x),
$$

where Q_c is given in Theorem 2.[1.](#page-2-0) By the blow-up alternative, we see that $\lambda_n \to 0$ as $n \to \infty$. Moreover, $||v_n||_{L^2} = ||u(t_n)||_{L^2} = ||u_0||_{L^2} = M_{gs},$ (4.5)

and

$$
||v_n||_{\dot{H}_c^1} = \lambda_n ||u(t_n)||_{\dot{H}_c^1} = ||Q_c||_{\dot{H}_c^1},
$$
\n(4.6)

and

$$
E(v_n) = \lambda_n^2 E(u(t_n)) = \lambda_n^2 E(u_0) \to 0,
$$

as $n \to \infty$. In particular,

$$
||v_n||_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2} \to \frac{d+2}{d} ||Q_c||_{\dot{H}_c^1}^2,
$$
\n(4.7)

as $n \to \infty$. Thus the sequence $(v_n)_{n \geq 1}$ satisfies conditions of Lemma [2](#page-6-1).4 with

$$
M^{2} = ||Q_{c}||_{\dot{H}_{c}^{1}}^{2}, \quad m^{\frac{4}{d}+2} = \frac{d+2}{d}||Q_{c}||_{\dot{H}_{c}^{1}}^{2}.
$$

Therefore, there exist $V \in H^1$ and a sequence $(x_n)_{n \geq 1}$ in \mathbb{R}^d such that up to a subsequence,

$$
v_n(\cdot + x_n) = \lambda_n^{\frac{d}{2}} u(t_n, \lambda_n \cdot + x_n) \to V
$$
 weakly in H^1 ,

as $n \to \infty$ and $||V||_{L^2} \ge ||Q_c||_{L^2} = M_{gs}$. Since $v_n(\cdot + x_n) \to V$ weakly in H^1 as $n \to \infty$, the semi-continuity of weak convergence and [\(4](#page-9-0)*.*5) imply

$$
M_{gs} \le ||V||_{L^2} \le \liminf_{n \to \infty} ||v_n||_{L^2} = M_{gs}.
$$

This shows that

$$
||V||_{L^2} = \lim_{n \to \infty} ||v_n||_{L^2} = M_{\text{gs}}.
$$
\n(4.8)

.

,

,

Therefore, $v_n(\cdot + x_n) \to V$ strongly in L^2 as $n \to \infty$. By the sharp Gagliardo-Nirenberg inequality (4.[1\)](#page-8-1), we also have that $v_n(\cdot + x_n) \to V$ strongly in $L^{\frac{4}{d}+2}$ as $n \to \infty$. Indeed,

$$
||v_n(\cdot + x_n) - V||_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2} \leq C_{GN}(c)||v_n(\cdot + x_n) - V||_{L^2}^{\frac{4}{d}}||v_n(\cdot + x_n) - V||_{\dot{H}_c^1}^2
$$

$$
\lesssim C_{GN}(c) \left(||Q_c||_{\dot{H}_c^1}^2 + ||V||_{\dot{H}_c^1}^2 \right) ||v_n(\cdot + x_n) - V||_{L^2}^{\frac{4}{d}} \to 0,
$$

as $n \to \infty$. Here we use

$$
||v_n(\cdot + x_n)||_{\dot{H}^1_x} \sim ||v_n(\cdot + x_n)||_{\dot{H}^1} = ||v_n||_{\dot{H}^1} \sim ||v_n||_{\dot{H}^1_c}
$$

in the second estimate. Moreover, using [\(4](#page-9-1)*.*7), (4*.*[8\)](#page-10-0) and the sharp Gagliardo-Nirenberg inequality (4*.*[1\)](#page-8-1), we get

$$
||Q_c||_{\dot{H}_c^1}^2 = \frac{d}{d+2} \lim_{n \to \infty} ||v_n||_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2} = \frac{d}{d+2} ||V||_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2} \le \left(\frac{||V||_{L^2}}{M_{\text{gs}}}\right)^{\frac{4}{d}} ||V||_{\dot{H}_c^1}^2 = ||V||_{\dot{H}_c^1}^2
$$

Thus the semi-continuity of weak convergence and (4*.*[6\)](#page-9-2) imply

$$
||Q_c||_{\dot{H}_c^1} \le ||V||_{\dot{H}_c^1} \le \liminf_{n \to \infty} ||v_n||_{\dot{H}_c^1} = ||Q_c||_{\dot{H}_c^1}.
$$

Hence

$$
||V||_{\dot{H}_c^1} = \lim_{n \to \infty} ||v_n||_{\dot{H}_c^1} = ||Q_c||_{\dot{H}_c^1}
$$

We next claim that

$$
v_n(\cdot + x_n) \to V \text{ strongly in } \dot{H}^1
$$

as $n \to \infty$. Since

$$
V\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2} = \frac{d+2}{d} \|Q_c\|_{\dot{H}_c^1}^2, \quad \|V\|_{\dot{H}_c^1} = \|Q_c\|_{\dot{H}_c^1},
$$

we see that $E(V) = 0$. It follows that there exists $V \in H^1$ such that

 $\|$

$$
||V||_{L^2} = M_{gs}, \quad E(V) = 0.
$$

The variational structure of ground states given in Lemma 4.3 shows that there exists $Q \in \mathcal{G}$ such that $V(x) = e^{i\theta} \lambda^{\frac{d}{2}} Q(\lambda x)$ for some $\theta \in \mathbb{R}$ and $\lambda > 0$. Thus,

$$
v_n(\cdot + x_n) = \lambda_n^{\frac{d}{2}} u(t_n, \lambda_n \cdot + x_n) \to V = e^{i\theta} \lambda^{\frac{d}{2}} Q(\lambda \cdot) \text{ strongly in } H^1,
$$

as $n \to \infty$. Redefining $\tilde{\lambda}_n := \lambda_n \lambda^{-1}$, we obtain

$$
e^{-i\theta} \tilde{\lambda}_n^{\frac{d}{2}} u(t_n, \tilde{\lambda}_n \cdot + x_n) \to Q
$$
 strongly in H^1

as $n \to \infty$. We now prove the claim. Since $v_n(\cdot+x_n) \to V$ weakly in H^1 . Set $r_n(x) := v_n(x) - V(x-x_n)$. We see that $r_n(\cdot + x_n) \to 0$ weakly in H^1 . By the same argument as in the proof of Proposition 2.[3,](#page-3-3) we have Z

$$
\int |x|^{-2} |v_n(x)|^2 dx = \int |x|^{-2} |V(x - x_n)|^2 dx + \int |x|^{-2} |r_n(x)|^2 dx + o_n(1).
$$

In particular, we have

$$
||v_n||_{\dot{H}_c^1}^2 = ||V(\cdot - x_n)||_{\dot{H}_c^1}^2 + ||r_n||_{\dot{H}_c^1}^2 + o_n(1),
$$

$$
||v_n||_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2} = ||V(\cdot - x_n)||_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2} + ||r_n||_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2} + o_n(1),
$$

.

as $n \to \infty$. Thus,

$$
E(v_n) = E(V(\cdot - x_n)) + E(r_n) + o_n(1),
$$
\n(4.9)

as $n \to \infty$. On the other hand, since $v_n(\cdot + x_n) \to V$ strongly in L^2 , it follows that $r_n(\cdot + x_n) \to 0$ strongly in L^2 . This implies in particular that $r_n \to 0$ strongly in L^2 and $r_n \to 0$ weakly in H^1 . The sharp Gagliardo-Nireberg inequality (4.1) (4.1) then implies $r_n \to 0$ strongly in $L^{\frac{4}{d}+2}$. By the semi-continuity of weak convergence,

$$
0 \le \frac{1}{2} \liminf_{n \to \infty} ||r_n||_{\dot{H}_c^1}^2 = \frac{1}{2} \liminf_{n \to \infty} ||r_n||_{\dot{H}_c^1}^2 - \frac{d}{2d+4} \liminf_{n \to \infty} ||r_n||_{\dot{L}_c^{\frac{4}{d}+2}}^{\frac{4}{d}+2}
$$

$$
\le \liminf_{n \to \infty} \left(\frac{1}{2} ||r_n||_{\dot{H}_c^1}^2 - \frac{d}{2d+4} ||r_n||_{\dot{L}_c^{\frac{4}{d}+2}}^{\frac{4}{d}+2} \right)
$$

$$
= \liminf_{n \to \infty} E(r_n).
$$

In particular,

$$
\liminf_{n \to \infty} E(V(\cdot - x_n)) \le \liminf_{n \to \infty} E(V(\cdot - x_n)) + \liminf_{n \to \infty} E(r_n)
$$

\n
$$
\le \liminf_{n \to \infty} (E(V(\cdot - x_n)) + E(r_n)) = \liminf_{n \to \infty} E(v_n) = 0.
$$

We also have from the sharp Gagliardo-Nirenberg inequality [\(4](#page-8-1).1) and the fact $||V(\cdot - x_n)||_{L^2} = ||V||_{L^2}$ M_{gs} that $E(V(\cdot - x_n)) \geq 0$ for all $n \geq 1$. Therefore, we must have

$$
\liminf_{n \to \infty} E(V(\cdot - x_n)) = 0.
$$

Taking lim inf both sides of (4.[9\)](#page-11-0), we obtain $\liminf_{n\to\infty} E(r_n) = 0$. Since $r_n \to 0$ strongly in $L^{\frac{4}{d}+2}$, we see that up to a subsequence, $\lim_{n\to\infty} ||r_n||_{\dot{H}_c^1} = 0$. Using the equivalence $|| \cdot ||_{\dot{H}_c^1} \sim || \cdot ||_{\dot{H}^1}$, we obtain $\lim_{n\to\infty} \|\nabla r_n\|_{L^2} = 0$. Thanks to the expansion

$$
\|\nabla v_n\|_{L^2}^2 = \|\nabla V\|_{L^2}^2 + \|\nabla r_n\|_{L^2}^2 + o_n(1),
$$

as $n \to \infty$, we obtain

$$
\lim_{n\to\infty} \|\nabla v_n\|_{L^2} = \|\nabla V\|_{L^2}.
$$

Since $v_n(\cdot + x_n) \to V$ weakly in H^1 , we infer that $v_n(\cdot + x_n) \to V$ strongly in \dot{H}^1 as $n \to \infty$. This proves the claim and the proof of the first item is complete.

We now consider the case $c < 0$. Let $(t_n)_{n \geq 1}$ be a sequence such that $t_n \uparrow T$. Denote

$$
\rho_n := \frac{\|Q_{c,\text{rad}}\|_{\dot{H}_c^1}}{\|u(t_n)\|_{\dot{H}_c^1}}, \quad v_n(x) := \rho_n^{\frac{d}{2}} u(t_n, \rho_n x),
$$

where $Q_{c,rad}$ is given in Theorem 2.[1.](#page-2-0) Since $u_0 \in H_{rad}^1$, we see that $u(t) \in H_{rad}^1$ for any t as long as the solution exists. By the blow-up alternative, it follows that $\rho_n \to 0$ as $n \to \infty$. We also have

$$
||v_n||_{L^2} = ||u(t_n)||_{L^2} = ||u_0||_{L^2} = M_{\text{gs,rad}}, \quad ||v_n||_{\dot{H}_c^1} = \rho_n ||u(t_n)||_{\dot{H}_c^1} = ||Q_{c,\text{rad}}||_{\dot{H}_c^1},\tag{4.10}
$$

and

$$
E(v_n) = \rho_n^2 E(u(t_n)) = \rho_n^2 E(u_0) \to 0,
$$

as $n \to \infty$. In particular,

$$
||v_n||_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2} \to \frac{d+2}{d} ||Q_{c,\text{rad}}||_{\dot{H}_c^1}^2,
$$
\n(4.11)

as $n \to \infty$. We thus obtain a bounded sequence $(v_n)_{n\geq 1}$ of H_{rad}^1 -functions which satisfies conditions of Lemma [2](#page-6-1)*.*4 with

$$
M^{2} = \|Q_{c,\text{rad}}\|_{\dot{H}_{c}^{1}}^{2}, \quad m^{\frac{4}{d}+2} = \frac{d+2}{d} \|Q_{c,\text{rad}}\|_{\dot{H}_{c}^{1}}^{2}.
$$

We learn from Lemma [2](#page-6-1).4 that there exists $V \in H_{rad}^1$ such that up to a subsequence

$$
v_n \rightharpoonup V
$$
 weakly in H^1 ,

as $n \to \infty$ and $||V||_{L^2} \ge ||Q_{c,\text{rad}}||_{L^2} = M_{\text{gs,rad}}$. The semi-continuity of weak convergence and (4.[10\)](#page-11-1) imply that

$$
M_{\rm gs, rad} \le \|V\|_{L^2} \le \liminf_{n \to \infty} \|v_n\|_{L^2} = M_{\rm gs, rad}.
$$
 (4.12)

We thus get

$$
||V||_{L^2} = \lim_{n \to \infty} ||v_n||_{L^2} = M_{\text{gs,rad}}.
$$

In particular, $v_n \to V$ strongly in L^2 as $n \to \infty$. This together with the sharp Gagliardo-Nirenberg inequality [\(4](#page-8-2).2) yield that $v_n \to V$ strongly in $L^{\frac{4}{d}+2}$ as $n \to \infty$. By (4.[11\)](#page-11-2) and (4.[12\)](#page-11-3), the sharp Gagliardo-Nirenberg inequality implies that

$$
\|Q_{c,\text{rad}}\|_{\dot H^1_c}^2 = \frac{d}{d+2} \lim_{n \to \infty} \|v_n\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2} = \frac{d}{d+2} \|V\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2} \le \left(\frac{\|V\|_{L^2}}{M_{\text{gs,rad}}}\right)^{\frac{4}{d}} \|V\|_{\dot H^1_c}^2 = \|V\|_{\dot H^1_c}^2.
$$

Using the above inequality, the semi-continuity of weak convergence and (4*.*[10\)](#page-11-1) imply

$$
||Q_{c,\text{rad}}||_{\dot{H}_c^1} \le ||V||_{\dot{H}_c^1} \le \liminf_{n \to \infty} ||v_n||_{\dot{H}_c^1} = ||Q_{c,\text{rad}}||_{\dot{H}_c^1}.
$$

Hence

$$
||V||_{\dot{H}_c^1} = \lim_{n \to \infty} ||v_n||_{\dot{H}_c^1} = ||Q_{c,\text{rad}}||_{\dot{H}_c^1}
$$

.

.

,

We now claim that

$$
v_n \to V \text{ strongly in } H^1
$$

as $n \to \infty$. To see this, we write

$$
v_n(x) = V(x) + r_n(x),
$$

with $r_n \rightharpoonup 0$ weakly in H^1 as $n \to \infty$. We easily verify that

$$
E(v_n) = E(V) + E(r_n) + o_n(1),
$$

as $n \to \infty$. Since $v_n \to V$ strongly in L^2 , we see that $r_n \to 0$ strongly in L^2 . The sharp Gagliardo-Nirenberg inequality [\(4](#page-8-2).2) then implies that $r_n \to 0$ strongly in $L^{\frac{4}{d}+2}$. Arguing as in the case $0 < c < \lambda(d)$, we get

$$
\liminf_{n \to \infty} E(r_n) \ge 0,
$$

and

$$
E(V) \le E(V) + \liminf_{n \to \infty} E(r_n) \le \liminf_{n \to \infty} (E(V) + E(r_n)) = \liminf_{n \to \infty} E(v_n) = 0.
$$

On the other hand, since $||V||_{L^2} = M_{gs,rad}$, the sharp Gagliardo-Nirenberg inequality (4.[2\)](#page-8-2) implies that $E(V) \geq 0$. Therefore, $E(V) = 0$. As a result, we obtain that

$$
\liminf_{n \to \infty} E(r_n) = 0.
$$

Since $r_n \to 0$ strongly in $L^{\frac{4}{d}+2}$, we see that up to a subsequence, $\lim_{n\to\infty} ||r_n||_{\dot{H}_c^1} = 0$. This implies in particular that $\lim_{n\to\infty} \|\nabla r_n\|_{L^2} = 0$. Using the fact

$$
\|\nabla v_n\|_{L^2}^2 = \|\nabla V\|_{L^2}^2 + \|\nabla r_n\|_{L^2}^2 + o_n(1),
$$

as $n \to \infty$, we obtain $\lim_{n \to \infty} ||\nabla v_n||_{L^2} = ||\nabla V||_{L^2}$. Since $v_n \to V$ weakly in H^1 as $n \to \infty$, it follows that $v_n \to V$ strongly in \dot{H}^1 as $n \to \infty$. This proves the claim.

We thus obtain $V \in H^1_{rad}$ such that

$$
||V||_{L^2} = M_{\rm gs, rad}, \quad E(V) = 0.
$$

The second equality follows from

$$
||V||_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2} = \frac{d+2}{d} ||Q_{c,\text{rad}}||_{\dot{H}_c^1}^2, \quad ||V||_{\dot{H}_c^1} = ||Q_{c,\text{rad}}||_{\dot{H}_c^1}
$$

The variational structure of radial ground states given in Lemma 4.3 implies that there exists $Q_{\text{rad}} \in \mathcal{G}_{\text{rad}}$ such that $V(x) = e^{i\vartheta} \rho^{\frac{d}{2}} Q(\rho x)$ for some $\vartheta \in \mathbb{R}$ and $\rho > 0$. We thus obtain

$$
v_n(\cdot) = \rho_n^{\frac{d}{2}} u(t_n, \rho_n \cdot) \to V = e^{i\vartheta} \rho^{\frac{d}{2}} Q_{\text{rad}}(\rho \cdot) \text{ strongly in } H^1,
$$

as $n \to \infty$. Redefining $\tilde{\rho}_n := \rho_n \rho^{-1}$, we obtain

 $e^{-i\vartheta} \tilde{\rho}_n^{\frac{d}{2}} u(t_n, \tilde{\rho}_n \cdot) \to Q_{\text{rad}}$ strongly in H^1 ,

as $n \to \infty$. The proof is complete.

In order to prove the characterization of finite time blow-up solutions with minimal mass, we need to recall basic facts related to [\(1](#page-0-0)*.*1). Let us start with the following Cauchy-Schwarz inequality due to Banica [\[1\]](#page-15-22).

Lemma 4.5. *If one of the following conditions holds true*

- \bullet *d* ≥ 3*,* 0 < *c* < *λ*(*d*) *and u* ∈ *H*¹ *is such that* $||u||_{L^2} = M_{gs}$,
- $d \geq 3$, $c < 0$ *and* $u \in H_{rad}^1$ *is such that* $||u||_{L^2} = M_{gs,rad}$,

then for any real valued function $\varphi \in C^1$ *satisfying* $\nabla \varphi$ *is bounded, we have*

$$
\left| \int \nabla \varphi \cdot \text{Im} \left(u \nabla \overline{u} \right) dx \right| \leq \sqrt{2E(u)} \left(\int |\nabla \varphi|^2 |u|^2 dx \right)^{1/2} . \tag{4.13}
$$

Note that by sharp Gagliardo Nirenberg inequalities [\(4](#page-8-1)*.*1) and (4*.*[2\)](#page-8-2), the above assumptions imply $E(u)$ is non-negative.

We also need the following virial identity (see e.g. [\[8,](#page-15-12) Lemma 5.3] or [\[7,](#page-15-1) Lemma 3, p.124]).

Lemma 4.6 (Virial identity). Let $d \geq 3$ and $c \neq 0$ be such that $c < \lambda(d)$. Let $u_0 \in H^1$ be such that $|x|u_0 \in L^2$ *and* $u: I \times \mathbb{R}^d \to \mathbb{C}$ *the corresponding solution to* (1.[1\)](#page-0-0)*. Then* $|x|u \in C(I, L^2)$ *and for any* $t \in I$,

$$
\frac{d^2}{dt^2}||xu(t)||_{L^2}^2 = 16E(u_0).
$$
\n(4.14)

In particular, we have for any $t \in I$ *,*

$$
\int |x|^2 |u(t)|^2 dx = \int |x|^2 |u_0|^2 dx - 4t \int x \cdot \text{Im} (u_0 \nabla \overline{u}_0) dx + 8t^2 E(u_0)
$$

= $8t^2 E \left(e^{i \frac{|x|^2}{4t}} u_0 \right).$ (4.15)

Proof. We refer the reader to [\[8,](#page-15-12) Lemma 5.3] or [\[7,](#page-15-1) Lemma 3, p.124] for the proof of (4.[14\)](#page-13-0). The first identity in (4*.*[15\)](#page-13-1) follows by integrating (4*.*[14\)](#page-13-0) over the time *t*. The second identity in (4*.*[15\)](#page-13-1) follows from a direct computation using the fact that

$$
\left| \nabla \left(e^{i \frac{|x|^2}{4t}} u_0 \right) \right| = \frac{1}{4t^2} |x|^2 |u_0|^2 - \frac{1}{t} x \cdot \text{Im} \left(u_0 \nabla \overline{u}_0 \right) + |\nabla u_0|^2.
$$

The proof is complete.

We are now able to prove the characterization of finite time blow-up solutions with minimal mass given in Theorem 1*.*4.

Proof of Theorem 1.4. Let us firstly consider the case $0 < c < \lambda(d)$. Let $(t_n)_{n>1}$ be such that $t_n \uparrow T$. By Theorem 4.4, we see that up to a subsequence, there exists $Q \in \mathcal{G}$ such that

$$
e^{i\theta_n} \lambda_n^{\frac{d}{2}} u(t_n, \lambda_n \cdot + x_n) \to Q \text{ strongly in } H^1,
$$
\n(4.16)

as $n \to \infty$, where $(\theta_n)_{n \geq 1} \subset \mathbb{R}$, $(x_n)_{n \geq 1} \subset \mathbb{R}^d$ and $\lambda_n \to 0$ as $n \to \infty$. From this, we infer that

$$
|u(t_n, x)|^2 dx - ||Q||_{L^2}^2 \delta_{x=x_n} \rightharpoonup 0,
$$
\n(4.17)

as $n \to \infty$.

Up to subsequence, we may assume that $x_n \to x_0 \in \{0, \infty\}$. Now let φ be a smooth non-negative radial compactly supported function satisfying

$$
\varphi(x) = |x|^2 \text{ if } |x| < 1, \text{ and } |\nabla \varphi(x)|^2 \le C\varphi(x),
$$

for some constant $C > 0$. For $R > 1$, we define

$$
\varphi_R(x) := R^2 \varphi(x/R), \quad U_R(t) := \int \varphi_R(x) |u(t,x)|^2 dx.
$$

Using the Cauchy-Schwarz inequality (4.[13\)](#page-13-2) and the fact $|\nabla \varphi_R|^2 \le C |\varphi_R|$, we have

$$
|U'_R(t)| = 2 \left| \int \nabla \varphi_R \cdot \text{Im} (u(t) \nabla \overline{u}(t)) dx \right|
$$

\n
$$
\leq 2 \sqrt{2E(u_0)} \left(\int |u(t)|^2 |\nabla \varphi_R|^2 dx \right)^{1/2}
$$

\n
$$
\leq C(u_0) \sqrt{U_R(t)}.
$$

Integrating with respect to *t*, we obtain

$$
\left|\sqrt{U_R(t)} - \sqrt{U_R(t_n)}\right| \le C(u_0)|t_n - t|.\tag{4.18}
$$

Thanks to (4.[17\)](#page-13-3), we see that $U_R(t_n) \to 0$ as $n \to \infty$. Indeed, if $|x_n| \to 0$, then $U_R(t_n) \to ||Q||_{L^2}^2 \varphi_R(0) = 0$ as $n \to \infty$. If $|x_n| \to \infty$, then $U_R(t_n) \to 0$ since φ_R is compactly supported. Letting $n \to \infty$ in (4.[18\)](#page-13-4), we obtain

$$
U_R(t) \leq C(u_0)(T-t)^2.
$$

Now fix $t \in [0, T)$, letting $R \to \infty$, we have

$$
8t^2 E\left(e^{i\frac{|x|^2}{4t}}u_0\right) = \int |x|^2 |u(t,x)|^2 dx \le C(u_0)(T-t)^2,
$$
\n(4.19)

where the first equality follows from Lemma 4.[6.](#page-13-5) Note that we have from (4.19) (4.19) that $u(t) \in L^2(|x|^2 dx)$ for any $t \in [0, T)$. We also have from (4.17) (4.17) and (4.19) (4.19) that

$$
\liminf_{n \to \infty} |x_n|^2 ||Q||_{L^2}^2 \le C(u_0)T^2.
$$

Thus x_n cannot go to infinity, hence x_n converges to zero. Letting t tends to T, we learn from (4.[19\)](#page-14-0) that

$$
E\left(e^{i\frac{|x|^2}{4T}}u_0\right) = 0.
$$

We also have

$$
\left\|e^{i\frac{|x|^2}{4T}}u_0\right\|_{L^2} = \|u_0\|_{L^2} = M_{\text{gs}}.
$$

Therefore, Lemma 4.3 shows that there exists $\tilde{Q} \in \mathcal{G}$ such that

$$
e^{i\frac{|x|^2}{4T}}u_0(x) = e^{i\tilde{\theta}}\tilde{\lambda}^{\frac{d}{2}}\tilde{Q}(\tilde{\lambda}x).
$$

Redefining $\tilde{\lambda} = \frac{\lambda}{T}$ and $\tilde{\theta} = \theta + \frac{\lambda^2}{T}$ $\frac{\lambda^2}{T}$, we obtain

$$
u_0(x) = e^{i\theta} e^{i\frac{\lambda^2}{T}} e^{-i\frac{|x|^2}{4T}} \left(\frac{\lambda}{T}\right)^{\frac{d}{2}} \tilde{Q}\left(\frac{\lambda x}{T}\right).
$$

This shows (1.[7\)](#page-2-6). By the uniqueness of solution to (1.[1\)](#page-0-0), we find that $u(t) = S_{\tilde{O},T,\theta,\lambda}(t)$ for any $t \in [0,T)$. This completes the proof of the case $0 < c < \lambda(d)$.

Let us now consider the case $c < 0$. Let $(t_n)_{n \geq 1}$ be such that $t_n \uparrow T$. We have from Theorem 4.4 that up to a subsequence, there exists $Q_{\text{rad}} \in \mathcal{G}_{\text{rad}}$ such that

$$
e^{i\vartheta_n} \rho_n^{\frac{d}{2}} u(t_n, \rho_n \cdot) \to Q_{\text{rad}}
$$
 strongly in H^1 ,

as $n \to \infty$, where $(\vartheta_n)_{n \geq 1} \subset \mathbb{R}$ and $\rho_n \to 0$ as $n \to \infty$. This implies that

$$
|u(t_n, x)|^2 dx - ||Q_{\text{rad}}||_{L^2}^2 \delta_{x=0} \to 0,
$$
\n(4.20)

as $n \to \infty$. By the same argument as in the case $0 < c < \lambda(d)$, we learn that

$$
\left|\sqrt{U_R(t)} - \sqrt{U_R(t_n)}\right| \le C(u_0)|t_n - t|.
$$

Here $U_R(t_n) \to 0$ as $n \to \infty$. Indeed, by (4.[20\)](#page-14-1), $U_R(t_n) \to ||Q_{rad}||^2_{L^2} \varphi_R(0) = 0$ as $n \to \infty$. Therefore, letting $n \to \infty$, we obtain

$$
U_R(t) \le C(u_0)(T-t)^2.
$$

Fix $t \in [0, T)$, letting $R \to \infty$, we obtain

$$
8t^2E\left(e^{i\frac{|x|^2}{4t}}u_0\right) = \int |x|^2|u(t,x)|^2dx \leq C(u_0)(T-t)^2.
$$

Letting $t \uparrow T$, we get

$$
E\left(e^{i\frac{|x|^2}{4T}}u_0\right) = 0,
$$

and also

$$
\left\| e^{i\frac{|x|^2}{4T}} u_0 \right\|_{L^2} = \|u_0\|_{L^2} = M_{\text{gs,rad}}.
$$

By Lemma 4.3, there exists $\tilde{Q}_{\text{rad}} \in \mathcal{G}_{\text{rad}}$ such that

$$
e^{i\frac{|x|^2}{4T}}u_0(x)=e^{i\tilde{\vartheta}}\tilde{\rho}^{\frac{d}{2}}\tilde{Q}_{\mathrm{rad}}(\tilde{\rho}x).
$$

Redefining $\tilde{\rho} = \frac{\rho}{T}$ and $\tilde{\vartheta} = \vartheta + \frac{\rho^2}{T}$ $\frac{\rho}{T}$, we get

$$
u_0 = e^{i\vartheta} e^{i\frac{\rho^2}{T}} e^{-i\frac{|x|^2}{4T}} \left(\frac{\rho}{T}\right)^{\frac{d}{2}} \tilde{Q}_{\text{rad}} \left(\frac{\rho x}{T}\right).
$$

This shows (1.8) . The proof is complete.

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