## MASS CONCENTRATION AND CHARACTERIZATION OF FINITE TIME BLOW-UP SOLUTIONS FOR THE NONLINEAR SCHRÖDINGER EQUATION WITH INVERSE-SQUARE POTENTIAL

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ABSTRACT. We consider the  $L^2$ -critical NLS with inverse-square potential

$$i\partial_t u + \Delta u + c|x|^{-2}u = -|u|^{\frac{4}{d}}u, \quad u(0) = u_0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d,$$

where  $d \ge 3$  and  $c \ne 0$  satisfies  $c < \lambda(d) := \left(\frac{d-2}{2}\right)^2$ . We extend the mass concentration of finite time blow-up solutions established by the first author in [2] to  $c < \lambda(d)$ . Using the profile decomposition, we give a short and simple proof of a limiting profile theorem that yields the same characterization of finite time blow-up solutions with minimal mass obtained by Csobo-Genoud in [7]. We also extend the characterization obtained by Csobo-Genoud to  $c < \lambda(d)$ .

### 1. INTRODUCTION

Consider the Cauchy problem for the focusing  $L^2$ -critical nonlinear Schrödinger equation with inversesquare potential

$$\begin{cases} i\partial_t u + \Delta u + c|x|^{-2}u &= -|u|^{\frac{4}{d}}u, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d, \\ u(0) &= u_0, \end{cases}$$
(1.1)

where  $d \ge 3$ ,  $u : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{C}$ ,  $u_0 : \mathbb{R}^d \to \mathbb{C}$  and  $c \ne 0$  satisfies  $c < \lambda(d) := \left(\frac{d-2}{2}\right)^2$ . The Schrödinger equation with inverse-square potential appears in a variety of physical settings, such as in quantum field equations or black hole solutions of the Einstein's equations (see e.g. [5] or [10]). The study of nonlinear Schrödinger equation with inverse-square potential and power-type nonlinearity has attracted a lot of interests in the last several years (see e.g. [10, 4, 19, 23, 20, 12, 13, 11, 14, 7, 8, 2] and references therein).

Denote  $P_c$  the self-adjoint extension of  $-\Delta - c|x|^{-2}$ . It is known (see e.g. [10]) that in the range  $\lambda(d) - 1 < c < \lambda(d)$ , the extension is not unique. In this case, we do make a choice among possible extensions, such as Friedrichs extension. The restriction  $c < \lambda(d)$  comes from the sharp Hardy inequality

$$\lambda(d) \int |x|^{-2} |f(x)|^2 dx \le \int |\nabla f(x)|^2 dx, \quad \forall f \in H^1,$$
(1.2)

which ensures that  $P_c$  is a positive operator. We define the homogeneous Sobolev space  $\dot{H}_c^1$  as a completion of  $C_0^{\infty}(\mathbb{R}^d \setminus \{0\})$  under the norm

$$\|f\|_{\dot{H}^{1}_{c}} := \|\sqrt{P_{c}}f\|_{L^{2}} = \left(\int |\nabla f(x)|^{2} - c|x|^{-2}|f(x)|^{2}dx\right)^{1/2}.$$
(1.3)

The sharp Hardy inequality implies that for  $c < \lambda(d)$ ,  $||f||_{\dot{H}^1_c} \sim ||f||_{\dot{H}^1}$ , and the homogeneous Sobolev space  $\dot{H}^1_c$  is equivalent to the usual homogeneous Sobolev space  $\dot{H}^1$ .

The local well-posedness for (1.1) was established by Okazawa-Suzuki-Yokota [19].

**Theorem 1.1** (Local well-posedness [19]). Let  $d \ge 3$  and  $c \ne 0$  be such that  $c < \lambda(d)$ . Then for any  $u_0 \in H^1$ , there exists  $T \in (0, +\infty]$  and a maximal solution  $u \in C([0, T), H^1)$  of (1.1). The maximal time of existence satisfies either  $T = +\infty$  or  $T < +\infty$  and  $\lim_{t\uparrow T} \|\nabla u(t)\|_{L^2} = \infty$ . Moreover, the local solution enjoys the conservation of mass and energy

$$M(u(t)) = \int |u(t,x)|^2 dx = M(u_0),$$
  

$$E(u(t)) = \frac{1}{2} \int |\nabla u(t,x)|^2 dx - \frac{c}{2} \int |x|^{-2} |u(t,x)|^2 dx - \frac{d}{2d+4} \int |u(t,x)|^{\frac{4}{d}+2} dx$$

for any  $t \in [0, T)$ .

We refer the reader to [19, Theorem 5.1] for the proof of the above local well-posedness result. Note that the existence of local solutions is based on a refined energy method, and the uniqueness follows from Strichartz estimates which are shown by Burq-Planchon-Stalker-Zadeh in [4].

The main purpose of this paper is to study dynamical properties of blow-up solutions to (1.1), including mass concentration, limiting profile and the characterization of finite time blow-up solutions with minimal mass. Such phenomena were extensively studied in the last decades especially for the mass-critical nonlinear Schrödinger equation (NLS) (i.e. c = 0 in (1.1)). For the mass-critical NLS, the mass concentration was first established by Tsutsumi [17] and Merle-Tsutsumi [18]. The limiting profile of finite time blow-up solutions was obtained by Weinstein in [22]. The characterization of finite time blow-up solutions with minimal mass was obtained by Merle in [16]. Based on a refined compactness lemma, Hmidi-Keraani in [9] gave much simpler proofs of all the aforementioned results. It is their approach that we are going to pursue in the sequel.

Following the idea of Hmidi-Keraani in [9], to study dynamical properties of finite time blow-up solutions for (1.1), we first need the profile decomposition of bounded sequences in  $H^1$  related to (1.1). This profile decomposition was proved recently by the first author in [2]. Thanks to this profile decomposition, a refined version of compactness lemma related to (1.1) was shown. With the help of this refined compactness lemma, we are able to study dynamical properties of finite time blow-up solutions for (1.1).

The mass concentration for non-radial blow-up solutions was established by the first author in [2] for the case  $0 < c < \lambda(d)$ . Here we extend this result to  $c < \lambda(d)$ . We also give an improvement of the mass concentration for radial blow-up solutions in the case c < 0. This improvement is due to the sharp radial Gagliardo-Nirenberg inequality related to (1.1) for c < 0. More precisely, we prove the following result.

**Theorem 1.2** (Mass concentration). Let  $d \ge 3$ ,  $c \ne 0$  and  $c < \lambda(d)$ . Let  $u_0 \in H^1$  be such that the corresponding solution u to (1.1) blows up at finite time  $0 < T < +\infty$ . Let a(t) > 0 be such that

$$a(t)\|\nabla u(t)\|_{L^2} \to \infty,\tag{1.4}$$

as  $t \uparrow T$ . Then there exists  $x(t) \in \mathbb{R}^d$  such that

$$\liminf_{t\uparrow T} \int_{|x-x(t)| \le a(t)} |u(t,x)|^2 dx \ge \|Q_{\overline{c}}\|_{L^2}^2, \tag{1.5}$$

where  $\overline{c} = \max\{c, 0\}$ . Moreover, in the case c < 0, if we assume in addition that  $u_0$  is radial, then (1.5) can be improved to

$$\liminf_{t\uparrow T} \int_{|x|\le a(t)} |u(t,x)|^2 dx \ge \|Q_{c,\mathrm{rad}}\|_{L^2}^2.$$
(1.6)

Here  $Q_c$  and  $Q_{c,rad}$  are given in Theorem 2.1.

**Remark 1.3.** • By using a standard argument of Merle-Raphaël [15], we have the following blowup rate: if u is a solution to (1.1) blows up at finite time  $0 < T < +\infty$ , then there exists C > 0such that

$$\|\nabla u(t)\|_{L^2} > \frac{C}{\sqrt{T-t}}$$

• Rewriting

we see t

$$\frac{1}{a(t)\|\nabla u(t)\|_{L^2}} = \frac{\sqrt{T-t}}{a(t)} \frac{1}{\sqrt{T-t}} \frac{1}{\|\nabla u(t)\|_{L^2}} < C\frac{\sqrt{T-t}}{a(t)},$$
  
hat any function  $a(t) > 0$  satisfying  $\frac{\sqrt{T-t}}{a(t)} \to 0$  as  $t \to T$  fulfills (1.4).

The characterization of finite time blow-up solutions for (1.1) with minimal mass was recently established by Csobo-Genoud in [7] in the case  $0 < c < \lambda(d)$ . They showed that up to symmetries of the equation, the only finite time blow-up solutions for (1.1) with minimal mass are the pseudo-conformal transformation of ground state standing waves. Note that since the uniqueness of ground states for (1.1) is not yet known, one needs to define properly a notion of ground states for (1.1). The proof of their result is based on the concentration-compactness lemma (see e.g. [6, Proposition 1.7.6]). The key point is the limiting profile result (see [7, Proposition 4, p.120]). In this paper, we aim to give a simple proof for the above result of Csobo-Genoud in the case  $0 < c < \lambda(d)$ . Our approach is based on the profile decomposition of [2]. This allows us to give a simple version of the limiting profile compared to the one of [7]. We also extend Csobo-Genoud's result to negative values of c. Since the sharp non-radial Gagliardo-Nirenberg inequality for c < 0 is never attained for c < 0. We need to restrict our attention only to finite time radial blow-up solutions. More precisely, we prove the following result.

**Theorem 1.4** (Characterization of finite time blow-up solutions with minimal mass). • Let  $d \ge 3$ and  $0 < c < \lambda(d)$ . Let  $u_0 \in H^1$  be such that  $||u_0||_{L^2} = M_{gs}$ . Suppose that the corresponding solution u to (1.1) blows up at finite time  $0 < T < +\infty$ . Then there exist  $Q \in \mathcal{G}$ ,  $\theta \in \mathbb{R}$  and  $\lambda > 0$  such that

$$u_0(x) = e^{i\theta} e^{i\frac{\lambda^2}{T}} e^{-i\frac{|x|^2}{4T}} \left(\frac{\lambda}{T}\right)^{\frac{d}{2}} Q\left(\frac{\lambda x}{T}\right).$$
(1.7)

In particular,  $u(t,x) = S_{Q,T,\theta,\lambda}(t,x)$ , where

$$S_{Q,T,\theta,\lambda}(t,x) := e^{i\theta} e^{i\frac{\lambda^2}{T-t}} e^{-i\frac{|x|^2}{4(T-t)}} \left(\frac{\lambda}{T-t}\right)^{\frac{d}{2}} Q\left(\frac{\lambda x}{T-t}\right).$$

• Let  $d \ge 3$  and c < 0. Let  $u_0 \in H^1_{rad}$  be such that  $||u_0||_{L^2} = M_{gs,rad}$ . Suppose that the corresponding solution u to (1.1) blows up at finite time  $0 < T < +\infty$ . Then there exist  $Q_{rad} \in \mathcal{G}_{rad}$ ,  $\vartheta \in \mathbb{R}$  and  $\rho > 0$  such that

$$u_0(x) = e^{i\vartheta} e^{i\frac{\rho^2}{T}} e^{-i\frac{|x|^2}{4T}} \left(\frac{\rho}{T}\right)^{\frac{d}{2}} Q_{\rm rad}\left(\frac{\rho x}{T}\right).$$
(1.8)

In particular,  $u(t,x) = S_{Q_{rad},T,\vartheta,\rho}(t,x)$ , where

$$S_{Q_{\mathrm{rad}},T,\vartheta,\rho}(t,x) := e^{i\vartheta} e^{i\frac{\rho^2}{T-t}} e^{-i\frac{|x|^2}{4(T-t)}} \left(\frac{\rho}{T-t}\right)^{\frac{d}{2}} Q_{\mathrm{rad}}\left(\frac{\rho x}{T-t}\right).$$

We refer the reader to Section 4 for the notations  $M_{\rm gs}$ ,  $\mathcal{G}$ ,  $M_{\rm gs,rad}$  and  $\mathcal{G}_{\rm rad}$ .

The paper is organized as follows. In Section 2, we recall sharp Gagliardo-Nirenberg inequalities and the compactness lemma related to (1.1). In Section 3, we give the proof of the mass concentration given in Theorem 1.2. In Section 4, we prove a simple version of the limiting profile result compared to the one in [7]. Using this limiting profile, we give the proof of the characterization of finite time blow-up solutions with minimal mass given in Theorem 1.4.

#### 2. Preliminaries

2.1. Sharp Gagliardo-Nirenberg inequalities. In this subsection, we recall sharp Gagliardo-Nirenberg inequalities related to (1.1). Let us start with the sharp non-radial Gagliardo-Nirenberg inequality

$$\|u\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2} \le C_{\rm GN}(c) \|u\|_{L^2}^{\frac{4}{d}} \|u\|_{\dot{H}^1_c}^2, \tag{2.1}$$

where the sharp constant  $C_{\rm GN}(c)$  is defined by

$$C_{\rm GN}(c) := \sup \left\{ J_c(u) : u \in H^1 \setminus \{0\} \right\}.$$

Here  $J_c(u)$  is the Weinstein functional

$$J_{c}(u) := \|u\|_{L^{\frac{4}{4}+2}}^{\frac{4}{4}+2} \div \left[ \|u\|_{L^{2}}^{\frac{4}{4}} \|u\|_{\dot{H}^{1}_{c}}^{2} \right].$$

$$(2.2)$$

We also recall the sharp radial Gagliardo-Nirenberg inequality

$$\|u\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2} \le C_{\rm GN}(c, \operatorname{rad}) \|u\|_{L^{2}}^{\frac{4}{d}} \|u\|_{\dot{H}^{1}_{c}}^{2},$$
(2.3)

where the sharp constant  $C_{\rm GN}(c, \rm rad)$  is defined by

$$C_{\mathrm{GN}}(c, \mathrm{rad}) := \sup \left\{ J_c(u) : u \in H^1_{\mathrm{rad}} \setminus \{0\} \right\},\$$

where  $H_{\rm rad}^1$  is the space of radial  $H^1$  functions. When c = 0, Weinstein in [21] proved that the sharp constant  $C_{\rm GN}(0)$  is attained by the fuction  $Q_0$  which is the unique (up to symmetries) positive radial solution of

$$\Delta Q_0 - Q_0 + |Q_0|^{\frac{4}{d}} Q_0 = 0.$$
(2.4)

We have the following result (see [11] and also [8]).

**Theorem 2.1** (Sharp Gagliardo-Nirenberg inequalities). Let  $d \ge 3$  and  $c \ne 0$  be such that  $c < \lambda(d)$ . Then  $C_{\text{GN}}(c) \in (0, \infty)$  and

• if  $0 < c < \lambda(d)$ , then the equality in (2.1) is attained by a function  $Q_c \in H^1$  which is a positive radial solution to the elliptic equation

$$\Delta Q_c + c|x|^{-2}Q_c - Q_c + |Q_c|^{\frac{4}{d}}Q_c = 0.$$
(2.5)

• if c < 0, then  $C_{GN}(c) = C_{GN}(0)$  and the equality in (2.1) is never attained. However, the equality in (2.3) is attained by a function  $Q_{c,rad} \in H^1_{rad}$  which is a positive solution to the elliptic equation

$$\Delta Q_{c,\text{rad}} + c|x|^{-2}Q_{c,\text{rad}} - Q_{c,\text{rad}} + |Q_{c,\text{rad}}|^{\frac{4}{d}}Q_{c,\text{rad}} = 0.$$
(2.6)

We refer the reader to [11, Theorem 3.1] (see also [8, Theorem 4.1]) for the proof of the above result.

**Remark 2.2.** • In the case  $0 < c < \lambda(d)$ , Theorem 2.1 shows that there exist positive radial solutions to the elliptic equation (2.5). However, unlike the case c = 0, the uniqueness up to symmetries of these solutions is not known yet. We also have the following Pohozaev's identities 1.

$$\|Q_c\|_{L^2}^2 = \frac{2}{d} \|Q_c\|_{\dot{H}_c^1}^2 = \frac{2}{d+2} \|Q_c\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2}$$

In particular,

$$C_{\rm GN}(c) = \frac{d+2}{d} \frac{1}{\|Q_c\|_{L^2}^{\frac{4}{d}}}.$$

• Since the above identities still hold true for c = 0, we get from Theorem 2.1 that for any  $c < \lambda(d)$ ,

$$C_{\rm GN}(c) = \frac{d+2}{d} \frac{1}{\|Q_{\overline{c}}\|_{L^2}^{\frac{d}{d}}},\tag{2.7}$$

where  $\overline{c} = \max\{c, 0\}$ .

• In the case c < 0, we also have

$$\|Q_{c,\mathrm{rad}}\|_{L^2}^2 = \frac{2}{d} \|Q_{c,\mathrm{rad}}\|_{\dot{H}^1_c}^2 = \frac{2}{d+2} \|Q_{c,\mathrm{rad}}\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2}.$$

In particular,

$$C_{\rm GN}(c, \rm rad) = \frac{d+2}{d} \frac{1}{\|Q_{c,\rm rad}\|_{L^2}^{\frac{4}{d}}}.$$
(2.8)

Note that since  $C_{\text{GN}}(c, \text{rad}) < C_{\text{GN}}(c)$ , we see that for any c < 0,

$$||Q_0||_{L^2} < ||Q_{c,\mathrm{rad}}||_{L^2}.$$

2.2. Profile decomposition. In this subsection, we recall the profile decomposition related to the nonlinear Schrödinger equation with inverse-square potential. This profile decomposition was established recently by the first author in [2] for  $0 < c < \lambda(d)$ . There is no difficulty to extend this result for negative values of c.

**Proposition 2.3** (Profile decomposition). Let  $d \ge 3$  and  $c < \lambda(d)$ . Let  $(v_n)_{n\ge 1}$  be a bounded sequence in  $H^1$ . Then there exist a subsequence still denoted by  $(v_n)_{n\ge 1}$ , a family  $(x_n^j)_{n\ge 1}$  of sequences in  $\mathbb{R}^d$  and a sequence  $(V^j)_{j\ge 1}$  of  $H^1$ -functions such that

i) for every  $j \neq k$ ,

as  $n \to \infty$ :

$$|x_n^j - x_n^k| \to \infty, \tag{2.9}$$

ii) for every  $l \geq 1$  and every  $x \in \mathbb{R}^d$ , we have

$$v_n(x) = \sum_{j=1}^{l} V^j(x - x_n^j) + v_n^l(x)$$

with

$$\limsup_{n \to \infty} \|v_n^l\|_{L^q} \to 0, \tag{2.10}$$

as  $l \to \infty$  for every  $2 < q < \frac{2d}{d-2}$ . Moreover, for every  $l \ge 1$ ,

$$\|v_n\|_{L^2}^2 = \sum_{j=1}^l \|V^j\|_{L^2}^2 + \|v_n^l\|_{L^2}^2 + o_n(1),$$
(2.11)

$$\|v_n\|_{\dot{H}^1_c}^2 = \sum_{j=1}^l \|V^j(\cdot - x_n^j)\|_{\dot{H}^1_c} + \|v_n^l\|_{\dot{H}^1_c} + o_n(1), \qquad (2.12)$$

as  $n \to \infty$ .

<sup>&</sup>lt;sup>1</sup>These identities can be proved rigorously by using the technique of [3, Proposition 1]: first, considering Pohozaev's identities in  $\Omega_{r,R} := \{x : r < |x| < R\}$ , and then showing the boundary term (on  $\partial \Omega_{r,R}$ ) to converge to 0 as  $r \to 0$  and  $R \to +\infty$ .

*Proof.* For reader's convenience, we recall some details. Since  $H^1$  is a Hilbert space, we denote  $\Omega(v_n)$  the set of functions obtained as weak limits of sequences of the translated  $v_n(\cdot + x_n)$  with  $(x_n)_{n\geq 1}$  a sequence in  $\mathbb{R}^d$ . Denote

$$\eta(v_n) := \sup\{\|v\|_{L^2} + \|\nabla v\|_{L^2} : v \in \Omega(v_n)\}.$$

Clearly,

$$\eta(v_n) \le \limsup_{n \to \infty} \|v_n\|_{L^2} + \|\nabla v_n\|_{L^2}.$$

We shall prove that there exist a sequence  $(V^j)_{j\geq 1}$  of  $\Omega(v_n)$  and a family  $(x_n^j)_{j\geq 1}$  of sequences in  $\mathbb{R}^d$  such that for every  $k\neq j$ ,

$$|x_n^k - x_n^j| \to \infty$$
, as  $n \to \infty$ ,

and up to a subsequence, the sequence  $(v_n)_{n\geq 1}$  can be written, for every  $l\geq 1$  and every  $x\in \mathbb{R}^d$ , as

$$v_n(x) = \sum_{j=1}^{l} V^j(x - x_n^j) + v_n^l(x),$$

with  $\eta(v_n^l) \to 0$  as  $l \to \infty$ . Moreover, the identities (2.11) and (2.12) hold as  $n \to \infty$ .

Indeed, if  $\eta(v_n) = 0$ , then we can take  $V^j = 0$  for all  $j \ge 1$ . Otherwise we choose  $V^1 \in \Omega(v_n)$  such that

$$||V^1||_{L^2} + ||\nabla V^1||_{L^2} \ge \frac{1}{2}\eta(v_n) > 0.$$

By the definition of  $\Omega(v_n)$ , there exists a sequence  $(x_n^1)_{n\geq 1} \subset \mathbb{R}^d$  such that up to a subsequence,

$$v_n(\cdot + x_n^1) \rightharpoonup V^1$$
 weakly in  $H^1$ .

Set  $v_n^1(x) := v_n(x) - V^1(x - x_n^1)$ . We see that  $v_n^1(\cdot + x_n^1) \to 0$  weakly in  $H^1$  and thus  $\|v_n\|_{2,0}^2 = \|V^1\|_{2,0}^2 + \|v_n^1\|_{2,0}^2 + o_n(1)$ 

$$\|\nabla v_n\|_{L^2}^2 = \|\nabla V^1\|_{L^2}^2 + \|\nabla v_n\|_{L^2}^2 + o_n(1),$$

as  $n \to \infty$ . We next show that

$$\int |x|^{-2} |v_n(x)|^2 dx = \int |x|^{-2} |V^1(x - x_n^1)|^2 dx + \int |x|^{-2} |v_n^1(x)|^2 dx + o_n(1),$$

as  $n \to \infty$ . Using the fact

$$|v_n(x)|^2 = |V^1(x - x_n^1)|^2 + |v_n^1(x)|^2 + 2\operatorname{Re}\left(V^1(x - x_n^1)\overline{v}_n^1(x)\right),$$

it suffices to show that

$$\int |x|^{-2} V^1(x - x_n^1) \overline{v}_n^1(x) dx \to 0,$$
(2.13)

as  $n \to \infty$ . Without loss of generality, we may assume that  $V^1$  is continuous and compactly supported. Moreover, up to a subsequence, we assume that  $|x_n^1| \to \{0, \infty\}$  as  $n \to \infty$ .

• Case 1:  $|x_n^1| \to \infty$ . Since  $|x_n^1| \to \infty$  as  $n \to \infty$ , we see that  $|x + x_n^1| \ge 1$  for all  $x \in \text{supp}(V^1)$  and all  $n \ge n_0$  with  $n_0$  large enough. Therefore, for  $n \ge n_0$ ,

$$\begin{split} \left| \int |x|^{-2} V^1(x - x_n^1) \overline{v}_n^1(x) dx \right| &= \int_{\mathrm{supp}(V^1)} |x + x_n^1|^{-2} |V^1(x)| |v_n^1(x + x_n^1)| dx \\ &\leq \int |V^1(x)| |v_n^1(x + x_n^1)| dx. \end{split}$$

Since  $v_n^1(\cdot + x_n^1) \to 0$  in  $H^1$  as  $n \to \infty$ , the last term tends to zero as  $n \to \infty$ . • Case 2:  $|x_n^1| \to 0$ . Let  $\epsilon > 0$ . For  $\eta > 0$  small to be chosen later, we split

$$\begin{split} \int_{\mathrm{supp}(V^1)} |x + x_n^1|^{-2} |V^1(x)| |v_n^1(x + x_n^1)| dx &= \int_{B(0,\eta)} |x + x_n^1|^{-2} |V^1(x)| |v_n^1(x + x_n^1)| dx \\ &+ \int_{\mathrm{supp}(V^1) \setminus B(0,\eta)} |x + x_n^1|^{-2} |V^1(x)| |v_n^1(x + x_n^1)| dx. \end{split}$$
(2.14)

Since  $|x_n^1| \to 0$ , we see that for all  $n \ge n_1$  with  $n_1$  large enough,  $|x+x_n^1| \ge \eta/2$  for all  $x \in \text{supp}(V^1) \setminus B(0,\eta)$ . Thus

$$\int_{\mathrm{supp}(V^1)\backslash B(0,\eta)} |x+x_n^1|^{-2} |V^1(x)| |v_n^1(x+x_n^1)| dx \lesssim \eta^{-2} \int |V^1(x)| |v_n^1(x+x_n^1)| dx$$

We next learn from the fact  $v_n^1(\cdot + x_n^1) \rightarrow 0$  in  $H^1$  as  $n \rightarrow \infty$  that for  $n \ge n_1$  (increasing  $n_1$  if necessary),

$$\int_{\mathrm{supp}(V^1)\setminus B(0,\eta)} |x+x_n^1|^{-2} |V^1(x)| |v_n^1(x+x_n^1)| dx < \frac{\epsilon}{2}.$$
(2.15)

We next use the Cauchy-Schwarz inequality, Hardy's inequality (1.2) and the fact  $(v_n^1)_{n\geq 1}$  is bounded in  $H^1$  to get

$$\int_{B(0,\eta)} |x + x_n^1|^{-2} |V^1(x)| |v_n^1(x + x_n^1)| dx \leq \left( \int_{B(0,\eta)} |x + x_n^1|^{-2} |V^1(x)|^2 dx \right)^{1/2} \\
\times \left( \int |x + x_n^1|^{-2} |v_n^1(x + x_n^1)|^2 dx \right)^{1/2} \\
\lesssim \left( \int_{B(0,\eta)} |x + x_n^1|^{-2} |V^1(x)|^2 dx \right)^{1/2} ||\nabla v_n^1||_{L^2} \\
\lesssim \left( \int_{B(0,\eta)} |x + x_n^1|^{-2} |V^1(x)|^2 dx \right)^{1/2}.$$
(2.16)

Since  $|V^1(x)|^2$  is continuous on the compact set  $\overline{B}(0, 3\eta)$ , hence it is uniformly continuous on  $\overline{B}(0, 3\eta)$ . Thus, there exists  $\delta > 0$  such that for all  $x, y \in \overline{B}(0, 3\eta)$  satisfying  $|x - y| < \delta$ , we have

$$||V^{1}(x)|^{2} - |V^{1}(y)|^{2}| < \frac{\epsilon^{2}}{8K(\eta)},$$

where

$$K(\eta) := \int_{B(0,2\eta)} |x|^{-2} dx = \frac{(2\eta)^{d-2}}{d-2} |\mathbb{S}^{d-1}|.$$

Note that we can take  $\delta \in (0, \eta)$ . Since  $|x_n^1| \to 0$ , we have for  $n \ge n_2$  with  $n_2$  large enough that

$$|x_n^1| < \delta < \eta$$

This implies that for all  $x \in B(0, 2\eta)$  and all  $n \ge n_2$ ,

$$||V^{1}(x - x_{n}^{1})|^{2} - |V^{1}(x)|^{2}| < \frac{\epsilon^{2}}{8K(\eta)}.$$
(2.17)

Since  $B(x_n^1, \eta) \subset B(0, 2\eta)$  for all  $n \ge n_2$ , we use (2.17) to get

$$\begin{split} \int_{B(0,\eta)} |x+x_n^1|^{-2} |V^1(x)|^2 dx &= \int_{B(x_n^1,\eta)} |x|^{-2} |V^1(x-x_n^1)|^2 dx \\ &\leq \int_{B(0,2\eta)} |x|^{-2} |V^1(x)|^2 dx + \int_{B(0,2\eta)} |x|^{-2} \frac{\epsilon^2}{8K(\eta)} dx \\ &= \int_{B(0,2\eta)} |x|^{-2} |V^1(x)|^2 dx + \frac{\epsilon^2}{8}. \end{split}$$

Using Hardy's inequality (1.2) with  $V^1 \in H^1$ , the dominated convergence allows to choose  $\eta > 0$  small enough so that

$$\int_{B(0,2\eta)} |x|^{-2} |V^1(x)|^2 dx < \frac{\epsilon^2}{8}$$

We thus obtain

$$\int_{B(0,\eta)} |x+x_n^1|^{-2} |V^1(x)|^2 dx < \frac{\epsilon^2}{4}$$

which together with (2.16) yield for  $n \ge n_2$ ,

$$\int_{B(0,\eta)} |x + x_n^1|^{-2} |V^1(x)| |v_n^1(x + x_n^1)| dx < \frac{\epsilon}{2}.$$
(2.18)

Combining (2.14), (2.15) and (2.18), we have for  $n \ge \max\{n_1, n_2\}$ ,

$$\int_{\text{supp}(V^1)} |x + x_n^1|^{-2} |V^1(x)| |v_n^1(x + x_n^1)| dx < \epsilon.$$

Therefore, (2.13) is proved in both cases.

We now replace  $(v_n)_{n\geq 1}$  by  $(v_n^1)_{n\geq 1}$  and repeat the same process. If  $\eta(v_n^1) = 0$ , then we choose  $V^j = 0$  for all  $j \geq 2$ . Otherwise there exist  $V^2 \in \Omega(v_n^1)$  and a sequence  $(x_n^2)_{n\geq 1} \subset \mathbb{R}^d$  such that

$$\|V^2\|_{L^2} + \|\nabla V^2\|_{L^2} \ge \frac{1}{2}\eta(v_n^1) > 0,$$

and

$$\begin{aligned} v_n^1(\cdot + x_n^2) &\rightharpoonup V^2 \text{ weakly in } H^1. \end{aligned}$$
  
Set  $v_n^2(x) := v_n^1(x) - V^2(x - x_n^2). \end{aligned}$  We thus have  $v_n^2(\cdot + x_n^2) \rightharpoonup 0$  weakly in  $H^1$  and  
 $\|v_n^1\|_{L^2}^2 = \|V^2\|_{L^2}^2 + \|v_n^2\|_{L^2}^2 + o_n(1), \\ \|v_n^1\|_{\dot{H}_c^1}^2 = \|V^2\|_{\dot{H}_c^1}^2 + \|v_n^2\|_{\dot{H}_c^1}^2 + o_n(1), \end{aligned}$ 

as  $n \to \infty$ . We claim that

$$|x_n^1 - x_n^2| \to \infty$$
, as  $n \to \infty$ .

In fact, if it is not true, then up to a subsequence,  $x_n^1 - x_n^2 \to x_0$  as  $n \to \infty$  for some  $x_0 \in \mathbb{R}^d$ . Since  $v_{\pm}^{1}(x+x_{\pm}^{2}) = v_{\pm}^{1}(x+(x_{\pm}^{2}-x_{\pm}^{1})+x_{\pm}^{1}).$ 

converges weakly to 0, we see that 
$$V^2 = 0$$
. This implies that  $\eta(v_n^1) = 0$  and it gument of iteration and orthogonal extraction allows us to construct the family

and  $v_n^1(\cdot + x_n^1)$ is a contradiction. An arg  $y(x_n^j)_{j>1}$  of sequences in  $\mathbb{R}^d$  and the sequence  $(V^j)_{j\geq 1}$  of  $H^1$  functions satisfying the claim above. Furthermore, the convergence of the series  $\sum_{j\geq 1}^{\infty} \|V^j\|_{L^2}^2 + \|\nabla V^j\|_{L^2}^2$  implies that

$$\|V^j\|_{L^2}^2 + \|\nabla V^j\|_{L^2}^2 \to 0, \text{ as } j \to \infty.$$

By construction, we have

$$\eta(v_n^j) \le 2 \left( \|V^{j+1}\|_{L^2} + \|\nabla V^{j+1}\|_{L^2} \right)$$

which proves that  $\eta(v_n^j) \to 0$  as  $j \to \infty$ . The proof of 2.3 follows by the same lines as in [9, Proposition 2.3]. We thus omit the details.

2.3. Compactness lemma. In this subsection, we recall a compactness lemma related to the nonlinear Schrödinger equation with inverse-square potential.

**Lemma 2.4** (Compactness lemma). Let  $d \ge 3$ ,  $c \ne 0$  and  $c < \lambda(d)$ . Let  $(v_n)_{n \ge 1}$  be a bounded sequence in  $H^1$  such that

$$\limsup_{n \to \infty} \|v_n\|_{\dot{H}^1_c} \le M, \quad \limsup_{n \to \infty} \|v_n\|_{L^{\frac{4}{d}+2}} \ge m.$$
(2.19)

Then there exists  $(x_n)_{n\geq 1}$  in  $\mathbb{R}^d$  such that up to a subsequence,  $v_n(\cdot + x_n) \rightharpoonup V$  weakly in  $H^1$  for some  $V \in H^1$  satisfying

$$\|V\|_{L^2}^{\frac{4}{d}} \ge \frac{d}{d+2} \frac{m^{\frac{4}{d}+2}}{M^2} \|Q_{\overline{c}}\|_{L^2}^{\frac{4}{d}}$$

where  $\overline{c} = \max\{c, 0\}$  and  $Q_c$  is given in Theorem 2.1. Moreover, in the case c < 0, if we assume in addition  $(v_n)_{n\geq 1}$  are radially symmetric, then up to a subsequence  $v_n \rightharpoonup V$  weakly in  $H^1$  for some  $V \in H^1_{\mathrm{rad}}$  satisfying

$$\|V\|_{L^2}^{\frac{4}{d}} \ge \frac{d}{d+2} \frac{m^{\frac{4}{d}+2}}{M^2} \|Q_{c,\mathrm{rad}}\|_{L^2}^{\frac{4}{d}},$$

where  $Q_{c,rad}$  is also given in Theorem 2.1.

*Proof.* In the case  $c < \lambda(d)$  and  $v_n$  non-radial, the proof is given in [2, Lemma 5] using the profile decomposition, the sharp Gagliardo-Nirenberg inequality (2.1) and (2.7).

Let us now consider the case c < 0 and  $(v_n)_{n \ge 1}$  a bounded sequence in  $H^1_{rad}$  satisfying (2.19). Thanks to the fact

$$H^1_{\rm rad} \hookrightarrow L^{\frac{4}{d}+2}$$
 compactly,

we see that there exists  $V \in H^1_{\text{rad}}$  such that up to a subsequence,  $v_n \rightharpoonup V$  weakly in  $H^1$  as well as strongly in  $L^{\frac{4}{d}+2}$ . In particular, we have from the second condition in (2.19) that  $m \leq \|V\|_{L^{\frac{4}{d}+2}}$ . By the sharp radial Gagliardo-Nirenberg inequality and (2.8), we have

$$m^{\frac{4}{d}+2} \le \|V\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2} \le \frac{d+2}{d} \frac{1}{\|Q_{c,\mathrm{rad}}\|_{L^{2}}^{\frac{4}{d}}} \|V\|_{L^{2}}^{\frac{4}{d}} \|V\|_{\dot{H}^{1}_{c}}^{2}$$

By the lower semi continuity of Hardy's functional, the first condition in (2.19) implies

$$\|V\|_{\dot{H}^{1}_{c}} \leq \limsup_{n \to \infty} \|v_{n}\|_{\dot{H}^{1}_{c}} \leq M$$

We thus obtain

$$\|V\|_{L^2}^{\frac{4}{d}} \ge \frac{d}{d+2} \frac{m^{\frac{4}{d}+2}}{M^2} \|Q_{c,\mathrm{rad}}\|_{L^2}^{\frac{4}{d}}.$$

The proof is complete.

#### 3. Mass concentration

In this short section, we give the proof of the mass concentration given in Theorem 1.2. *Proof of Theorem* 1.2. The proof is similar to the one of [2, Theorem 1]. For the sake of completeness, we recall some details. Let  $(t_n)_{n\geq 1}$  be a time sequence such that  $t_n \uparrow T$  as  $n \to \infty$ . Set

$$\lambda_n := \frac{\|Q_{\overline{c}}\|_{\dot{H}^1_c}}{\|u(t_n)\|_{\dot{H}^1_c}}, \quad v_n(x) := \lambda_n^{\frac{d}{2}} u(t_n, \lambda_n x).$$

By the local well-posedness theory given in Theorem 1.1 and the equivalence between  $\dot{H}_c^1$  and  $\dot{H}^1$ , we see that  $\lambda_n \to 0$  as  $n \to \infty$ . Moreover, a direct computation combined with the conservation of mass and energy show

$$\|v_n\|_{L^2} = \|u(t_n)\|_{L^2} = \|u_0\|_{L^2}, \quad \|v_n\|_{\dot{H}^1_c} = \lambda_n \|u(t_n)\|_{\dot{H}^1_c} = \|Q_{\overline{c}}\|_{\dot{H}^1_c},$$

and

$$E(v_n) = \lambda_n^2 E(u(t_n)) = \lambda_n^2 E(u_0) \to 0$$

as  $n \to \infty$ . In particular,

$$\|v_n\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2} \to \frac{d+2}{d} \|Q_{\overline{c}}\|_{\dot{H}^1_c}^2,$$

as  $n \to \infty$ . This implies in particular that  $(v_n)_{n\geq 1}$  satisfies conditions of Lemma 2.4 with

$$m^{\frac{4}{d}+2} = \frac{d+2}{d} \|Q_{\overline{c}}\|_{\dot{H}^1_c}^2, \quad M^2 = \|Q_{\overline{c}}\|_{\dot{H}^1_c}^2.$$

Therefore, there exist a sequence  $(x_n)_{n\geq 1}$  in  $\mathbb{R}^d$  and  $V\in H^1$  such that up to a subsequence

$$u_n(\cdot + x_n) = \lambda_n^{\frac{\pi}{2}} u(t_n, \lambda_n \cdot + x_n) \rightharpoonup V$$
 weakly in  $H^1$ ,

as  $n \to \infty$  with  $||V||_{L^2} \ge ||Q_{\overline{c}}||_{L^2}$ . This implies for every R > 0,

$$\liminf_{n \to \infty} \int_{|x| \le R} \lambda_n^d |u(t_n, \lambda_n x + x_n)|^2 dx \ge \int_{|x| \le R} |V(x)|^2 dx,$$

hence

$$\liminf_{n \to \infty} \int_{|x-x_n| \le R\lambda_n} |u(t_n, x)|^2 dx \ge \int_{|x| \le R} |V(x)|^2 dx$$

Since

$$a(t_n) \|\nabla u(t_n)\|_{L^2} = \frac{a(t_n)}{\lambda_n} \frac{\|\nabla u(t_n)\|_{L^2}}{\|u(t_n)\|_{\dot{H}^1_c}} \|Q_{\overline{c}}\|_{\dot{H}^1_c}$$

the equivalence  $\|\nabla u(t_n)\|_{L^2} \sim \|u(t_n)\|_{\dot{H}^1_c}$  and the condition (1.4) yield  $\frac{a(t_n)}{\lambda_n} \to \infty$  as  $n \to \infty$ . We thus get for every R > 0,

$$\liminf_{n \to \infty} \sup_{y \in \mathbb{R}^d} \int_{|x-y| \le a(t_n)} |u(t_n, x)|^2 dx \ge \int_{|x| \le R} |V(x)|^2 dx,$$

which means that

$$\liminf_{n \to \infty} \sup_{y \in \mathbb{R}^d} \int_{|x-y| \le a(t_n)} |u(t_n, x)|^2 dx \ge \int |V(x)|^2 dx \ge \int |Q_{\overline{c}}(x)|^2 dx$$

Since the sequence  $(t_n)_{n\geq 1}$  is arbitrary, we infer that

$$\liminf_{t\uparrow T} \sup_{y\in\mathbb{R}^d} \int_{|x-y|\leq a(t)} |u(t,x)|^2 dx \ge \int |Q_{\overline{c}}(x)|^2 dx$$

Moreover, since for every  $t \in (0,T)$ , the function  $u \mapsto \int_{|x-y| \le a(t)} |u(t,x)|^2 dx$  is continuous and goes to zero at inifinity, there exists  $x(t) \in \mathbb{R}^d$  such that

$$\sup_{y \in \mathbb{R}^d} \int_{|x-y| \le a(t)} |u(t,x)|^2 dx = \int_{|x-x(t)| \le a(t)} |u(t,x)|^2 dx$$

This completes the first part of Theorem 1.2.

We now consider the case c < 0 and assume  $u_0 \in H^1_{\text{rad}}$ . It is well-known that the corresponding solution u(t) to (1.1) with initial data  $u_0$  is also in  $H^1_{\text{rad}}$  for any t in the existence time. Let  $(t_n)_{n\geq 1}$  be such that  $t_n \uparrow T$  as  $n \to \infty$ . Denote

$$\rho_n := \frac{\|Q_{c, \text{rad}}\|_{\dot{H}^1_c}}{\|u(t_n)\|_{\dot{H}^1_c}}, \quad v_n(x) := \rho_n^{\frac{d}{2}} u(t_n, \rho_n x).$$

As above, the blow-up alternative implies  $\rho_n \to 0$  as  $n \to \infty$ . We also have

$$\|v_n\|_{L^2} = \|u_0\|_{L^2}, \quad \|v_n\|_{\dot{H}^1_c} = \rho_n \|u(t_n)\|_{\dot{H}^1_c} = \|Q_{c,\mathrm{rad}}\|_{\dot{H}^1_c},$$

and

$$E(v_n) = \rho_n^2 E(u(t_n)) = \rho_n^2 E(u_0) \to 0,$$

as  $n \to \infty$ . This implies in particular that

$$\|v_n\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2} \to \frac{d+2}{d} \|Q_{c,\mathrm{rad}}\|_{\dot{H}^1_c}^2,$$

as  $n \to \infty$ . We thus obtain a bounded sequence  $(v_n)_{n\geq 1}$  in  $H^1_{rad}$  satisfying conditions of Lemma 2.4 with

$$m^{\frac{4}{d}+2} = \frac{d+2}{d} \|Q_{c,\mathrm{rad}}\|_{\dot{H}^1_c}^2, \quad M^2 = \|Q_{c,\mathrm{rad}}\|_{\dot{H}^1_c}^2.$$

Thus, there exists  $V \in H^1_{rad}$  such that

 $v_n \rightharpoonup V$  weakly in  $H^1$ ,

as  $n \to \infty$  with  $||V||_{L^2} \ge ||Q_{c,rad}||_{L^2}$ . The rest of the proof follows by the same argument as in the first case. The proof is complete.

### 4. CHARACTERIZATION OF FINITE TIME BLOW-UP SOLUTIONS WITH MINIMAL MASS

In this section, we give the proof of the characterization of finite time blow-up solutions with minimal mass given in Theorem 1.4. Let us start with the following variational structure of ground states.

4.1. Variational structure of ground states. In this subsection, we show the variational structure of ground states which is neccessary in the study of limiting profile of finite time blow-up solutions with minimal mass. To successfully study the variational structure of ground states, we need to define a proper notion of ground states. To do this, we follow the idea of Csobo-Genoud in [7].

- **Definition 4.1** (Ground states). In the case  $0 < c < \lambda(d)$ , we call **ground states** the maximizers of  $J_c$  (see (2.2)) which are positive radial solutions to the elliptic equation (2.5). The set of ground states is denoted by  $\mathcal{G}$ .
  - In the case c < 0, we call **radial ground states** the maximizers of  $J_c$  which are positive radial solutions to the elliptic equation (2.6). The set of radial ground states is denoted by  $\mathcal{G}_{rad}$ .

# **Remark 4.2.** • The reason for introducing the above notion of ground states is that the uniqueness (up to symmetries) of positive radial solutions to (2.5) and (2.6) are not yet known.

- By definition, the function  $Q_c$  (resp.  $Q_{c,rad}$ ) given in Theorem 2.1 belongs to  $\mathcal{G}$  (resp.  $\mathcal{G}_{rad}$ ).
- It follows from the proof of Theorem 2.1 and (2.7) that all ground states have the same mass. Hence, there exists  $M_{\rm gs} > 0$  such that  $||Q||_{L^2} = M_{\rm gs}$  for all  $Q \in \mathcal{G}$ . The constant  $M_{\rm gs}$  is called **minimal mass**.
- Similarly, it follows from the proof of Theorem 2.1 and (2.8) that all radial ground states have the same mass. Hence there exists  $M_{\text{gs,rad}} > 0$  such that  $\|Q_{\text{rad}}\|_{L^2} = M_{\text{gs,rad}}$  for all  $Q_{\text{rad}} \in \mathcal{G}_{\text{rad}}$ . The constant  $M_{\text{gs,rad}}$  is called **radial minimal mass**.

Using Definition 4.1, we have the following sharp Gagliardo-Nirenberg inequality: for  $0 < c < \lambda(d)$ ,

$$\|u\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2} \le C_{\rm GN}(c) \|u\|_{L^2}^{\frac{4}{d}} \|u\|_{\dot{H}^1_c}^2, \tag{4.1}$$

for any  $u \in H^1 \setminus \{0\}$ , where

$$C_{\rm GN}(c) = \frac{d+2}{d} \frac{1}{M_{\rm gs}^{\frac{4}{d}}}$$

and also the following sharp radial Gagliardo-Nirenberg inequality: for c < 0,

$$\|u\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2} \le C_{\rm GN}(c, \operatorname{rad}) \|u\|_{L^{2}}^{\frac{4}{d}} \|u\|_{\dot{H}^{1}_{c}}^{2}, \tag{4.2}$$

for any  $u \in H^1_{\text{rad}} \setminus \{0\}$ , where

$$C_{\rm GN}(c, {\rm rad}) = \frac{d+2}{d} \frac{1}{M_{\sigma {\rm grad}}^{\frac{d}{d}}}.$$

We have the following variational structure of ground states.

**Lemma 4.3** (Variational structure of ground states). • Let  $d \ge 3$  and  $0 < c < \lambda(d)$ . If  $v \in H^1$  satisfies

$$||v||_{L^2} = M_{\rm gs}, \quad E(v) = 0,$$

then there exists  $Q \in \mathcal{G}$  such that v is of the form

$$u(x) = e^{i\theta} \lambda^{\frac{d}{2}} Q(\lambda x),$$

for some  $\theta \in \mathbb{R}$  and  $\lambda > 0$ .

• Let 
$$d \ge 3$$
 and  $c < 0$ . If  $v \in H^1_{\text{rad}}$  satisfies

$$||v||_{L^2} = M_{\text{gs,rad}}, \quad E(v) = 0,$$

then there exists  $Q_{rad} \in \mathcal{G}_{rad}$  such that v is of the form

$$v(x) = e^{i\vartheta} \rho^{\frac{a}{2}} Q_{\rm rad}(\rho x),$$

for some  $\vartheta \in \mathbb{R}$  and  $\rho > 0$ .

*Proof.* In the case  $0 < c < \lambda(d)$ , the proof of the above result is given in [7, Proposition 3, p.119]. The one for c < 0 is similar. We thus omit the details.

4.2. Limiting profile of finite time minimal mass blow-up solutions. Using the variational structure of ground states given in Lemma 4.3, we obtain the following limiting profile of finite time blow-up solutions with minimal mass. This limiting profile plays a same role as the one proved by Csobo-Genoud in [7, Proposition 4, p.120]. With the help of this limiting profile, we show the classification of finite time blow-up solutions with minimal mass for (1.1).

**Theorem 4.4** (Limiting profile with minimal mass). • Let  $d \ge 3$  and  $0 < c < \lambda(d)$ . Let  $u_0 \in H^1$ be such that  $||u_0||_{L^2} = M_{gs}$ . Suppose that the corresponding solution u to (1.1) blows up at finite time  $0 < T < +\infty$ . Then for any time sequence  $(t_n)_{n\ge 1}$  satisfying  $t_n \uparrow T$ , there exist a subsequence still denoted by  $(t_n)_{n\ge 1}$ , a function  $Q \in \mathcal{G}$ , sequences of  $\theta_n \in \mathbb{R}$ ,  $\lambda_n > 0$ ,  $\lambda_n \to 0$  and  $x_n \in \mathbb{R}^d$  such that

$$e^{it\theta_n}\lambda_n^{\frac{a}{2}}u(t_n,\lambda_n\cdot+x_n)\to Q \text{ strongly in } H^1,$$

$$(4.3)$$

as  $n \to \infty$ .

• Let  $d \geq 3$  and c < 0. Let  $u_0 \in H^1_{\text{rad}}$  satisfy  $||u_0||_{L^2} = M_{\text{gs,rad}}$ . Suppose that the corresponding solution u to (1.1) blows up at finite time  $0 < T < +\infty$ . Then for any time sequence  $(t_n)_{n\geq 1}$  satisfying  $t_n \uparrow T$ , there exist a subsequence still denoted by  $(t_n)_{n\geq 1}$ , a function  $Q_{\text{rad}} \in \mathcal{G}_{\text{rad}}$ , sequences of  $\vartheta_n \in \mathbb{R}$  and  $\rho_n > 0$ ,  $\rho_n \to 0$  such that

$$e^{it\vartheta_n}\rho_n^{\frac{2}{2}}u(t_n,\rho_n\cdot) \to Q_{\mathrm{rad}} \ strongly \ in \ H^1,$$

$$(4.4)$$

as  $n \to \infty$ .

*Proof.* Let us firstly consider the case  $0 < c < \lambda(d)$ . Let  $(t_n)_{n \geq 1}$  be a sequence such that  $t_n \uparrow T$ . Set

$$\lambda_n := \frac{\|Q_c\|_{\dot{H}_c^1}}{\|u(t_n)\|_{\dot{H}_c^1}}, \quad v_n(x) := \lambda_n^{\frac{d}{2}} u(t_n, \lambda_n x),$$

where  $Q_c$  is given in Theorem 2.1. By the blow-up alternative, we see that  $\lambda_n \to 0$  as  $n \to \infty$ . Moreover,

$$|v_n||_{L^2} = ||u(t_n)||_{L^2} = ||u_0||_{L^2} = M_{\rm gs}, \tag{4.5}$$

and

$$\|v_n\|_{\dot{H}^1_c} = \lambda_n \|u(t_n)\|_{\dot{H}^1_c} = \|Q_c\|_{\dot{H}^1_c}, \tag{4.6}$$

and

$$E(v_n) = \lambda_n^2 E(u(t_n)) = \lambda_n^2 E(u_0) \to 0$$

as  $n \to \infty$ . In particular,

$$\|v_n\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2} \to \frac{d+2}{d} \|Q_c\|_{\dot{H}^1_c}^2, \tag{4.7}$$

as  $n \to \infty$ . Thus the sequence  $(v_n)_{n \ge 1}$  satisfies conditions of Lemma 2.4 with

$$M^{2} = \|Q_{c}\|_{\dot{H}^{1}_{c}}^{2}, \quad m^{\frac{4}{d}+2} = \frac{d+2}{d} \|Q_{c}\|_{\dot{H}^{1}_{c}}^{2}.$$

Therefore, there exist  $V \in H^1$  and a sequence  $(x_n)_{n \ge 1}$  in  $\mathbb{R}^d$  such that up to a subsequence,

$$v_n(\cdot + x_n) = \lambda_n^{\frac{\alpha}{2}} u(t_n, \lambda_n \cdot + x_n) \rightharpoonup V$$
 weakly in  $H^1$ ,

as  $n \to \infty$  and  $\|V\|_{L^2} \ge \|Q_c\|_{L^2} = M_{gs}$ . Since  $v_n(\cdot + x_n) \rightharpoonup V$  weakly in  $H^1$  as  $n \to \infty$ , the semi-continuity of weak convergence and (4.5) imply

$$M_{\rm gs} \le \|V\|_{L^2} \le \liminf_{n \to \infty} \|v_n\|_{L^2} = M_{\rm gs}$$

This shows that

$$\|V\|_{L^2} = \lim_{n \to \infty} \|v_n\|_{L^2} = M_{\rm gs}.$$
(4.8)

Therefore,  $v_n(\cdot + x_n) \to V$  strongly in  $L^2$  as  $n \to \infty$ . By the sharp Gagliardo-Nirenberg inequality (4.1), we also have that  $v_n(\cdot + x_n) \to V$  strongly in  $L^{\frac{4}{d}+2}$  as  $n \to \infty$ . Indeed,

$$\begin{aligned} \|v_{n}(\cdot+x_{n})-V\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2} &\leq C_{\mathrm{GN}}(c)\|v_{n}(\cdot+x_{n})-V\|_{L^{2}}^{\frac{4}{d}}\|v_{n}(\cdot+x_{n})-V\|_{\dot{H}_{c}^{1}}^{2}\\ &\lesssim C_{\mathrm{GN}}(c)\left(\|Q_{c}\|_{\dot{H}_{c}^{1}}^{2}+\|V\|_{\dot{H}_{c}^{1}}^{2}\right)\|v_{n}(\cdot+x_{n})-V\|_{L^{2}}^{\frac{4}{d}}\to 0,\end{aligned}$$

as  $n \to \infty$ . Here we use

$$\|v_n(\cdot + x_n)\|_{\dot{H}^1_x} \sim \|v_n(\cdot + x_n)\|_{\dot{H}^1} = \|v_n\|_{\dot{H}^1} \sim \|v_n\|_{\dot{H}^1_x}$$

in the second estimate. Moreover, using (4.7), (4.8) and the sharp Gagliardo-Nirenberg inequality (4.1), we get

$$\|Q_c\|_{\dot{H}^1_c}^2 = \frac{d}{d+2} \lim_{n \to \infty} \|v_n\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2} = \frac{d}{d+2} \|V\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2} \le \left(\frac{\|V\|_{L^2}}{M_{\rm gs}}\right)^{\frac{1}{d}} \|V\|_{\dot{H}^1_c}^2 = \|V\|_{\dot{H}^1_c}^2$$

Thus the semi-continuity of weak convergence and (4.6) imply

$$\|Q_c\|_{\dot{H}^1_c} \le \|V\|_{\dot{H}^1_c} \le \liminf_{n \to \infty} \|v_n\|_{\dot{H}^1_c} = \|Q_c\|_{\dot{H}^1_c}$$

Hence

$$\|V\|_{\dot{H}^{1}_{c}} = \lim_{n \to \infty} \|v_{n}\|_{\dot{H}^{1}_{c}} = \|Q_{c}\|_{\dot{H}^{1}_{c}}$$

We next claim that

$$v_n(\cdot + x_n) \to V$$
 strongly in  $\dot{H}^1$ 

as  $n \to \infty$ . Since

$$V\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2} = \frac{d+2}{d} \|Q_c\|_{\dot{H}^1_c}^2, \quad \|V\|_{\dot{H}^1_c} = \|Q_c\|_{\dot{H}^1_c}$$

we see that E(V) = 0. It follows that there exists  $V \in H^1$  such that

$$||V||_{L^2} = M_{\rm gs}, \quad E(V) = 0.$$

The variational structure of ground states given in Lemma 4.3 shows that there exists  $Q \in \mathcal{G}$  such that  $V(x) = e^{i\theta}\lambda^{\frac{d}{2}}Q(\lambda x)$  for some  $\theta \in \mathbb{R}$  and  $\lambda > 0$ . Thus,

$$v_n(\cdot + x_n) = \lambda_n^{\frac{2}{2}} u(t_n, \lambda_n \cdot + x_n) \to V = e^{i\theta} \lambda^{\frac{d}{2}} Q(\lambda \cdot) \text{ strongly in } H^1,$$

as  $n \to \infty$ . Redefining  $\tilde{\lambda}_n := \lambda_n \lambda^{-1}$ , we obtain

$$e^{-i\theta}\tilde{\lambda}_n^{\frac{n}{2}}u(t_n,\tilde{\lambda}_n\cdot+x_n)\to Q$$
 strongly in  $H^1$ 

as  $n \to \infty$ . We now prove the claim. Since  $v_n(\cdot + x_n) \rightharpoonup V$  weakly in  $H^1$ . Set  $r_n(x) := v_n(x) - V(x - x_n)$ . We see that  $r_n(\cdot + x_n) \rightharpoonup 0$  weakly in  $H^1$ . By the same argument as in the proof of Proposition 2.3, we have

$$\int |x|^{-2} |v_n(x)|^2 dx = \int |x|^{-2} |V(x - x_n)|^2 dx + \int |x|^{-2} |r_n(x)|^2 dx + o_n(1).$$

In particular, we have

$$\begin{aligned} \|v_n\|_{\dot{H}_c^1}^2 &= \|V(\cdot - x_n)\|_{\dot{H}_c^1}^2 + \|r_n\|_{\dot{H}_c^1}^2 + o_n(1), \\ \|v_n\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2} &= \|V(\cdot - x_n)\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2} + \|r_n\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2} + o_n(1), \end{aligned}$$

as  $n \to \infty$ . Thus,

$$E(v_n) = E(V(\cdot - x_n)) + E(r_n) + o_n(1),$$
(4.9)

as  $n \to \infty$ . On the other hand, since  $v_n(\cdot + x_n) \to V$  strongly in  $L^2$ , it follows that  $r_n(\cdot + x_n) \to 0$ strongly in  $L^2$ . This implies in particular that  $r_n \to 0$  strongly in  $L^2$  and  $r_n \to 0$  weakly in  $H^1$ . The sharp Gagliardo-Nireberg inequality (4.1) then implies  $r_n \to 0$  strongly in  $L^{\frac{4}{d}+2}$ . By the semi-continuity of weak convergence,

$$0 \leq \frac{1}{2} \liminf_{n \to \infty} \|r_n\|_{\dot{H}^1_c}^2 = \frac{1}{2} \liminf_{n \to \infty} \|r_n\|_{\dot{H}^1_c}^2 - \frac{d}{2d+4} \liminf_{n \to \infty} \|r_n\|_{L^{\frac{4}{4}+2}}^4$$
$$\leq \liminf_{n \to \infty} \left(\frac{1}{2} \|r_n\|_{\dot{H}^1_c}^2 - \frac{d}{2d+4} \|r_n\|_{L^{\frac{4}{4}+2}}^4\right)$$
$$= \liminf_{n \to \infty} E(r_n).$$

In particular,

$$\liminf_{n \to \infty} E(V(\cdot - x_n)) \le \liminf_{n \to \infty} E(V(\cdot - x_n)) + \liminf_{n \to \infty} E(r_n)$$
$$\le \liminf_{n \to \infty} (E(V(\cdot - x_n)) + E(r_n)) = \liminf_{n \to \infty} E(v_n) = 0.$$

We also have from the sharp Gagliardo-Nirenberg inequality (4.1) and the fact  $||V(\cdot - x_n)||_{L^2} = ||V||_{L^2} = M_{gs}$  that  $E(V(\cdot - x_n)) \ge 0$  for all  $n \ge 1$ . Therefore, we must have

$$\liminf_{n \to \infty} E(V(\cdot - x_n)) = 0.$$

Taking lim inf both sides of (4.9), we obtain  $\liminf_{n\to\infty} E(r_n) = 0$ . Since  $r_n \to 0$  strongly in  $L^{\frac{4}{d}+2}$ , we see that up to a subsequence,  $\lim_{n\to\infty} ||r_n||_{\dot{H}^1_c} = 0$ . Using the equivalence  $||\cdot||_{\dot{H}^1_c} \sim ||\cdot||_{\dot{H}^1}$ , we obtain  $\lim_{n\to\infty} ||\nabla r_n||_{L^2} = 0$ . Thanks to the expansion

$$\nabla v_n \|_{L^2}^2 = \|\nabla V\|_{L^2}^2 + \|\nabla r_n\|_{L^2}^2 + o_n(1),$$

as  $n \to \infty$ , we obtain

$$\lim_{n\to\infty} \|\nabla v_n\|_{L^2} = \|\nabla V\|_{L^2}.$$

Since  $v_n(\cdot + x_n) \rightarrow V$  weakly in  $H^1$ , we infer that  $v_n(\cdot + x_n) \rightarrow V$  strongly in  $\dot{H}^1$  as  $n \rightarrow \infty$ . This proves the claim and the proof of the first item is complete.

We now consider the case c < 0. Let  $(t_n)_{n \ge 1}$  be a sequence such that  $t_n \uparrow T$ . Denote

$$\rho_n := \frac{\|Q_{c, \text{rad}}\|_{\dot{H}_c^1}}{\|u(t_n)\|_{\dot{H}_c^1}}, \quad v_n(x) := \rho_n^{\frac{d}{2}} u(t_n, \rho_n x),$$

where  $Q_{c,\text{rad}}$  is given in Theorem 2.1. Since  $u_0 \in H^1_{\text{rad}}$ , we see that  $u(t) \in H^1_{\text{rad}}$  for any t as long as the solution exists. By the blow-up alternative, it follows that  $\rho_n \to 0$  as  $n \to \infty$ . We also have

$$\|v_n\|_{L^2} = \|u(t_n)\|_{L^2} = \|u_0\|_{L^2} = M_{\text{gs,rad}}, \quad \|v_n\|_{\dot{H}^1_c} = \rho_n \|u(t_n)\|_{\dot{H}^1_c} = \|Q_{c,\text{rad}}\|_{\dot{H}^1_c}, \tag{4.10}$$

and

$$E(v_n) = \rho_n^2 E(u(t_n)) = \rho_n^2 E(u_0) \to 0,$$

as  $n \to \infty$ . In particular,

$$\|v_n\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2} \to \frac{d+2}{d} \|Q_{c,\mathrm{rad}}\|_{\dot{H}^1_c}^2, \tag{4.11}$$

as  $n \to \infty$ . We thus obtain a bounded sequence  $(v_n)_{n\geq 1}$  of  $H^1_{\text{rad}}$ -functions which satisfies conditions of Lemma 2.4 with

$$M^{2} = \|Q_{c,\mathrm{rad}}\|_{\dot{H}^{1}_{c}}^{2}, \quad m^{\frac{4}{d}+2} = \frac{d+2}{d} \|Q_{c,\mathrm{rad}}\|_{\dot{H}^{1}_{c}}^{2}.$$

We learn from Lemma 2.4 that there exists  $V \in H^1_{rad}$  such that up to a subsequence

$$v_n \rightharpoonup V$$
 weakly in  $H^1$ ,

as  $n \to \infty$  and  $||V||_{L^2} \ge ||Q_{c,rad}||_{L^2} = M_{gs,rad}$ . The semi-continuity of weak convergence and (4.10) imply that

$$M_{\rm gs,rad} \le \|V\|_{L^2} \le \liminf_{n \to \infty} \|v_n\|_{L^2} = M_{\rm gs,rad}.$$
 (4.12)

We thus get

$$||V||_{L^2} = \lim_{n \to \infty} ||v_n||_{L^2} = M_{\text{gs,rad}}$$

In particular,  $v_n \to V$  strongly in  $L^2$  as  $n \to \infty$ . This together with the sharp Gagliardo-Nirenberg inequality (4.2) yield that  $v_n \to V$  strongly in  $L^{\frac{4}{d}+2}$  as  $n \to \infty$ . By (4.11) and (4.12), the sharp Gagliardo-Nirenberg inequality implies that

$$\|Q_{c,\mathrm{rad}}\|_{\dot{H}^1_c}^2 = \frac{d}{d+2} \lim_{n \to \infty} \|v_n\|_{L^{\frac{4}{d}+2}}^4 = \frac{d}{d+2} \|V\|_{L^{\frac{4}{d}+2}}^4 \le \left(\frac{\|V\|_{L^2}}{M_{\mathrm{gs,rad}}}\right)^{\frac{1}{d}} \|V\|_{\dot{H}^1_c}^2 = \|V\|_{\dot{H}^1_c}^2.$$

Using the above inequality, the semi-continuity of weak convergence and (4.10) imply

$$||Q_{c,\mathrm{rad}}||_{\dot{H}^1_c} \le ||V||_{\dot{H}^1_c} \le \liminf_{n \to \infty} ||v_n||_{\dot{H}^1_c} = ||Q_{c,\mathrm{rad}}||_{\dot{H}^1_c}.$$

Hence

$$\|V\|_{\dot{H}^{1}_{c}} = \lim_{n \to \infty} \|v_{n}\|_{\dot{H}^{1}_{c}} = \|Q_{c, \text{rad}}\|_{\dot{H}^{1}_{c}}$$

We now claim that

$$v_n \to V$$
 strongly in  $H^1$ 

as  $n \to \infty$ . To see this, we write

$$v_n(x) = V(x) + r_n(x),$$

with  $r_n \rightarrow 0$  weakly in  $H^1$  as  $n \rightarrow \infty$ . We easily verify that

$$E(v_n) = E(V) + E(r_n) + o_n(1),$$

as  $n \to \infty$ . Since  $v_n \to V$  strongly in  $L^2$ , we see that  $r_n \to 0$  strongly in  $L^2$ . The sharp Gagliardo-Nirenberg inequality (4.2) then implies that  $r_n \to 0$  strongly in  $L^{\frac{4}{d}+2}$ . Arguing as in the case  $0 < c < \lambda(d)$ , we get

$$\liminf_{n \to \infty} E(r_n) \ge 0$$

and

$$E(V) \le E(V) + \liminf_{n \to \infty} E(r_n) \le \liminf_{n \to \infty} (E(V) + E(r_n)) = \liminf_{n \to \infty} E(v_n) = 0.$$

On the other hand, since  $||V||_{L^2} = M_{gs,rad}$ , the sharp Gagliardo-Nirenberg inequality (4.2) implies that  $E(V) \ge 0$ . Therefore, E(V) = 0. As a result, we obtain that

$$\liminf_{n \to \infty} E(r_n) = 0.$$

Since  $r_n \to 0$  strongly in  $L^{\frac{4}{d}+2}$ , we see that up to a subsequence,  $\lim_{n\to\infty} ||r_n||_{\dot{H}^1_c} = 0$ . This implies in particular that  $\lim_{n\to\infty} ||\nabla r_n||_{L^2} = 0$ . Using the fact

$$\|\nabla v_n\|_{L^2}^2 = \|\nabla V\|_{L^2}^2 + \|\nabla r_n\|_{L^2}^2 + o_n(1),$$

as  $n \to \infty$ , we obtain  $\lim_{n\to\infty} \|\nabla v_n\|_{L^2} = \|\nabla V\|_{L^2}$ . Since  $v_n \to V$  weakly in  $H^1$  as  $n \to \infty$ , it follows that  $v_n \to V$  strongly in  $\dot{H}^1$  as  $n \to \infty$ . This proves the claim.

We thus obtain  $V \in H^1_{\text{rad}}$  such that

$$||V||_{L^2} = M_{\text{gs,rad}}, \quad E(V) = 0$$

The second equality follows from

$$\|V\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2} = \frac{d+2}{d} \|Q_{c,\mathrm{rad}}\|_{\dot{H}^{1}_{c}}^{2}, \quad \|V\|_{\dot{H}^{1}_{c}} = \|Q_{c,\mathrm{rad}}\|_{\dot{H}^{1}_{c}}^{1}$$

The variational structure of radial ground states given in Lemma 4.3 implies that there exists  $Q_{\text{rad}} \in \mathcal{G}_{\text{rad}}$ such that  $V(x) = e^{i\vartheta} \rho^{\frac{d}{2}} Q(\rho x)$  for some  $\vartheta \in \mathbb{R}$  and  $\rho > 0$ . We thus obtain

$$v_n(\cdot) = \rho_n^{\frac{2}{2}} u(t_n, \rho_n \cdot) \to V = e^{i\vartheta} \rho^{\frac{d}{2}} Q_{\text{rad}}(\rho \cdot)$$
 strongly in  $H^1$ ,

as  $n \to \infty$ . Redefining  $\tilde{\rho}_n := \rho_n \rho^{-1}$ , we obtain

$$e^{-i\vartheta}\tilde{\rho}_n^{\frac{\mu}{2}}u(t_n,\tilde{\rho}_n\cdot) \to Q_{\mathrm{rad}} \text{ strongly in } H^1,$$

as  $n \to \infty$ . The proof is complete.

In order to prove the characterization of finite time blow-up solutions with minimal mass, we need to recall basic facts related to (1.1). Let us start with the following Cauchy-Schwarz inequality due to Banica [1].

Lemma 4.5. If one of the following conditions holds true

- $d \ge 3, \ 0 < c < \lambda(d)$  and  $u \in H^1$  is such that  $||u||_{L^2} = M_{gs}$ ,
- $d \ge 3$ , c < 0 and  $u \in H^1_{\text{rad}}$  is such that  $||u||_{L^2} = M_{\text{gs,rad}}$ ,

then for any real valued function  $\varphi \in C^1$  satisfying  $\nabla \varphi$  is bounded, we have

$$\left| \int \nabla \varphi \cdot \operatorname{Im} (u \nabla \overline{u}) dx \right| \le \sqrt{2E(u)} \left( \int |\nabla \varphi|^2 |u|^2 dx \right)^{1/2}.$$
(4.13)

Note that by sharp Gagliardo Nirenberg inequalities (4.1) and (4.2), the above assumptions imply E(u) is non-negative.

We also need the following virial identity (see e.g. [8, Lemma 5.3] or [7, Lemma 3, p.124]).

**Lemma 4.6** (Virial identity). Let  $d \geq 3$  and  $c \neq 0$  be such that  $c < \lambda(d)$ . Let  $u_0 \in H^1$  be such that  $|x|u_0 \in L^2$  and  $u : I \times \mathbb{R}^d \to \mathbb{C}$  the corresponding solution to (1.1). Then  $|x|u \in C(I, L^2)$  and for any  $t \in I$ ,

$$\frac{d^2}{dt^2} \|xu(t)\|_{L^2}^2 = 16E(u_0).$$
(4.14)

In particular, we have for any  $t \in I$ ,

$$\int |x|^2 |u(t)|^2 dx = \int |x|^2 |u_0|^2 dx - 4t \int x \cdot \operatorname{Im} (u_0 \nabla \overline{u}_0) dx + 8t^2 E(u_0)$$

$$= 8t^2 E\left(e^{i\frac{|x|^2}{4t}}u_0\right).$$
(4.15)

*Proof.* We refer the reader to [8, Lemma 5.3] or [7, Lemma 3, p.124] for the proof of (4.14). The first identity in (4.15) follows by integrating (4.14) over the time t. The second identity in (4.15) follows from a direct computation using the fact that

$$\left| \nabla \left( e^{i \frac{|x|^2}{4t}} u_0 \right) \right| = \frac{1}{4t^2} |x|^2 |u_0|^2 - \frac{1}{t} x \cdot \operatorname{Im} (u_0 \nabla \overline{u}_0) + |\nabla u_0|^2.$$

The proof is complete.

We are now able to prove the characterization of finite time blow-up solutions with minimal mass given in Theorem 1.4.

Proof of Theorem 1.4. Let us firstly consider the case  $0 < c < \lambda(d)$ . Let  $(t_n)_{n \ge 1}$  be such that  $t_n \uparrow T$ . By Theorem 4.4, we see that up to a subsequence, there exists  $Q \in \mathcal{G}$  such that

$$e^{i\theta_n}\lambda_n^{\frac{d}{2}}u(t_n,\lambda_n\cdot+x_n)\to Q$$
 strongly in  $H^1$ , (4.16)

as  $n \to \infty$ , where  $(\theta_n)_{n \ge 1} \subset \mathbb{R}, (x_n)_{n \ge 1} \subset \mathbb{R}^d$  and  $\lambda_n \to 0$  as  $n \to \infty$ . From this, we infer that

$$|u(t_n, x)|^2 dx - ||Q||_{L^2}^2 \delta_{x=x_n} \rightharpoonup 0,$$
(4.17)

as  $n \to \infty$ .

Up to subsequence, we may assume that  $x_n \to x_0 \in \{0, \infty\}$ . Now let  $\varphi$  be a smooth non-negative radial compactly supported function satisfying

$$\varphi(x) = |x|^2$$
 if  $|x| < 1$ , and  $|\nabla \varphi(x)|^2 \le C \varphi(x)$ ,

for some constant C > 0. For R > 1, we define

$$\varphi_R(x) := R^2 \varphi(x/R), \quad U_R(t) := \int \varphi_R(x) |u(t,x)|^2 dx$$

Using the Cauchy-Schwarz inequality (4.13) and the fact  $|\nabla \varphi_R|^2 \leq C |\varphi_R|$ , we have

$$\begin{aligned} |U_R'(t)| &= 2 \left| \int \nabla \varphi_R \cdot \operatorname{Im} (u(t) \nabla \overline{u}(t)) dx \right| \\ &\leq 2\sqrt{2E(u_0)} \left( \int |u(t)|^2 |\nabla \varphi_R|^2 dx \right)^{1/2} \\ &\leq C(u_0) \sqrt{U_R(t)}. \end{aligned}$$

Integrating with respect to t, we obtain

$$\left|\sqrt{U_R(t)} - \sqrt{U_R(t_n)}\right| \le C(u_0)|t_n - t|.$$

$$(4.18)$$

Thanks to (4.17), we see that  $U_R(t_n) \to 0$  as  $n \to \infty$ . Indeed, if  $|x_n| \to 0$ , then  $U_R(t_n) \to ||Q||_{L^2}^2 \varphi_R(0) = 0$  as  $n \to \infty$ . If  $|x_n| \to \infty$ , then  $U_R(t_n) \to 0$  since  $\varphi_R$  is compactly supported. Letting  $n \to \infty$  in (4.18), we obtain

$$U_R(t) \le C(u_0)(T-t)^2.$$

Now fix  $t \in [0, T)$ , letting  $R \to \infty$ , we have

$$8t^{2}E\left(e^{i\frac{|x|^{2}}{4t}}u_{0}\right) = \int |x|^{2}|u(t,x)|^{2}dx \le C(u_{0})(T-t)^{2},$$
(4.19)

where the first equality follows from Lemma 4.6. Note that we have from (4.19) that  $u(t) \in L^2(|x|^2 dx)$  for any  $t \in [0, T)$ . We also have from (4.17) and (4.19) that

$$\liminf_{n \to \infty} |x_n|^2 ||Q||_{L^2}^2 \le C(u_0) T^2.$$

Thus  $x_n$  cannot go to infinity, hence  $x_n$  converges to zero. Letting t tends to T, we learn from (4.19) that

$$E\left(e^{i\frac{|x|^2}{4T}}u_0\right) = 0.$$

We also have

$$\left\| e^{i\frac{|x|^2}{4T}} u_0 \right\|_{L^2} = \|u_0\|_{L^2} = M_{\rm gs}$$

Therefore, Lemma 4.3 shows that there exists  $\tilde{Q} \in \mathcal{G}$  such that

$$e^{i\frac{|x|^2}{4T}}u_0(x) = e^{i\tilde{\theta}}\tilde{\lambda}^{\frac{d}{2}}\tilde{Q}(\tilde{\lambda}x).$$

Redefining  $\tilde{\lambda} = \frac{\lambda}{T}$  and  $\tilde{\theta} = \theta + \frac{\lambda^2}{T}$ , we obtain

$$u_0(x) = e^{i\theta} e^{i\frac{\lambda^2}{T}} e^{-i\frac{|x|^2}{4T}} \left(\frac{\lambda}{T}\right)^{\frac{3}{2}} \tilde{Q}\left(\frac{\lambda x}{T}\right)$$

This shows (1.7). By the uniqueness of solution to (1.1), we find that  $u(t) = S_{\tilde{Q},T,\theta,\lambda}(t)$  for any  $t \in [0,T)$ . This completes the proof of the case  $0 < c < \lambda(d)$ .

Let us now consider the case c < 0. Let  $(t_n)_{n \ge 1}$  be such that  $t_n \uparrow T$ . We have from Theorem 4.4 that up to a subsequence, there exists  $Q_{\text{rad}} \in \mathcal{G}_{\text{rad}}$  such that

$$e^{i\vartheta_n}\rho_n^{\overline{2}}u(t_n,\rho_n\cdot) \to Q_{\mathrm{rad}}$$
 strongly in  $H^1$ ,

as  $n \to \infty$ , where  $(\vartheta_n)_{n \ge 1} \subset \mathbb{R}$  and  $\rho_n \to 0$  as  $n \to \infty$ . This implies that

$$|u(t_n, x)|^2 dx - ||Q_{\text{rad}}||_{L^2}^2 \delta_{x=0} \rightharpoonup 0, \qquad (4.20)$$

as  $n \to \infty$ . By the same argument as in the case  $0 < c < \lambda(d)$ , we learn that

$$\left|\sqrt{U_R(t)} - \sqrt{U_R(t_n)}\right| \le C(u_0)|t_n - t|.$$

Here  $U_R(t_n) \to 0$  as  $n \to \infty$ . Indeed, by (4.20),  $U_R(t_n) \to ||Q_{\rm rad}||^2_{L^2} \varphi_R(0) = 0$  as  $n \to \infty$ . Therefore, letting  $n \to \infty$ , we obtain

$$U_R(t) \le C(u_0)(T-t)^2$$

Fix  $t \in [0, T)$ , letting  $R \to \infty$ , we obtain

$$8t^{2}E\left(e^{i\frac{|x|^{2}}{4t}}u_{0}\right) = \int |x|^{2}|u(t,x)|^{2}dx \leq C(u_{0})(T-t)^{2}.$$

Letting  $t \uparrow T$ , we get

$$E\left(e^{i\frac{|x|^2}{4T}}u_0\right) = 0,$$

and also

$$\left\| e^{i\frac{|x|^2}{4T}} u_0 \right\|_{L^2} = \|u_0\|_{L^2} = M_{\text{gs,rad}}.$$

By Lemma 4.3, there exists  $\tilde{Q}_{rad} \in \mathcal{G}_{rad}$  such that

$$e^{i\frac{|x|^2}{4T}}u_0(x) = e^{i\tilde{\vartheta}}\tilde{\rho}^{\frac{d}{2}}\tilde{Q}_{\rm rad}(\tilde{\rho}x)$$

Redefining  $\tilde{\rho} = \frac{\rho}{T}$  and  $\tilde{\vartheta} = \vartheta + \frac{\rho^2}{T}$ , we get

$$u_0 = e^{i\vartheta} e^{i\frac{\rho^2}{T}} e^{-i\frac{|x|^2}{4T}} \left(\frac{\rho}{T}\right)^{\frac{d}{2}} \tilde{Q}_{\rm rad}\left(\frac{\rho x}{T}\right)$$

This shows (1.8). The proof is complete.

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