

# Enhanced group classification of nonlinear diffusion–reaction equations with gradient-dependent diffusivity

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We carry out the enhanced group classification of a class of (1+1)-dimensional nonlinear diffusion–reaction equations with gradient-dependent diffusivity using the two-step version of the method of furcate splitting. For simultaneously finding the equivalence groups of an unnormalized class of differential equations and a collection of its subclasses, we suggest an optimized version of the direct method. The optimization includes the preliminary study of admissible transformations within the entire class and the successive splitting of the corresponding determining equations with respect to arbitrary elements and their derivatives depending on auxiliary constraints associated with each of required subclasses. In the course of applying the suggested technique to subclasses of the class under consideration, we construct, for the first time, a nontrivial example of finite-dimensional effective generalized equivalence group. Using the method of Lie reduction and the generalized separation of variables, exact solutions of some equations under consideration are found.

## 1 Introduction

The recent researches in biology showed that diffusion–reaction equations draw an attention as a prototype model for pattern formation. The various patterns, such as fronts, spirals, targets etc., can be found in various types of diffusion–reaction systems depending on large discrepancies [24, 25]. The latest application of diffusion–reaction processes are connected to process of morphogenesis as well as can be relevant to animal coats and skin pigmentation [16]. Among other applications are ecological invasions, spread of epidemics, tumour growth and wound healing [45].

The initial purpose of the present paper was the group classification of the class  $\mathcal{R}$  of (1+1)-dimensional diffusion–reaction equations with a gradient-dependent diffusivity,

$$u_t = f(u_x)u_{xx} + g(u), \quad (1)$$

where  $f = f(u_x)$  and  $g = g(u)$  are smooth functions of their arguments with  $f \neq 0$ . This problem had been considered in [8] with several weaknesses (see the conclusion of the present paper), which made necessary to accurately study it once more. Using the classical method of Lie reduction and other techniques, we also planned to construct exact solutions of equations from the regular subclass of the class  $\mathcal{R}$  that are the most interesting from the Lie-symmetry point of view.

According to the formalized definition of a class of differential equations [35, 39], the complete system of auxiliary equations and inequalities for the arbitrary elements  $f$  and  $g$  of the class  $\mathcal{R}$  is given by

$$\begin{aligned} f_t = f_x = f_u = f_{u_t} = f_{u_{tt}} = f_{u_{tx}} = f_{u_{xx}} = 0, \quad f \neq 0, \\ g_t = g_x = g_{u_t} = g_{u_x} = g_{u_{tt}} = g_{u_{tx}} = g_{u_{xx}} = 0. \end{aligned} \quad (2)$$

The structure of the class  $\mathcal{R}$  is not nice from the point of view of equivalence transformations and Lie symmetries. This is why it is convenient to represent this class as a union of four subclasses,

$$\mathcal{R} = \mathcal{H} \cup \mathcal{L} \cup \mathcal{F} \cup \mathcal{C}.$$

The subclass  $\mathcal{H}$  of semilinear equations (called “nonlinear *heat* equations with source”) is singled out by the constraint  $f_{u_x} = 0$ . This class includes all linear equations from the class  $\mathcal{R}$ , which additionally satisfy the constraint  $g_{uu} = 0$  and are reduced by simple point transformations to the linear heat equation  $u_t = u_{xx}$ . The complete group classification of the subclass  $\mathcal{H}$  was carried out in [10] in the course of the group classification of the wider class of diffusion–reaction equations of the general form  $u_t = f(u)u_{xx} + g(u)$  with  $f \neq 0$ . See also [3, pp. 133–136] for an enhanced representation of these results.

The subclass  $\mathcal{L}$  consists of equations from the class  $\mathcal{R}$ , where the values of the arbitrary element  $f$  satisfy the constraint  $(u_x^2 f)_{u_x} = 0$ . The subclass  $\mathcal{L}$  is special because each equation from it can be *linearized* to a Kolmogorov equation. More specifically, the hodograph transformation  $\tilde{t} = t$ ,  $\tilde{x} = u$ ,  $\tilde{u} = x$  with  $(\tilde{t}, \tilde{x})$  and  $\tilde{u}$  being the new independent and dependent variables, respectively, maps an equation of the form (1), where  $f = cu_x^{-2}$  with  $c = \text{const} \neq 0$ , to the Kolmogorov equation

$$\tilde{u}_{\tilde{t}} = c\tilde{u}_{\tilde{x}\tilde{x}} - g(\tilde{x})\tilde{u}_{\tilde{x}}. \quad (3)$$

In particular, this means that the subclass  $\mathcal{L}$  has completely different point-transformation and Lie-symmetry properties in comparison with the other subclasses, and its group classification reduces to the group classification of the class of Kolmogorov equations, which was presented, e.g., in [40, Corollary 7] up to general point equivalence.

The subclass  $\mathcal{F}$  is singled out by the constraint  $g_u = 0$ . The singularity of the subclass  $\mathcal{F}$  is exhibited by properties of its equivalence transformations, see Section 2. In particular, the subclass  $\mathcal{F}$  admits an extension of equivalence group in comparison with the entire class  $\mathcal{R}$ , and it is mapped by a family of its equivalence transformations to its subclass  $\mathcal{F}'$  associated with the additional constraint  $g = 0$ . Thus, the group classification of the subclass  $\mathcal{F}$  reduces to the group classification of the subclass  $\mathcal{F}'$ . The latter subclass consists of “nonlinear *filtration* equations”, which are of the form (1) with  $f \neq 0$  and  $g = 0$ . The group classification of the class  $\mathcal{F}'$  was carried out in [1, 2]. Lists of inequivalent Lie-symmetry extensions in this class up to its complete equivalence group and up to a proper subgroup of this group can also be singled out from the corresponding lists for potential diffusion–convection equations, presented in [38], by selecting cases with the zero convection coefficient.

Each of the additional auxiliary constraints associated with subclasses  $\mathcal{H}$ ,  $\mathcal{L}$  and  $\mathcal{F}$  is related to a special case of solving the group classification problem for the class  $\mathcal{R}$ . This is why we call the *complement*  $\mathcal{C}$  of  $\mathcal{H} \cup \mathcal{L} \cup \mathcal{F}$  in  $\mathcal{R}$  the *regular subclass* of  $\mathcal{R}$ . It is associated, as a subclass of  $\mathcal{R}$ , with the system of the inequalities  $f_{u_x} \neq 0$ ,  $(u_x^2 f)_{u_x} \neq 0$  and  $g_u \neq 0$ .

Note that the union of the subclasses  $\mathcal{H}$ ,  $\mathcal{L}$  and  $\mathcal{F}$  is not disjoint since there are two nonempty intersections among the pairwise intersections of these subclasses,  $\mathcal{H} \cap \mathcal{F}$  and  $\mathcal{L} \cap \mathcal{F}$ . Unfortunately, there is no partition of the class  $\mathcal{R}$  into subclasses that is convenient for group classification. For example, the equivalence group of the subclass  $\mathcal{F} \setminus (\mathcal{H} \cup \mathcal{L})$  is merely a proper subgroup of the equivalence group of the subclass  $\mathcal{F}$ , which essentially complicates the group classification of  $\mathcal{F} \setminus (\mathcal{H} \cup \mathcal{L})$  in comparison with  $\mathcal{F}$ .

Although the group classifications for the subclasses  $\mathcal{H}$ ,  $\mathcal{L}$  and  $\mathcal{F}$  are in fact known, the complete exclusion of these subclasses from the consideration, as this was done in [8], is not natural. Thus, there are point transformations mapping equations from the subclass  $\mathcal{F}$  to equations from the subclass  $\mathcal{C}$ , and related Lie-symmetry extensions in  $\mathcal{F}$  are simpler than their counterparts in  $\mathcal{C}$ . The same claim is true for the subclass pair  $(\mathcal{L}, \mathcal{H})$ . Therefore, to solve the group classification problem for the class  $\mathcal{R}$ , we carry out the group classification of the regular subclass  $\mathcal{C}$  using the technique of furcate splitting [17, 26, 48], combine the result with the known group classification of the subclass  $\mathcal{F}$  and supplement it with the additional inequivalent cases of Lie-symmetry extension from the subclasses  $\mathcal{H}$  and  $\mathcal{L}$ .

As mentioned above, the initial purpose of the present paper was to correctly solve the group classification problem for the class  $\mathcal{R}$  but it was changed after the careful analysis of admissible and equivalence transformations within this class. The generalized equivalence group  $\hat{G}_{\mathcal{F}}^{\sim}$  of the subclass  $\mathcal{F}$  turns out to be nontrivial, and using it as a conditional generalized equivalence group of the class  $\mathcal{R}$  simplifies the computation in the course of the group classification of this class and makes it more consistent with the subclass hierarchy considered for the class  $\mathcal{R}$ . We also constructed an effective generalized equivalence group  $\hat{G}_{\mathcal{F}}^{\sim}$  of the subclass  $\mathcal{F}$ , which is perhaps the most interesting and unexpected result of the paper since it gives the first example of nontrivial finite-dimensional effective generalized equivalence group in the literature. Moreover, the group  $\hat{G}_{\mathcal{F}}^{\sim}$  is a proper but not normal subgroup of the group  $\bar{G}_{\mathcal{F}}^{\sim}$ , and hence it is not a unique effective generalized equivalence group  $\hat{G}_{\mathcal{F}}^{\sim}$  the class  $\mathcal{F}$ . One more interesting feature of the class  $\mathcal{F}$  is that its usual equivalence group is contained in no effective generalized equivalence group of this class.

The rest of the paper is organized as follows. In Section 2 we simultaneously compute the equivalence groups of the entire class  $\mathcal{R}$  and of the above subclasses of this class using an original optimized version of the direct method. The classification of Lie symmetries of equations from the class  $\mathcal{R}$  is presented in Section 3. Section 4 contains the first comprehensive description of the method of furcate splitting. The two-step version of this method is used in Section 5 to solve, as a part of the group classification problem for the entire class  $\mathcal{R}$ , the group classification problem for its regular subclass  $\mathcal{C}$ . In Section 6, we select the three most interesting cases in the classification list obtained for the subclass  $\mathcal{C}$  and construct exact solutions of the related equations using Lie reduction or the generalized separation of variables. Results of the paper are discussed in Section 7.

## 2 Equivalence transformations

For simultaneously finding the equivalence groups of an unnormalized class of differential equations and a collection of its subclasses, we suggest an optimized version of the direct method, which involves the preliminary study of admissible transformations within the entire class and the successive splitting of the determining equations for these transformations with respect to the corresponding arbitrary elements and their derivatives, depending on auxiliary constraints associated with each of required subclasses.

By definition, equivalence transformations for the class  $\mathcal{R}$  and its subclasses are point transformations in the joint space of the independent variables  $(t, x)$ , the dependent variable  $u$ , its first- and second-order derivatives and the arbitrary elements  $f$  and  $g$ . Due to specific form of the arbitrary elements  $f$  and  $g$ , these transformations can be defined on spaces with a smaller number of coordinates; cf. [28]. Since the arbitrary element  $f$  depends on  $u_x$ , this derivative should be among the coordinates of such a space although the corresponding transformation components can still be computed from the  $x$ - and  $u$ -components using the chain rule. Due to the evolution form of equations, the derivative  $u_t$  is not involved in the transformation components for  $u_x$  and thus can be excluded from the coordinates of such a space. As a result, the minimal list of coordinates for a space underlying equivalence transformations for the classes  $\mathcal{R}$ ,

$\mathcal{L}$ ,  $\mathcal{F}$  and  $\mathcal{C}$  is  $(t, x, u, u_x, f, g)$ . (The  $u_x$ -component of equivalence transformations was missed in [8, Theorem 2].) Since the subclass  $\mathcal{H}$  is associated with the constraint  $f_{u_x} = 0$ , the coordinate  $u_x$  can be neglected when defining equivalence transformations of this subclass but for unification we will not use this possibility. At the same time, in view of the constraint  $g = 0$  it is convenient to exclude the  $g$ -component when defining equivalence transformations of the subclass  $\mathcal{F}'$ .

In order to compute the equivalence groups of the class  $\mathcal{R}$  and its subclasses in a uniform way, we begin this computation with the preliminary study of admissible transformations in this class. Since the class  $\mathcal{R}$  consists of (1+1)-dimensional second-order evolution equations whose right hand sides are affine in the derivative  $u_{xx}$ , any contact admissible transformation in this class is a prolongation of a point admissible transformation [41]. Moreover, the  $t$ -component of any admissible transformation within  $\mathcal{R}$  depends only on  $t$ ; see the related assertions in [19] and [18] for contact and point transformations between (1+1)-dimensional evolution equations, respectively. Therefore, we consider a point transformation of independent and dependent variables of the form

$$\tilde{t} = T(t), \quad \tilde{x} = X(t, x, u), \quad \tilde{u} = U(t, x, u), \quad (4)$$

where  $T_t(X_x U_u - X_u U_x) \neq 0$ , that connects the source and the target equations from the class  $\mathcal{R}$ ,

$$u_t = f(u_x)u_{xx} + g(u) \quad \text{and} \quad \tilde{u}_{\tilde{t}} = \tilde{f}(\tilde{u}_{\tilde{x}})\tilde{u}_{\tilde{x}\tilde{x}} + \tilde{g}(\tilde{u}). \quad (5)$$

Proceeding with the direct method of finding the equivalence groupoid of a class of equations, we write the differentiation operators  $\partial_{\tilde{t}}$  and  $\partial_{\tilde{x}}$  with respect to new independent variables in terms of old ones as

$$\partial_{\tilde{t}} = \frac{1}{T_t} \left( D_t - \frac{D_t X}{D_x X} D_x \right), \quad \partial_{\tilde{x}} = \frac{1}{D_x X} D_x,$$

where  $D_t = \partial_t + u_t \partial_u + u_{tt} \partial_{u_t} + u_{tx} \partial_{u_x} + \dots$  and  $D_x = \partial_x + u_x \partial_u + u_{tx} \partial_{u_t} + u_{xx} \partial_{u_x} + \dots$  are the total derivative operators with respect to  $t$  and  $x$ , respectively. We substitute the expressions for  $\tilde{u}$ ,  $\tilde{u}_{\tilde{t}}$ ,  $\tilde{u}_{\tilde{x}}$  and  $\tilde{u}_{\tilde{x}\tilde{x}}$  in terms of old variables into the target equation,

$$\tilde{u}_{\tilde{x}} = V := \frac{D_x U}{D_x X}, \quad \tilde{u}_{\tilde{t}} = \frac{1}{T_t} (D_t U - V D_t X), \quad \tilde{u}_{\tilde{x}\tilde{x}} = \frac{D_x V}{D_x X},$$

replace  $u_t$  by  $f u_{xx} + g$  and successively split the equation obtained with respect to  $u_{xx}$ . This results in “expressions” for the target arbitrary elements  $\tilde{f}$  and  $\tilde{g}$ ,

$$\tilde{f}(V) = \frac{(D_x X)^2}{T_t} f(u_x), \quad (6)$$

$$\tilde{g}(U) = \frac{\Delta}{T_t D_x X} g(u) - \frac{D_x X}{T_t} (V_x + u_x V_u) f(u_x) + \frac{U_t D_x X - X_t D_x U}{T_t D_x X}, \quad (7)$$

where  $\Delta := X_x U_u - X_u U_x \neq 0$ . Since both the source and target arbitrary-element tuples satisfy the auxiliary system (2), the equations (6) and (7) imply further determining equations for admissible transformations in the class  $\mathcal{R}$ . There are two ways for deriving these determining equations.

The first way is to express the operators  $\partial_{\tilde{t}}$ ,  $\partial_{\tilde{x}}$ ,  $\partial_{\tilde{u}}$  and  $\partial_{\tilde{u}_{\tilde{x}}}$  in terms of the operators  $\partial_t$ ,  $\partial_x$ ,  $\partial_u$  and  $\partial_{u_x}$ , which act on functions of  $(t, x, u, u_x)$ , using the equalities implied by the chain rule for these operators:

$$\begin{aligned} \partial_t &= T_t \partial_{\tilde{t}} + X_t \partial_{\tilde{x}} + U_t \partial_{\tilde{u}} + V_t \partial_{\tilde{u}_{\tilde{x}}}, & \partial_x &= X_x \partial_{\tilde{x}} + U_x \partial_{\tilde{u}} + V_x \partial_{\tilde{u}_{\tilde{x}}}, & \partial_{u_x} &= V_{u_x} \partial_{\tilde{u}_{\tilde{x}}}. \\ \partial_u &= X_u \partial_{\tilde{x}} + U_u \partial_{\tilde{u}} + V_u \partial_{\tilde{u}_{\tilde{x}}}, \end{aligned}$$

Then one acts by the operators  $\partial_{\tilde{t}}$ ,  $\partial_{\tilde{x}}$  and  $\partial_{\tilde{u}}$  on the equation (6) and the operators  $\partial_{\tilde{t}}$ ,  $\partial_{\tilde{x}}$  and  $\partial_{\tilde{u}_{\tilde{x}}}$  on the equation (7). The determining equations derived in this way are appropriate

in order to solve the problem of describing the equivalence groupoid  $\mathcal{G}_{\mathcal{R}}^{\sim}$  of the class  $\mathcal{R}$ . This problem is reduced to the classification of admissible transformations, which is similar to but more complicated than the classification of Lie symmetries in Sections 3 and 5 below. It is not a subject of the present paper although we will need a partial classification of admissible transformations for proving Theorem 13.

We use the other way, which gives more compact determining equations for equivalence transformations. We solve the equations (6) and (7) with respect to  $f$  and  $g$ , respectively,

$$f(u_x) = \frac{T_t}{(D_x X)^2} \tilde{f}(V), \quad (8)$$

$$g(u) = \frac{T_t D_x X}{\Delta} \tilde{g}(U) + \frac{(D_x X)^2}{\Delta} (V_x + u_x V_u) f(u_x) - \frac{U_t D_x X - X_t D_x U}{\Delta}, \quad (9)$$

separately differentiate the equation (8) with respect to  $t$ ,  $x$  and  $u$ , then separately differentiate the equation (9) with respect to  $t$  and  $x$  and successively substitute for  $f$  in view of (8) as well as substitute the expression (8) for  $f$  into (9) and then differentiate the resulting equation with respect to  $u_x$ . This leads to the following classifying equations for admissible transformations within the class  $\mathcal{R}$ :

$$\frac{T_t V_z}{(D_x X)^2} \tilde{f}_{\tilde{u}_x}(V) + \left( \frac{T_t}{(D_x X)^2} \right)_z \tilde{f}(V) = 0, \quad z \in \{t, x, u\}, \quad (10)$$

$$\begin{aligned} \frac{T_t D_x X}{\Delta} U_y \tilde{g}_{\tilde{u}}(U) + \left( \frac{T_t D_x X}{\Delta} \right)_y \tilde{g}(U) + \left( \frac{(D_x X)^2}{\Delta} (V_x + u_x V_u) \right)_y \frac{T_t}{(D_x X)^2} \tilde{f}(V) \\ - \left( \frac{U_t D_x X - X_t D_x U}{\Delta} \right)_y = 0, \quad y \in \{t, x\}, \end{aligned} \quad (11)$$

$$\frac{X_u}{\Delta} \tilde{g}(U) + \frac{V_x + u_x V_u}{(D_x X)^2} \tilde{f}_{\tilde{u}_x}(V) + (V_u + V_{xu_x} + u_x V_{uu_x}) \frac{\tilde{f}(V)}{\Delta} - \frac{U_t X_u - X_t U_u}{T_t \Delta} = 0. \quad (12)$$

(We divided the last equation by  $T_t$ .) Since the  $t$ -,  $x$ - and  $u$ -components of usual equivalence transformations do not depend on the arbitrary elements  $f$  and  $g$ , in the course of computing the usual equivalence groups of the class  $\mathcal{R}$  and its subclasses we can split the classifying equations (10)–(12) with respect to all parametric derivatives of the target arbitrary elements including  $\tilde{f}$  and  $\tilde{g}$  themselves. At the same time, the complete system of classifying equations for admissible transformations within a subclass of the class  $\mathcal{R}$  may be more restrictive than the equations (10)–(12); see the proofs of propositions below.

**Remark 1.** Each equivalence transformation of any subclass of the class  $\mathcal{R}$  is completely defined by its the  $t$ -,  $x$ - and  $u$ -components. Indeed, if these components are known, then the  $u_x$ -component is computed by the chain rule, and the expressions for  $f$ - and  $g$ -components follow from the equations (6) and (7). This is why we do not discuss the derivation of the latter expressions below.

**Proposition 2.** *The usual equivalence group  $G_{\mathcal{R}}^{\sim}$  of the class  $\mathcal{R}$  coincides with the usual equivalence groups of its subclasses  $\mathcal{H}$ ,  $\mathcal{L}$  and  $\mathcal{C}$  and consists of the point transformations in the space with the coordinates  $(t, x, u, u_x, f, g)$ , whose components are of the form*

$$\begin{aligned} \tilde{t} &= T_1 t + T_0, & \tilde{x} &= X_1 x + X_0, & \tilde{u} &= U_2 u + U_0, & \tilde{u}_x &= \frac{U_2}{X_1} u_x, \\ \tilde{f} &= \frac{X_1^2}{T_1} f, & \tilde{g} &= \frac{U_2}{T_1} g, \end{aligned} \quad (13)$$

where  $T$ 's,  $X$ 's and  $U$ 's are arbitrary constants with  $T_1 X_1 U_2 \neq 0$ .

*Proof.* For each of the classes  $\mathcal{R}$ ,  $\mathcal{H}$ ,  $\mathcal{L}$  and  $\mathcal{C}$ , the arbitrary element  $\tilde{g}$  and its derivative  $\tilde{g}_{\tilde{u}}$  are not constrained. Collecting the coefficients of  $\tilde{g}_{\tilde{u}}$  and  $\tilde{g}$  in the equations (11) and (12), respectively, gives the determining equations  $U_t = U_x = 0$  and  $X_u = 0$ . Therefore,  $X_x U_u \neq 0$  and  $V = u_x U_u / X_x$ . Then we split the equation (11) with  $y = t$  with respect to  $\tilde{g}$  and derive  $T_{tt} = 0$ . The arbitrary element  $\tilde{f}$  can also be assumed as a parametric value for splitting, and its derivative  $\tilde{f}_{\tilde{u}_{\tilde{x}}}$  is either unconstrained or equal to zero or  $-2\tilde{f}/\tilde{u}_{\tilde{x}}$ . This is why we can select terms without arbitrary elements and their derivatives in the equation (12), which leads to  $X_t = 0$ .

Now we only need to obtain the equations  $X_{xx} = 0$  and  $U_{uu} = 0$ , which is done by considering separately cases with different constraints for  $\tilde{f}$ . Then the  $t$ -,  $x$ - and  $u$ -components of usual equivalence transformations have precisely the form (13), and further we follow Remark 1. All the constructed transformations preserve each of the systems of auxiliary constraints for the arbitrary elements that are associated with the classes  $\mathcal{R}$ ,  $\mathcal{H}$ ,  $\mathcal{L}$  and  $\mathcal{C}$ .

In particular, the values of  $\tilde{f}$  and  $\tilde{f}_{\tilde{u}_{\tilde{x}}}$  are not constrained by equations in the classes  $\mathcal{R}$  and  $\mathcal{C}$ . This allows us to split with respect to these values and get the equations  $V_z = 0$ , which are expanded, for  $z = x$  and  $z = u$  to the requested equations  $X_{xx} = 0$  and  $U_{uu} = 0$ , respectively.

In view of the constraint  $\tilde{f}_{\tilde{u}_{\tilde{x}}} = 0$  for the class  $\mathcal{H}$ , the equation (10) with  $z = x$  and the equation (12) respectively reduce to  $X_{xx} = 0$  and  $V_u + V_{xu_x} + u_x V_{uu_x} = 0$ . The expansion of the last equation implies  $U_{uu} = 0$ .

Since  $\tilde{f}_{\tilde{u}_{\tilde{x}}} = -2\tilde{f}/\tilde{u}_{\tilde{x}} \neq 0$  within the class  $\mathcal{L}$ , the equation (10) with  $z = u$  is then equivalent to  $V_u = 0$ , i.e.,  $U_{uu} = 0$ , and the equation (12) reduces to  $X_{xx} = 0$ .  $\square$

Solving the additional auxiliary equation  $(u_x^2 f)_{u_x} = 0$  for the arbitrary element  $f$  in the subclass  $\mathcal{L}$ , we obtain the representation  $f = cu_x^{-2}$  with an arbitrary nonzero constant  $c$ . If we reparameterize the subclass  $\mathcal{L}$  by taking this constant as a new arbitrary element instead of  $f$ , then the corresponding transformation component is  $\tilde{c} = U_2^2 T_1^{-1} c$ .

**Proposition 3.** *The usual equivalence group  $G_{\mathcal{F}}^{\sim}$  of the class  $\mathcal{F}$  is constituted by the point transformations in the space with the coordinates  $(t, x, u, u_x, f, g)$ , whose components are of the form*

$$\begin{aligned} \tilde{t} &= T_1 t + T_0, & \tilde{x} &= X_1 x + X_0, & \tilde{u} &= U_1 x + U_2 u + U_3 t + U_0, & \tilde{u}_{\tilde{x}} &= \frac{U_1 + U_2 u_x}{X_1}, \\ \tilde{f} &= \frac{(X_1)^2}{T_1} f, & \tilde{g} &= \frac{U_2}{T_1} g + \frac{U_3}{T_1}, \end{aligned} \quad (14)$$

where  $T$ 's,  $X$ 's and  $U$ 's are arbitrary constants with  $T_1 X_1 U_2 \neq 0$ .

*Proof.* For the class  $\mathcal{F}$  we should extend the system of the classifying equations (10)–(12) with one more equation by replacing the subscript  $y$  by  $z$  in the equation (11), which takes into account the additional auxiliary constraint  $g_u = 0$  of this class. Then we substitute  $\tilde{g}_{\tilde{u}} = 0$  into the extended system and split it with respect to the varying values  $\tilde{g}$ ,  $\tilde{f}$  and  $\tilde{f}_{\tilde{u}_{\tilde{x}}}$ . Thus, vanishing the coefficient of  $\tilde{g}$  and the term without the varying values in the equation (12) results in the equations  $X_u = 0$  (and hence  $X_x U_u \neq 0$ ) and  $X_t = 0$ . From the equation (10) we derive  $V_z = 0$  and  $T_{tt} = 0$ . We successively expand the equations  $V_z = 0$  for  $z = t$ ,  $z = u$  and  $z = x$  and split them with respect to  $u_x$ , obtaining

$$U_{tx} = U_{tu} = 0, \quad U_{xu} = U_{uu} = 0 \quad \text{and} \quad X_{xx} = U_{xx} = 0,$$

respectively. Then terms in the equation (11) without the varying values merely give  $U_{tt} = 0$ . The obtained equations for the transformations components constitute the complete system of determining equations for usual equivalence transformations of the class  $\mathcal{F}$  since the extended version of the system (10)–(12) is identically satisfied in view of the collection of these equations. Therefore, the components of all transformations from the group  $G_{\mathcal{F}}^{\sim}$  are of the form (14) and each transformation whose components are of the form (14) belongs to this group.  $\square$

The group  $G_{\mathcal{F}}^{\sim}$  is a nontrivial conditional usual equivalence group of the class  $\mathcal{R}$  under the condition  $g_u = 0$  since the usual equivalence group  $G_{\mathcal{R}}^{\sim}$  of the class  $\mathcal{R}$  is a proper subgroup of  $G_{\mathcal{F}}^{\sim}$ , which is singled out by the constraints  $U_1 = U_3 = 0$  for group parameters. Elements of  $G_{\mathcal{F}}^{\sim}$  with  $(U_1, U_3) \neq (0, 0)$  are purely conditional equivalence transformations for the class  $\mathcal{R}$  under the condition  $g_u = 0$ .

The family of transformations from  $G_{\mathcal{F}}^{\sim}$  with  $(t, x, u)$ -components  $\tilde{t} = t$ ,  $\tilde{x} = x$ ,  $\tilde{u} = u - gt$ , which is parameterized by the arbitrary element  $g$ , maps the class  $\mathcal{F}$  onto its subclass  $\mathcal{F}'$  singled out by the constraint  $g = 0$ . The usual equivalence group of the class  $\mathcal{F}'$  was found in [1, 2]. We can easily prove this result using the classifying equations (10)–(12).

**Proposition 4.** *The usual equivalence group  $G_{\mathcal{F}'}^{\sim}$  of the class  $\mathcal{F}'$  consists of the point transformations in the space with the coordinates  $(t, x, u, u_x, f)$ , whose components are of the form*

$$\begin{aligned} \tilde{t} &= T_1 t + T_0, & \tilde{x} &= X_1 x + X_2 u + X_0, & \tilde{u} &= U_1 x + U_2 u + U_0, & \tilde{u}_x &= \frac{U_1 + U_2 u_x}{X_1 + X_2 u_x}, \\ \tilde{f} &= \frac{(X_1 + X_2 u_x)^2}{T_1} f, \end{aligned}$$

where  $T$ 's,  $X$ 's and  $U$ 's are arbitrary constants with  $T_1(X_1 U_2 - X_2 U_1) \neq 0$ .

*Proof.* Restricted to the subclass  $\mathcal{F}'$  by the substitution  $g = 0$  and  $\tilde{g} = 0$ , the determining equation (9) itself becomes classifying for admissible transformations in this subclass. Therefore, it replaces its differential consequences, including the equations (11) and (12). The splitting of the equations (9) and (10) with respect  $\tilde{f}$  and  $\tilde{f}_{\tilde{u}_x}$  merely implies the equations

$$V_z = 0, \quad \left( \frac{T_t}{(D_x X)^2} \right)_z = 0, \quad z \in \{t, x, u\}, \quad U_t D_x X - X_t D_x U = 0, \quad (15)$$

which can be further split with respect to  $u_x$ . The second and the first equations of (15) with  $z \in \{x, u\}$  successively imply the equations  $X_{xx} = X_{ux} = X_{uu} = 0$  and  $U_{xx} = U_{ux} = U_{uu} = 0$ . The last equation of (15) splits into the system  $X_z U_t - U_z X_t = 0$  with  $z \in \{x, u\}$ , which has, as a homogeneous nondegenerate linear system of algebraic equations with respect to  $(X_t, U_t)$ , the zero solution only, i.e.,  $X_t = U_t = 0$ . Then the second equation with  $z = t$  is equivalent to  $T_{tt} = 0$ . The derived equations exhaustively define the  $(t, x, u)$ -components of equivalence transformations of the class  $\mathcal{F}'$ . In view of Remark 1, this completes the proof.  $\square$

The group  $G_{\mathcal{F}'}^{\sim}$  is a nontrivial conditional usual equivalence group of both the classes  $\mathcal{R}$  and  $\mathcal{F}$  under the condition  $g = 0$ . Elements of  $G_{\mathcal{F}'}^{\sim}$  with  $X_2 \neq 0$  have no counterparts in  $G_{\mathcal{R}}^{\sim}$  and  $G_{\mathcal{F}}^{\sim}$  and hence they are purely conditional equivalence transformations for the classes  $\mathcal{R}$  and  $\mathcal{F}$  under the condition  $g = 0$ .

The subgroup of  $G_{\mathcal{F}}^{\sim}$  preserving the subclass  $\mathcal{F}'$  of  $\mathcal{F}$  is associated with constraint  $U_3 = 0$ , and the projection to the space with the coordinates  $(t, x, u, u_x, f)$  maps this subgroup to a proper subgroup of  $G_{\mathcal{F}'}^{\sim}$ . This is why the group classification of the class  $\mathcal{F}$  up to  $G_{\mathcal{F}}^{\sim}$ -equivalence does not reduce to the group classification of the class  $\mathcal{F}'$  up to  $G_{\mathcal{F}'}^{\sim}$ -equivalence under the above map of the class  $\mathcal{F}$  onto its subclass  $\mathcal{F}'$ . To make the group classifications of the classes  $\mathcal{F}$  and  $\mathcal{F}'$  consistent, we should consider the stronger equivalence in the class  $\mathcal{F}$  that is associated with generalized equivalence group of this class.

**Proposition 5.** *The generalized equivalence group  $\bar{G}_{\mathcal{F}}^{\sim}$  of the class  $\mathcal{F}$  is constituted by the point transformations in the space with the coordinates  $(t, x, u, u_x, f, g)$ , whose components are of the form*

$$\begin{aligned} \tilde{t} &= \bar{T}^1 t + \bar{T}^0, & \tilde{x} &= \bar{X}^1 x + \bar{X}^2 u - g \bar{X}^2 t + \bar{X}^0, & \tilde{u} &= \bar{U}^1 x + \bar{U}^2 u + (\bar{T}^1 \bar{F} - g \bar{U}^2) t + \bar{U}^0, \\ \tilde{u}_x &= \frac{\bar{U}^1 + \bar{U}^2 u_x}{\bar{X}^1 + \bar{X}^2 u_x}, & \tilde{f} &= \frac{(\bar{X}^1 + \bar{X}^2 u_x)^2}{\bar{T}^1} f, & \tilde{g} &= \bar{F}, \end{aligned}$$

where  $\bar{T}$ 's,  $\bar{X}$ 's,  $\bar{U}$ 's and  $\bar{F}$  are arbitrary smooth functions of  $g$  with  $\bar{T}^1(\bar{X}^1 \bar{U}^2 - \bar{X}^2 \bar{U}^1) \bar{F}_g \neq 0$ .

*Proof.* By the definition of generalized equivalence transformations [20, 21, 28, 39], their  $t$ -,  $x$ - and  $u$ -components may in general depend on arbitrary elements. At the same time, these components of transformations from  $\tilde{G}_{\mathcal{F}}$  can depend only on  $g$  by the following reasons. The arbitrary element  $f$  depends on  $u_x$ . Each generalized equivalence transformation generates a family of admissible transformations parameterized by the arbitrary elements [28]. Each contact admissible transformation in the class  $\mathcal{F}$  is the prolongation of a point admissible transformation to the first-order derivatives of  $u$ , i.e., the class  $\mathcal{F}$  possesses no nontrivial contact admissible transformations.

We follow the proof of Proposition 3 but, due to potential dependence of the  $t$ -,  $x$ - and  $u$ -components of generalized equivalence transformations on  $g$ , here we cannot split with respect to this arbitrary element. This is the only essential difference with the proof of Proposition 3. Similarly to the proof of Proposition 4, the splitting of the equation (10) with respect  $\tilde{f}$  and  $\tilde{f}_{\tilde{u}_x}$  merely implies the equations

$$V_z = 0, \quad \left( \frac{T_t}{(D_x X)^2} \right)_z = 0, \quad z \in \{t, x, u\}. \quad (16)$$

The further splitting of the second and the first equations of (16) with  $z \in \{x, u\}$  with respect to  $u_x$  successively yields the equations  $X_{xx} = X_{ux} = X_{uu} = 0$  and  $U_{xx} = U_{ux} = U_{uu} = 0$ . Thus,  $\Delta_x = \Delta_u = 0$ . Then the equation (9) reduces to  $g\Delta + U_t D_x X - X_t D_x U = \tilde{g} T_t D_x X$ . Differentiating it with respect to  $x$  and  $u$  and splitting with respect to  $u_x$ , we obtain two systems,  $X_z U_{xt} - U_z X_{xt} = 0$  and  $X_z U_{ut} - U_z X_{ut} = 0$ , where  $z \in \{x, u\}$ , which are equivalent, in view of the condition  $\Delta \neq 0$ , to the equations  $X_{xt} = U_{xt} = 0$  and  $X_{ut} = U_{ut} = 0$ , respectively. Hence  $\Delta_t = 0$ , and differentiating the equation (9) with respect to  $t$  gives  $X_z U_{tt} - U_z X_{tt} = 0$  with  $z \in \{x, u\}$ , i.e., we also have  $X_{tt} = U_{tt} = 0$ . Taking into account the derived equations  $X_{xt} = X_{ut} = 0$  in the second equation of (16) with  $z = t$  immediately gives  $T_{tt} = 0$ .

In view of the constructed equations for admissible transformations, the  $(t, x, u)$ -components of generalized equivalence transformations within the class  $\mathcal{F}$  should be of the form

$$\tilde{t} = \bar{T}^1 t + \bar{T}^0, \quad \tilde{x} = \bar{X}^1 x + \bar{X}^2 u + \bar{X}^3 t + \bar{X}^0, \quad \tilde{u} = \bar{U}^1 x + \bar{U}^2 u + \bar{U}^3 t + \bar{U}^0,$$

where  $\bar{T}$ 's,  $\bar{X}$ 's and  $\bar{U}$ 's are smooth functions of  $g$  with  $\bar{T}^1 \neq 0$  and  $\bar{X}^1 \bar{U}^2 - \bar{X}^2 \bar{U}^1 = \Delta \neq 0$ . We represent the result of the splitting of the equation (9) with respect to  $u_x$  as the system

$$\bar{X}^1 \bar{U}^3 - \bar{U}^1 \bar{X}^3 = \bar{X}^1 T_t \tilde{g} - g\Delta, \quad \bar{X}^2 \bar{U}^3 - \bar{U}^2 \bar{X}^3 = \bar{X}^2 T_t \tilde{g}. \quad (17)$$

It is clear from this system that the  $g$ -component of any generalized equivalence transformation within the class  $\mathcal{F}$  is a function  $\bar{F}$  of  $g$  only, and we can choose this function as a parameter function merely constrained by the inequality  $\bar{F}_g \neq 0$ , which is needed for the transformation nondegeneracy. Then the system (17) considered as a linear system of algebraic equations with respect to  $(\bar{X}^3, \bar{U}^3)$  possesses a unique solution resulting in the form of transformations from proposition's statement.  $\square$

The usual equivalence group  $G_{\mathcal{F}}$  is a (finite-dimensional) subgroup of the generalized equivalence group  $\tilde{G}_{\mathcal{F}}$  that is singled out from  $\tilde{G}_{\mathcal{F}}$  by the following system of constraints for the group parameters:

$$\bar{T}_g^0 = \bar{T}_g^1 = 0, \quad \bar{X}_g^0 = \bar{X}_g^1 = 0, \quad \bar{X}^2 = 0, \quad \bar{U}_g^0 = \bar{U}_g^1 = \bar{U}_g^2 = 0, \quad \bar{T}^1 \bar{F}_g = \bar{U}^2.$$

Denote by  $\mathcal{G}_{\mathcal{F}}$  the equivalence groupoid of the class  $\mathcal{F}$  and by  $\mathcal{S}_{\mathcal{F}}$  the subgroupoid of  $\mathcal{G}_{\mathcal{F}}$  generated by the generalized equivalence group  $\tilde{G}_{\mathcal{F}}$ . The subgroupoid of  $\mathcal{G}_{\mathcal{F}}$  generated by the usual equivalence group  $G_{\mathcal{F}}$  is a proper subgroupoid of  $\mathcal{S}_{\mathcal{F}}$ . Hence the group  $\tilde{G}_{\mathcal{F}}$  is an example of a nontrivial generalized equivalence group, and it is also a nontrivial conditional generalized equivalence group of the class  $\mathcal{R}$ , where the specializing attribute "nontrivial" is related to both



the attributes “conditional” and “generalized”. The dependence of group parameters on  $g$  is needless for generating admissible transformations in the class  $\mathcal{F}$  and is merely a manifestation of the fact that the arbitrary element  $g$  is constant within the subclass  $\mathcal{F}$ . This is why we need to consider an effective generalized equivalence group of the class  $\mathcal{F}$ , which is a minimal subgroup of  $\tilde{G}_{\mathcal{F}}$  generating the subgroupoid  $\mathcal{S}_{\mathcal{F}}$  of  $\mathcal{G}_{\mathcal{F}}$ . See [28] for related definitions. The only dependence on  $g$  that is essential for generalized equivalence is the explicit involvement of  $g$  in the  $t$ -coefficient of the  $x$ -component. At the same time, setting the group parameters  $\bar{T}$ 's,  $\bar{X}$ 's,  $\bar{U}$ 's and  $\bar{T}^1\bar{F} - \bar{U}^2$  to be constants singles out the subset of elements from  $\tilde{G}_{\mathcal{F}}$  that is not a subgroup of  $\tilde{G}_{\mathcal{F}}$  although this subset is minimal among subsets of  $\tilde{G}_{\mathcal{F}}$  generating  $\mathcal{S}_{\mathcal{F}}$ . The construction of an effective generalized equivalence group of the class  $\mathcal{F}$  is in fact more tricky.

**Proposition 6.** *An effective generalized equivalence group  $\hat{G}_{\mathcal{F}}$  of the subclass  $\mathcal{F}$  is constituted by the point transformations*

$$\begin{aligned}\tilde{t} &= T_1 t + T_0, & \tilde{x} &= X_1 x + X_2 u - X_2 g t + X_0, \\ \tilde{u} &= U_1 x + U_2 u + (1 - U_2) g t + U_3 t + \frac{T_0}{T_1} g + U_0, & \tilde{u}_x &= \frac{U_1 + U_2 u_x}{X_1 + X_2 u_x}, \\ \tilde{f} &= \frac{(X_1 + X_2 u_x)^2}{T_1} f, & \tilde{g} &= \frac{g + U_3}{T_1},\end{aligned}$$

where  $T$ 's,  $X$ 's and  $U$ 's are arbitrary constants with  $T_1(X_1 U_2 - X_2 U_1) \neq 0$ .

*Proof.* Consider the set  $H_1$  of the point transformations in the space with the coordinates  $(t, x, u, u_x, f, g)$ , whose components are of the form

$$\begin{aligned}\tilde{t} &= T_1 t + T_0, \\ \tilde{x} &= X_1 x + X_2 u + (A_{11} g + A_{10}) t + B_{11} g + B_{10}, \\ \tilde{u} &= U_1 x + U_2 u + (A_{21} g + A_{20}) t + B_{21} g + B_{20}, \\ \tilde{u}_x &= \frac{U_1 + U_2 u_x}{X_1 + X_2 u_x}, & \tilde{f} &= \frac{(X_1 + X_2 u_x)^2}{T_1} f, & \tilde{g} &= \frac{C_1 g + C_0}{T_1},\end{aligned}\tag{18}$$

where  $T$ 's,  $X$ 's,  $U$ 's,  $A$ 's,  $B$ 's and  $C$ 's are arbitrary constants with  $T_1(X_1 U_2 - X_2 U_1) C_1 \neq 0$ . It is obvious that this set is closed with respect to the composition of transformations and taking the inverse, i.e., it is a (local) transformation group with  $\dim H_1 = 16$ . Then the intersection  $H_0 := H_1 \cap \tilde{G}_{\mathcal{F}}$  of  $H_1$  with  $\tilde{G}_{\mathcal{F}}$ , which is singled out from  $H_1$  by the constraints  $A_{10} = 0$ ,  $A_{11} = -X_2$ ,  $A_{20} = C_0$  and  $A_{21} = C_1 - U_2$ , is also a group, and  $\dim H_0 = 12$ . The subgroup  $H_0$  of  $\tilde{G}_{\mathcal{F}}$  generates the entire subgroupoid  $\mathcal{S}_{\mathcal{F}}$  of  $\mathcal{G}_{\mathcal{F}}$ , which is generated by  $\tilde{G}_{\mathcal{F}}$ . At the same time, for each fixed pair of the arbitrary elements  $(f, g)$ , the subgroupoid  $\mathcal{S}_{\mathcal{F}}$  contains a precisely nine-parameter family of admissible transformations with the source  $(f, g)$ . This is why we should try to find three more constraints for group parameters of the group  $H_1$  in order to construct a nine-dimensional subgroup of  $H_0$  that still generates the entire  $\mathcal{S}_{\mathcal{F}}$ .

We analyze the composition of two arbitrary elements from the group  $H_0$ ,  $\hat{\mathcal{T}} = \tilde{\mathcal{T}}\mathcal{T}$  with  $\tilde{\mathcal{T}}, \mathcal{T} \in H_0$ . These generalized equivalence transformations have the general form (18), where group parameters satisfy the above constraints for the subgroup  $H_0$ . We additionally reparameterize  $H_0$  with replacing the parameter  $B_{21}$  by  $B'_{21} + T_0/T_1$  and mark the group-parameter values corresponding to  $\hat{\mathcal{T}}$  and  $\tilde{\mathcal{T}}$  by hats and tildes, respectively. We obtain, in particular, the following expressions for group-parameter values of the composition  $\hat{\mathcal{T}}$ :

$$\hat{C}_1 = \tilde{C}_1 C_1, \quad \hat{B}_{11} = \tilde{X}_1 B_{11} + \tilde{X}_2 B'_{21} + \frac{\tilde{B}_{11}}{T_1}, \quad \hat{B}'_{21} = \tilde{U}_1 B_{11} + \tilde{U}_2 B'_{21} + \frac{\tilde{B}'_{21}}{T_1},$$

which imply that the constrains  $C_1 = 1$ ,  $B_{11} = B'_{21} = 0$  singling out  $\hat{G}_{\mathcal{F}}$  from the subgroup  $H_0$  are preserved by the composition of transformations and taking the inverse in  $H_0$ . Therefore,  $\hat{G}_{\mathcal{F}}$  is really a group. It generates the entire subgroupoid  $\mathcal{S}_{\mathcal{F}}$  of  $\mathcal{G}_{\mathcal{F}}$ , and any its proper subset does not possess this property, i.e., it is a minimal subgroup of  $\tilde{G}_{\mathcal{F}}$  with this property.  $\square$

The usual equivalence group  $G_{\mathcal{F}}^{\sim}$  of the subclass  $\mathcal{F}$  is not contained in the effective generalized equivalence group  $\hat{G}_{\mathcal{F}}^{\sim}$  constructed in Proposition 6. The intersection  $G_{\mathcal{F}}^{\sim} \cap \hat{G}_{\mathcal{F}}^{\sim}$  is singled out from  $G_{\mathcal{F}}^{\sim}$  by the constraints  $T_0 = 0$  and  $U_2 = 1$ .

To prove an assertion generalizing the above claim, we need to consider the infinitesimal counterparts of related groups. For convenience, we introduce the following dual notation for relevant vector fields on the space with the coordinates  $(t, x, u, u_x, f, g)$ :

$$\begin{aligned} Q^1 &= P^t = \partial_t, & Q^2 &= D^t = t\partial_t - f\partial_f - g\partial_g, \\ Q^3 &= P^x = \partial_x, & Q^4 &= D^x = x\partial_x - u_x\partial_{u_x} + 2f\partial_f, \\ Q^5 &= P^u = \partial_u, & Q^6 &= D^u = u\partial_u + u_x\partial_{u_x} + g\partial_g, \\ Q^7 &= Z^t = t\partial_u + \partial_g, & Q^8 &= Z^x = x\partial_u + \partial_{u_x}, & Q^9 &= R = (u - gt)\partial_x - u_x^2\partial_{u_x} + 2u_x f\partial_f. \end{aligned}$$

Up to the anticommutativity of the Lie bracket, the nonzero commutation relations between these vector fields are exhausted by

$$\begin{aligned} [P^t, D^t] &= P^t, & [P^x, D^x] &= P^x, & [P^u, D^u] &= P^u, & [P^t, Z^t] &= P^u, & [P^x, Z^x] &= P^u, \\ [Z^t, D^t] &= -Z^t, & [Z^x, D^x] &= -Z^x, & [Z^t, D^u] &= Z^t, & [Z^x, D^u] &= Z^x, \\ [P^t, R] &= -gP^x, & [P^u, R] &= P^x, & [D^x, R] &= -R, & [D^u, R] &= R, & [Z^x, R] &= D^x - D^u + gZ^t. \end{aligned}$$

The Lie algebras  $\mathfrak{g}_{\mathcal{F}}^{\sim}$ ,  $\bar{\mathfrak{g}}_{\mathcal{F}}^{\sim}$  and  $\hat{\mathfrak{g}}_{\mathcal{F}}^{\sim}$  of the groups  $G_{\mathcal{F}}^{\sim}$ ,  $\hat{G}_{\mathcal{F}}^{\sim}$  and  $\bar{G}_{\mathcal{F}}^{\sim}$  are naturally called the usual equivalence algebra, the generalized equivalence algebra and an effective generalized equivalence algebra of the class  $\mathcal{F}$ , respectively. Each of them is merely the set of infinitesimal generators of one-parameter subgroups of the corresponding group. In order to construct all such generators, we successively take one of the group parameter in the respective general form of group elements to depend on a continuous subgroup parameter  $\delta$  and set the other parameter-functions to their values corresponding to the identity transformations, which are  $T_1 = X_1 = U_2 = 1$  and  $T_0 = X_0 = X_2 = U_0 = U_1 = U_3 = 0$  for the groups  $G_{\mathcal{F}}^{\sim}$  and  $\hat{G}_{\mathcal{F}}^{\sim}$  (the parameter  $X_2$  is relevant only for  $\hat{G}_{\mathcal{F}}^{\sim}$ ) and similarly  $\bar{T}^1 = \bar{X}^1 = \bar{U}^2 = 1$ ,  $\bar{T}^0 = \bar{X}^0 = \bar{X}^2 = \bar{U}^0 = \bar{U}^1 = 0$  and  $\bar{F} = g$  for the group  $\bar{G}_{\mathcal{F}}^{\sim}$ . Then we differentiate the transformation components with respect to  $\delta$  and evaluate the result at  $\delta = 0$ . As a result, we derive that

$$\begin{aligned} \mathfrak{g}_{\mathcal{F}}^{\sim} &= \langle Q^1, \dots, Q^8 \rangle, & \bar{\mathfrak{g}}_{\mathcal{F}}^{\sim} &= \left\{ \sum_{i=1}^9 \vartheta^i(g) Q^i \right\}, \\ \hat{\mathfrak{g}}_{\mathcal{F}}^{\sim} &= \langle P^t + gP^u, D^t, P^x, D^x, P^u, D^u - gZ^t, Z^t, Z^x, R \rangle, \end{aligned}$$

where the coefficients  $\vartheta$ 's run through the set of smooth functions of  $g$ , i.e., the algebra  $\bar{\mathfrak{g}}_{\mathcal{F}}^{\sim}$  is the module over the ring of smooth functions of  $g$  with basis  $(Q^1, \dots, Q^9)$  equipped with the Lie bracket of vector fields.

**Theorem 7.** *Any effective generalized equivalence group of the class  $\mathcal{F}$  does not contain the usual equivalence group  $G_{\mathcal{F}}^{\sim}$  of this class.*

*Proof.* We prove the following re-formulated assertion: Suppose that a subgroup of the generalized equivalence group  $\bar{G}_{\mathcal{F}}^{\sim}$  of the class  $\mathcal{F}$  contains the usual equivalence group  $G_{\mathcal{F}}^{\sim}$  of this class and generates the same subgroupoid of the equivalence groupoid  $\mathcal{G}_{\mathcal{F}}^{\sim}$  as the entire group  $\bar{G}_{\mathcal{F}}^{\sim}$  does. Then this subgroup is not an effective generalized equivalence group of the class  $\mathcal{F}$ .

A complete list of discrete usual equivalence transformations of the class  $\mathcal{F}$  that are independent up to combining with each other and with continuous usual equivalence transformations of this class is exhausted by the involutions  $I^t$ ,  $I^x$  and  $I^u$  alternating the signs of  $(t, f, g)$ ,  $(x, u_x)$  and  $(u, u_x, g)$ , respectively. Among generalized equivalence transformations, there is one more independent discrete transformation  $I^g$ :  $(\tilde{t}, \tilde{x}, \tilde{u}, \tilde{u}_x, \tilde{f}, \tilde{g}) = (t, x, u - 2gt, u_x, f, -g)$ . Discrete equivalence transformations play an auxiliary role in the course of the proof.

It suffices to prove the infinitesimal counterpart of the above assertion, which states the following. Let a subalgebra  $\mathfrak{h}$  of  $\widetilde{\mathfrak{g}}_{\mathcal{F}}$  contain  $\mathfrak{g}_{\mathcal{F}}$  and a vector field  $Q = \sum_{i=1}^9 \zeta^i Q^i$ , where  $\zeta^i = \zeta^i(g)$  are smooth functions of  $g$  with  $\zeta^9 \neq 0$ , be invariant with respect to discrete transformations in  $G_{\mathcal{F}}$ ,  $I_*^t \mathfrak{h}, I_*^x \mathfrak{h}, I_*^u \mathfrak{h} \subseteq \mathfrak{h}$ , and be associated with a transformation (pseudo)group. Then this subalgebra properly contains another subalgebra  $\mathfrak{s}$  among whose elements there are  $K^j = \sum_{i=1}^9 \chi^{ij} Q^i$ , where  $\chi^{ij}, i, j = 1, \dots, 9$ , are smooth functions of  $g$  with  $\det(\chi^{ij}) \neq 0$ , and which is also invariant with respect to  $I_*^t, I_*^x$  and  $I_*^u$  and is associated with a transformation (pseudo)group. Here the subscript “\*” combined with the notation of a point transformation denotes pushing forward vector fields on the same manifold by this transformation.

If the algebra  $\mathfrak{h}$  contains the pure vector field  $R$ , then we commute  $R$  with elements of  $\widetilde{\mathfrak{g}}_{\mathcal{F}}$  and successively obtain that

$$[R, P^t] = gP^x \in \mathfrak{h}, \quad [gP^x, Z^x] = gP^u \in \mathfrak{h}, \quad [Z^x, R] = D^t - D^u + gZ^t \in \mathfrak{h}.$$

Hence  $gZ^t \in \mathfrak{h}$ , i.e.,  $\mathfrak{h} \supseteq \widetilde{\mathfrak{g}}_{\mathcal{F}} + \langle gP^x, gP^u, gZ^t \rangle \supseteq \widehat{\mathfrak{g}}_{\mathcal{F}}$ . We can choose  $\mathfrak{s} = \widehat{\mathfrak{g}}_{\mathcal{F}}$ . Then we also have  $I_*^t \mathfrak{s} = I_*^x \mathfrak{s} = I_*^u \mathfrak{s} = I_*^g \mathfrak{s} = \mathfrak{s}$ .

Otherwise, we compute the commutators

$$\begin{aligned} [Q, D^x] &= \zeta^9 R - \zeta^8 Z^x + \zeta^3 P^x \in \mathfrak{h}, \\ [\zeta^9 R - \zeta^8 Z^x + \zeta^3 P^x, D^x] &= \zeta^9 R + \zeta^8 Z^x + \zeta^3 P^x \in \mathfrak{h}, \\ [\zeta^9 R - \zeta^8 Z^x + \zeta^3 P^x, D^t + D^u] &= -\zeta^9 R - \zeta^8 Z^x \in \mathfrak{h}, \end{aligned}$$

and thus derive that  $\zeta^9 R \in \mathfrak{h}$ , and  $\zeta^9 \neq \text{const.}$  In the same way, we can show that for any element  $\sum_{i=1}^9 \vartheta^i(g) Q^i \in \mathfrak{h}$ , the element  $\vartheta^3 P^x$  and thus the element  $\vartheta^3 P^u = [\vartheta^3 P^x, Z^x]$  also belong to  $\mathfrak{h}$ . Taking two more commutators,

$$\begin{aligned} [Z^x, \zeta^9 R] &= \zeta^9 (D^x - D^u + gZ^t) \in \mathfrak{h}, \\ [Z^x, \zeta^9 (D^x - D^u + gZ^t)] &= -2\zeta^9 Z^x \in \mathfrak{h}, \end{aligned}$$

we get  $\zeta^9 Z^x \in \mathfrak{h}$ . Consider the span

$$\mathfrak{s} = \langle P^t, D^t, Z^t, D^x, D^u, \beta P^x, \beta P^u, \alpha R, \alpha(D^x - D^u + gZ^t), \alpha Z^x \mid \alpha R, \beta P^x \in \mathfrak{h} \rangle.$$

It is a subalgebra of  $\mathfrak{h}$ . Since the entire algebra  $\mathfrak{h}$  is invariant with respect to  $I_*^t, I_*^x$  and  $I_*^u$  and is associated with a transformation (pseudo)group, the subalgebra  $\mathfrak{s}$  has the same properties. In view of  $R \notin \mathfrak{h}$ , the parameter function  $\alpha$  does not take constant values. Hence  $Z^x \notin \mathfrak{s}$ , i.e.,  $\mathfrak{s} \subsetneq \mathfrak{h}$ . As the required elements  $K^j, j = 1, \dots, 9$ , we can choose  $P^t, D^t, Z^t, D^x, D^u, P^x, P^u, \zeta^9 R$  and  $\zeta^9 Z^x$ .

Therefore, the algebra  $\mathfrak{h}$  is not an effective generalized equivalence algebra of the class  $\mathcal{F}$ .  $\square$

### 3 Classification of Lie symmetries

In order to compute the maximal Lie invariance algebras of equations from the class  $\mathcal{R}$ , we employ the infinitesimal method [27, 31]. The infinitesimal generator of a one-parameter Lie-symmetry group of an equation from the class  $\mathcal{R}$  is a vector field  $Q = \tau(t, x, u)\partial_t + \xi(t, x, u)\partial_x + \eta(t, x, u)\partial_u$  on the space with coordinates  $(t, x, u)$  with the components  $\tau, \xi$  and  $\eta$  being the smooth functions of these coordinates. The infinitesimal invariance criterion requires that

$$Q^{(2)}(u_t - f(u_x)u_{xx} - g(u)) \Big|_{u_t=f(u_x)u_{xx}+g(u)} = 0, \quad (19)$$

where  $Q^{(2)}$  is the second prolongation of the vector field  $Q$  defined by the well-known prolongation formula [27],  $Q^{(2)} = Q + \eta^{(1,0)}\partial_{u_t} + \eta^{(0,1)}\partial_{u_x} + \eta^{(2,0)}\partial_{u_{tt}} + \eta^{(1,1)}\partial_{u_{tx}} + \eta^{(0,2)}\partial_{u_{xx}}$ , with the

coefficients  $\eta$ 's defined as  $\eta^\alpha = D^\alpha (\eta - \tau u_t - \xi u_x) + \tau u_{\alpha+\delta_1} + \xi u_{\alpha+\delta_2}$ . Here  $\alpha = (\alpha_1, \alpha_2)$  is a multi-index,  $D^\alpha = D_t^{\alpha_1} D_x^{\alpha_2}$ ,  $D_t = \partial_t + \sum_\alpha u_{\alpha+\delta_1} \partial_{u_\alpha}$  and  $D_x = \partial_x + \sum_\alpha u_{\alpha+\delta_2} \partial_{u_\alpha}$  are the total derivative operators with respect to  $t$  and  $x$ , respectively,  $\delta_1 = (1, 0)$  and  $\delta_2 = (0, 1)$ .

Since the class  $\mathcal{R}$  consists of evolution equations, the  $t$ -component of  $Q$  does not depend on  $x$  and  $u$ , i.e.,  $\tau = \tau(t)$  [18, 19].

We expand the expression in the left hand side of (19), substitute  $f(u_x)u_{xx} + g(u)$  for  $u_t$  and then split the resulting equation with respect to  $u_{xx}$ . This gives the system of determining equations for the components of Lie-symmetry vector fields of equations from the class  $\mathcal{R}$ ,

$$(-\xi_u u_x^2 + (\eta_u - \xi_x)u_x + \eta_x) f_{u_x} + (-2\xi_u u_x + \tau_t - 2\xi_x) f = 0, \quad (20a)$$

$$\begin{aligned} &(-\xi_{uu} u_x^3 + (\eta_{uu} - 2\xi_{xu})u_x^2 + (2\eta_{xu} - \xi_{xx})u_x + \eta_{xx}) f + (\xi_t + \xi_u g)u_x \\ &+ \eta g_u + (\tau_t - \eta_u)g - \eta_t = 0. \end{aligned} \quad (20b)$$

Thus, the problem of group classification for the class  $\mathcal{R}$  reduces to the classification of solutions of the system (20), depending on values of the arbitrary elements  $f$  and  $g$  up to  $G_{\mathcal{R}}^\sim$ -equivalence or up to the general point equivalence.

To find the kernel Lie invariance algebras of the class  $\mathcal{R}$  and of its subclasses  $\mathcal{H}$ ,  $\mathcal{L}$ ,  $\mathcal{F}$ ,  $\mathcal{F}'$  and  $\mathcal{C}$  (each of these algebras is the intersection of the maximal Lie invariance algebras of equations from the corresponding (sub)class), we successively split the equations (20a) and (20b) with respect to the arbitrary elements and their derivatives and with respect to  $u_x$ , taking into account the associated auxiliary equations for arbitrary elements. See also the papers [8], [10] and [1, 2] for the kernel Lie invariance algebras of the classes  $\mathcal{R}$ ,  $\mathcal{H}$  and  $\mathcal{F}'$ , respectively.

**Proposition 8.** *The kernel Lie invariance algebra  $\mathfrak{g}_{\mathcal{R}}^\square$  of the class  $\mathcal{R}$  coincides with the kernel Lie invariance algebras  $\mathfrak{g}_{\mathcal{H}}^\square$  and  $\mathfrak{g}_{\mathcal{C}}^\square$  of its subclasses  $\mathcal{H}$  and  $\mathcal{C}$ , and it is spanned by the vector fields  $\partial_t$  and  $\partial_x$ ,*

$$\mathfrak{g}_{\mathcal{R}}^\square = \mathfrak{g}_{\mathcal{H}}^\square = \mathfrak{g}_{\mathcal{C}}^\square = \langle \partial_t, \partial_x \rangle.$$

*The kernel Lie invariance algebras of the subclasses  $\mathcal{L}$ ,  $\mathcal{F}$  and  $\mathcal{F}'$  are respectively*

$$\mathfrak{g}_{\mathcal{L}}^\square = \langle \partial_t, \partial_x, x\partial_x \rangle, \quad \mathfrak{g}_{\mathcal{F}}^\square = \langle \partial_t, \partial_x, \partial_u \rangle, \quad \mathfrak{g}_{\mathcal{F}'}^\square = \langle \partial_t, \partial_x, \partial_u, 2t\partial_t + x\partial_x + u\partial_u \rangle.$$

Accurately merging the group classifications of the subclasses  $\mathcal{C}$ ,  $\mathcal{F}'$ ,  $\mathcal{H}$  and  $\mathcal{L}$  that is presented in Lemma 14 below, that were carried out in [1, 2] and in [10], and that can be obtained by mapping with the hodograph transformation  $\tilde{t} = t$ ,  $\tilde{x} = u$ ,  $\tilde{u} = x$  from the group classification of the class of Kolmogorov equations given in [40], respectively, we get the following assertion.

**Theorem 9.** *A complete list of inequivalent Lie-symmetry extensions in the class  $\mathcal{R}$  is exhausted by the cases given in Table 1, where we use  $G_{\mathcal{R}}^\sim$ -,  $\bar{G}_{\mathcal{F}'}^\sim$ -,  $G_{\mathcal{R}}^\sim$ - and  $\mathcal{G}_{\mathcal{L}}^\sim$ -equivalence for equations from the subclasses  $\mathcal{C}$ ,  $\mathcal{F}$ ,  $\mathcal{H}$  and  $\mathcal{L}$ , respectively.*

Here  $\mathcal{G}_{\mathcal{L}}^\sim$  denotes the equivalence groupoid of the class  $\mathcal{L}$ .

**Notation for Table 1.** Numbers with the same Arabic numerals and different Roman letters correspond to cases that are equivalent with respect to additional equivalence transformations. Explicit formulas for these transformations are presented in the footnote of Table 1. The cases that are numbered with different numbers in Arabic numerals are reciprocally inequivalent with respect to point transformations. Lie invariance algebras presented in Cases 0, 1 and 2 are maximal only if the corresponding tuples of the arbitrary elements  $(f, g)$  are  $G_{\mathcal{R}}^\sim$ -inequivalent to those from other cases.

The (usual) equivalence groups of the subclasses  $\mathcal{H}$  and  $\mathcal{C}$  coincide with each other and with the group  $G_{\mathcal{R}}^\sim$ . This is why within the class  $\mathcal{R}$ , Cases 3' and 4a' can be attached to Cases 3 and 4a by allowing the value  $n = 0$  in the latter cases.

Table 1: Group classification of the class  $\mathcal{R}$ 

no.	$f(u_x)$	$g(u)$	Vector fields spanning the maximal Lie invariance algebra
the subclass $\mathcal{C}$ , up to $G_{\mathcal{R}}^{\sim}$ -equivalence			
0	$\forall$	$\forall$	$\partial_t, \partial_x$
1	$\forall$	$u$	$\partial_t, \partial_x, e^t \partial_u$
2	$\forall$	$u^{-1}$	$\partial_t, \partial_x, 2t\partial_t + x\partial_x + u\partial_u$
3	$ u_x ^n$	$\varepsilon e^u$	$\partial_t, \partial_x, (n+2)t\partial_t + x\partial_x - (n+2)\partial_u$
4a	$ u_x ^n$	$ u ^m$	$\partial_t, \partial_x, (1-m)(n+2)t\partial_t + (n+1-m)x\partial_x + (n+2)u\partial_u$
4b	$ u_x ^n$	$ u ^{n+1} + \varepsilon u$	$\partial_t, \partial_x, e^{-\varepsilon n t}(\varepsilon \partial_t + u\partial_u)$
5	$(u_x + 1)^{-1}$	$\varepsilon u$	$\partial_t, \partial_x, e^{\varepsilon t} \partial_u, e^{\varepsilon t}(\partial_t + \varepsilon(u+x)\partial_u)$
6a	$u_x$	$u^2$	$\partial_t, \partial_x, t\partial_t - u\partial_u, t^2\partial_t - (2tu+1)\partial_u$
6b	$u_x$	$u^2 + 1$	$\partial_t, \partial_x, \cos 2t(\partial_t + 2\partial_u) + 2u \sin 2t \partial_u, \sin 2t(\partial_t + 2\partial_u) - 2u \cos 2t \partial_u$
6c	$u_x$	$u^2 - 1$	$\partial_t, \partial_x, e^{2t}(\partial_t - 2(u+1)\partial_u), e^{-2t}(\partial_t + 2(u-1)\partial_u)$
7	$u_x(u_x + 1)^{-3}$	$\varepsilon u$	$\partial_t, \partial_x, e^{\varepsilon t} \partial_u, e^{-\varepsilon t}(\partial_t - \varepsilon u\partial_x + \varepsilon u\partial_u)$
9b	$ u_x ^n$	$\varepsilon u$	$\partial_t, \partial_x, e^{\varepsilon t} \partial_u, nx\partial_x + (n+2)u\partial_u, e^{-\varepsilon n t}(\partial_t + \varepsilon u\partial_u)$
the subclass $\mathcal{F}$ , up to $G_{\mathcal{F}}^{\sim}$ -equivalence			
8	$\forall$	0	$\partial_t, \partial_x, \partial_u, 2t\partial_t + x\partial_x + u\partial_u$
9a	$ u_x ^n$	0	$\partial_t, \partial_x, \partial_u, 2t\partial_t + x\partial_x + u\partial_u, nt\partial_t - u\partial_u$
10	$e^{u_x}$	0	$\partial_t, \partial_x, \partial_u, 2t\partial_t + x\partial_x + u\partial_u, t\partial_t - x\partial_u$
11	$\frac{e^m \arctan u_x}{u_x^2 + 1}$	0	$\partial_t, \partial_x, \partial_u, 2t\partial_t + x\partial_x + u\partial_u, mt\partial_t + u\partial_x - x\partial_u$
12a	1	0	$\partial_t, \partial_x, \partial_u, 2t\partial_t + x\partial_x, u\partial_u, 2t\partial_x - xu\partial_u,$ $4t^2\partial_t + 4tx\partial_x - (x^2 + 2t)u\partial_u, h\partial_u$
the subclass $\mathcal{H}$ , up to $G_{\mathcal{H}}^{\sim}$ -equivalence, only cases additional to Case 12a			
3'	1	$\varepsilon e^u$	$\partial_t, \partial_x, 2t\partial_t + x\partial_x - 2\partial_u$
4a'	1	$ u ^m$	$\partial_t, \partial_x, 2(1-m)t\partial_t + (1-m)x\partial_x + 2u\partial_u$
13	1	$\varepsilon u \ln  u $	$\partial_t, \partial_x, e^{\varepsilon t}(2\partial_x - \varepsilon xu\partial_u), e^{\varepsilon t}u\partial_u$
12b	1	$\varepsilon u$	$\partial_t, \partial_x, 2t\partial_t + x\partial_x + 2\varepsilon tu\partial_u, u\partial_u, 2t\partial_x - xu\partial_u,$ $4t^2\partial_t + 4tx\partial_x - (x^2 + 2t - 4\varepsilon t^2)u\partial_u, h\partial_u$
12c	1	1	$\partial_t, \partial_x, 2t\partial_t + x\partial_x + 2t\partial_u, (u-t)\partial_u, 2t\partial_x - x(u-t)\partial_u,$ $4t^2\partial_t + 4tx\partial_x - ((x^2+2t)(u-t) - 4t^2)\partial_u, h\partial_u$
the subclass $\mathcal{L}$ , up to $G_{\mathcal{L}}^{\sim}$ -equivalence			
14	$u_x^{-2}$	$\forall$	$\partial_t, \tilde{h}\partial_x, x\partial_x$
15	$u_x^{-2}$	$\mu u^{-1}$	$\partial_t, \tilde{h}\partial_x, x\partial_x, 4t\partial_t + 2u\partial_u, 4t^2\partial_t + 4tu\partial_u - (u^2 + 2(1+\nu)t)x\partial_x$
16	$u_x^{-2}$	$\frac{1 - 2\nu \tan(\nu \ln  u )}{u}$	$\partial_t, \tilde{h}\partial_x, x\partial_x, 2t\partial_t + u\partial_u - xug\partial_x, 4t^2\partial_t + 4tu\partial_u - (u^2 + 2t + 2tug)x\partial_x$
12d	$u_x^{-2}$	0	$\partial_t, \tilde{h}\partial_x, x\partial_x, 2t\partial_t + u\partial_u, 4t^2\partial_t + 4tu\partial_u - (u^2 + 2t)x\partial_x, 2t\partial_u - xu\partial_x$

$n \in \mathbb{R} \setminus \{0, -2\}$ ,  $m \in \mathbb{R}$ .  $\varepsilon \neq 0$ ,  $\varepsilon = \pm 1 \pmod{G_{\mathcal{R}}^{\sim}}$ .  $m \neq -1, 0, 1$  and  $(n, m) \neq (1, 2)$  in Case 4a.  $m \neq 0, 1$  in Case 4a'.  $n \neq \pm 1$  in Case 4b.  $n \geq -1 \pmod{G_{\mathcal{F}}^{\sim}}$  in Case 9a.  $m \geq 0 \pmod{G_{\mathcal{R}}^{\sim}}$  in Case 11. In Cases 12a, 12b and 12c, the parameter function  $h = h(t, x)$  runs through the solution set of the corresponding equation,  $h_t = h_{xx}$ ,  $h_t = h_{xx} + \varepsilon h$  and  $h_t = h_{xx} + \varepsilon$ , respectively. In Cases 14–16 and 12d, the real constant parameters  $\mu$  and  $\nu$  satisfy, up to point transformations, the constraints  $\mu \geq 1$ ,  $\mu \neq 2$  and  $\nu > 0$ , and the parameter function  $\tilde{h} = \tilde{h}(t, u)$  runs through the solution set of the corresponding linear equation,  $\tilde{h}_t = \tilde{h}_{uu} - g(u)\tilde{h}_u$ .

Additional equivalence transformations between cases of the table are exhausted by the following:

6b  $\rightarrow$  6a:  $\tilde{t} = \arctan t$ ,  $\tilde{x} = x$ ,  $\tilde{u} = (t^2 + 1)u + t$ ;    6c  $\rightarrow$  6a:  $\tilde{t} = \frac{1}{2} \ln \left| \frac{t-1}{t+1} \right|$ ,  $\tilde{x} = x$ ,  $\tilde{u} = (t^2 - 1)u + t$ ;

4b  $\rightarrow$  4a $_{m=n+1}$ , 9b  $\rightarrow$  9a:  $\tilde{t} = e^{\varepsilon n t} / (\varepsilon n)$ ,  $\tilde{x} = x$ ,  $\tilde{u} = e^{-\varepsilon t} u$ ;

12b  $\rightarrow$  12a:  $\tilde{t} = t$ ,  $\tilde{x} = x$ ,  $\tilde{u} = e^{-\varepsilon t} u$ ;    12c  $\rightarrow$  12a:  $\tilde{t} = t$ ,  $\tilde{x} = x$ ,  $\tilde{u} = u - t$ ;    12d  $\rightarrow$  12a:  $\tilde{t} = t$ ,  $\tilde{x} = u$ ,  $\tilde{u} = x$ .

Crossing out the numbers of Cases 12c and 12d indicates that they are implicitly presented in the course of listing Lie-symmetry extensions for the subclass  $\mathcal{F}$  since both of them are  $\tilde{G}_{\mathcal{F}}$ -equivalent to Case 12a.

**Remark 10.** We should carefully treat arguments of logarithm and bases of powers with non-integer exponents. The condition of positivity is justified for  $u$  by physical arguments but this is not the case for  $u_x$ . Moreover, assuming  $u$  positive makes impossible the change of the sign of  $u$ , which results in the disappearance of some discrete equivalence transformations. This is why we use the absolute values of  $u$ ,  $u_x$  and similar expressions as arguments of logarithm and as bases of related powers with general real exponents. This is not necessary for some particular exponents (more specifically, integer exponents and rational exponents with odd denominators) but replacing an absolute value by the value itself in a power base may lead to modifying the gauging of the coefficient of this power since such a replacement may affect the action of discrete equivalence transformations on power coefficients.

**Remark 11.** To construct an exhaustive list of  $G_{\mathcal{R}}$ -inequivalent Lie-symmetry extensions for the subclass  $\mathcal{F}$ , we should extend Cases 8–12 using transformations from the generalized equivalence group  $\tilde{G}_{\mathcal{F}}$  of  $\mathcal{F}$  (or, equivalently, from the effective generalized equivalence group  $\hat{G}_{\mathcal{F}}$ , which is more convenient) and then gauge arising parameters by elements of  $G_{\mathcal{R}}$ . In each of the  $G_{\mathcal{R}}$ -inequivalent cases constructed, we can set  $g \in \{0, 1\} \bmod G_{\mathcal{R}}$ , and a complete list of  $G_{\mathcal{R}}$ -inequivalent values of the arbitrary element  $f$  is exhausted by the general value (the extended Case 8) and

$$|u_x + \delta|^n, \quad \frac{|u_x + \mu|^n}{|u_x + \nu|^{n+2}}; \quad e^{u_x}, \quad \frac{e^{(u_x + \lambda)^{-1}}}{(u_x + \lambda)^2}; \quad \frac{e^{m \arctan(u_x + \lambda)}}{(u_x + \lambda)^2 + 1}; \quad 1, \quad (u_x + \delta)^{-2}.$$

Here  $n \in \mathbb{R} \setminus \{0, -2\}$ ,  $\delta \in \{0, 1\} \bmod G_{\mathcal{R}}$ ,  $\lambda, \mu, \nu, m \in \mathbb{R}$ ,  $\mu \neq \nu$ , and one nonzero constant among  $\mu$  and  $\nu$  can be set to be equal 1 modulo  $G_{\mathcal{R}}$ -equivalence,  $m \geq 0 \bmod G_{\mathcal{R}}$ . Semicolons in the above displayed equation separate the extensions of Cases 9, 10, 11 and 12, respectively. In the course of implementing the above procedure for Case 11, we should take into account the formula for sum of arctangents,

$$\arctan y + \arctan z = \begin{cases} \arctan \frac{y+z}{1-yz} + \frac{\pi}{2}(\operatorname{sgn} y)(1 + \operatorname{sgn}(1-yz)), & \text{if } yz \neq 1, \\ \frac{\pi}{2} \operatorname{sgn} y, & \text{if } yz = 1. \end{cases}$$

**Remark 12.** Recall that the class  $\mathcal{L}$  is similar to the class  $\mathcal{K}$  of Kolmogorov equations of the general form (3) with respect to a point transformation, which is the hodograph transformation  $\tilde{t} = t$ ,  $\tilde{x} = u$ ,  $\tilde{u} = x$ . This is why the groups classifications of the class  $\mathcal{L}$  up to  $\mathcal{G}_{\mathcal{L}}$ - and  $G_{\mathcal{L}}$ -equivalences reduce to their counterparts for the class  $\mathcal{K}$ ; see the theoretical background on mappings between classes of differential equations that are generated by point transformations in [47]. For the part of Table 1 related to the class  $\mathcal{L}$ , we use a complete list of inequivalent (up to general point equivalence) Lie-symmetry extensions that is constructed for the general class of (1+1)-dimensional Kolmogorov equations in [40, Corollary 7] and coincides with such a list for the class  $\mathcal{K}$ . The reason for this is that a similar list for the group classification of the class  $\mathcal{K}$  up to the equivalence generated by its equivalence group is too cumbersome [29].

**Theorem 13.** *A complete list of  $G_{\mathcal{R}}$ -inequivalent (i.e., inequivalent up point transformations) cases of Lie-symmetry extensions in the class  $\mathcal{R}$  is exhausted by Cases 1, 2, 3  $\cup$  3', 4a  $\cup$  4a', 5, 6a, 7, 8, 9a, 10, 11, 12a, and 13–16 of Table 1.*

*Proof.* To prove the  $G_{\mathcal{R}}$ -inequivalence of the cases listed in this theorem, we first use the fact that the maximal Lie invariance algebras of similar equations are similar realizations of isomorphic Lie algebras. In particular, such algebras are of the same dimension. Since the  $t$ -component of

any admissible transformation in the class  $\mathcal{R}$  depends only on  $t$ , the dimension of the projection of the maximal Lie invariance algebra  $\mathfrak{g}$  of an equation from the class  $\mathcal{R}$  to the space with coordinate  $t$ ,  $\dim \text{pr}_t \mathfrak{g}$ , is one more invariant of  $\mathcal{G}_{\mathcal{R}}^{\sim}$ . We identify the structure of the associated maximal Lie invariance algebras, when they are finite dimensional, according to Mubarakzhanov's classification of real low-dimensional Lie algebras [22, 23]. We use the notation of these algebras from [6, 36], additionally reparameterizing families of algebras by homogeneous parameters if this is convenient.

$\dim \mathfrak{g} = 2$ . Case 0:  $\mathfrak{g} \simeq 2A_1$ ;

$\dim \mathfrak{g} = 3$ . Case 1:  $\mathfrak{g} \simeq A_{2,1} \oplus A_1$ ,  $\dim \text{pr}_t \mathfrak{g} = 1$ ; Case 2:  $\mathfrak{g} \simeq A_{3,4}^{[2;1]}$ ; Case 3:  $\mathfrak{g} \simeq A_{3,3}$  if  $n = -1$ , and  $\mathfrak{g} \simeq A_{3,4}^{[(n+2);1]}$  otherwise; Case 4:  $\mathfrak{g} \simeq A_{2,1} \oplus A_1$ ,  $\dim \text{pr}_t \mathfrak{g} = 2$  if  $m = n + 1$ ,  $\mathfrak{g} \simeq A_{3,3}$  if  $(1-m)(n+2) = n+1-m$ , and  $\mathfrak{g} \simeq A_{3,4}^{[(1-m)(n+2);(n+1-m)]}$  otherwise;

$\dim \mathfrak{g} = 4$ . Case 5:  $\mathfrak{g} \simeq A_{4,8}^0$ ,  $\dim \text{pr}_t \mathfrak{g} = 2$ ; Case 6:  $\mathfrak{g} \simeq \mathfrak{sl}_2(\mathbb{R}) \oplus A_1$ ; Case 7:  $\mathfrak{g} \simeq A_{3,4}^{[-1;1]} \oplus A_1$ ; Case 8:  $\mathfrak{g} \simeq A_{4,5}^{[2;1;1]}$ ; Case 13:  $\mathfrak{g} \simeq A_{4,8}^0$ ,  $\dim \text{pr}_t \mathfrak{g} = 1$ ;

$\dim \mathfrak{g} = 5$ . Case 9:  $\mathfrak{g} \simeq A_{5,33}^{n+2,-n}$ ; Case 10:  $\mathfrak{g} \simeq A_{5,34}^2$ ; Case 11:  $\mathfrak{g} \simeq A_{5,35}^{2,m}$ .

$\dim \mathfrak{g} = \infty$ . Cases 12, 14, 15 and 16 are then inequivalent to other cases. The inequivalence of these cases to each other and the impossibility of further gauging of the parameters  $\mu$  and  $\nu$  follows from the same properties of similar Kolmogorov equations [40, Corollary 7].

We discuss only unobvious  $\mathcal{G}_{\mathcal{R}}^{\sim}$ -inequivalences.

Admissible transformations of the class  $\mathcal{R}$  preserve the  $t$ -direction. This is why the order of parameters is essential in each of the above algebras from the family  $A_{3,4}$ . Since  $n \notin \{0, -2\}$  and, in Case 4,  $m \neq -1$ , Cases 3 and 4 cannot be mapped by elements of  $\mathcal{G}_{\mathcal{R}}^{\sim}$  to Case 2.

Algebras  $A_{5,35}^{2,m}$  and  $A_{5,35}^{2,m'}$  with different to each other nonnegative  $m$  and  $m'$  are non-isomorphic. Hence the equations of Case 11 with such  $m$ 's are  $\mathcal{G}_{\mathcal{R}}^{\sim}$ -inequivalent.

It is still necessary to prove that equations from different Cases  $3 \cup 3'$  and  $4a \cup 4a'$  are  $\mathcal{G}_{\mathcal{R}}^{\sim}$ -inequivalent to each other, and different values of  $n$  in Case  $3 \cup 3'$  and different values of  $(n, m)$  in Case  $4a \cup 4a'$  are  $\mathcal{G}_{\mathcal{R}}^{\sim}$ -inequivalent. In Cases  $3 \cup 3'$ , 5, 7 and 13, the values  $\varepsilon = 1$  and  $\varepsilon = -1$  should be  $\mathcal{G}_{\mathcal{R}}^{\sim}$ -inequivalent. We also should check that in Case 9a, different values of  $n$  in  $[-1, +\infty) \setminus \{0, 2\}$  are  $\mathcal{G}_{\mathcal{R}}^{\sim}$ -inequivalent. Further we use the direct method.

Consider a point transformation of independent and dependent variables of the general form (4) that connects the source and the target equations from the class  $\mathcal{R}$ , (5) with  $f(u_x) = |u_x|^n$  and  $\tilde{f}(\tilde{u}_{\tilde{x}}) = |\tilde{u}_{\tilde{x}}|^{\tilde{n}}$ , where  $n, \tilde{n} \in \mathbb{R} \setminus \{0, -2\}$ . We substitute these values of the arbitrary element  $f$  into the equation (8) and act on the obtained equation by the operator  $\partial_{u_x} - nu_x^{-1}$ . Rearranging the result, we derive the equation

$$\tilde{n} \frac{\Delta}{D_x U} - n \frac{D_x X}{u_x} - 2X_u = 0,$$

whose left hand side is rational in  $u_x$ . There are two cases depending on the value of  $U_u$ :

1.  $U_u = 0$ . Then  $X_u U_x \neq 0$ ,  $X_x = 0$  and  $\tilde{n} = -n - 2$ .
2.  $U_u \neq 0$ . Then  $U_x = 0$ ,  $X_x \neq 0$ ,  $X_u = 0$  and  $\tilde{n} = n$ .

In the first case, the equation (8) reduces to  $T_t |X_u|^n / |U_x|^{n+2} = 1$ , which further implies that  $X_{uu} = U_{xx} = 0$ . Then  $V_x = V_u = 0$ , and the equation (9) splits with respect to  $u_x$  into  $g = -X_t / X_u$  and  $\tilde{g} = U_t / T_t$ , i.e.,  $g$  and  $\tilde{g}$  are affine in  $u$ . Therefore, the source and the target equations fit into Cases 9a or 9b up to  $\mathcal{G}_{\mathcal{R}}^{\sim}$ -equivalence and merely into Case 9a up to  $\mathcal{G}_{\mathcal{R}}^{\sim}$ -equivalence. This completes the proof for Case 9. Notes that algebras  $A_{5,33}^{a,b}$  and  $A_{5,33}^{a',b'}$  are isomorphic if and only if

$$(a', b') \in \{(a, b), (b, a)\} \cup \left\{ \left( \frac{1}{a}, -\frac{b}{a} \right), \left( -\frac{b}{a}, \frac{1}{a} \right) \right\}_{\text{if } a \neq 0} \cup \left\{ \left( -\frac{a}{b}, \frac{1}{b} \right), \left( \frac{1}{b}, -\frac{a}{b} \right) \right\}_{\text{if } b \neq 0}.$$

Hence inequivalence of equations within Case 9 cannot be checked algebraically with comparing structure of the corresponding Lie invariance algebras.

The second case is analyzed in a similar way. The equation (8) reduces to the equation  $T_t|U_u|^n/|X_x|^{n+2} = 1$  implying  $X_{xx} = U_{uu} = 0$  and  $T_t > 0$ . Then again  $V_x = V_u = 0$ , and the equation (9) splits with respect to  $u_x$  into  $X_t = 0$  and  $g = (T_t\tilde{g} - U_t)/U_u$ . It follows from the last equation that point transformations cannot switch an exponential value of  $g$  to a power one and conversely as well as they cannot change the value of the exponent  $m$  when the arbitrary element  $g$  is of power form  $g = |u|^m$ . Moreover, if  $g = \varepsilon e^u$  and  $\tilde{g} = \tilde{\varepsilon} e^{\tilde{u}}$ , then  $U_u = 1$  and one cannot alternate the sign of  $\varepsilon$ . In other words, there are no more additional equivalence transformations related to equations of Cases 3 and 4a.

Up to extending the argument tuples of  $g$  and  $\tilde{g}$ , the equations (8) and (9) preserve their forms if we consider a wider class of differential equations, allowing for the arbitrary element  $g$  to additionally depend on  $t$ ,  $x$  and  $u_x$ . This is why for simplifying the analysis of Cases 5 and 7, we map the associated equations,  $u_t = (u_x + 1)^{-1}u_{xx} + \varepsilon u$  and  $u_t = u_x(u_x + 1)^{-3}u_{xx} + \varepsilon u$ , to simpler equations,  $u_t = u_x^{-1}u_{xx} + \varepsilon(u - x)$  and  $u_t = u_x u_{xx} - \varepsilon u u_x$ , by the point transformations  $\tilde{t} = t$ ,  $\tilde{x} = x$ ,  $\tilde{u} = u + x$  and  $\tilde{t} = \varepsilon^{-1}e^{\varepsilon t}$ ,  $\tilde{x} = x + u$ ,  $\tilde{u} = e^{-\varepsilon t}u$ , respectively, where tildes indicate variables of target equations. (The same mapping of Case 7 is used in Section 6, cf. the equation (41).) Looking for a point transformation between target equations of the same form with different values of  $\varepsilon$ , well fits into the above second case. For the first target equation, we have  $n = \tilde{n} = -1$ ,  $g = \varepsilon(u - x)$  and  $\tilde{g} = \tilde{\varepsilon}(\tilde{u} - \tilde{x})$ , and collecting the coefficients of  $u$  in the equation (9) leads to the equation  $\tilde{\varepsilon} = T_t\varepsilon$  implying the impossibility of alternating between the values  $\varepsilon = 1$  and  $\varepsilon = -1$  since  $T_t > 0$ . For the second target equation, we similarly have  $n = \tilde{n} = 1$ ,  $g = -\varepsilon u u_x$  and  $\tilde{g} = -\tilde{\varepsilon} \tilde{u} \tilde{u}_x$ , and collecting the coefficients of  $u u_x$  in the equation (9) gives the equation  $\tilde{\varepsilon} = T_t U_u X_x^{-1} \varepsilon$ . Jointly with the consequence  $T_t U_u = X_x^3$  of the equation (8), this again implies the impossibility of alternating the sign of  $\varepsilon$ .

Let now  $f = 1$  and  $\tilde{f} = 1$ . Then the equation (8) splits with respect to  $u_x$  into the equations  $X_u = 0$  and  $T_t = X_x^2$ , which imply  $X_x U_u \neq 0$ ,  $T_t > 0$  and  $X_{xx} = 0$ . Collecting coefficients of different powers of  $u_x$  in the equation (9) results in the equations  $U_{uu} = 0$ ,  $2X_x U_{xu} + X_t U_u = 0$  and  $U_u g = T_t \tilde{g} + U_{xx} - U_t + X_t U_x / X_x$ . We differentiate the last equation twice with respect to  $u$  in order to derive its differential consequence  $g_{uu} = T_t U_u \tilde{g}_{\tilde{u}\tilde{u}}$ . Therefore, elements of  $\mathcal{G}_{\mathcal{R}}^{\sim}$  preserve each of Cases 3', 4a' and 13 modulo  $G_{\mathcal{R}}^{\sim}$ -equivalence including the value of  $m$ . Moreover, if  $g = \varepsilon e^u$  and  $\tilde{g} = \tilde{\varepsilon} e^{\tilde{u}}$ , then  $U_u = 1$  and  $\tilde{\varepsilon} = T_t \varepsilon e^{u-U}$ , i.e., one cannot alternate the sign of  $\varepsilon$ . Analogously, for  $g = \varepsilon u \ln |u|$  and  $\tilde{g} = \tilde{\varepsilon} \tilde{u} \ln |\tilde{u}|$  we have  $\tilde{\varepsilon} = T_t \varepsilon$ . This means that the set of Cases 3', 4a' and 13 also admits no additional equivalence transformations.  $\square$

## 4 Method of furcate splitting

Although the literature dedicated to group classification of classes (of systems) of differential equations is vast, a considerable part thereof is not trustworthy at all. The main problem lies in the fact that often a brute-force approach is applied to tackle group classification problems. Albeit this approach is definitely applicable (but certainly inefficient) for some simple classes of differential equations, it requires a very thorough inspection to keep track of various cases that arise upon integrating the corresponding determining equations for Lie symmetries, otherwise it leads to missed, overlapping, repeated and/or incorrect classification cases. These flaws happened in the majority of papers where the brute-force approach is used for group classification of differential equations. Moreover, since this approach is often cumbersome, such papers do not contain details of solutions, which makes pretty hard to find flaws in the computations afterwards.

While the class  $\mathcal{C}$  does not possess a necessary structure to employ the powerful algebraic method of group classification, a brute force is still not the last resort. The best choice here is the method of *furcate splitting*. This method was suggested and applied for the first time in [26]



but its name appeared later in [17]. See also examples of its application in [37, 46, 48, 49]. It is basically a refinement of the direct approach to group classification. In contrast to the chaotical integration of the classifying equations and the irregular selection of cases with Lie-symmetry extensions within the framework of the direct approach, the method of furcate splitting provides an algorithm for systematically finding such cases. It is worth to note that the method is effective for classes with arbitrary elements depending on a few arguments. So far, it was used for classes with arbitrary elements depending on at most two arguments [26]. This limit for manually using the method seems to be reasonable in view of the necessity of verifying the compatibility condition of arising equations, which becomes the more difficult problem the more arguments the arbitrary elements depend on.

The method of furcate splitting can be easily understood by examples but the description of its standard version in the general setting is quite tedious, not to mention its various modifications. Nevertheless, below we present this description, and in Section 5 we illustrate possible complications for using the method if arbitrary elements of the class under study depend on different arguments, which is the case for the class  $\mathcal{R}$ ; see also the related discussion in the end of Section 7.

Consider a class of (systems of) differential equations

$$\mathcal{L}|_{\mathcal{S}} = \{\mathcal{L}_{\theta}: L(x, u_{(r)}, \theta(x, u_{(r)})) = 0 \mid \theta \in \mathcal{S}\}.$$

Here and below in this section the notation differs from the other sections of the paper:  $x$  and  $u$  are the tuples of independent and dependent variables, respectively, constituting the coordinates for a coordinate space,  $u_{(r)}$  denotes the tuple of derivatives of  $u$  with respect to  $x$  up to order  $r$ , which also includes the  $u$ 's as the derivatives of order zero, and  $L$  is a tuple of differential functions in  $u$ , parameterized by the tuple of functions  $\theta = (\theta^1(x, u_{(r)}), \dots, \theta^p(x, u_{(r)}))$ , called the arbitrary elements of the class  $\mathcal{L}|_{\mathcal{S}}$ . The arbitrary-element tuple runs through the solution set  $\mathcal{S}$  of an auxiliary system AS of differential equations and inequalities in  $\theta$ ,  $S(x, u_{(r)}, \theta_{(q)}(x, u_{(r)})) = 0$  and, e.g.,  $\Sigma(x, u_{(r)}, \theta_{(q)}(x, u_{(r)})) \neq 0$ , where  $(x, u_{(r)})$  is assumed to be the tuple of independent variables, and the notation  $\theta_{(q)}$  encompasses the partial derivatives of the arbitrary elements  $\theta$  up to order  $q$  with respect to both  $x$  and  $u_{(r)}$ . See [35, 39] for more detailed explanations on the notion of class of differential equations.

Denote by  $\mathfrak{g}_{\theta}$  the maximal Lie invariance algebra of a system  $\mathcal{L}_{\theta} \in \mathcal{L}|_{\mathcal{S}}$ . The system of determining equations DE $[\theta]$  for components of vector fields in  $\mathfrak{g}_{\theta}$  may split into two subsystems. The equations of one of these subsystems, CE $[\theta]$ , essentially involve arbitrary elements and are called *classifying equations*, and the other subsystem, SE, does not involve  $\theta$  and is thus shared by all  $\mathcal{L}_{\theta} \in \mathcal{L}|_{\mathcal{S}}$ . The subsystem SE can be integrated directly, thus specifying the general form of Lie-symmetry vector fields of systems in  $\mathcal{L}|_{\mathcal{S}}$ . The subsystem CE $[\theta]$  with a fixed value of the arbitrary-element tuple  $\theta$  is the system of specific equations for the components of vector fields in  $\mathfrak{g}_{\theta}$ . The solution set of the joint system DE $[\theta]$  is a linear space in view of the fact that DE $[\theta]$  is a homogeneous linear system of differential equations with respect to the components of Lie-symmetry vector fields. Conversely, fixing a vector field  $Q$  on the space with coordinate  $(x, u)$ , we obtain a system of differential equations for the values of  $\theta$  for which the corresponding system  $\mathcal{L}_{\theta}$  admits  $Q$  as a Lie-symmetry vector field.

The system AS as a rule includes equations implying the independence of  $\theta$  on certain independent or dependent variables or, more generally, on certain functional combinations of these variables. It is convenient to replace the coordinates  $(x, u_{(r)})$  in the  $r$ th order jet space by the (local-)coordinate tuple  $z$  that is split into two subtuples,  $\hat{z}$  and  $\check{z}$ , such that  $\hat{z}$  is a maximal tuple of coordinates not involved in  $\theta$ , and each of the tuples  $\hat{z}$  and  $\check{z}$  is split into two subtuples,  $\hat{z}'$ ,  $\hat{z}''$  and  $\check{z}'$ ,  $\check{z}''$ , where  $\hat{z}'$  and  $\check{z}'$  are maximal subtuples of  $\hat{z}$  and  $\check{z}$ , respectively, that appear in arguments of functions parameterizing the general solution of SE. We choose the tuple  $\check{z}'$  of functions of  $(x, u)$  such that the joint tuple  $(\hat{z}', \check{z}', \check{z}'')$  gives new coordinates in the space of  $(x, u)$ . In most of practical computations by the method of furcate splitting, the optimal choice of  $\hat{z}'$ ,  $\hat{z}''$ ,  $\check{z}'$  and  $\check{z}''$  is obvious, cf. Section 5.

We substitute the expressions for the components of Lie-symmetry vector fields that are specified by SE into CE $[\theta]$  and split the obtained system with respect to  $\hat{z}''$ . This leads to the system CE' $[\theta]$  for the constants and functions parameterizing these expressions, which is supposed to be of the form

$$\sum_{i=i_{s-1}+1}^{i_s} \psi^i F^i = 0, \quad s = 1, \dots, |\text{CE}'|,$$

where  $0 = i_0 < i_1 < \dots < i_{|\text{CE}'|} = N$  with  $N \in \mathbb{N}$ , and  $|\text{CE}'|$  denotes the number of (independent) equations in CE' $[\theta]$ . The functions  $F^i$ ,  $i = 1, \dots, N$ , are known differential functions of  $\theta$  with  $\hat{z}$  assumed to be the tuple of independent variables. The coefficients  $\psi^i$ ,  $i = 1, \dots, N$ , may depend on  $(\hat{z}', \hat{z}', \hat{z}')$  and are in fact parameterized by an element of  $\mathfrak{g}_\theta$ .

Let us imagine that the algebra  $\mathfrak{g}_\theta$  is known for each fixed  $\theta$ . Substituting values of the above parameters corresponding to a vector field in  $\mathfrak{g}_\theta$  and of the variables  $\hat{z}'$  into the system CE' $[\theta]$  and varying the vector field within  $\mathfrak{g}_\theta$  and the variables  $\hat{z}'$  give a family (denoted by TFF) of systems for  $\theta$  of the same general form called the *template form*,

$$\sum_{i=i_{s-1}+1}^{i_s} a^i F^i = 0, \quad s = 1, \dots, |\text{CE}'|,$$

where coefficients  $a^i$ ,  $i = 1, \dots, N$ , may depend on  $(\hat{z}', \hat{z}')$ . In general, these coefficients are not precisely known at this stage but they may satisfy known constraints.

Let  $k$  be the maximal number of systems in TFF such that the rank of the collection of the coefficient tuples  $\bar{a}^q = (a^{q1}, \dots, a^{qN})$  of these systems equals their number,  $\text{rank } A = k$  with  $A := (a^{qi})_{\substack{i=1, \dots, N \\ q=1, \dots, k}}$ . It is obvious that  $0 \leq k \leq N$ . The condition of consistency of systems within TFF and of these systems with AS additionally majorizes the possible values of  $k$ .

Supposing a certain value for  $k$  within the range of possible values of  $k$ , we study the consistency of TFF. The consideration can be partitioned into cases by choosing a sequence of  $k \times k$  minors of the matrix  $A$  in a way consistent with the relevant equivalence within the class  $\mathcal{L}|_{\mathfrak{g}}$  and by successively supposing that a minor in the sequence does not vanish and all the preceding minors are zero. Some of the coefficients  $a^{qi}$  can be gauged using equivalence transformations in view of the made supposition or derived constraints at this or further steps. For each of the cases, one should

- derive the conditions implied by the consistency of TFF on the coefficients  $a^{qi}$  and additionally gauge these coefficients by equivalence transformations (if possible),
- split the system CE' $[\theta]$  on the solution set of the joint system of AS and TFF with respect to parametric derivatives of  $\theta$ ,
- solve the obtained system of additional determining equations for still unspecified parameters in Lie symmetry vector fields,
- derive the precise form of TFF for the computed algebra of vector fields and check its consistency with the supposed value of  $k$  and other suppositions,
- if the consistency holds, solve the specified system TFF with respect to  $\theta$ .

One has  $k = 0$  if and only if the classifying equations are identically satisfied in view of SE, which corresponds to the general case without Lie-symmetry extension.

The value  $k = 1$  is associated with minimal Lie-symmetry extensions. For this value of  $k$ , the second step of the above procedure is convenient to be carried out in a special way since it means that the tuple  $\bar{\psi} = (\psi^1, \dots, \psi^N)$  is proportional to  $\bar{a}^1$ ,  $\psi^i = \lambda a^{1i}$ . Here the multiplier  $\lambda$  may depend on  $(\hat{z}', \hat{z}', \hat{z}')$  or, equivalently, on  $(x, u)$ , and this dependence can be easily specified in view of the form of  $\bar{\psi}$  and  $\bar{a}^1$ .

## 5 Solution of group classification problem for regular subclass

In this section we prove the following assertion.

**Proposition 14.** *A complete list of  $G_{\mathcal{R}}^{\sim}$ -inequivalent Lie-symmetry extensions in the class  $\mathcal{C}$  is exhausted by Cases 1, 2, 3, 4a, 4b, 5, 6a, 6b, 6c, 7 and 9b of Table 1.*

Recall that the  $G_{\mathcal{C}}^{\sim}$ -equivalence coincides with the restriction of the  $G_{\mathcal{R}}^{\sim}$ -inequivalence to the class  $\mathcal{C}$ .

It turns out that additionally to  $\tau_x = \tau_u = 0$ , components of Lie-symmetry vector fields of equations in the class  $\mathcal{C}$  satisfy more determining equations not involving the arbitrary elements  $f$  and  $g$ .

**Lemma 15.** *Under the conditions  $f_{u_x} \neq 0$ ,  $(u_x^2 f)_{u_x} \neq 0$  and  $g_u \neq 0$ , the system (20) implies*

$$\xi_{xx} = \xi_{xu} = \xi_{uu} = \eta_{xx} = \eta_{xu} = \eta_{uu} = 0. \quad (21)$$

*Proof.* We consider two obvious differential consequences of the system (20). One of the consequences is derived by acting the operators  $u_x \partial_u + \partial_x$  and  $-\partial_{u_x}$  on the equations (20a) and (20b), respectively, and summing the obtained equations, which gives

$$(\xi_{uu} u_x^2 - 2\eta_{uu} u_x - 2\eta_{xu} - \xi_{xx})f = \xi_t + \xi_u g. \quad (22)$$

The other consequence is the sum of the equation (20b) with the equation (22) multiplied by  $u_x$ ,

$$((\eta_{uu} + 2\xi_{xu})u_x^2 + 2\xi_{xx}u_x - \eta_{xx})f = \eta g_u + (\tau_t - \eta_u)g - \eta_t. \quad (23)$$

The equations (22) and (23) hint that the proof is partitioned into three cases, depending on whether these equations imply conditions on  $f$  and what structures of such conditions are,

$$1. (1/f)_{u_x u_x u_x} \neq 0, \quad 2. (1/f)_{u_x u_x u_x} = 0, (1/f)_{u_x u_x} \neq 0, \quad \text{and} \quad 3. (1/f)_{u_x u_x} = 0.$$

In the first case, we can split the above differential consequences with respect to both  $f$  and  $u_x$ , which directly leads to the required equations.

The third condition means that  $f = (u_x + \gamma)^{-1} \bmod G_{\mathcal{R}}^{\sim}$ . (Recall that we suppose the inequality  $f_{u_x} \neq 0$ .) Substituting the expression for  $f$  into (20a) and splitting with respect to  $u_x$  give the equations  $\xi_u = 0$ ,  $\eta_u = \gamma(\tau_t - \xi_x)$  and  $\eta_x = \tau_t - \xi_x$ , which imply  $\eta_{xu} = \eta_{uu} = 0$ . Using derived equations, we simplify the equation (22) and then treat it in the same way, which in particular gives  $\xi_{xx} = 0$ , and thus also  $\eta_{xx} = 0$ .

In the second case, which is much more complicated than the other cases, modulo  $G_{\mathcal{R}}^{\sim}$ -equivalence we can set  $f = (u_x^2 + 2\beta u_x + \gamma)^{-1}$ , where  $(\beta, \gamma) \neq (0, 0)$  since  $(u_x^2 f)_{u_x} \neq 0$ . Substituting the expression for  $f$  into (20a), splitting with respect to  $u_x$  and arranging the obtained equations, we get

$$\eta_u = -\beta \xi_u + \frac{1}{2} \tau_t, \quad \eta_x = -\beta \xi_x + \frac{\beta}{2} \tau_t + (\beta^2 - \gamma) \xi_u, \quad (\beta^2 - \gamma)(2\xi_x - \beta \xi_u - \tau_t) = 0. \quad (24)$$

The cross differentiation of the two first equations in (24) implies  $(\beta^2 - \gamma)\xi_{uu} = 0$ . Then the separate differentiations of the last equation in (24) with respect to  $x$  and  $u$  successively give  $(\beta^2 - \gamma)\xi_{xu} = 0$  and  $(\beta^2 - \gamma)\xi_{xx} = 0$ . In view of these equations for  $\xi$ , the same differentiation of the two first equations in (24) results in  $\eta_{xx} = -\beta \xi_{xx}$ ,  $\eta_{xu} = -\beta \xi_{xu}$  and  $\eta_{uu} = -\beta \xi_{uu}$ . Therefore, if  $\gamma \neq \beta^2$ , then we have the required determining equations.

Suppose that  $\gamma = \beta^2$ . Then  $\beta \neq 0$ , and modulo  $G_{\mathcal{R}}^{\sim}$ -equivalence we can set  $\beta = 1$ , i.e.,  $f = (u_x + 1)^{-2}$ . The general solution of the system  $\eta_x = -\xi_x + \tau_t/2$ ,  $\eta_u = -\xi_u + \tau_t/2$  with

respect to  $\eta$  is  $\eta = -\xi + \tau_t(u+x)/2 + \eta^0(t)$ , where  $\eta^0$  is an arbitrary smooth function of  $t$ . For these values of  $f$  and  $\eta$ , the determining equation (20b) takes the form

$$((\xi_t + \xi_u g)u_x + \eta g_u + (\tau_t - \eta_u)g - \eta_t)(u_x + 1) = \xi_{uu}u_x^2 + 2\xi_{xu}u_x + \xi_{xx}$$

and splits with respect to  $u_x$  into the system

$$\xi_t + \xi_u g = \xi_{uu}, \quad \eta g_u + (\tau_t - \eta_u)g - \eta_t = \xi_{xx}, \quad \xi_{uu} - 2\xi_{xu} + \xi_{xx} = 0. \quad (25)$$

The last equation can be represented as  $(\partial_x - \partial_u)^2 \xi = 0$ , and hence its general solution is  $\xi = \xi^1(t, \omega)u + \xi^0(t, \omega)$ , where  $\xi^1$  and  $\xi^0$  are arbitrary smooth functions of  $t$  and  $\omega = x + u$ . In view of the derived expressions for  $\xi$  and  $\eta$ , the system of the first two equations of (25) reduces to the system

$$\xi_t^1 u + \xi_t^0 + (\xi_\omega^1 u + \xi^1 + \xi_\omega^0)g = \xi_{\omega\omega}^1 u + 2\xi_\omega^1 + \xi_{\omega\omega}^0, \quad (26)$$

$$(-2\xi^1 u - 2\xi^0 + \tau_t \omega + 2\eta^0)g_u + \tau_t g + 4\xi_\omega^1 - \tau_{tt} \omega - 2\eta_t^0 = 0. \quad (27)$$

We assume  $(t, \omega, u)$  to be the tuple of independent variables in the last system.

If the arbitrary element  $g$  is not a fractional linear function, then we can split the equation (26) simultaneously with respect to  $g$  and  $u$  and obtain the equations  $\xi_\omega^1 = 0$  and  $\xi_\omega^0 = -\xi^1$ , whose consequence is  $\xi_{\omega\omega}^0 = 0$ . Otherwise (i.e., in the case of nonconstant fractional linear  $g$ , since  $g_u \neq 0$  by a lemma's assumption), we differentiate the equation (27) with respect to  $\omega$  and split the derived differential consequence with respect to  $u$ , which again leads to the equations  $\xi_\omega^1 = 0$  and  $\xi_{\omega\omega}^0 = 0$ . Therefore, in the third case we also have the required equations  $\xi_{xx} = \xi_{xu} = \xi_{uu} = 0$  and, therefore,  $\eta_{xx} = \eta_{xu} = \eta_{uu} = 0$ .  $\square$

In other words, the system of determining equations for Lie symmetries of equations in the class  $\mathcal{C}$  in fact reduces to the system of the equations  $\tau_x = \tau_u = 0$ , (20a), (21) and

$$\xi_t + \xi_u g = 0, \quad (28)$$

$$\eta g_u + (\tau_t - \eta_u)g - \eta_t = 0. \quad (29)$$

The system (21) is equivalent to the following representation for the components  $\xi$  and  $\eta$ :

$$\xi = \xi^1(t)x + \xi^2(t)u + \xi^0(t), \quad \eta = \eta^1(t)x + \eta^2(t)u + \eta^0(t),$$

where  $\xi^0, \xi^1, \xi^2, \eta^0, \eta^1$ , and  $\eta^2$  are smooth functions of  $t$ .

**Lemma 16.** (i) For all equations in the class  $\mathcal{C}$ , except those  $G_{\mathcal{R}}^{\sim}$ -equivalent to equations of Case 7, the  $x$ -components of admitted Lie-symmetry vector fields satisfy the determining equation  $\xi_u = 0$ . (ii) For all equations in the class  $\mathcal{C}$ , except those  $G_{\mathcal{R}}^{\sim}$ -equivalent to equations of Case 5, the  $u$ -components of admitted Lie-symmetry vector fields satisfy the determining equation  $\eta_x = 0$ .

*Proof.* Suppose that an equation  $\mathfrak{E}$  in the class  $\mathcal{C}$  admits a Lie-symmetry vector field  $Q$  with  $(\xi^2, \eta^1) \neq (0, 0)$ , and  $(f, g)$  is the associated value of the arbitrary-element tuple.

If  $\xi_u = \xi^2 \neq 0$  (resp.  $\eta_x = \eta^1 \neq 0$ ) for  $Q$ , then the equation (28) (resp. the differential consequence of (29) obtained with the action by the operator  $\partial_x$ ) implies  $g_{uu} = 0$ , and hence  $g = u \bmod G_{\mathcal{R}}^{\sim}$  since  $g_u \neq 0$  in the class  $\mathcal{C}$ . For this value of the arbitrary element  $g$ , the equations (28) and (29) respectively split with respect to  $(x, u)$  into the equations

$$\xi_t^0 = \xi_t^1 = 0, \quad \xi_t^2 = -\xi^2 \quad \text{and} \quad \eta_t^0 = \eta^0, \quad \eta_t^1 = \eta^1, \quad \eta_t^2 = \tau_t. \quad (30)$$

Now we apply the furcate splitting with respect to  $f$ . Here the template form of equations for  $f$  is

$$(a_1 u_x^2 + a_2 u_x + a_3) f_{u_x} + (2a_1 u_x + a_4) f = 0, \quad (31)$$

where  $a_1, \dots, a_4$  are constants. Template-form equations for  $f$  are obtained by fixing  $t$  in the classifying equation (20a). Denote by  $k$  the number of linearly independent tuples  $(a_1, a_2, a_3, a_4)$  among those associated with template-form equations for  $f$ . We have  $k > 0$  since  $(\xi^2, \eta^1) \neq (0, 0)$  for the Lie-symmetry vector field  $Q$ .

If  $k \geq 2$ , then  $f = \varepsilon(u_x + 1)^{-2} \bmod G_{\mathcal{R}}^{\sim}$  with  $\varepsilon = \pm 1$  since  $f_{u_x} \neq 0$  and  $(u_x^2 f)_{u_x} \neq 0$  for equations in the class  $\mathcal{C}$ . Splitting the equation (20a) for this value of  $f$  with respect to  $u_x$ , we derive the equations  $\tau_t = 2\eta^1 + 2\xi^1$  and  $R := \xi^1 - \xi^2 + \eta^1 - \eta^2 = 0$ . Then in view of the system (30), we have  $\tau_{tt} = 2\eta^1$ ,  $R_{tt} = -\xi^2 + \eta^1 = 0$  and  $R_{ttt} = \xi^2 + \eta^1 = 0$ , i.e.,  $\xi^2 = \eta^1 = 0$  for Lie-symmetry vector fields of the equation  $\mathfrak{E}$ , which contradicts the condition  $(\xi^2, \eta^1) \neq (0, 0)$  for  $Q$ .

Therefore,  $k = 1$ . We fix a template-form equation, and let  $(a_1, a_2, a_3, a_4)$  be the corresponding (nonzero) coefficient tuple. Since  $k = 1$ , the coefficient tuple  $(-\xi^2, \eta^2 - \xi^1, \eta^1, \tau_t - 2\xi^1)$  of the equation (20a) is proportional to  $(a_1, a_2, a_3, a_4)$ ,

$$-\xi^2 = \lambda a_1, \quad \eta^2 - \xi^1 = \lambda a_2, \quad \eta^1 = \lambda a_3, \quad \tau_t - 2\xi^1 = \lambda a_4, \quad (32)$$

where  $\lambda = \lambda(t)$  is a nonvanishing smooth function of  $t$ , and  $(a_1, a_3) \neq (0, 0)$  since  $(\xi^2, \eta^1) \neq (0, 0)$  for  $Q$ . This implies that  $a_1\eta^1 + a_3\xi^2 = 0$ . The differentiation of this equation with respect to  $t$  gives, in view of (30), the equation  $a_1\eta^1 - a_3\xi^2 = 0$ , i.e.,  $a_1\eta^1 = a_3\xi^2 = 0$  and thus  $\xi^2\eta^1 = a_1a_3 = 0$ .

If  $a_1 \neq 0$ , then due to the possibility of simultaneous opposite scalings of  $(a_1, a_2, a_3, a_4)$  and  $\lambda$  we can assume without loss of generality that  $a_1 = 1$  and  $\lambda = -\xi^2 \neq 0$ . Hence  $\eta^1 = 0$  and  $a_3 = 0$ . The other equations following from (32) are  $\eta^2 = -a_2\xi^2 + \xi^1$  and  $\tau_t = -a_4\xi^2 + 2\xi^1$ . We combine them with equations of (30) to get  $\tau_t = \eta_t^2 = a_2\xi^2$ , which leads to  $a_4 = -a_2$  and  $\xi^1 = 0$ . We have  $a_2 \neq 0$  since otherwise the fixed template-form equation for  $f$  contradicts the auxiliary inequality  $(u_x^2 f)_{u_x} \neq 0$  for arbitrary elements of equations in the class  $\mathcal{C}$ . This is why modulo  $G_{\mathcal{R}}^{\sim}$ -equivalence we can set  $a_2 = 1$  and obtain, after alternating the sign of  $t$  if  $\varepsilon = -1$ , Case 7.

For the case  $a_3 \neq 0$ , the consideration is similar. Due to the possibility of simultaneous opposite scalings of  $(a_1, a_2, a_3, a_4)$  and  $\lambda$  we can assume without loss of generality that  $a_3 = 1$  and  $\lambda = \eta^1 \neq 0$ . Hence  $\xi^2 = 0$  and  $a_1 = 0$ . The other equations following from (32) are  $\eta^2 = a_2\eta^1 + \xi^1$  and  $\tau_t = a_4\eta^1 + 2\xi^1$ . We combine them with equations of (30) to obtain  $\tau_t = \eta_t^2 = a_2\eta^1$ , which leads to  $a_4 = a_2$  and  $\xi^1 = 0$ . We have  $a_2 \neq 0$  since otherwise the fixed template-form equation for  $f$  contradicts the auxiliary inequality  $f_{u_x} \neq 0$  for arbitrary elements of equations in the class  $\mathcal{C}$ . This is why modulo  $G_{\mathcal{R}}^{\sim}$ -equivalence we can set  $a_2 = 1$  and obtain, after alternating the sign of  $t$  if  $\varepsilon = -1$ , Case 5.  $\square$

Therefore, in the rest of the proof we can exclude equations falling, modulo  $G_{\mathcal{R}}^{\sim}$ -equivalence, into Cases 5 or 7 and additionally set  $\xi_t = \xi_u = \eta_x = 0$ , i.e.,

$$\xi = c_1x + c_2, \quad \eta = \eta^2(t)u + \eta^0(t),$$

where  $c_1$  and  $c_2$  are constants. The unsolved determining equations are exhausted by the reduced form of the equations (20a) and (29),

$$(\eta^2 - c_1)u_x f_{u_x} + (\tau_t - 2c_1)f = 0, \quad (33)$$

$$(\eta^2 u + \eta^0)g_u + (\tau_t - \eta^2)g = \eta_t^2 u + \eta_t^0. \quad (34)$$

We complete the group classification of the class  $\mathcal{C}$  by the method of furcate splitting. The system of the reduced classifying equations (33) and (34) is decoupled, and the arbitrary elements  $f$  and  $g$  are unary functions of different arguments. Thus, we need to apply two-step furcate splitting, with respect to  $f$  and then with respect to  $g$ . Here the template forms of equations for  $f$  and  $g$  are respectively

$$a_2 u_x f_{u_x} + a_4 f = 0, \quad (35)$$

$$(b_1 u + b_2)g_u + b_3 g = b_4 u + b_5, \quad (36)$$

where  $a_2, a_4$  and  $b_1, \dots, b_5$  are constants. (We number the constants  $a_2$  and  $a_4$  in the way consistent with the template form (31).) Template-form equations for  $f$  and  $g$  are obtained by fixing  $t$  in the classifying equations (33) and (34), respectively. Denote by  $k$  (resp.  $l$ ) the number of linearly independent tuples  $(a_2, a_4)$  (resp.  $(b_1, \dots, b_5)$ ) among those associated with template-form equations for  $f$  (resp.  $g$ ). We have  $k < 2$  since  $f \neq 0$ .

**$k = 0$ .** This means that the equation (33) is an identity with respect to  $f$ , i.e.,  $\eta^2 = c_1$  and  $\tau_t = 2c_1$ . Then  $b_3 = b_1$  and  $b_4 = 0$  in the template form (36) of equations for  $g$ . The number  $l$  of linearly independent tuples among possible values of the coefficient tuple  $(b_1, b_2, b_5)$  cannot exceed one since otherwise the corresponding system of template-form equations either is inconsistent or has only constant solutions, which contradicts the auxiliary inequality  $g_u \neq 0$  for arbitrary elements of equations in the class  $\mathcal{C}$ . If  $l = 0$ , i.e., the classifying equation (34) is also an identity, then we get the general classification Case 0 of Table 1 with no Lie symmetry extension. For  $l = 1$ , depending on whether or not the coefficient  $b_1$  in a nonidentical template-form equation vanishes, we obtain that either  $g = u \bmod G_{\mathcal{R}}^{\sim}$  or  $g = u^{-1} + \nu \bmod G_{\mathcal{R}}^{\sim}$ . The first option corresponds to Case 1. The second option leads to Case 2 in view of the condition  $\nu = 0$  since for  $\nu \neq 0$  we only obtain the kernel invariance algebra  $\mathfrak{g}_{\mathcal{C}}^{\square} = \langle \partial_t, \partial_x \rangle$ , which corresponds to the condition  $l = 0$  contradicting the supposed condition  $l = 1$ .

**$k = 1$ .** We fix a (nonidentical) template-form equation for  $f$ , and let  $(a_2, a_4)$  be the corresponding (nonzero) coefficient tuple. The inequality  $f \neq 0$  implies  $a_2 \neq 0$ , and thus we can set  $a_2 = 1$ . Denote  $n := -a_4$ . Then the equation (35) implies that  $f = |u_x|^n \bmod G_{\mathcal{R}}^{\sim}$ , where  $n \neq 0$  and  $n \neq -2$  according to the auxiliary inequalities  $f_{u_x} \neq 0$  and  $(u_x^2 f)_{u_x} \neq 0$  for arbitrary elements of equations in the class  $\mathcal{C}$ . Since  $k = 1$ , the coefficient tuple  $(\eta^2 - c_1, \tau_t - 2c_1)$  of the equation (33) is proportional to  $(1, -n)$ . In other words,  $\eta^2 - c_1 = \lambda$ ,  $\tau_t - 2c_1 = -n\lambda$ , where  $\lambda = \lambda(t)$  is a nonvanishing smooth function of  $t$ . Hence  $\tau_t = -n\eta^2 + (n+2)c_1$ , and the equation (34) takes the form

$$(\eta^2 u + \eta^0)g_u + ((n+2)c_1 - (n+1)\eta^2)g = \eta_t^2 u + \eta_t^0. \quad (37)$$

We again apply the furcate splitting with respect to  $g$  using the number  $l$  for marking different cases. We have  $l > 0$  since otherwise the equation (37) would be an identity with respect to  $g$  and thus  $\eta^0 = \eta^2 = 0$  and, in view of  $n \neq -2$ ,  $c_1 = 0$ , which would lead to the kernel invariance algebra  $\mathfrak{g}_{\mathcal{C}}^{\square} = \langle \partial_t, \partial_x \rangle$  and would thus give  $k = 0$ , contradicting the condition  $k = 1$ . The further consideration splits into two cases,  $l \geq 2$  and  $l = 1$ .

$l \geq 2$ . Note that three is the maximal value of  $l$  since for greater values of  $l$ , systems of template-form equations are not consistent.

Suppose that for two template-form equations with linearly independent tuples of coefficients  $(b_1, \dots, b_5)$  and  $(b'_1, \dots, b'_5)$ , the conditions  $b_2 b'_1 - b'_2 b_1 \neq 0$ ,  $b_3 b'_1 - b'_3 b_1 = 0$  and  $b_4 b'_1 - b'_4 b_1 \neq 0$  hold. Then the consistency of these template-form equations implies that  $g_{uuu} = 0$  and  $g_{uu} \neq 0$ , i.e.,  $g = u^2 + \delta \bmod G_{\mathcal{R}}^{\sim}$  with  $\delta \in \{-1, 0, 1\}$ . Splitting the equation (37) for the obtained value of the arbitrary element  $g$  with respect to  $u$ , we derive the equations  $(n+2)c_1 = (n-1)\eta^2$ ,  $\eta_t^0 = -2\delta\eta^2$  and  $\eta_t^2 = 2\eta^0$ . These equations imply  $n = 1$  since otherwise  $\eta^2 = (n+2)c_1/(n-1)$ ,  $\eta^0 = 0$ ,  $\delta c_1 = 0$  and thus  $l \leq 1$ , which contradicts the assumed condition  $l \geq 2$ . Then  $c_1 = 0$ , and we obtain Cases 6a, 6b and 6c depending on the value of  $\delta$ .

If there exists a pair of template-form equations with other conditions on the associated linearly independent tuples of coefficients  $b$ 's and  $b'$ 's, then we get  $g_{uu} = 0$ , i.e.,  $g = \varepsilon u \bmod G_{\mathcal{R}}^{\sim}$  with  $\varepsilon \in \{-1, 1\}$  in view of the auxiliary inequality  $g_u \neq 0$  within the class  $\mathcal{C}$ . The equation (37) with this value of  $g$  splits into the equations  $\eta_t^2 = -\varepsilon n \eta^2 + \varepsilon(n+2)c_1$  and  $\eta_t^0 = \varepsilon \eta^0$ . As a result, we get Case 9b, for which in fact  $l = 3$ .

$l = 1$ . Then the coefficient tuple of the equation (37) is proportional to the (nonvanishing) coefficient tuple of a template-form equation,

$$\eta^2 = \chi b_1, \quad \eta^0 = \chi b_2, \quad (n+2)c_1 - (n+1)\eta^2 = \chi b_3, \quad -\eta_t^2 = \chi b_4, \quad -\eta_t^0 = \chi b_5.$$

$\chi = \chi(t)$  is a nonvanishing smooth function of  $t$ . Moreover, we have the inequality  $g_{uu} \neq 0$  since otherwise  $l = 2$ . We consider separately two cases,  $(b_1, b_4) \neq (0, 0)$  and  $b_1 = b_4 = 0$ .

Suppose at first that  $(b_1, b_4) \neq (0, 0)$ . We recombine the above equations to the equations  $b_1\chi_t + b_4\chi = 0$  and  $b_2\chi_t + b_5\chi = 0$  whose consistency as algebraic equations with respect to  $(\chi_t, \chi)$  implies that the pair  $(b_2, b_5)$  is proportional to  $(b_1, b_4)$ . Therefore, modulo  $G_{\mathcal{R}}^{\sim}$ -equivalence (more specifically, modulo shifts of  $u$ ) we can set  $b_2 = b_5 = 0$ , and thus  $\eta^0 = 0$ . Then it becomes obvious that  $b_1 \neq 0$  since otherwise  $b_3b_4 \neq 0$ , which implies  $g_{uu} = 0$ . Therefore, due to the possibility of simultaneous opposite scalings of  $(b_1, b_3, b_4)$  and  $\chi$  we can assume without loss of generality that  $b_1 = 1$  and  $\chi = \eta^2 \neq 0$ . Hence  $(n+2)c_1 = (n+1+b_3)\eta^2$  and  $\eta_t^2 = -b_4\eta^2$ . For  $b_4 = 0$  we directly obtain Case 4a. If  $b_4 \neq 0$ , then  $c_1 = 0$  and  $b_3 = -n-1$ , which gives Case 4b.

Let  $b_1 = b_4 = 0$ . Then the inequality  $g_{uu} \neq 0$  implies  $b_2 \neq 0$  and  $b_5 = 0$ . To derive the last equality, we use the fact that otherwise  $\eta_t^0 = b_2\chi_t \neq 0$  and hence  $b_3 = 0$ , which contradicts the inequality  $g_{uu} \neq 0$ . Therefore,  $g = \varepsilon e^u \bmod G_{\mathcal{R}}^{\sim}$  with  $\varepsilon \in \{-1, 1\}$ , which gives Case 3.

The proof of Proposition 14 is completed.

## 6 Lie reductions and exact solutions

Exact solutions of equations for the class  $\mathcal{H}$  were constructed by various methods in many papers, including classical Lie reductions [10], nonclassical reductions using conditional symmetries [4, 9, 12, 44, 47] and generalized separation of variables [13]. See also [3, Section 10.5], [34, Chapter 5] and references therein.

The construction of exact solutions of equations from the class  $\mathcal{F}$  is reduced by the family of transformations from  $G_{\mathcal{F}}^{\sim}$  with  $(t, x, u)$ -components  $\tilde{t} = t$ ,  $\tilde{x} = x$ ,  $\tilde{u} = u - gt$ , which is parameterized by the arbitrary element  $g$ , to the same problem for the subclass  $\mathcal{F}'$ , cf. Section 2. Equations from the subclass  $\mathcal{F}'$ , which are nonlinear filtration equations, are potential equations of nonlinear diffusion equations. Lie reductions of the latter equations were studied in [30]. Exact solutions of both nonlinear diffusion equations and nonlinear filtration equations were found using Lie and quasiloal (potential) symmetries [1, 2], generalized separation of variables [7, 11], nonclassical reductions and nonclassical potential reductions [5, 15, 42] and generalized conditional symmetries [43]. See also [3, Section 10.2], [34, Chapter 5] and references therein.

The optimal way for constructing exact solutions of equations from the class  $\mathcal{L}$  is to use their linearization by the hodograph transformation  $\tilde{t} = t$ ,  $\tilde{x} = u$ ,  $\tilde{u} = x$  with  $(\tilde{t}, \tilde{x})$  and  $\tilde{u}$  being the new independent and dependent variables, respectively, to Kolmogorov equations; see the introduction.

This is why it is justified to look for exact solutions of equations from the class  $\mathcal{C}$  only, excluding equations related to Case 9b, which is similar to Case 9a with respect to additional equivalence transformations. We select all  $G_{\mathcal{R}}^{\sim}$ -inequivalent cases of Table 1 with four-dimensional maximal Lie invariance algebras as the most interesting, which are Cases 5, 6a and 7. They are the obvious contenders for allowing one to construct nontrivial explicit solutions. We identify the structure of the associated maximal Lie algebras according Mubarakzhanov's classification of real four-dimensional Lie algebras [22]. Optimal lists of subalgebras of such algebras (i.e., complete lists of subalgebras that are inequivalent up to inner automorphisms of the corresponding algebras) were constructed by Patera and Winternitz in [32]. Since we carry out Lie reductions of partial differential equations with two independent variables to ordinary differential equations, we only need optimal lists of one-dimensional subalgebras. For each of the selected classification cases, we additionally gauge, whenever it is possible, subalgebra parameters using outer automorphisms of the corresponding algebras that are induced by discrete symmetries of related equations. See also [36] for a collection of various characteristics of low-dimensional Lie algebras and of classifications of related objects. It is convenient to present the results of Lie reductions as Table 2, each row of which includes a basis element of the canonical representative of an equivalence class of one-dimensional subalgebras, an associated ansatz for the dependent vari-

Table 2: Lie reductions of Cases 5 and 6a and the equation (41) to ODEs

no.	Subalgebra basis element	Ansatz for $u$	$\omega$	Reduced equation
5.1	$\partial_x$	$u = \phi$	$t$	$\phi_\omega = \varepsilon\phi$
5.2	$\partial_t + \kappa\partial_x$	$u = \phi$	$x - \kappa t$	$\phi_{\omega\omega} = -(\kappa\phi_\omega + \varepsilon\phi)(\phi_\omega + 1)$
5.3	$e^{\varepsilon t}(\partial_t + \varepsilon(u+x)\partial_u)$	$u = \phi e^{\varepsilon t} - x$	$x$	$\phi_{\omega\omega} = \varepsilon\omega\phi_\omega$
5.4	$e^{\varepsilon t}(\partial_t + \varepsilon(u+x)\partial_u) + \partial_x$	$u = \phi e^{\varepsilon t} - x - e^{-\varepsilon t}/(2\varepsilon)$	$x + e^{-\varepsilon t}/\varepsilon$	$\phi_{\omega\omega} = \phi_\omega(\varepsilon\omega - \phi_\omega)$
6.1	$\partial_x$	$u = \phi$	$t$	$\phi_\omega = \phi^2$
6.2	$\partial_t$	$u = \phi$	$x$	$\phi_\omega\phi_{\omega\omega} + \phi^2 = 0$
6.3	$\partial_t + \partial_x$	$u = \phi$	$x - t$	$\phi_\omega\phi_{\omega\omega} + \phi^2 + \phi_\omega = 0$
6.4	$t\partial_t - u\partial_u + \kappa\partial_x$	$u = \phi/t$	$x - \kappa \ln  t $	$\phi_\omega\phi_{\omega\omega} + \phi^2 + \kappa\phi_\omega + \phi = 0$
6.5	$(t^2+1)\partial_t - (2tu+1)\partial_u + \kappa\partial_x$	$u = (\phi - t)/(t^2 + 1)$	$x - \kappa \arctan t$	$\phi_\omega\phi_{\omega\omega} + \phi^2 + \kappa\phi_\omega + 1 = 0$
$\tilde{7}.1$	$\partial_x$	$u = \phi$	$t$	$\phi_\omega = 0$
$\tilde{7}.2$	$t\partial_x + \varepsilon^{-1}\partial_u$	$u = \phi + \varepsilon^{-1}x/t$	$t$	$\phi_\omega + \omega^{-1}\phi = 0$
$\tilde{7}.3$	$\partial_t$	$u = \phi$	$x$	$\phi_\omega(\phi_{\omega\omega} - \varepsilon\phi) = 0$
$\tilde{7}.4$	$\partial_t + t\partial_x + \varepsilon^{-1}\partial_u$	$u = \phi + \varepsilon^{-1}t$	$x - t^2/2$	$\phi_\omega(\phi_{\omega\omega} - \varepsilon\phi) = \varepsilon^{-1}$
$\tilde{7}.5$	$t\partial_t - u\partial_u + \kappa\partial_x$	$u = (\phi + \varepsilon^{-1}\kappa)/t$	$x - \kappa \ln  t $	$\phi_\omega(\phi_{\omega\omega} - \varepsilon\phi) + \phi = -\varepsilon^{-1}\kappa$

able, the invariant independent variable  $\omega$  and the corresponding reduced equation, respectively. Here and below  $c$ 's are arbitrary constants.

**Case 5.** The maximal Lie invariance algebra  $\mathfrak{g}_5$  of the equation

$$u_t = \frac{u_{xx}}{u_x + 1} + \varepsilon u, \quad \varepsilon = \pm 1 \text{ mod } G_{\mathcal{R}}, \quad (38)$$

is isomorphic to the algebra  $\mathfrak{g}_{4,8}$  with  $h = 0$  from Mubarakzyanov's classification, which is denoted by  $A_{4,8}^0$  in [36] and by  $A_{4,9}^0$  in [32]. To obtain canonical commutation relations, basis elements can be chosen in the following way:

$$e_1 = e^{\varepsilon t}\partial_u, \quad e_2 = e^{\varepsilon t}(\partial_t + \varepsilon(u+x)\partial_u), \quad e_3 = \varepsilon^{-1}\partial_x, \quad e_4 = \varepsilon^{-1}\partial_t.$$

An optimal list of one-dimensional subalgebras of  $\mathfrak{g}_5$  is exhausted by the subalgebras  $\langle e_1 \rangle$ ,  $\langle e_2 \rangle$ ,  $\langle e_3 \rangle$ ,  $\langle e_2 + \varepsilon' e_3 \rangle$  and  $\langle e_4 + \kappa e_3 \rangle$ , where  $\varepsilon' = \pm 1$  and  $\kappa$  is an arbitrary constant. The equation (38) also admits the discrete point symmetry  $I_{x,u}$  of simultaneously alternating signs of  $(x, u)$ , which generates an outer automorphism of the algebra  $\mathfrak{g}_5$  with the matrix  $\text{diag}(-1, 1, -1, 1)$  in the chosen basis. Using this automorphism we can set  $\varepsilon' = 1$  in the fourth subalgebra. The first subalgebra does not satisfy the transversality condition and thus cannot be used for Lie reduction. Ansatzes constructed with the other subalgebras and the corresponding reduced equations are listed in Table 2. The general solutions of reduced equations 5.1, 5.3 and 5.4 are respectively

$$\phi = c_1 e^{\varepsilon t}, \quad \phi = c_1 \int e^{\varepsilon\omega^2/2} d\omega + c_2, \quad \phi = \ln \left| \int e^{\varepsilon\omega^2/2} d\omega + c_1 \right| + c_2,$$

where  $c$ 's are arbitrary constants, and integrals denote fixed antiderivatives. The reduced equation 5.2 with  $\kappa = 0$  is integrated once to  $\phi_\omega - \ln |\phi_\omega + 1| = -\varepsilon\phi^2/2 + c_1$ . The general solution of the last equation is represented in a parametric form with quadrature, where  $\phi_\omega$  plays the role of the parameter.

**Case 6a.** The maximal Lie invariance algebra  $\mathfrak{g}_{6a}$  of the equation  $u_t = u_x u_{xx} + u^2$  is a realization of the algebra  $\mathfrak{g}_{3,5} + \mathfrak{g}_1$  from Mubarakzyanov's classification, which is denoted by  $A_{3,8} \oplus A_1$  in [32] and is merely  $\mathfrak{sl}(2, \mathbb{R}) \oplus A_1$  [36]. We choose the basis

$$e_1 = \partial_t, \quad e_2 = t\partial_t - u\partial_u, \quad e_3 = t^2\partial_t - (2tu + 1)\partial_u, \quad e_4 = \partial_x.$$



An optimal list of one-dimensional subalgebras of  $\mathfrak{g}_{6_a}$  is exhausted by the subalgebras  $\langle e_1 \rangle$ ,  $\langle e_4 \rangle$ ,  $\langle e_2 + \varepsilon' e_4 \rangle$ ,  $\langle e_2 + \kappa e_4 \rangle$  and  $\langle e_1 + e_3 + \kappa e_4 \rangle$ , where  $\varepsilon' = \pm 1$ , and  $\kappa$  is an arbitrary nonnegative constant and an arbitrary constant in the fourth and the fifth subalgebras, respectively. The equation  $u_t = u_x u_{xx} + u^2$  also admits the discrete point symmetry  $I_{t,u}$  of simultaneously alternating signs of  $(t, u)$ , which generates the outer automorphism of the algebra  $\mathfrak{g}_{6_a}$  with the matrix  $\text{diag}(-1, 1, -1, 1)$  in the chosen basis. Using this automorphism we can set  $\varepsilon' = 1$  in the third subalgebra. Ansatzes constructed with the above subalgebras and the corresponding reduced equations are listed in Table 2.

The solutions of reduced equation 6.1 are exhausted by the general solution  $\phi = -(t + c_1)^{-1}$  and the singular solution  $\phi = 0$ .

Reduced equation 6.2 is integrated to  $-\int(\phi^3 + c_1)^{-1/3}d\phi = x + c_2$ . For the value  $c_1 = 0$ , we obtain the one-parameter singular solution family  $\phi = \tilde{c}_2 e^{-x}$ . If  $c_1 \neq 0$ , then the expression  $(\phi^3 + c_1)^{-1/3}d\phi$  is a differential binomial  $\phi^m(a + b\phi^n)^pd\phi$  with  $m = 0$ ,  $n = 3$ ,  $p = -1/3$ ,  $a = c_1$  and  $b = 1$ , i.e.,  $(m + 1)/n + p = 0$  is an integer. According the Chebyshev theorem on the integration of differential binomials, the integral of this differential binomial is reduced to the integral of a rational function by the substitution  $c_1\phi^{-3} + 1 = t^3$  and thus it can be expressed by elementary functions. Finally, we construct the general solution of reduced equation 6.2 in the implicit form

$$\frac{1}{2} \ln |(\phi^3 + c_1)^{1/3} - \phi| + \frac{1}{\sqrt{3}} \arctan \frac{2(\phi^3 + c_1)^{1/3} + \phi}{\sqrt{3}\phi} = x + c_2.$$

Reduced equations 6.3–6.5 are of the general form

$$\phi_\omega \phi_{\omega\omega} + \phi^2 + \kappa \phi \phi_\omega + \mu \phi + \nu = 0, \quad (39)$$

where  $(\kappa, \mu, \nu) = (1, 0, 0)$ ,  $(\mu, \nu) = (1, 0)$  and  $(\mu, \nu) = (0, 1)$  for reduced equation 6.3, 6.4 and 6.5. By the standard substitution lowering orders of autonomous equations,  $\phi_\omega = p(y)$ , where  $y = \phi$  plays the role of the new independent variable, the equation (39) is reduced to the first-order ordinary differential equation  $p^2 p_y + y^2 + \kappa p + \mu y + \nu = 0$ . The point transformation  $\tilde{y} = p + y$ ,  $\tilde{p} = y$  maps the last equation to an Abel equation of the second kind,

$$((2\tilde{y} + \mu - \kappa)\tilde{p} - \tilde{y}^2 + \kappa\tilde{y} + \nu)\tilde{p}_{\tilde{y}} + (\tilde{p} - \tilde{y})^2 = 0,$$

see, e.g., [33, substitution 1.4.4-1.3°]. If  $\nu = 0$ , one can also use the point transformation  $\tilde{y} = p/y$ ,  $\tilde{p} = 1/y$  resulting to the simpler Abel equation of the second kind  $((\kappa\tilde{y} + \mu)\tilde{p} + \tilde{y}^3 + 1)\tilde{p}_{\tilde{y}} = \tilde{y}^2\tilde{p}$ . The above Abel equations are further reduced by point transformations to Abel equations of the second (or first) kind in the canonical form.

**Case 7.** Up to  $G_{\mathcal{R}}^\sim$ -equivalence, this is the only case among cases of Lie-symmetry extensions within the subclass  $\mathcal{C}$  that admits non-fiber-preserving point symmetry transformations. (Such symmetry transformations are typical for equations from the subclasses  $\mathcal{F}$  and  $\mathcal{L}$ .) The indication, in infinitesimal terms, of the presence of such symmetry transformations is that the  $x$ -component of the Lie-symmetry vector field  $e^{-\varepsilon t}(\partial_t - \varepsilon u \partial_x + \varepsilon u \partial_u)$  of this case depends on  $u$ . To avoid Lie reductions with implicit ansatzes and to simplify the Lie reduction procedure in total, it is convenient to map the corresponding equation

$$u_t = \frac{u_x u_{xx}}{(u_x + 1)^3} + \varepsilon u, \quad \varepsilon = \pm 1 \text{ mod } G_{\mathcal{R}}^\sim, \quad (40)$$

by the point transformation  $\tilde{t} = \varepsilon^{-1} e^{\varepsilon t}$ ,  $\tilde{x} = x + u$ ,  $\tilde{u} = e^{-\varepsilon t} u$  to the equation

$$\tilde{u}_{\tilde{t}} = \tilde{u}_{\tilde{x}} \tilde{u}_{\tilde{x}\tilde{x}} - \varepsilon \tilde{u} \tilde{u}_{\tilde{x}}, \quad (41)$$

which we denote by ‘‘Case  $\tilde{7}$ ’’. We omit tildes of  $t$ ,  $x$  and  $u$  below. The maximal Lie invariance algebra  $\mathfrak{g}_{\tilde{7}}$  of the equation (41) is spanned by the vector fields  $e_1 = \partial_x$ ,  $e_2 = \partial_t$ ,  $e_3 = t\partial_x + \varepsilon^{-1}\partial_u$  and  $e_4 = t\partial_t - u\partial_u$ . It is a realization of the algebra  $\mathfrak{g}_{4,8}$  with  $h = -1$  from Mubarakzhanov’s classification, which is denoted by  $A_{4,8}$  in [32] and by  $A_{4,8}^{-1}$  in [36]. An optimal list of one-dimensional subalgebras of  $\mathfrak{g}_{\tilde{7}}$  is exhausted by the subalgebras  $\langle e_1 \rangle$ ,  $\langle e_3 \rangle$ ,  $\langle e_2 \rangle$ ,  $\langle e_2 + \varepsilon' e_3 \rangle$  and  $\langle e_4 + \kappa e_1 \rangle$ , where  $\varepsilon' = \pm 1$  and  $\kappa$  is an arbitrary constant. The equation (41) admits the discrete point symmetries  $I_{t,u}$  and  $I_{x,u}$  of simultaneously alternating signs of  $(t, u)$  and of  $(x, u)$ , respectively. The involution  $I_{x,u}$  generates the outer automorphism of the algebra  $\mathfrak{g}_{\tilde{7}}$  with the matrix  $\text{diag}(-1, 1, -1, 1)$  in the fixed basis. Using this automorphism we can set  $\varepsilon' = 1$  in the fourth subalgebra. Ansatzes constructed with the above subalgebras and the corresponding reduced equations are marked in Table 2 by  $\tilde{7}$ , where tildes of  $t$ ,  $x$  and  $u$  are still omitted.

Reduced equations  $\tilde{7}.1$  and  $\tilde{7}.2$  are trivially integrated to  $\phi = c_0$  and  $\phi = c_0/t$ , respectively.

The solution set of reduced equation  $\tilde{7}.3$  splits into two families. The first family  $\phi = c_0$  is common for both the values of  $\varepsilon$ . The second family depends on  $\varepsilon$ ,  $\phi = c_1 e^\omega + c_2 e^{-\omega}$  if  $\varepsilon = 1$  and  $\phi = c_1 \cos \omega + c_2 \sin \omega$  if  $\varepsilon = -1$ .

Reduced equation  $\tilde{7}.4$  has the first integral  $\phi_\omega^2 - \varepsilon \phi^2 - 2\varepsilon^{-1} \omega = c_1$ . This equation is mapped by the substitution  $1/\phi_\omega + \phi = p(y)$ , where  $y = -1/\phi_\omega$  plays the role of the new independent variable, to the Abel equation of the second kind in the canonical form  $pp_y - p = \varepsilon/y^3$ .

Similarly, reduced equation  $\tilde{7}.5$  is mapped by the substitution  $p(y) = \phi$ , where  $y = \phi_\omega$  plays the role of the new independent variable, to the equation  $((1 - \varepsilon y)p + \varepsilon^{-1} \kappa)p_y = -y^2$ , which is an Abel equation of the second kind if  $\kappa \neq 0$  and an equation with separable variables if  $\kappa = 0$ . In the second case, we obtain the equation  $\phi^2 = \varepsilon^{-1} \phi_\omega^2 + 2\varepsilon^{-2} \phi_\omega + 2\varepsilon^{-3} \ln |\phi_\omega - \varepsilon^{-1}| + c_1$  whose general solution is represented in a parametric form with quadrature, where  $\phi_\omega$  plays the role of the parameter. If  $\kappa \neq 0$ , the corresponding Abel equation is reduced to the canonical form  $\bar{p}\bar{p}_{\bar{y}} - \bar{p} = \varepsilon^{-3}(\bar{y} + \varepsilon^{-1} \kappa)^2/\bar{y}^3$  by the point transformation  $\bar{y} = \varepsilon^{-1} \kappa/(1 - \varepsilon y)$ ,  $\bar{p} = p + \varepsilon^{-1} \kappa/(1 - \varepsilon y)$ .

The equation (41) is of the form admitting the generalized separation of variables [14]. The nonlinear differential operator  $\tilde{u}_{\tilde{x}} \tilde{u}_{\tilde{x}\tilde{x}} - \varepsilon \tilde{u} \tilde{u}_{\tilde{x}}$  from the right hand side of this equation preserves the exponential linear space  $\langle 1, e^{\tilde{x}}, e^{-\tilde{x}} \rangle$  or trigonometric linear spaces  $\langle 1, \cos \tilde{x}, \sin \tilde{x} \rangle$  if  $\varepsilon = 1$  or  $\varepsilon = -1$ , respectively, cf. [13]. Using the ansatz  $\tilde{u} = \tau^0(\tilde{t}) + \tau^1(\tilde{t})e^{\tilde{x}} + \tau^2(\tilde{t})e^{-\tilde{x}}$  for  $\varepsilon = 1$  or  $\tilde{u} = \tau^0(\tilde{t}) + \tau^1(\tilde{t})\cos \tilde{x} + \tau^2(\tilde{t})\sin \tilde{x}$  for  $\varepsilon = -1$ , where  $\tau$ ’s are the new unknown functions, we construct three-parameter families of exact solutions of the equation (41),

$$\tilde{u} = c_0 + c_1 e^{\tilde{x} + c_0 \tilde{t}} + c_2 e^{-\tilde{x} - c_0 \tilde{t}} \quad \text{if } \varepsilon = 1 \quad \text{and} \quad \tilde{u} = c_0 + c_1 \cos(\tilde{x} + c_0 \tilde{t} + c_2) \quad \text{if } \varepsilon = -1.$$

At the same time, these solutions are Lie invariant solutions, which can be constructed using reduction  $\tilde{7}.3$  and the action of Galilean boosts generated by the Lie-symmetry vector field  $e_3$ . No further extension of these families of exact solutions of the equation (41) with its Lie symmetries is possible.

Substituting  $\tilde{t} = \varepsilon^{-1} e^{\varepsilon t}$ ,  $\tilde{x} = x + u$ ,  $\tilde{u} = e^{-\varepsilon t} u$  into the above solutions of the equation (41), we construct implicit solutions of the equation (40).

Note that the further hodograph transformation  $\hat{t} = \tilde{t}$ ,  $\hat{x} = \tilde{u}$ ,  $\hat{u} = \tilde{x}$  reduces the equation (41) to the equation  $\hat{u}_{\hat{t}} = \hat{u}_{\hat{x}}^{-3} \hat{u}_{\hat{x}\hat{x}} + \varepsilon \hat{x}$ , which is the potential equation for the nonlinear diffusion–reaction equations  $\check{u}_{\hat{t}} = (\check{u}^{-3} \check{u}_{\hat{x}})_{\hat{x}} + \varepsilon$ , where  $\check{u} = \hat{u}_{\hat{x}}$ . See the first paragraph of this section for references on symmetry analysis of the last equation.

## 7 Conclusion

Carrying out group classification of the class  $\mathcal{R}$  of (1+1)-dimensional nonlinear diffusion–reaction equations with gradient-dependent diffusivity in the present paper, we have corrected and enhanced results of [8] in several aspects.

We have justified the necessity of studying the entire class  $\mathcal{R}$  and carefully analyzed its structure. In the notation of the present paper, the authors of [8] considered the group classification problem only for the subclass  $\mathcal{C}$  with respect to its equivalence group, having discarded other subclasses of  $\mathcal{R}$ . They include the subclass  $\mathcal{H}$  with known group classification, the subclass  $\mathcal{F}$  whose group classification can be easily derived from known group classification of the subclass  $\mathcal{F}'$  and the subclass  $\mathcal{L}$  of linearizable equations with transformational properties essentially different from other equations in the class  $\mathcal{R}$ . Nevertheless, the consideration of the subclass  $\mathcal{C}$  alone is not natural since some equations in  $\mathcal{C}$  are related by point transformations to equations in  $\mathcal{F}$ , and the subclass  $\mathcal{F}$  intersects the subclasses  $\mathcal{H}$  and  $\mathcal{L}$ .

We have computed the usual equivalence groups of the class  $\mathcal{R}$  and all the above subclasses as well as the generalized equivalence group  $\bar{G}_{\mathcal{F}}^{\sim}$  and the effective generalized equivalence group  $\hat{G}_{\mathcal{F}}^{\sim}$  of the subclass  $\mathcal{F}$ . We have also checked the consistency of these groups for using in the course of group classification. The group  $\hat{G}_{\mathcal{F}}^{\sim}$  gives the first nontrivial example of *finite-dimensional effective generalized equivalence group* in the literature. Moreover, the class  $\mathcal{F}$  has another unexpected property formulated in Theorem 7: any effective generalized equivalence group of the class  $\mathcal{F}$  does not contain the usual equivalence group of this class. This phenomenon had not been observed before, and finding out it is the most interesting result of the present paper although it was obtained as a by-product. Since  $\hat{G}_{\mathcal{F}}^{\sim}$  is not a normal subgroup of  $\bar{G}_{\mathcal{F}}^{\sim}$ , it is obvious that  $\hat{G}_{\mathcal{F}}^{\sim}$  is not a unique effective generalized equivalence group of the subclass  $\mathcal{F}$ . Whether this group is unique up to the subgroup similarity within  $\bar{G}_{\mathcal{F}}^{\sim}$  is still an open problem.

In the context of the group classification of the subclass  $\mathcal{C}$  given in [8], the most significant and also unexpected enhancement is discovering Case 7, which was missed in [8]. It essentially differs from the other classification cases in the subclass  $\mathcal{C}$  since only equations fitting in Case 7 up to  $G_{\mathcal{R}}^{\sim}$ -equivalence admits non-fiber-preserving Lie-symmetry transformations. An explanation of the presence of such transformations is that these equations are reduced by non-fiber-preserving point transformations to a potential nonlinear diffusion–reaction equation.

The third direction of enhancements concerns additional equivalences between classification cases. We have constructed additional equivalence transformations relating Cases 6a–6c, which were missed in [8]. It is obvious that there also exist similar transformations between multidimensional counterparts of Cases 6a–6c, cf. Cases 6–8 of [8, Table 2]. Using relevant equivalence relations, we have properly gauged all constants parameterizing classification cases and first proved the completeness of the additional equivalence transformations presented in the footnote of Table 1, which leads to the group classification of the class  $\mathcal{R}$  up to the general point equivalence, see Theorem 13.

The complex structure of the class  $\mathcal{R}$  required the application of various advanced techniques of group analysis of differential equations. Moreover, we needed to combine and develop them for an efficient solution of the group classification problem for the class  $\mathcal{R}$ . At the same time, the simple form of equations in this class has allowed us to present the obtained results in a clear way.

Results of Section 2 on various equivalence groups of the class  $\mathcal{R}$  and of its subclasses  $\mathcal{H}$ ,  $\mathcal{L}$ ,  $\mathcal{F}$ ,  $\mathcal{F}'$  and  $\mathcal{C}$  and the classification of Lie symmetries of equations from these classes that is presented in Section 3 jointly imply that these classes are not normalized in both the usual and the generalized sense. (See [28, 35, 39] for related definitions.) The existence of additional equivalence transformations among cases of Lie-symmetries extension in the subclasses  $\mathcal{H}$ ,  $\mathcal{L}$  and  $\mathcal{C}$  means that these subclasses as well as the entire class  $\mathcal{R}$  are even not semi-normalized. This is why in the course of computing the above equivalence groups, we have suggested and applied an optimized version of the direct method. This version involves preliminary study of admissible transformations within the entire class and the successive splitting of the determining equations for these transformations with respect to the corresponding arbitrary elements and their derivatives, depending on auxiliary constraints associated with each of the subclasses under consideration separately. In order to make the group classifications of the subclasses  $\mathcal{F}$  and  $\mathcal{F}'$

consistent, we have found the generalized equivalence group of  $\mathcal{F}$  and its effective counterpart. The group classification of the class  $\mathcal{L}$  has been obtained from the known group classification of Kolmogorov equations up to general point equivalence [40, Corollary 7] using a technique based on mappings between classes of differential equations via point transformations [47].

The arbitrary elements  $f$  and  $g$  depend on different arguments,  $u_x$  and  $u$ , respectively. This is why in the course of solving the group classification problem for the subclass  $\mathcal{C}$ , we needed to use the double furcate splitting with respect to two pairs “(an argument, an arbitrary element)”,  $(u_x, f)$  and  $(u, g)$ . Moreover, it is impossible to directly apply furcate splitting to the classifying system (20) within the subclass  $\mathcal{C}$ , not to mention the entire class  $\mathcal{R}$ . The subsystem of non-classifying determining equations for Lie symmetries of equations from the class  $\mathcal{R}$  is exhausted by two equations, which are common for all (1+1)-dimensional evolution equations and constrain only the  $t$ -components of Lie-symmetry vector fields,  $\tau_x = \tau_u = 0$ . The dependence of the corresponding  $x$ - and  $u$ -components on  $u$  is not specified and hence the furcate splitting with respect to  $(u, g)$  cannot be initiated without additional constraints on the arbitrary elements and without a preliminary preparation of the classifying equations. One more complication is that the classifying system (20) is (partially) coupled in  $(f, g)$ . Under the auxiliary inequalities  $f_{u_x} \neq 0$ ,  $(u_x^2 f)_{u_x} \neq 0$  and  $g_u \neq 0$  singling out the subclass  $\mathcal{C}$ , the system (20) implies more equations, (21), not involving the arbitrary elements  $(f, g)$  and thus reduces to an uncoupled system for  $(f, g)$ ; see Lemma 15 and the discussion after it. An inconvenience of the reduced system is that it includes two equations for  $g$ , (28) and (29). Then we have found out a property of Lie symmetry algebras that singles out two classification cases being singular in the subclass  $\mathcal{C}$ , Cases 5 and 7. In the course of computing these cases, we use the auxiliary furcate splitting with respect  $(u_x, f)$ . For other classification cases, there are very restrictive determining equations,  $\xi_t = \xi_u = \eta_x = 0$ , which simplify the system of classifying equations to the uncoupled systems of only two equations (33) and (34). The last system is perfectly appropriate for the double furcate splitting with respect to  $(u_x, f)$  and  $(u, g)$ .

For effectively constructing additional equivalence transformations or for proving nonexistence of such transformations, we have combined the direct method of computing point transformations between similar equations with the algebraic method based on comparing the structure of the corresponding Lie invariance algebras.

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## References

- [1] Akhatov I.S., Gazizov R.K. and Ibragimov N.Kh., Group classification of equations of nonlinear filtration, *Dokl. Akad. Nauk SSSR* **293** (1987), 1033–1035, (in Russian); translated in *Soviet Math. Dokl.* **35** (1987), 1033–1035.
- [2] Akhatov I.S., Gazizov R.K. and Ibragimov N.Kh., Nonlocal symmetries. A heuristic approach, in *Current Problems in Mathematics. Newest Results, Vol. 34*, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1989, pp. 3–83. (in Russian); translated in *J. Soviet Math.* **55** (1991), 1401–1450.
- [3] Ames W.F., Anderson R.L., Dorodnitsyn V.A., Ferapontov E.V., Gazizov R.K., Ibragimov N.H. and Svirshchevskii S.R., *CRC Handbook of Lie Group Analysis of Differential Equations. Vol. 1. Symmetries, Exact Solutions and Conservation Laws*, CRC Press, Boca Raton, FL, 1994.

- [4] Arrigo D.J., Hill J.M. and Broadbridge P., Nonclassical symmetry reductions of the linear diffusion equation with a nonlinear source, *IMA J. Appl. Math.* **52** (1994), 1–24.
- [5] Bluman G.W. and Yan Z., Nonclassical potential solutions of partial differential equations, *European J. Appl. Math.* **16** (2005), 239–261.
- [6] Boyko V., Patera J. and Popovych R., Computation of invariants of Lie algebras by means of moving frames, *J. Phys. A* **39** (2006), 5749–5762, [arXiv:math-ph/0602046](#).
- [7] Broadbridge P. and Vassiliou P., The role of symmetry and separation in surface evolution and curve shortening, *SIGMA* **7** (2011), 052, 19 pp., [arXiv:1106.0092](#).
- [8] Cherniha R., King J.R. and Kovalenko S., Lie symmetry properties of nonlinear reaction-diffusion equations with gradient-dependent diffusivity, *Commun. Nonlinear Sci. Numer. Simul.* **36** (2016), 98–108, [arXiv:1507.01893](#).
- [9] Clarkson P.A. and Mansfield E.L., Symmetry reductions and exact solutions of a class of nonlinear heat equations, *Phys. D* **70** (1994), 250–288.
- [10] Dorodnitsyn V.A., On invariant solutions of the equation of non-linear heat conduction with a source, *USSR Comput. Math. Math. Phys.* **22** (1982), 115–122.
- [11] Doyle P.W. and Vassiliou P.J., Separation of variables for the 1-dimensional non-linear diffusion equation, *Internat. J. Non-Linear Mech.* **33** (1998), 315–326.
- [12] Fushchich V.I. and Serov N.I., Conditional invariance and reduction of a nonlinear heat equation, *Dokl. Akad. Nauk Ukrain. SSR Ser. A* (1990), no. 7, 24–27, (in Russian).
- [13] Galaktionov V. and Svirshchevskii S., *Exact Solutions and Invariant Subspaces of Nonlinear Partial Differential Equations in Mechanics and Physics*, Chapman & Hall/CRC Applied Mathematics and Nonlinear Science Series. Chapman & Hall/CRC, Boca Raton, FL, 2007.
- [14] Galaktionov V.A., Posashkov S.A. and Svirshchevskii S.R., Generalized separation of variables for differential equations with polynomial nonlinearities, *Differentsial'nye Uravneniya* **31** (1995), 253–261, (in Russian); translated in *Differential Equations* **31** (1995), 233–240.
- [15] Gandarias M.L., New symmetries for a model of fast diffusion, *Phys. Lett. A* **286** (2001), 153–160.
- [16] Grindrod P., *The Theory and Applications of Reaction-Diffusion Equations. Patterns and Waves*, Oxford Applied Mathematics and Computing Science Series. The Clarendon Press, Oxford University Press, New York, 2nd ed., 1996.
- [17] Ivanova N.M., Popovych R.O. and Sophocleous C., Group analysis of variable coefficient diffusion-convection equations. I. Enhanced group classification, *Lobachevskii J. Math.* **31** (2010), 100–122, [arXiv:0710.2731](#).
- [18] Kingston J.G., On point transformations of evolution equations, *J. Phys. A* **24** (1991), L769–L774.
- [19] Magadeev B.A., Group classification of nonlinear evolution equations, *Algebra i Analiz* **5** (1993), 141–156, (in Russian); English translation in *St. Petersburg Math. J.* **5** (1994), 345–359.
- [20] Meleshko S.V., Group classification of equations of two-dimensional gas motions, *Prikl. Mat. Mekh.* **58** (1994), 56–62, (in Russian); translation in *J. Appl. Math. Mech.*, **58** (1994), 629–635.
- [21] Meleshko S.V., Generalization of the equivalence transformations, *J. Nonlinear Math. Phys.* **3** (1996), 170–174.
- [22] Mubarakzhanov G., On solvable Lie algebras, *Izv. Vys. Ucheb. Zaved. Matematika* (1963), no. 1(32), 114–123, (in Russian).
- [23] Mubarakzhanov G., The classification of the real structure of five-dimensional Lie algebras, *Izv. Vys. Ucheb. Zaved. Matematika* (1963), no. 3(34), 99–106, (in Russian).
- [24] Murray J.D., *Mathematical Biology. I. An Introduction*, vol. 17 of *Interdisciplinary Applied Mathematics*, Springer-Verlag, New York, 3rd ed., 2002.
- [25] Murray J.D., *Mathematical Biology. II. Spatial Models and Biomedical Applications*, vol. 17 of *Interdisciplinary Applied Mathematics*, Springer-Verlag, New York, 3rd ed., 2003.
- [26] Nikitin A.G. and Popovych R.O., Group classification of nonlinear Schrödinger equations, *Ukr. Mat. Zh.* **53** (2001) 1053–1060 (in Ukrainian); translated in *Ukrainian Math. J.* **53** (2001), 1255–1265, [arXiv:math-ph/0301009](#).
- [27] Olver P.J., *Applications of Lie Groups to Differential Equations*, vol. 107 of *Graduate Texts in Mathematics*, Springer-Verlag, New York, 2nd ed., 1993.
- [28] Opanasenko S., Bihlo A. and Popovych R.O., Group analysis of general Burgers–Korteweg–de Vries equations, *J. Math. Phys.* **58** (2017), 081511, 37 pp., [arXiv:1703.06932](#).
- [29] Opanasenko S. and Popovych R.O., Admissible transformations and Lie symmetries of Fokker–Planck equations (2019), in preparation.

- [30] Ovsiannikov L.V., Group relations of the equation of non-linear heat conductivity, *Dokl. Akad. Nauk SSSR* **125** (1959), 492–495.
- [31] Ovsiannikov L.V., *Group Analysis of Differential Equations*, Academic Press, Inc., New York – London, 1982.
- [32] Patera J. and Winternitz P., Subalgebras of real three- and four-dimensional Lie algebras, *J. Math. Phys.* **18** (1977), 1449–1455.
- [33] Polyanin A.D. and Zaitsev V.F., *Handbook of Exact Solutions for Ordinary Differential Equations*, Chapman & Hall/CRC, Boca Raton, FL, 2003.
- [34] Polyanin A.D. and Zaitsev V.F., *Handbook of Nonlinear Partial Differential Equations*, Chapman and Hall/CRC Press, Boca Raton – London, 2012.
- [35] Popovych R.O., Classification of admissible transformations of differential equations, in *Collection of Works of Institute of Mathematics*, vol. 3, no. 2, Institute of Mathematics, Kyiv, 2006, pp. 239–254.
- [36] Popovych R.O., Boyko V., Nesterenko M. and Lutfullin M., Realizations of real low-dimensional Lie algebras, 2005, [arXiv:math-ph/0301029](https://arxiv.org/abs/math-ph/0301029)v7, extended version of *J. Phys. A* **36** (2003), 7337–7360.
- [37] Popovych R.O. and Ivanova N., New results on group classification of nonlinear diffusion–convection equations, *J. Phys. A* **37** (2004), 7547–7565, [arXiv:math-ph/0306035](https://arxiv.org/abs/math-ph/0306035).
- [38] Popovych R.O. and Ivanova N., Potential equivalence transformations for nonlinear diffusion–convection equations, *J. Phys. A* **38** (2005), 3145–3155, [arXiv:math-ph/0402066](https://arxiv.org/abs/math-ph/0402066).
- [39] Popovych R.O., Kunzinger M. and Eshraghi H., Admissible transformations and normalized classes of nonlinear Schrödinger equations, *Acta Appl. Math.* **109** (2010), 315–359, [arXiv:math-ph/0611061](https://arxiv.org/abs/math-ph/0611061).
- [40] Popovych R.O., Kunzinger M. and Ivanova N.M., Conservation laws and potential symmetries of linear parabolic equations, *Acta Appl. Math.* **100** (2008), 113–185, [arXiv:0706.0443](https://arxiv.org/abs/0706.0443).
- [41] Popovych R.O. and Samoilenko A.M., Local conservation laws of second-order evolution equations, *J. Phys. A* **41** (2008), 362002, 11 pp., [arXiv:0806.2765](https://arxiv.org/abs/0806.2765).
- [42] Popovych R.O., Vaneeva O.O. and Ivanova N.M., Potential nonclassical symmetries and solutions of fast diffusion equation, *Phys. Lett. A* **362** (2007), 166–173, [arXiv:math-ph/0506067](https://arxiv.org/abs/math-ph/0506067).
- [43] Qu C., Exact solutions to nonlinear diffusion equations obtained by a generalized conditional symmetry method, *IMA J. Appl. Math.* **62** (1999), 283–302.
- [44] Serov N.I., Conditional invariance and exact solutions of a nonlinear heat equation, *Ukrain. Mat. Zh.* **42** (1990), 1370–1376, (in Russian); translated in *Ukrainian Math. J.* **42** (1990), 1216–1222.
- [45] Smoller J., *Shock Waves and Reaction-Diffusion Equations*, vol. 258 of *Grundlehren der Mathematischen Wissenschaften*, Springer-Verlag, New York, 2nd ed., 1994.
- [46] Vaneeva O.O., Kuriksha O. and Sophocleous C., Enhanced group classification of Gardner equations with time-dependent coefficients, *Commun. Nonlinear Sci. Numer. Simul.* **22** (2015), 1243–1251, [arXiv:1407.8488](https://arxiv.org/abs/1407.8488).
- [47] Vaneeva O.O., Popovych R.O. and Sophocleous C., Enhanced group analysis and exact solutions of variable coefficient semilinear diffusion equations with a power source, *Acta Appl. Math.* **106** (2009), 1–46, [arXiv:0708.3457](https://arxiv.org/abs/0708.3457).
- [48] Vaneeva O.O., Popovych R.O. and Sophocleous C., Extended group analysis of variable coefficient reaction-diffusion equations with exponential nonlinearities, *J. Math. Anal. Appl.* **396** (2012), 225–242, [arXiv:1111.5198](https://arxiv.org/abs/1111.5198).
- [49] Vaneeva O.O., Sophocleous C. and Leach P., Lie symmetries of generalized Burgers equations: application to boundary-value problems, *J. Engrg. Math.* **91** (2015), 165–176, [arXiv:1303.3548](https://arxiv.org/abs/1303.3548).